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# ON THE BOREL ENVELOPING $C^*$-ALGEBRA OF A $C^*$-ALGEBRA

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Silviu TELEMAN

In this paper we shall associate to any $C^*$-algebra A the Borel enveloping $C^*$-algebra over A, denoted $\mathcal{B}(A)$, and defined as being the smallest $C^*$-subalgebra of $A^{**}$, possessing the following properties:

a) it contains the set $(A_{sa})^m$ of all (lower) semi-continuous elements in $A^{**}$ over A;

b) its self-adjoint part $\mathcal{B}(A)_{sa}$ is closed in $A^{**}_{sa}$, with respect to sequential monotone convergence.

We shall show that $\mathcal{B}(A)$ has a good behaviour with respect to the representations of A and to their irreducible disintegrations.

If A is commutative, then $\mathcal{B}(A)$ can be identified with the $C^*$-algebra of all bounded Borel measurable complex functions on the maximal spectrum of A. Of course, $\mathcal{B}(A)$ is, in general, different from the Baire enveloping $C^*$-algebra $\mathcal{B}_o(A)$ of A, denoted $\mathcal{B}(A)$ in [8], and defined as being the smallest $C^*$-subalgebra of $A^{**}$, which contains A and has property b) (see, also [15]).

In order to make the paper more readable, we have included, sometimes with full proofs, some results from other sources. We shall also use results obtained in [17] , [18] , [19] , [20] , [21], [22], to which we shall frequently refer (see, also [1]). We shall slightly alter the notations and the names for the various objects we encounter, in comparaison with previous uses, hoping that by so doing we achieve an improvement.

We shall probably not always refer to the original sources. Theorems 1; 2; 3; 4, c); 5 and 6 are new; also, some of the Lemmas and Propositions. The others are included for the reader's convenience.

§1. Let A be an arbitrary $C^*$-algebra, $A^*$ its Banach space dual, $E_o(A)=\{f\in A^*;\ f\geq 0,\ \|f\|\leq 1\}$ the $\sigma(A^*;A)$- compact convex set of the quasi-states of A, $E(A)=\{f\in E_o(A);\ \|f\|=1\}$ the convex set of the states of A. We have

$$\text{ex } E_o(A) = P(A)\cup\{0\} ,$$

where P(A) is the set of the pure states of A (see [5], Proposition 2.5.5). E(A) is $\sigma(A^*;A)$-compact if, and only if, A has a unit element.

Let $A^*_+\subset A^*$ be the convex cone consisting of the positive linear functionals in $A^*$; then for any $f\in A^*_+$ we shall denote by $H_f$ the Hilbert space, by $\xi^o_f\in H_f$ the canonical cyclic vector, by $\pi_f: A\to\mathcal{L}(H_f)$ the representation and by $\theta_f: A\to H_f$ the canonical mapping corresponding to the Gelfand-Naimark-Segal construction. We have $\theta_f(a)=\pi_f(a)\xi^o_f$ , $f(a)=(\pi_f(a)\xi^o_f|\xi^o_f)$ and $\|\xi_f\|^2=\|f\|$ , $f\in A^*_+$, $a\in A$ (see [5], Proposition 2.4.4).

By Sherman's Theorem, the second dual $A^{**}$ can be endowed with a structure of a $C^*$-algebra, which extends its Banach space structure, such that the natural imbedding $j: A\to A^{**}$ is a $C^*$-algebra homomorphism (see [5], Corollaire 12.1.3.).

We shall frequently identify ~~A~~ with j(A) by this isomorphism.

By Kadison's Theorem, the restriction mapping

$$A^{**}\ni a\mapsto a|E_o(A)\in A^b(E_o(A);0)$$

is a linear norm-continuous isomorphism of $A^{**}$ onto the Banach space $A_o^b(E_o(A);\mathbb{C})$ of all bounded affine complex functions on $E_o(A)$, vanishing at 0, endowed with the sup-norm; its restriction to the self-adjoint part $A_{sa}^{**}$ of $A^{**}$ is a real linear isometric isomorphism onto the real Banach subspace $A_o^b(E_o(A))\subset A_o^b(E_o(A);\mathbb{C})$, consisting of all bounded affine real functions on $E_o(A)$, vanishing at 0.

