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A STRUCTURE THEOREM OF FORMALLY SMOOTH MORPHISMS IN POSITIVE CHARACTERISTIC

Vasile NICA and Dorin POPESCU

§ 1. INTRODUCTION

For proving results about local formally smooth morphisms $u:A\longrightarrow B$, the case $\dim A=\dim B$, it is sometimes easier. If $\dim A<\dim B$ the question arises to find a noetherian local A-algebra $\tilde{A}$ and a local A-morphism $\tilde{u}:\tilde{A}\longrightarrow B$ such that

1) $\dim \tilde{A} = \dim B$,

2) $\tilde{u}$ formally smooth

3) $\tilde{A}$ has a "good" structure, for example to be a localization of a polynomial A-algebra.

The purpose of our paper is to prove the following structure theorem:

(1.1) Theorem. Let $u:A\longrightarrow B$ be a local, formally smooth morphism of noetherian local rings and $k\subset K$ the residue field extension induced by u. Suppose that $p:=\operatorname{char} k>0$ and that one of the following conditions holds:

i) K has a separate p-basis over k (i.e. a p-basis $x=(x_i)_{i\in I}$ such that K is separable over $k(x)$, see (2.4)), or

ii) B is a complete ring.

Then there exist a noetherian local A-algebra $\tilde{A}$ and a local A-morphism $\tilde{u}:\tilde{A}\longrightarrow B$ such that

1) $\tilde{u}$ is formally smooth,

2) $\dim \tilde{A} =\dim B$,

3) $\tilde{A}$ is a filtered inductive limit of localizations of some polynomial A-

algebras (each of them in a finite number of indeterminates). Moreover in the case i), we can choose $\tilde{A}$ to be simply a localization of a polynomial A-algebra.

If the residue field extension $k\subset K$ is separable, the construction of $\tilde{A}$ and $\tilde{u}$ is well known and not difficult (see Lemma (3.7)). For example, if A=k is a field and B is a local, formally smooth, one dimensional k-algebra, i.e. a discrete valuation ring then if $\sigma : k[T] \longrightarrow B$ is given by $T \mapsto t$, t being a local parameter of B, we can take $\tilde{A} := k[T]_{\sigma^{-1}(tB)}$.

In the nonseparable case (but finite type extension case), the proof is difficult enough even when B is a discrete valuation ring (see [P_1]). If K/k is not separable, $K=k(K^p)$ and B is complete, the result was already obtained in [P_3] and, in fact, the proof of Theorem (1.1) uses the same methods.

In Section 2 we give some preliminaries about field extensions of positive characteristic, some of them new (as Theorem (2.13)), others already known (Theorem (6.3) from [P_3] plays an important role in our construction and for the sake of completeness we have included its proof in Theorem (2.20)). Next section contains the proof of Theorem (1.1). Here we get a different proof of Theorem (22.2.6) from [EGA] (see Corollary (3.6) which was suggested to us by Professor N.Radu to whom we owe thanks). As an application of Theorem (1.1) we obtain in the last section another proof of Theorem (7.3) from [P_3]. Throughout this paper, all rings are supposed to be commutative with identity, all fields are of positive characteristic and a local morphism $u:A \longrightarrow B$ is called unramified if the maximal ideal of A generates the maximal ideal of B.

§ 2. THE STRUCTURE OF FIELD EXTENSIONS OF POSITIVE CHARACTERISTIC

(2.1) For the beginning, we list some facts from the homology theory of commutative algebras, which can be found in [A] (see also [M] for the field case). Denote by $H_n(A,B,-)$ the n-th homology functor of an A-algebra B. It is defined on the category of B-modules and has its values in the same category.

(2.1.1) $H_n(A,B,-)=0$, $n\geq 1$, for a polynomial A-algebra

(2.1.2) $H_o(A,B,-)=\Omega_{B/A}\otimes_B -$, where $\Omega_{B/A}$ denotes the B-module of (absolute) differentials of the A-algebra B.

(2.1.3) If $B=A/_I$, where $I\subset A$ is an ideal, then $H_1(A,B,-)=I/_{I^2}\otimes_B -$

(2.1.4) All functors $H_n(A,B,-)$ commute - in the second variable B - with filtered inductive limits, are stable under localisations and, if B is flat over A, are also stable under base change.

(2.1.5) If $A\longrightarrow B\longrightarrow C$ are ring morphisms, and $W\in$Mod C, there exists a natural long exact sequence of C-modules, called the Jacobi-Zariski sequence:

$$\ldots\longrightarrow H_1(A,B,W)\longrightarrow H_1(A,C,W)\longrightarrow H_1(B,C,W)\longrightarrow\Omega_{B/A}\otimes_B W\longrightarrow$$
$$\longrightarrow\Omega_{C/A}\otimes_C W\rightarrow\Omega_{C/B}\otimes_C W\rightarrow 0$$

which prolongates the classical exact sequence of the modules of differentials.

(2.1.6) In low dimensions, vanishing of these functors characterises important properties of ring morphisms. For example:

i) If B is a finitely presented A-algebra, then B is smooth over A, iff $H_1(A,B,-)=0$ (the jacobian criterion of smoothness).

ii) If $u:A\longrightarrow B$ is a local flat morphism of noetherian local rings, then B is formally smooth over A iff $H_1(A,B,K)=0$, where K denotes the residue field of B (the jacobian criterion of formal smoothness).

iii) If $u:A\longrightarrow B$ is a flat morphism of noetherian rings, then u is regular, (i.e. all fibers of u are formally smooth) iff $H_1(A,B,-)=0$ (Andre's theorem [A]).

(2.1.7) If $k\subset K$ are fields, then $H_n(k,K,-)=0$ for $n\geq 2$ and $H_1(k,K,-)= =H_1(k,K,K)\otimes_K -$. Writing $\Gamma_{K/k}$ instead $H_1(k,K,K)$ as in [M] (39.A), the Jacobi Zariski sequence (2.1.5) associated to $k\subset L\subset K$ and $W=K$ becomes:

$$0\to \Gamma_{L/k}\otimes_L K\to \Gamma_{K/k}\to \Gamma_{K/L}\to \Omega_{L/k}\otimes_L K\xrightarrow{\alpha_{K/L/k}} \Omega_{K/k}\to \Omega_{K/L}\to 0$$

(2.1.8) An extension of fields $k\subset K$ is separable iff $\Gamma_{K/k}=0$

(2.2) Let $k\subset K$ be an arbitrarily given extension of characteristic $p>0$. Recall that a family $x=(x_i)_{i\in I}$ of elements from K is p-free over k (resp. is a p-basis of K over k) iff the family of differentials $d_{K/k}(x)=(d_{K/k}(x_i))_{i\in I}$ is linearly independent in $\Omega_{K/k}$ over K (resp. is a basis of $\Omega_{K/k}$ over K) cf. [M] Th.86.

(2.3) Proposition. In the above notations, let $x=(x_i)_{i\in I}$ be a family of elements from K, p-free over k, and $L=k(x)$. The following statements hold:

i) x is a p-basis of L over k,

ii) the sequence $0\to \Gamma_{L/k}\otimes_L K\to \Gamma_{K/k}\to \Gamma_{K/L}\to 0$ is exact,

iii) if $k\subset K$ is separable, then $L\subset K$ is separable,

iv) if x is a p-basis of K over k, then K is an unramified extension of L, i.e. $\Omega_{K/L}=0$,

v) if K is a finite type extension of k and x is a p-basis of K over k then $L\subset K$ is a finite separable extension.

Proof i) The family of differentials $d_{L/k}(x)$ generates $\Omega_{L/k}$ over L and $\alpha_{K/L/k}$ from (2.1.7) maps it on $d_{K/k}(x)$ which is linearly independent in $\Omega_{K/k}$ over K. Thus $d_{L/k}(x)$ is linearly independent over L.

ii) follow from (2.1.7), $\alpha_{K/L/k}$ being injective, cf. (2.2).

iii) is a consequence of ii) and (2.1.8).

iv) if x is a p-basis of K over k, then $\alpha_{K/L/k}$ is an isomorphism by (2.2) and then $\Omega_{K/L}=0$.

v) K being a finite type extension of L, $\Omega_{K/L}=0$ means exactly that K is finite and separable over L, cf. [M] Th.59.

(2.3.1) Remark . In zero characteristic case, $\Omega_{K/L}=0$ iff $L\subset K$ is algebra

If the caracteristic is $p>0$, then $\Omega_{K/L}=0$ means only $K=L(K^p)$ and if K is not finitely generated over L, this not involves necessarily the separability of K over L. For example, take L a nonperfect field and $K=L^{p^{-\infty}}$.

(2.4) Definition. In the notations of (2.3), we say that x is a separate p-basis of K over k if x is a p-basis of K over k and K is separable over $L=k(x)$. K is a separate extension of k if K has a separate p-basis over k (we shall see in Theorem (2.13) that if there exists one then any is separate).

(2.4.1) Remark. There exist field extensions without separate p-bases. Let k_o be a perfect field, T a variable, $k=k_o(T)$ and $K:=k^{p^{-\infty}}=k_o(T,T^{p^{-1}},\ldots$ $\ldots,T^{p^{-n}},\ldots)$. Then $\Omega_{K/k}=0$ and thus the empty set is the unique p-basis of K over k, but K is not separable over k. Note that rank $\Gamma_{K/k}=1$.

In the following, we shall consider only field extensions $k\subset K$ of positive characteristic p, with rank $\Gamma_{K/k}<\infty$. Our particular interest in the study of such extensions is motivated by the rest.

(2.5) Proposition [EGA] (Th.22.2.2). Let $u:A\longrightarrow B$ be a local, formally smooth morphism of noetherian local rings and $k\subset K$ the residue field extension induced by u. Then

$$\text{rank}\,\Gamma_{K/k}\leq \dim B-\dim A$$

(2.6) Lemma. Let $L\subset K$, $L\subset E$ be two field extensions. Suppose that $r:=\text{rank}\,\Gamma_{K/L}<\infty$. Then

$$\text{rank}\,\Gamma_{E(K)/E}\leq r.$$

(2.7) Lemma. Let $L\subset K$ be an unramified extension such that $r:=\text{rank}\,\Gamma_{K/L}<\infty$ and $E\subset L^{p^{-\infty}}$ a field extension of L. Then

$$\text{rank}\,\Gamma_{E(K)/E}=r$$

Proof. By the above Lemma we have

$$\operatorname{rank}\Gamma_{E(K)/E} \leq r.$$

First suppose that $L\subset E$ is finite. Then there exists a positive integer s such that $E\subset L_s:=L^{p^{-s}}$. Using again the above Lemma we get

$$\operatorname{rank}\Gamma_{L_s(K)/L_s} \leq \operatorname{rank}\Gamma_{E(K)/E} \leq r.$$

Since the extensions $L_s\subset L_s(K)$, $L\subset L(K^{p^s})$ are isomorphic and $K=L(K^p)$ by hypothesis, it follows

$$\operatorname{rank}\Gamma_{L_s(K)/L_s} = r,$$

which is enough.

If $L\subset E$ is not finite then express E as a filtered inductive union of finite field extensions over L and apply (2.1.4). □

(2.8) Lemma. Let $k\subset K$ be a field extension with $r:=\operatorname{rank}\Gamma_{K/k}<\infty$ and $(L_i)_{i\in I}$ a family of subfields of L, all containing k, filtered inductively by inclusion and such that $\bigcup_{i\in I} L_i=L$. Then there exists $i_o\in I$ such that $\operatorname{rank}\Gamma_{L_i/k}=r$ for $i\geq i_o$.

Proof. By (2.1.7) $\Gamma_{L_i/k}\otimes_{L_i} K\longrightarrow\Gamma_{K/k}$ are injective morphisms and by (2.1.4) $\Gamma_{K/k} = \varinjlim \Gamma_{L_i/k}\otimes_{L_i} K$. □

(2.9) Lemma. Let $k\subset L$ be a field extension generated by one of its p-bases x. Then there exist a finite subset $x'\subset x$ ~~such that~~ $x\setminus x'$ form an algebraically independent system over $k(x')$.

Proof. For every finite subset $J\subset I$ we put $L_J:=k(x_i)_{i\in J}$. Clearly, $L=\bigcup_J L_J$ and

the union is filtered inductively by inclusion. Lemma (2.8) assures the existence of a finite subset $J_o \subset I$ such that for $F=L_{J_o}$, rank $\Gamma_{F/k}$=rank $\Gamma_{L/k}$.
In the Jacobi-Zariski sequence:

$$0 \longrightarrow \Gamma_{F/k} \otimes_F L \longrightarrow \Gamma_{L/k} \longrightarrow \Gamma_{L/F} \longrightarrow \Omega_{F/k} \otimes_F L \xrightarrow{\alpha_{L/F/k}} \Omega_{L/k}$$

written for $k \subset F \subset L$, the first map is bijective and the last one is injective. It results $\Gamma_{L/F} = 0$, i.e. L is separable over F. Take $x' = (x_i)_{i \in J_o}$. Since $x'' := x \setminus x'$ is a p-basis of L over $k(x')$ it follows that x'' is algebraically independent over $k(x')$ (see [M] Th.89).