We have

$$x\in j(A)\iff \rho(x)\in A_o(E_o(A);\mathbb{C});$$

i.e., $\rho\circ j$ is a continuous linear isomorphism of A onto the Banach space $A_o(E_o(A);\mathbb{C})$ of all $\sigma(A^*;A)$- continuous affine complex functions on $E_o(A)$, vanishing at 0; its restriction to the real vector subspace $A_{sa}\subset A$ of all self-adjoint elements in A is a linear isometric isomorphism onto the real Banach space $A_o(E_o(A))$ of all $\sigma(A^*;A)$-continuous affine real functions on $E_o(A)$, vanishing at 0 (see [8], Theorem 3.10.3; [15], Ch.II, §9.1, Lemma 9.1.10.) (For any subset $M\subset A^{**}$ we denote by $M_{sa}=M\cap A_{sa}^{**}$ the set of all self-adjoint elements in M).

We can also consider the universal representation $\pi_u:A\to\mathcal{L}(H_u)$ given by $\pi_u=\bigoplus_{f\in E(A)}\pi_f$ in $H_u=\bigoplus_{f\in E(A)}H_f$, and its normal extension $\widetilde{\pi}_u:A^{**}\to\mathcal{L}(H_u)$, given by

$$(\widetilde{\pi}_u(a)\theta_f(b)\mid\theta_f(c))=(j(c)^*aj(b))(f),$$

for $f\in E(A)$, $a\in A^{**}$, $b,c\in A$. We have $\pi_u=\widetilde{\pi}_u\circ j$. Then $\widetilde{\pi}_u$ is a $*$-isomorphism of $A^{**}$ onto $\pi_u(A)''$ (see [5], Corollaire 12.1.3).

Of course, j(A) is $\sigma(A^{**};A^*)$-dense (i.e., ultraweakly dense) in $A^{**}$. It follows that

(1) $$(((j(A)_1^+)^m)_m)^m = (A^{**})_1^+ ,$$

where $(M)_1$ denotes the unit ball, $(M)_1^+$ the positive part of the unit ball of $M \subset A^{**}$; $(M)^m$ denotes the set of all limits of increasing nets of elements belonging to $M \subset A_{sa}^{**}$ and, similarly, $(M)_m$ denotes the set of all limits of decreasing nets of elements belonging to $M \subset A_{sa}^{**}$. By Vigier's Theorem, the limits are to be understood either as suprema, respectively infima, with respect to the usual order relation in $A_{sa}^{**}$, or with respect to the topology $\sigma(A^{**} : A^{*})$; by the isomorphism $\varrho$, the latter is identified with the topology of the pointwise convergence on $E_o(A)$ (see [15], Ch.II, §9.1, Lemma 9.1.10; [16], Ch.II, Theorem 4.24; [25], Ch.I, §1 Theorem 1.8).

We remark that in (1) the limits are to be taken, in general, over uncountable nets. This fact is cumbersome, especially in problems where integration processes are involved. For this reason one is led to consider intermediate (real) vector spaces $\mathcal{M}$, such that

$$A_{sa} \subset \mathcal{M} \subset A_{sa}^{**} ,$$

and such that $\varrho(\mathcal{M}) \subset A_o^b(E_o(A))$ should consist of bounded affine real functions, having good integration properties. Moreover, from an algebraic point of view, it would be desirable that the set

$$\mathcal{M} + i\mathcal{M} \subset A^{**}$$

be a $C^*$-subalgebra.

§2. The situation is best illustrated by the case of a commutative $C^*$-algebra A, possessing a unit element. In this case, A is isomorphic with the $C^*$-algebra $C(X)$ of all continuous complex functions defined on a suitable compact space X (the maximal spectrum of A); whereas $E_o(A)$ identifies with the $\sigma(C(X)^*;C(X))$-compact convex space $M(X)_1^+$ of all positive Radon measures on X, of norm $\leq 1$. Moreover, $P(A)$ identifies with X by the evaluation mapping $X \ni x \mapsto \delta_x \in P(C(X)) = \text{ex } E(C(X)) \subset E(C(X))$.