(2.10) <u>Proposition.</u> Let $k \subset K$ be a field extension with rank $\Gamma_{K/k} < \infty$. The following statements are equivalent:

i) $k \subset K$ is separate (see def.(2.4)),

ii) there exists a finite type extension F of k, contained in K such that K is separable over F,

iii) there exists a finite type extension F of k, contained in K and generated over k by one of its p-bases, such that K is separable over F.

<u>Proof</u> ii) $\Rightarrow$ iii). Let $k \subset F'$ be a finite type field subextension of $k \subset K$ such that $F' \subset K$ is separable. Let x be a (finite) p-basis of F' over k and $F=k(x)$. By (2.3) v), $F \subset F'$ is (finite) separable and thus $F \subset K$ is separable.

i) $\Rightarrow$ ii) Let $x=(x_i)_{i \in I}$ be a separate p-basis of K over k and $L=k(x)$. We have rank $\Gamma_{L/k} < \infty$ and $\Gamma_{L/k} \otimes_L K \longrightarrow \Gamma_{K/k}$ is injective. By the above Lemma there exist a finite subset $x' \subset x$ such that $k(x) \subset L$ is separable. But $L \subset K$ is separable and so K is separable over $F:=k(x')$, too.

iii) $\Rightarrow$ i) Suppose given the field extension $k \subset F \subset K$, the first of finite type, the second separable and $y=(y_i)_{i \in I}$ a (finite) p-basis of F over k such that $F=k(y)$. Let $z=(z_j)_{j \in J}$ be a p-basis of K over F and $L=k(y,z)$.

By (2.3) iii) and iv), $L\subset K$ is separable and $\Omega_{K/L}=0$. Thus $\Omega_{L/k}\otimes_L K\simeq \Omega_{K/k}$ which shows that a p-basis of L over k is a p-basis of K over k, too. From the exact sequence

$$0\to\Omega_{F/k}\otimes_F L\to\Omega_{L/k}\to\Omega_{L/F}\to 0$$

we deduce that $y\cup z$ is a p-basis of L over k. □

(2.11) Lemma. Let $k\subset L$ be a field extension generated by one of its p-bases x. Then there exist a finite extension k' of k, $k'\subset k^{p^{-\infty}}$ and a subset $\tilde{x}\subset x$ such that:

i) $\tilde{x}$ is algebraically independent over k',

ii) $k'(\tilde{x})\subset k'(L)$ is finite separable.

Proof. By Lemma (2.9) there exists a finite subset $x'\subset x$ such that $x'':=x\setminus x'$ form an algebraically independent system over $F:=k(x')$. Using [N](39.10) there exists a finite field extension k' of k, $k'\subset k^{p^{-\infty}}$ such that $k'(F)$ is separable over k'. We can choose a subset $\tilde{x}'\subset x'$ which form a p-basis in $k'(F)$ over k'. Then $k'(\tilde{x}')\subset k'(F)$ is finite separable by (2.3) v) and so we get ii) for $\tilde{x}:=\tilde{x}'\cup x''$. By [M]Th.89 the system $\tilde{x}'$ is algebraically independent over k and thus i) holds from above. □

(2.12) Proposition. Let $k\subset K$ be a field extension such that rank $\Gamma_{K/k}<\infty$, $K_i:=k(K^{p^i})$ and n a positive integer such that $t:=\text{rank}\,\Gamma_{K_i/k}$ is constant for $i\geq n$. Then every p-basis x of K over k holds

$$\text{rank}\,\Gamma_{K/k(x)}=t.$$

Proof. Fix a p-basis x. By Lemma (2.11) there exist a finite field extension $k\subset k'$, $k'\subset k^{p^{-\infty}}$ and a subset $\tilde{x}\subset x$ such that

1) $\tilde{x}$ is algebraically independent over k',

2) $k'(\tilde{x})\subset k'(x)$ is finite separable.

Choose a positive integer $s \geq n$ such that $k' \subset k_s := k^{p^{-s}}$. By 2) and (2.3) iv) the extensions $k_s(\tilde{x}) \subset k_s(x)$, $k_s(x) \subset k_s(K)$ are unramified and so $k_s(\tilde{x}) \subset k_s(K)$ is too. Thus $\tilde{x}$ is a p-base of $k_s(K)$ over k_s and it follows

$$\operatorname{rank} \Gamma_{k_s(K)/k_s(\tilde{x})} = \operatorname{rank} \Gamma_{k_s(K)/k_s} = \operatorname{rank} \Gamma_{K_s/k} = t$$

by (2.3) ii), the extensions $k_s \subset k_s(K)$, $k \subset K_s$ - being isomorphic. Using the following exact sequence

$$0 \to \Gamma_{k_s(x)/k_s(\tilde{x})} \otimes k_s(K) \longrightarrow \Gamma_{k_s(K)/k_s(\tilde{x})} \longrightarrow \Gamma_{k_s(K)/k_s(x)} \longrightarrow 0$$

we get:

$$\operatorname{rank} \Gamma_{k_s(K)/k_s(x)} = t$$

by 2). Since $L := k(x) \subset K$ is an unramified extension the result is a consequence of Lemma (2.7) applied for $E := k_s(x)$. □

(2.13) <u>Theorem.</u> Let $k \subset K$ be a field extension with $\operatorname{rank} \Gamma_{K/k} < \infty$. The following statements are equivalent:

i) there exists a separate p-basis of K over k (i.e. $k \subset K$ is separate),

ii) there exists a finite type field extension F of k, contained in K such that K is separable over F,

iii) there exists a finite field extension E of k, $E \subset k^{p^{-\infty}}$ such that $E \subset E(K)$ is separable,

iv) there exists a positive integer s such that $k^{p^{-s}} \subset k^{p^{-s}}(K)$ is separable.

v) every p- basis of K over k is separate.

<u>Proof</u> i) $\Rightarrow$ ii) follows from Proposition (2.10). ii) $\Rightarrow$ iii) By [N] (39.10) there exists a finite field extension E of k, $E \subset k^{p^{-\infty}}$ such that $E \subset E(F)$ is separable. Since $F \subset K$ is separable we get $E \subset E(K)$ separable too.

iii) $\Rightarrow$ iv) and v) $\Rightarrow$ i) are trivial.

$$\operatorname{rank} \Gamma_{K_i/k} = \operatorname{rank} \Gamma_{k^{p^{-i}}(K)/k^{p^{-i}}} \leq$$

$$\leq \operatorname{rank} \Gamma_{k^{p^{-s}}(K)/k^{p^{-s}}} = 0$$

for all $i \geq s$ (see Lemma (2.6)).□

(2.14) Proposition. Let $k \subset K$ be a field extension, with $\operatorname{rank} \Gamma_{K/k} < \infty$. Then there exist some subfields $E \subset F \subset k$ such that

1) $E \subset K$ is a separable extension,
2) $E \subset F$ is a finite type and purely transcedental extension,
3) $F \subset k$ is an etale extension, i.e. is separable and $\Omega_{k/F} = 0$.

Proof. Let $(e_i)_{i \in I}$ be a p-basis of k over its prime subfield k_o and

$$0 \to \Gamma_{K/k} \to \Omega_k \otimes_k K \xrightarrow{\alpha_{K/k}} \Omega_K \to 0$$

the Jacobi-Zariski sequence written for $k_o \subset k \subset K$, where Ω_k, Ω_K and $\alpha_{K/k}$ stand for Ω_{k/k_o}, Ω_{K/k_o} and $\alpha_{K/k/k_o}$ respectively.

Since $\operatorname{rank} \Gamma_{K/k}$ is finite, there exists a finite subset $J \subset I$ such that the image of $\Gamma_{K/k}$ in $\Omega_k \otimes_k K$ is contained in the subspace generated by $(d_k(e_i) \otimes 1)_{i \in J}$. We put $E = k_o(e_i)_{i \in I-J}$ and $F = k_o(e_i)_{i \in I}$.

Since $k_o \subset k$ is separable, $(e_i)_{i \in I}$ is algebraically free over k_o, cf. [M], Th.89 and thus, $F = E(e_i)_{i \in J}$ is of finite type and purely transcedental over E. In the Jacobi-Zariski sequence:

$$0 \to \Gamma_{K/E} \to \Omega_E \otimes_E K \xrightarrow{\alpha_{K/E}} \Omega_K$$

written for $k_o \subset E \subset K$, $\alpha_{K/E}$ is injective by construction and then $\Gamma_{K/E} = 0$, i.e. K is separable over E. Finally, 3) is a consequence of (2.3) iii) and iv).□

(2.15) The last part of the section is devoted to those extensions $k \subset K$ which are not separate, (see Remark (2.4.1)). As we shall see in the following, such an extension contains a field F such that:

1) K/F is separable,

2) F is a filtered inductive limit of finite type extensions of k, each of them being generated over k by one of its p-bases (compare with (2.10)).

We need some preparations.

Let p>1 be an integer and I an arbitrarily given set . For every integer r>0 let $\Lambda_r(I)$ be the set of those multiindexes $\lambda=(\lambda_i)_{i\in I}\in \mathbb{N}^I$ with finite support such that $0\leq\lambda_i<p^r$, $i\in I$. If r=1 we put $\Lambda(I)$ instead $\Lambda_1(I)$. The addition of multiindexes and the multiplication of a multiindex with a nonnegative integer are defined componentwise.

Let $x=(x_i)_{i\in I}$ be a family of elements from a ring and $\lambda\in\Lambda_r(I)$. We put:

$$x^\lambda=\prod_{i\in I}x_i^{\lambda_i}$$

x^λ is well-defined, because λ has finite support.

Let k be a field of characteristic p>0. For every extension K of k and all integers $r\geq 0$, we put $K_r=k(K^{p^r})$. Clearly: $k\subset\ldots\subset K_r\subset K_{r-1}\subset\ldots\subset K_1\subseteq K_0=K$.

First, note the following simple:

(2.16) <u>Lemma</u>. Suppose given a field extension $k\subset K$ of positive characteristic p>0, with $t=\operatorname{rank}\Gamma_{K/k}<\infty$ and $x=(x_i)_{i\in I}$ a family of elements from k. Then:

i) if x (viewed in K) remains p-free over K^p and it is a maximal family in k with this property, then there exist $y=(y_1,\ldots,y_t)$ from k such that $x\cup y$ is a p-basis of k over k^p (i.e. over its prime subfield).

ii) conversely, if x remains p-free in K over K^p and there exist $y=(y_1,\ldots$ $\ldots,y_t)$ from k such that $x\cup y$ is a p-basis of k over k^p then x is a maximal family in k with the above property.

(2.17) <u>Lemma.</u> Let $k\subset E\subset K$ be fields of characteristic p>0, with $t=\operatorname{rank}\Gamma_{K/k}<\infty$ and $x=(x_i)_{i\in I}$, $y=(y_1,\ldots,y_t)$ be some elements from k such that:

1) x is p-free over K^p

2) $x \cup y$ is a p-basis of k over k^p.

Then, for every integer $r>0$, the following statements are equivalent:

i) $\operatorname{rank} \Gamma_{E_i/k} = t \quad 0 \leq i < r \quad (E_i := k(E^{p^i})$ see (2.9)).

ii) there exists an unique family $z=(z_{u\lambda})$, $1 \leq u \leq t$, $\lambda \in \Lambda_r(1)$ of elements from E, with finite support, such that

$$(2.17.1)\; y_u = \sum_{\lambda \in \Lambda_r(1)} z_{u\lambda}^{p^r} x^{\lambda} \qquad 1 \leq u \leq t.$$

Proof. i)$\Rightarrow$ii) The hypotheses and the lemma (2.16) with E_{r-1} instead K show that x is maximal among the families from k which remain p-free over E^p_{r-1}; consequently $y_u \in E^p_{r-1}(x)$, $1 \leq u \leq t$. But $k = k^{p^{r-1}}(x,y)$ and so we get $E^p_{r-1}(x) = E^{p^r}(y^p, x)$. Then $y_u \in E^{p^r}(y^p, x)$ $1 \leq u \leq t$ and step by step it results that $y_u \in E^{p^r}(x)$, $1 \leq u \leq t$

ii)$\Rightarrow$ i) For every i we have the exact sequence

$$0 \to \Gamma_{E_i/k} \longrightarrow \Omega_k \otimes_k E_i \to \Omega_{E_i} \longrightarrow 0$$

By hypothesis $d_k(x) \cup d_k(y)$ is a basis of Ω_k over k, and $d_{E_i}(x)$ is linearly independent in Ω_{E_i}. The relations (2.17.1) show that $d_{E_i}(y)$ depends linearly on $d_{E_i}(x)$ for every $0 \leq i < r$. Now it is easy to see that $\operatorname{rank} \Gamma_{E_i/k} = t$ for all $0 \leq i < r$. $\square$

(2.18) Corollary. If i) from the above lemma holds and in addition $\operatorname{rank}_{E_r} \Gamma_{E_r/k} = t$, then in (2.17.1) we have: $z_{u\lambda} \in E^p(x)$ for all u, λ.