It is easy to prove that the real vector space $\mathcal{B}_o^b(X;\mathbb{R})$ of all bounded Baire measurable real functions on X coincides with the smallest subset (it turns out to be a real vector subspace) of $\mathbb{R}^X$, which contains the space $C(X;\mathbb{R})$ of all continuous real functions on X and is closed with respect to sequential bounded monotone point-wise convergence on X. Similarly, the real vector space $\mathcal{B}^b(X;\mathbb{R})$ of all bounded Borel measurable real functions on X coincides with the smallest real vector subspace of $\mathbb{R}^X$, which contains the bounded (lower) semi-continuous real functions on X, and is closed with respect to sequential bounded monotone point-wise convergence on X. We obviously have the inclusions

$$C(X;\mathbb{R}) \subset \mathcal{B}_o^b(X;\mathbb{R}) \subset \mathcal{B}^b(X;\mathbb{R})$$

and the mapping

$$T_\varphi : M(X)_1^+ \ni \mu \mapsto \mu(\varphi)$$

extends $\varphi \in \mathcal{B}^b(X;\mathbb{R})$ from X to $M(X)_1^+$, as a bounded, affine Borel measurable real function on $M(X)_1^+$, endowed with the topology $\sigma(C(X)^*; C(X))$; if $\varphi \in \mathcal{B}_o^b(X; \ )$, then the corresponding affine function on $M(X)_1^+$ is Baire measurable.

An example due to Choquet (see [2]; [10], §12) corresponding to the choice $X=[0,1]$, shows that, in general, the range of the restriction of T to $\mathcal{B}_o^b(X;\mathbb{R})$ is strictly included in the real vector space of all bounded affine Baire measurable real functions on $M(X)_1^+$, vanishing at 0. Moreover, the (boundary) barycentric calculus does not hold for all functions belonging to this latter class, but it does hold for all functions in the range of T (see

This shows that one has to be careful when trying to extend the notion of a Baire, or Borel, element over a $C^*$-algebra, from the commutative case to the case of an arbitrary $C^*$-algebra A.

§3. As far as the Baire elements are concerned, their theory was developed especially by Kadison and Pedersen (see [6];[8], Ch.IV, §5). Namely, let $\mathcal{B}_o(A)_{sa} \subset A^{**}_{sa}$ be the smallest real vector subspace (equivalently, subset) of $A^{**}_{sa}$, which contains $j(A_{sa}) = j(A)_{sa} = A_{sa}$, and is closed with respect to sequential bounded monotone convergence in $A^{**}_{sa}$, either with respect to the order relation or, equivalently, point-wise on $E_o(A)$.

Then $\mathcal{B}_o(A) = \mathcal{B}_o(A)_{sa} + i\mathcal{B}_o(A)_{sa}$ is a $C^*$-algebra, which behaves well with respect to representations (see [8], Theorem 4.5.4, Theorem 4.5.9;[22]).

Remark. It is obvious that $\mathcal{B}_o(A)_{sa}$ is the self-adjoint part of $\mathcal{B}_o(A)$. This fact gives an a posteriori justification of the notation.

It is natural to call $\mathcal{B}_o(A)$ the Baire enveloping $C^*$-algebra of A, and to say that the elements of $\mathcal{B}_o(A)$ are the Baire elements of $A^{**}$ over A.

The situation is no more so simple when trying to obtain the Borel enveloping $C^*$-algebra over A.

§4. The notion of a universally measurable (bounded) function on a (locally) compact space has been extended to the case of arbitrary $C^*$-algebras by Pedersen (see [8], Ch.IV, §3). We first consider the subset $(A_{sa})^m \subset A^{**}_{sa}$, consisting of all elements in $A^{**}_{sa}$, which are suprema of (bounded) increasing nets in $A_{sa}(=j(A_{sa}))$. Of course, for any $a \in (A_{sa})^m$, the function

$\rho(a)\in A_o^b(E_o(A))$ is lower semi-continuous on $E_o(A)$.

It is easy to see that $(A_{sa})^m\subset A_{sa}^{**}$ is a convex cone, and $1\in(A_{sa})^m$ (here 1 denotes the unit element of $A^{**}$; see [8], Proposition 3.11.5); we have $\rho(1)(f)=\|f\|$, $f\in E_o(A)$.

If $\varphi\in A_o^b(E_o(A))$ is lower semi-continuous on $E_o(A)$, then $\rho^{-1}(\varphi)\in((A_{sa})^m)^-$; conversely, for any $a\in((A_{sa})^m)^-$, the function $\rho(a)\in A_o^b(E_o(A))$ is lower semi-continuous on $E_o(A)$ (Here $(M)^-$ denotes the norm closure of the subset $M\subset A_{sa}^{**}$ in $A_{sa}^{**}$).

We shall denote by $\mathcal{B}_{sa}^o(A)$ the smallest <u>real vector subspace</u> of $A_{sa}^{**}$, which contains $(A_{sa})^m$ and is closed with respect to the sequential bounded monotone convergence in $A_{sa}^{**}$; here convergence is to be understood either with respect to the order relation in $A_{sa}^{**}$ or, equivalently, strongly in the space $H_u$ of the universal representation of A, where $A^{**}$ can be identified with $\pi_u(A)''$; again, this is equivalent to the sequential bounded monotone point-wise convergence on $E_o(A)$, modulo the functional representation $\rho$.

It is not known whether $\mathcal{B}^o(A)=\mathcal{B}_{sa}^o(A)+i\mathcal{B}_{sa}^o(A)$ is a $C^*$-algebra; in any case, we obviously have

$$A\subset\mathcal{B}_o(A)\subset\mathcal{B}^o(A),$$

and we shall prove that $\mathcal{B}^o(A)$ is a $\mathcal{B}_o(A)$-bimodule.

We shall call the elements $x\in\mathcal{B}^o(A)$ the <u>strongly Borel elements</u> of $A^{**}$ over A (see, also [15], Ch.II, §10, where the space $\mathcal{B}_{sa}^o(A)$ is denoted $\mathcal{B}o^h(A)$). It is known that $\mathcal{B}_{sa}^o(A)$ is a JC-algebra (see [15], Ch.II, §10, Proposition 10.3).

The (self-adjoint, strongly) <u>universally measurable</u> elements x in $A_{sa}^{**}$ (over A) are defined as follows:

for any $f\in E(A)$ and any $\varepsilon>0$ there exist $h,k\in(A_{sa})^m$, such that

$$-k\leq x\leq h \quad \text{and} \quad f(h+k)<\varepsilon .$$

(see [8], 4.3.11, p.104).

By $\mathcal{U}(A)$ we shall denote the norm-closed real vector subspace of $A^{**}_{sa}$, consisting of all (self-adjoint strongly) universally measurable elements over $A$ (see [8], Proposition 4.3.13). Since $\mathcal{U}(A)$ is sequentially monotone closed in $A^{**}_{sa}$ (see [8], Lemma 4.5.12), from the inclusion $(A_{sa})^m\subset\mathcal{U}(A)$, we infer that

$$A_{sa}\subset\mathcal{B}_o(A)_{sa}\subset\mathcal{B}^o_{sa}(A)\subset\mathcal{U}(A).$$

§5. In this section we shall prove that $\mathcal{B}^o(A)$ is a $\mathcal{B}_o(A)$-bimodule. This will be done by an attentive analysis of the proof of Theorem 4.5.4 from [8].

<u>Lemma 1.</u> <u>Let $\mathcal{V}\subset A^{**}_{sa}$ be a real vector subspace, such that $1\in\mathcal{V}$ and $\mathcal{V}$ is closed with respect to sequential monotone convergence in $A^{**}_{sa}$. Then $\mathcal{V}$ is norm closed in $A^{**}_{sa}$.</u>

<u>Proof.</u> Let $(x_n)_{n\geq 0}$ be a norm convergent sequence in $\mathcal{V}$, converging to $x\in A^{**}_{sa}$. We may assume that $\|x_{n+1}-x_n\|\leq\frac{1}{2^n}$, for all $n\geq 0$. Consider the sequence $(x_n-\frac{1}{2^{n-1}}1)_{n\geq 0}$, which converges to $x$ and belongs to $\mathcal{V}$. Since

$$(x_{n+1}-\frac{1}{2^n}1)-(x_n-\frac{1}{2^{n-1}}1) = x_{n+1}-x_n+\frac{1}{2^n}1 \geq 0$$

the sequence is increasing and, therefore, its limit $x$ belongs to $\mathcal{V}$. Thus $\mathcal{V}$ is norm closed.

<u>Remark.</u> The proof is the almost verbatim repetition of the proof given in [8], Proof of Theorem 4.5.4, whereas the result is essentially due to Kadison (see [15], Ch.II, §6.2; [6]).