Proof. The hypotheses implies the existence of an unique family $(z'_{u\lambda'})$, $1 \leq u \leq$ $\lambda' \in \Lambda_{r+1}(1)$ of elements from E such that

$$y_u = \sum_{\lambda' \in \Lambda_{r+1}(1)} z'^{p^{r+1}}_{u\lambda'} \cdot x^{\lambda'} \qquad 1 \leq u \leq t$$

But every $\lambda' \in \Lambda_{r+1}(1)$ can be expressed in an unique way as $\lambda' = p^r\varepsilon + \lambda$ where $\varepsilon \in \Lambda(1)$ and $\lambda \in \Lambda_r(1)$. So, the above relations can be written in the following form

$$y_n = \sum_{\lambda \in \Lambda_r(1)} \Big(\sum_{\varepsilon \in \Lambda;\, \lambda' = p^r\varepsilon + \lambda} z'^{\,p}_{u\lambda'}\, x^{\varepsilon} \Big)^{p^r} x^{\lambda}$$

We deduce

$$z_{u\lambda} = \sum_{\varepsilon \in \Lambda;\, \lambda' = p^r\varepsilon + \lambda} z'^{\,p}_{u\lambda'}\, x^{\varepsilon} \in E^p(x)$$

by the uniqueness of $(z_{u\lambda})$. □

(2.19) Lemma. Let $k \subset K$ be fields of characteristic $p > 0$ with $t = \operatorname{rank} \Gamma_{K/k} < \infty$ and $r > 0$ an integer. Suppose $\operatorname{rank} \Gamma_{K_i/k} = t$, $0 \leq i < r$. Then there exists a subfield $F \subset K$, finite type extension of k such that:

i) if $E \subset K$ is an extension of k, then $\operatorname{rank} \Gamma_{E_i/k} = t$, $0 \leq i < r$ iff $F \subset E$

ii) every p-basis of F over k generates F over k.

Proof. Let $x = (x_i)_{i \in I}$ be a maximal family from k which remains p-free over K^p. Using (2.16.)i) there exist $y = (y_1, \ldots, y_t)$ from k such that $x \cup y$ is a p-basis of k over k^p. By (2.17) there exist $z = (z_{u\lambda})$, $1 \leq u \leq t$, $\lambda \in \Lambda_r(1)$ in K with finite support such that (2.17.1) holds. We put $F := k(z)$. Then i) is a consequence of the same lemma (2.17).

Now let $w = (w_j)_{j \in J}$ a (finite) p-basis of F over k and $L = k(w)$. Since F is a finite type extension of k, $L \subset F$ is finite and separable by (2.3)v). Moreover $F_i := k(F^{p^i})$ is finite and separable over $L_i := k(L^{p^i})$ for every i and so rank $\Gamma_{L_i/k} = \operatorname{rank} \Gamma_{F_i/k} = t$ for $0 \leq i < r$. Then $F \subset L$ by i) and thus $F = L$.

(2.19.1) Remark. It could be interesting to study the structure of an extension $k \subset F$ satisfying ii).

(2.20) Theorem $[P_3]$. Let $k \subset K$ be a field extension of characteristic $p > 0$ with

$t = \operatorname{rank} \Gamma$ ... $< \infty$. Suppose that for every $n \in \mathbb{N}$, rank Γ

where $K_n = k(K^{p^n})$. Then there exists an ascending chain of subfields of K, all containing k:

$$F_1 \subset F_2 \subset \ldots \subset F_n \subset \ldots$$

such that for every positive integer n :

1) F_n is a finite type field extension of k and rank $\Gamma_{F_n/k} = t$
2) every p-basis of F_n over k, generates F_n over k
3) $F_n \subset k(F^p_{n+1})$

Moreover, if $F = \bigcup_{n \in \mathbb{N}} F_n$, then:

i) $F = k(F^p)$ and rank $\Gamma_{F/k} = t$
ii) K is a separable extension of F.

Proof. For every $n \in \mathbb{N}$ let F_n be the field defined in (2.19) for r=n. Then F_n satisfies 1) and 2). Corollary (2.18) implies 3). $F=k(F^p)$ is a consequence of 3) and $\Gamma_{F/k} \simeq \varinjlim \Gamma_{F_n/k} \otimes_{F_n} F$ implies rank $\Gamma_{F/k} = t$. In the Jacobi-Zariski sequence written for $k \subset F \subset K$

$$0 \rightarrow \Gamma_{F/k} \otimes_F K \longrightarrow \Gamma_{K/k} \longrightarrow \Gamma_{K/F} \rightarrow \Omega_{F/k} \otimes_F K$$

the first morphism is an isomorphism and $\Omega_{F/k} = 0$ because $F=k(F^p)$. Then $\Gamma_{K/F} = 0$ i.e. $F \subset K$ is separable. □

(2.20.1) Remark. The hypothesis of the above Theorem are fulfilled, for example, in the case $K=k(K^p)$ i.e. when $\Omega_{K/k} = 0$.

§ 3 . CONSTRUCTIONS OF SOME FORMALLY SMOOTH ALGEBRAS

The following Lemma is an easy extension of [P_3] Lemma (7.1).

(3.1) Lemma. Let $A \xrightarrow{v} S \xrightarrow{w} B$ be two local morphisms of noetherian local rings, $k \subset F \subset K$ the residue field extensions induced by v,w and $\underline{m},\underline{q},\underline{n}$ the maximal ideals of A,S,B respectively. Assume that:

1) the composed map wv is formally smooth,

2) $S/\underline{m}S$ is regular,

3) $\dim S = \dim A + \operatorname{rank} \Gamma_{F/k}$ (the rank is finite, because $\operatorname{rank} \Gamma_{F/k} \leq \operatorname{rank} \Gamma_{K/k} \leq \dim B - \dim A$ by (2.5)).

Then w is a flat morphism and $B/w(q)B$ is a regular ring.

Proof. By [M] (20.G) it is sufficient to prove that the morphism $\overline{w}: S/\underline{m}S \longrightarrow B/\underline{m}B$ obtained by base change is flat. Consequently we can suppose A=k and so both rings B and S are regular. By [EGA] (17.3.3) it is enough to show that the canonical map

$$\psi : q/q^2 \otimes_F K \longrightarrow \underline{n}/\underline{n}^2$$

is injective. We consider the following commutative diagram:

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ H_1(k,S,K) & \longrightarrow & \Gamma_{F/k} \otimes_F K & \xrightarrow{\lambda} & q/q^2 \otimes_F K \\ & & \downarrow \nu & & \downarrow \psi \\ H_1(k,B,K) & \longrightarrow & \Gamma_{F/k} & \xrightarrow{\mu} & \underline{n}/\underline{n}^2 \end{array}$$

The upper (resp.bottom) row is a part of the Jacobi-Zariski sequence written for $k \subset S \to F$ (resp. $k \to B \to K$). By jacobian criterion of formal smoothness we have $H_1(k,B,K)=0$ and so μ is injective. Further $\mu\nu$ is injective hence λ is injective too. Moreover λ is an isomorphism because $\operatorname{rank} q/q^2 = \dim S = \operatorname{rank} \Gamma_{F/k}$. So ψ is injective. □