Corollary 1. The real-vector subspace $\mathcal{B}^{o}_{sa}(A)$ is norm closed in $A^{**}_{sa}$.

Let $\mathcal{S}(A) \subset A^{**}_{sa}$ be the real vector subspace of $A^{**}_{sa}$, generated by $(A_{sa})^{m}$, and let $\mathcal{S}_{sa}(A)^{-}$ be its norm closure in $A^{**}_{sa}$.

Corollary 2. $\mathcal{S}_{sa}(A)^{-} \subset \mathcal{B}^{o}_{sa}(A)$.

Proof. From the inclusion $(A_{sa})^{m} \subset \mathcal{B}^{o}_{sa}(A)$, we infer that $\mathcal{S}_{sa}(A) \subset \mathcal{B}^{o}_{sa}(A)$. The required inclusion now follows from Corollary 1.

Corollary 3. $\mathcal{B}^{o}_{sa}(A)$ is the smallest subset of $A^{**}_{sa}$, which contains $\mathcal{S}_{sa}(A)$ and is closed with respect to sequential monotone convergence in $A^{**}_{sa}$.

Proof. It is sufficient to prove that the smallest subset of $A^{**}_{sa}$, possessing the required properties, is a real vector subspace of $A^{**}_{sa}$. This is left to the reader.

We recall that for any Hilbert space H a real vector subspace $\mathcal{V} \subset \mathcal{L}(H)_{sa}$ is said to be Jordan algebra (of operators) if

$$a, b \in \mathcal{V} \Rightarrow \frac{1}{2}(ab+ba) \in \mathcal{V}.$$

Following D.Topping, $\mathcal{V}$ is said to be a JC-algebra if, moreover, $\mathcal{V}$ is norm closed in $\mathcal{L}(H)_{sa}$ (see [14], p.438; [15], Ch.II, §6.1, p.373).

Remark. The notions obviously extend to real vector subspaces of $A^{**}_{sa}$.

The following Lemma is translated from ([15], Ch.II, §6, Lemma 6.1.5), with slight modifications.

*Lemma 2.* i) *If* $\vartheta \subset A^{**}_{sa}$ *is a JC-algebra, such that* $1\in\vartheta$, *and if* $x\in\vartheta$ *is such that* $x\geq 0$ *and* $x^{-1}$ *exists in* $A^{**}_{sa}$, *then* $x^{-1}\in\vartheta$.

ii) *Let* $\Gamma\subset A^{**}_{sa}$ *be a convex cone, such that* $1\in\Gamma$ *and let* $\vartheta$ *be the norm closed real vector subspace of* $A^{**}_{sa}$, *generated by* $\Gamma$. *If*

$$a\in\Gamma,\ a\geq 0,\ a^{-1} \text{ exists in } A^{**}_{sa} \Rightarrow a^{-1}\in\vartheta,$$

*then* $\vartheta$ *is a JC-algebra.*

*Proof.* i) Since $\vartheta$ is a JC-algebra, any real polynomial in x belongs to $\vartheta$; hence, $x^{-1}$ belongs to $\vartheta$, by the spectral theory.

ii) If $x\in\Gamma$ and $\|x\|\leq 1$, then $1+tx\geq 0$ and $1+tx\in\Gamma$, for any $t\in[0,1)$; since $(1+tx)^{-1}$ exists in $A^{**}_{sa}$, for any $t\in[0,1)$, we infer that $(1+tx)^{-1}\in\vartheta$. From the norm convergence

$$x^2=\lim_{t\to 0}\frac{1}{t^2}\left[(1+tx)^{-1}-1+tx\right],$$

we infer that $x^2\in\vartheta$; hence, $x^2\in\vartheta$, for any $x\in\Gamma$.

Let now $\vartheta_0$ be the real vector subspace of $A^{**}_{sa}$, generated by $\Gamma$. Then $x\in\vartheta_0 \Rightarrow x=y-z$, where $y,z\in\Gamma$; it follows that $x^2=2y^2+$ $+2z^2-(y+z)^2\in\vartheta$. Since $\vartheta$ is the norm closure of $\vartheta_0$, in $A^{**}_{sa}$, we infer that $x\in\vartheta\Rightarrow x^2\in\vartheta$, and this implies that $\vartheta$ is a JC-algebra by easy computations. The Lemma is proved.