(3.2) Theorem. Let $(A,\underline{m})$ be a noetherian local ring, k its residue field and $k\subset K$, $K\subset E$ two field extensions, the first one being separate with $r:=$ $=\operatorname{rank} \Gamma_{K/k}<\infty$. Then there exist a noetherian local ring $(\tilde{A},\tilde{\underline{m}})$ and a local morphism $v:A\longrightarrow\tilde{A}$ such that:

i) the residue field extension $k\subset\tilde{K}$ induced by v is a subextension of $k\subset K$,

ii) $\tilde{K}\subset K$ is separable,

iii) $\dim\tilde{A} = \dim A+r$

iv) $\tilde{A}$ is a localization of a polynomial A-algebra in a finite number of variables. Moreover, if $(B,\underline{n})$ is a noetherian local ring and $u:A\longrightarrow B$ is a formally smooth, local morphism inducing the composed extension $k\subset E$ as residue field extension then, there exists a local flat morphism $\tilde{u}:\tilde{A}\longrightarrow B$ such that $B/\tilde{u}(\tilde{m})B$ is a regular ring and $u=\tilde{u}v$.

Proof. Let $\tilde{K}/k$ be a finite type field subextension of K/k which is generated by one of its p-basis $\overline{x}=(\overline{x}_1,\dots,\overline{x}_n)$ and such that $K/\tilde{K}$ is separable (see Proposition (2.10)). From the following exact sequence

$$0\rightarrow\Gamma_{\tilde{K}/k}\otimes K\longrightarrow\Gamma_{K/k}\longrightarrow\Gamma_{K/\tilde{K}} = 0$$

we get $r=\operatorname{rank}\Gamma_{\tilde{K}/k}$. Denote $R=A[X]$, $X=(X_1,\dots,X_n)$ and let $w:R\rightarrow K$ be the unique morphism lifting the composed morphism $A\rightarrow k\rightarrow K$ and maps X_i to $\overline{x}_i$. Denote $p:=\operatorname{Ker} w$, $\tilde{A}:=R_p$, $\tilde{m}:=p\tilde{A}$ and let $w':\tilde{A}\rightarrow K$ be the morphism induced by w under localization. Clearly $\tilde{A}$ is a noetherian local ring with residue field $\tilde{K}$, and $\tilde{A}/\underline{m}\tilde{A}$ is a localization of a polynomial algebra over k thus regular.

By dimension formula ([M] Th.23) we get

$$\dim\tilde{A} = \operatorname{ht} p = \dim A+|J| - \operatorname{trdeg}_k\tilde{K}$$

where $|J|$ denotes the cardinal of J. On the other hand, by (2.3) i), $|J| =$ $= \operatorname{rank}\Omega_{\tilde{K}/k}$. The Cartier equality ([M] ~~Th.92)~~ gives:

$$\operatorname{trdeg}_k\tilde{K} = \operatorname{rank}\Omega_{\tilde{K}/k} - \operatorname{rank}\Gamma_{\tilde{K}/k} = |J| - r$$

Consequently

$$\dim \tilde{A} = \dim A + r.$$

Given $(B,\underline{n})$, u, let $x=(x_1,\ldots,x_n)$ be a lifting of $\bar{x}$ in B. Let $\tilde{u}:\tilde{A}\longrightarrow B$ be the lifting of w' given by $X\leadsto x$. By Lemma (3.1) w is flat and $B/\tilde{u}(\tilde{m})B$ is regular.

(3.3) Corollary. Let $(A,\underline{m})$ be a noetherian local ring, k its residue field and $k\subset K$, $K\subset E$ two field extensions. Suppose rank $\Gamma_{K/k}<\infty$. Then there exists a noetherian local ring (S,q) and a local morphism $v:A\longrightarrow S$ such that:

i) the residue field extension $k\subset L$ induced by v is a subextension of $k\subset K$,

ii) K/L is unramified

iii) $\dim S = \dim A + \operatorname{rank}\Gamma_{L/k}$,

iv) S is a filtered inductive limit of localizations of polynomial A-algebras. Moreover, if $(B,\underline{n})$ is a noetherian local ring and $u:A\longrightarrow B$ is a formally smooth, local morphism inducing the composed extension $k\subset E$ as residue field extension then, there exists a local flat morphism $u':S\longrightarrow B$ such that $B/u'(q)B$ is a regular ring and $u=u'v$.

Proof. Let $\bar{z}$ be a p-basis of K/k and $L=k(\bar{z})$. Then $L\subset K$ is unramified and $r:=\operatorname{rank}\Gamma_{L/k}<\infty$ (see 2.3 iv), ii)). Applying the above Theorem for $k\subset L\subset K$ there exist a noetherian local ring $(\tilde{S},\tilde{q})$ and a local morphism $\tilde{v}:A\longrightarrow\tilde{S}$ such that

1) the residue field extension $k\subset\tilde{L}$, induced by v, is a subextension of $k\subset L$,

2) $\tilde{L}\subset L$ is separable

3) $\dim\tilde{S} = \dim A + r$

4) $\tilde{S}$ is a localization of a polynomial A-algebra in a finite number of variables

5) given u,B there exists a flat local morphism $\tilde{w}:\tilde{S}\longrightarrow B$ such that $B/w(\tilde{q})B$ is a regular ring and $u=w\tilde{v}$.

Let $\bar{x}=(\bar{x}_i)_{i\in I}$ be a p-basis of $L/\tilde{L}$, $X=(X_i)_{i\in I}$ some variables, $S:=\tilde{S}[X]_{\tilde{q}\tilde{S}[X]}$ and $q:=\tilde{q}S$. Then S is the filtered inductive limit of localizations of

$$\left\{\tilde{S}[X_J]\;\middle|\;X_J=(X_j)_{j\in J},\ J\subset I \text{ finite subset}\right\}.$$

Clearly dim $S = \dim \tilde{S}$.

Given $(B,\underline{n})$, u, let x be a lifting of $\bar{x}$ in B. The map $u':S \longrightarrow B$, $X \rightsquigarrow x$ induced by w is well defined because $\bar{x}$ is algebraically independent over L (see [M] Th.89). By [M] (20.G) u' is flat (w was already flat). Since $q:=\hat{q}S$ we are ready by 5).□

The following Lemma is inspired from $[P_3]$ Lemma (7.2).

(3.4) Lemma. Let $(A,\underline{m})$ be a noetherian local ring, k its residue field and K/k an unramified field extension such that $r:=\text{rank}\,\Gamma_{K/k}<\infty$. Then there exists a noetherian local ring $(\tilde{A},\tilde{m})$ and a local morphism $v:A\rightarrow\tilde{A}$ such that:

i) the residue field extension $k\subset\tilde{K}$ induced by v is a subextension of $k\subset K$,

ii) $\tilde{K}\subset K$ is separable,

iii) $\dim\tilde{A} = \dim A+r$,

iv) $\tilde{A}$ is a filtered inductive limit of localizations of some polynomial A-algebras.

Proof. Let

$$k \subset F_1 \subset \ldots \subset F_n \subset \ldots$$

be the chain of subfields of K given by Theorem (2.20) (see also (2.20.1). Thus for every $n \in \mathbb{N}$ we have:

1) F_n is a finite type field extension of k and $\text{rank}\,\Gamma_{F_n/k} = r$

2) every p-basis of F_n over k generates F_n over k,

3) $F_n \subset k(F^p_{n+1})$.

Moreover $\tilde{K}:=\bigcup_{n\in\mathbb{N}} F_n$ satisfies

4) $\text{rank}\,\Gamma_{\tilde{K}/k} = r$ and $\tilde{K}=k(\tilde{K}^p)$,

5) $\tilde{K}\subset K$ is separable.

Applying Corollary (3.3) to $k\subset F_n\subset K$ we get some noetherian local rings (S_n,q_n), $n\in\mathbb{N}$ and some local morphisms $v_n:A\longrightarrow S_n$, $n\in\mathbb{N}$ such that for every $n\in\mathbb{N}$:

6) the residue field extension $k\subset\tilde{F}_n$ induced by v_n is a subextension of $k\subset F_n$,

7) $\tilde{F}_n\subset F_n$ is unramified,

8) $\dim S_n = \dim A + \operatorname{rank}\Gamma_{\tilde{F}_n/k}$,

9) S_n is a localization of a polynomial A-algebra.

10) given a noetherian local ring C_{n+1} and a formally smooth, local morphism $u_{n+1}:A\rightarrow C_{n+1}$ inducing the extension $k\subset F_{n+1}$ as residue field extension then, there exists a flat local morphism $u'_{n+1}:S_n\rightarrow C_{n+1}$ such that $C/u'_{n+1}(q_n)C$ is regular and $u_{n+1}=u'_{n+1}v_n$.

Since $k\subset F_n$ is of finite type we get $\tilde{F}_n\subset F_n$ finite separable from 7), see (2.3) v). Then every p-basis of $\tilde{F}_n$ over k is still a p-basis in F_n over k and by 2) we get $\tilde{F}_n = F_n$. Clearly (v_n) are formally smooth by 9). Applying 10) for $C_{n+1}:=S_{n+1}, u_{n+1}:=v_{n+1}$ we get a flat local morphism $w_n:S_n\rightarrow S_{n+1}$ such that $S_{n+1}/w_n(q_n)S_{n+1}$ is regular and $v_{n+1}=w_nv_n$. Then $w_n(q_n)S_{n+1}$ is a prime ideal of height $\dim S_n$. Since

$$\dim S_n = \dim A + r = \text{constant}$$

we get

$$q_{n+1} = w_n(q_n)S_{n+1}.$$

Then the filtered inductive limit $(v,\tilde{A})$ of (v_n,S_n,w_n) is a noetherian local ring of dimension $\dim A+r$ and $\tilde{m}:=q_n\tilde{A}$ is its maximal ideal.□

(3.5) <u>Theorem</u>. Let $(A,\underline{m})$ be a noetherian local ring, k its residue field and K/k a field extension such that $r:=\operatorname{rank}\Gamma_{K/k}<\infty$. Then there exists a noetherian local ring $(\tilde{A},\tilde{m})$ and a local morphism $v:A\rightarrow\tilde{A}$ such that:

i) the residue field extension $k\subset\tilde{K}$ induced by v is a subextension $k\subset K$,

ii) $\tilde{K}\subset K$ is separable,

iii) $\dim\tilde{A} = \dim A + r$

iv) $\tilde{A}$ is a filtered inductive limit of localizations of some polynomial A-algebras. Moreover, if $(B,\underline{n})$ is a noetherian complete local ring and $u:A\rightarrow B$

is a formally smooth, local morphism inducing $k\subset K$ as residue field extension then, there exists a flat local morphism $\tilde{u}:\tilde{A}\to B$ such that $B/\tilde{u}(\tilde{m})B$ is a regular ring and $u=\tilde{u}v$.

Proof. Let (S,q), $v':A\to S$ be given by Corollary (3.3). Then we have

1) (S,q) noetherian local

2) the residue field extension $k\subset L$ induced by v' is a subextension of $k\subset K$,

3) $L\subset K$ is unramified,

4) $\dim S = \dim A + \operatorname{rank} \Gamma_{L/k}$

5) S is a filtered inductive limit of localizations of polynomial A-algebras

Now applying the above Lemma for the case (S,q), K/L there exist a noetherian local ring $(\tilde{A},\tilde{m})$ and a local morphism $\tilde{v}:S\to\tilde{A}$ such that:

6) the residue field extension $L\subset\tilde{K}$ induced by v is a subextension of $L\subset K$,

7) $\tilde{K}\subset K$ is separable,

8) $\dim\tilde{A} = \dim S + \operatorname{rank}\Gamma_{K/L}$

9) $\tilde{A}$ is a filtered inductive limit of localizations of some polynomial A-algebras.

Clearly $(\tilde{A},\tilde{m})$, $v:=\tilde{v}v'$ satisfy i)-iv) from above. The map v is formally smoot because it is a filtered inductive limit of formally smooth maps (v_n). As B is a complete local ring, the inclusion $\tilde{K}\subset K$ can be lifted to a local morphism $\tilde{u}:\tilde{A}\to B$ such that $u=\tilde{u}v$. Note that $\tilde{A}/m\tilde{A}$ is regular because v is formally smooth and so $k\otimes_A v$ is too. Applying Lemma (3.1) for $v,\tilde{u}$ we get $\tilde{u}$ flat and $B/\tilde{u}(\tilde{m})B$ regular. □

(3.6) Corollary (EGA (22.2.6)). Let $(A,\underline{m})$ be a noetherian local ring, k its residue field and K/k a field extension such that $r:=\operatorname{rank}\Gamma_{K/k}<\infty$. Then there exists a formally smooth noetherian complete local A-algebra $(B,\underline{n})$ such that

i) $\underline{n}\supset\underline{m}B$ and $B/\underline{n}\cong K$ over k,

ii) $\dim B = \dim A + r$.

Proof. By Theorem (3.5) there exists a formally smooth noetherian local A-al gebra $(\tilde{A},\tilde{m})$ such that

1) $\tilde{m}\supset m\tilde{A}$ and the residue field extension $k\subset\tilde{K}$ induced by $A\to\tilde{A}$ is a subextension of $k\subset K$,

2) $\tilde{K}\subset K$ is separable,

3) $\dim\tilde{A}=\dim A+r$.

Let $(B,\underline{n})$ be the the Cohen $\tilde{A}$-algebra of residue field K, i.e. the unique (up to an $\tilde{A}$-isomorphism formally smooth noetherian complete local $\tilde{A}$-algebra $(B,\underline{n})$ satisfying $\dim B=\dim\tilde{A}$ and $B/\underline{n}\simeq K$ as $\tilde{K}$-algebras. Clearly $(B,\underline{n})$ works.□

(3.7) Lemma. Let $u:A\to B$ be a flat local morphism of noetherian local rings. Let $\underline{m},\underline{n}$ be the maximal ideals and k,K the residue fields of A,B respectively. Assume that $B/\underline{m}B$ is regular. Then there exist a noetherian local A-algebra $(\tilde{A},\tilde{m})$ and a local A-morphism $\tilde{u}:\tilde{A}\to B$ such that:

1) $\tilde{A}$ is a localization of a polynomial A-algebra,

2) $\tilde{m}\supset m\tilde{A}$, $\tilde{u}$ is flat and unramified (in particular $\dim\tilde{A}=\dim B$).

If K is separable over k then $\tilde{u}$ is formally smooth.

Proof. Let $x=(x_1,\dots,x_n)$ be some elements from $\underline{n}$ which form modulo $\underline{m}B$ a regular system of parameters on $B/\underline{m}B$. Denote $R:=A[X]$, $X=(X_1,\dots,X_n)$ and let $\sigma:R\to B$ be the A-morphism given by $X\mapsto x$. Put $p:=\sigma^{-1}(\underline{n})$, $\tilde{A}:=R_p$, $\tilde{m}=p\tilde{A}$ and let $\tilde{u}:\tilde{A}\to B$ the map induced by localization from σ. Note that $p\cap A=m$ and so p is exactly the ideal generated in $\tilde{A}$ by $\underline{m},X$. Then $\tilde{A}/\tilde{m}\simeq k$ and $\tilde{u}$ is unramified. Flatness of $\tilde{u}$ is now a consequence of [M](20,G) and Th.81. The last statement follows from [EGA](19.7.1).

(3.8) Proof of Theorem (1.1) When $k\subset K$ is separate we apply Theorem (3.2) and so we find a noetherian local ring $(A',\underline{m}')$ and two local morphisms $v':A\to A'$, $u':A'\to B$ such that:

1) the residue field extension $k'\subset K$ induced by u' is separable,

2) $\dim A'=\dim A+\operatorname{rank}\Gamma_{K/k}$

3) A' is a localization of a polynomial A-algebra

4) u' is flat, $u=u'v'$ and $B/u'(\underline{m}')B$ is a regular ring.

Applying Lemma (3.7) for $u':A'\to B$ we finish this case.

Now suppose B complete. By Theorem (3.5) there exist a noetherian local ring $(A',\underline{m}')$ and two local morphisms $v':A\to A'$, $u':A'\to B$ such that:

5) A' is a filtered inductive limit of polynomial A-álgebras,

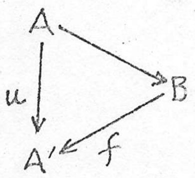
6) it holds 1), 2) and 4).

Applying Lemma (3.7) like above for u', we are ready. □

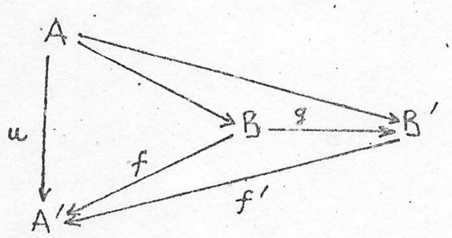
§ 4. AN APPLICATION

Trying to extend Néron desingularization for arbitrary regular morphisms we see ([P_3], § 8) that is enough to prove the following:

(4.1) Theorem. Let $u:A\to A'$ be a local, formally smooth morphism of noetherian local rings. Suppose given the commutative diagram

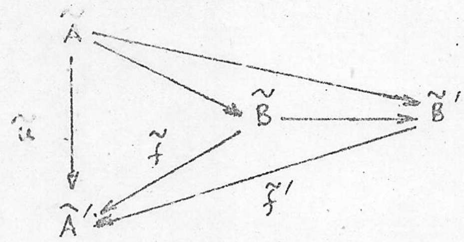


where B is a finite type A-algebra. Let $\underline{b}\subset B$ be the ideal defining the singular locus of B over A (i.e. B is smooth over A in $q\in \operatorname{Spec} B$ iff $q\not\supset \underline{b}$). Suppose that $f(\underline{b})A'$ is a $\underline{m}'$-primary ideal, where $\underline{m}'$ denotes the maximal ideal of A'. Then the above diagram can be embedded into a larger one:



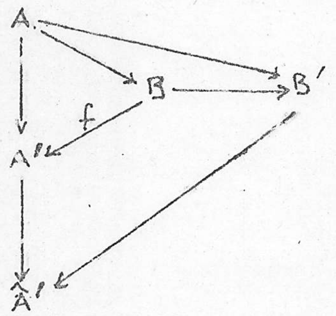
in which B' is a standard smooth A-algebra.

Proof. When dim A = dim A' then the residue field extension induced by u is separable and the above theorem is a consequence of [P_2] Th. (5.2). Otherwise, by theorem (1.1) there exist a noetherian local A-algebra $\widetilde{A}$ with the same dimension as A', which is a filtered inductive limit of localizations of polynomial A-algebras, and a formally smooth A-morphism $\widetilde{u}:\widetilde{A}\longrightarrow\widehat{A}'$ where $\widehat{A}'$ denotes the completion of A'. Then there exist a standard smooth $\widetilde{A}$ algebra $\widetilde{B}'$ and a commutative diagram:



where $\widetilde{B}=B\otimes_A\widetilde{A}$ and $\widetilde{f}:\widetilde{B}\rightarrow\widehat{A}'$ is given canonically by $\widetilde{u}$ and the composed morphism $B\xrightarrow{f}A'\rightarrow\widehat{A}'$.

Since $\widetilde{A}$ is a filtered inductive limit of standard smooth A-algebras we can find a standard smooth A-algebra B' and a commutative diagram:



This end our proof using [P_3] Lemma (7.3.2).

(4.2) Remark. The above proof lead to an easier proof for the [P_3] Desingularization Theorem which does not need Lemma (7.4) and Proposition (9.4) (and so [P_4] §§ 9-10).

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