The following Lemma slightly extends a result of F. Combes (see [15], Ch.II, Proposition 9.2.6, p.420).

*Lemma 3. Let* $a\in A^{**}_{sa}$, $a\geq 0$. *Then for any* $a\in((A_{sa})^{m})^{-}$ *we h* a is invertible $\Leftrightarrow \inf\{\rho(a)(p);\ p\in P(A)\} > 0$;

<u>in this case</u> $a^{-1}\in \mathcal{S}_{sa}(A)^{-}$.

<u>Proof</u>. Let us define, for any $a\in A_{sa}^{**}$, $a\geq 0$,

$$\lambda_a = \inf\{(a\xi|\xi);\ \xi\in H_u,\ \|\xi\| = 1\}.$$

Then we have

$$a \text{ is invertible} \Leftrightarrow \lambda_a > 0.$$

On the other hand, we have

$$\lambda_a = \inf\{\rho(a)(f);\ f\in E(A)\}, \quad a\in A_{sa}^{**}.$$

We shall prove that for $a\in((A_{sa})^m)^-$, if we define

$$\lambda'_a = \inf\{\rho(a)(p);\ p\in P(A)\},$$

we have $\lambda'_a=\lambda_a$. Indeed, it is obvious that $\lambda_a\leq\lambda'_a$. Let us consider the function $d:E_o(A)\to \mathbb{R}$, given by

$$d(f)=\rho(a)(f)-\lambda'_a\|f\|, \quad f\in E_o(A).$$

We obviously have $d|P(A)\cup\{0\}\geq 0$. From H.Bauer's Minimum Principle (see [17], Theorem 1.1; [20], Theorem 4), we infer that $d\geq 0$ on $E_o(A)$ and, therefore,

$$\rho(a)(f)\geq\lambda'_a, \quad f\in E(A);$$

this shows that $\lambda_a\geq\lambda'_a$, and the first assertion is proved.

Let us now assume that $a\in((A_{sa})^m)^-$, $a\geq 0$ and $a^{-1}$ exists in $A_{sa}^{**}$. From ([8], Proposition 3.11.6) we infer that

$$a + \varepsilon 1 \in (A_{sa}^{+})^{m}, \quad \forall \varepsilon > 0,$$

where $A_{sa}^{+} = \{a \in A_{sa};\ a \geq 0\}$. It follows that for any $\varepsilon > 0$ there exists an increasing net $(a_\alpha)_\alpha$, such that

$$a_\alpha \uparrow a + \varepsilon 1, \quad \text{where } a_\alpha \in A_{sa}^{+},$$

and, therefore,

$$\varepsilon 1 \leq a_\alpha + \varepsilon 1 \uparrow a + 2\varepsilon 1,$$

whence we infer that

$$(a_\alpha + \varepsilon 1)^{-1} \downarrow (a + 2\varepsilon 1)^{-1}.$$

If we define $b_\alpha = \frac{1}{\varepsilon} 1 - (a_\alpha + \varepsilon 1)^{-1}$, we have $b_\alpha \in A$, and $b_\alpha \uparrow \frac{1}{\varepsilon} 1 - (a + 2\varepsilon 1)^{-1}$. It follows that

$$\frac{1}{\varepsilon} 1 - (a + 2\varepsilon 1)^{-1} \in (A_{sa})^{m}, \quad \forall \varepsilon > 0,$$

and, therefore, $(a + 2\varepsilon 1)^{-1} \in \mathcal{S}_{sa}(A)$. Since we have

$$\lim_{\varepsilon \to 0} (a + 2\varepsilon 1)^{-1} = a^{-1}$$

in norm, we infer that $a^{-1} \in \mathcal{S}_{sa}(A)^{-}$, and the Lemma is proved.

The following result is due to Combes (see [4]; [15], Ch.II, §9.2, Proposition 9.2.7).

<u>Corollary 1.</u> <u>$\mathcal{S}_{sa}(A)^{-}$ is a JC-algebra.</u>

Proof. This follows from the preceding Lemma and from Lemma 2, in which we make $\Gamma=(A_{sa})^m$ and $\vartheta=\mathcal{G}_{sa}(A)^-$. The following Proposition is due to Combes (see [4];[15], Ch.II, §10, Proposition 10.3).

Proposition 1. $\mathcal{B}^o_{sa}(A)$ is a JC-algebra.

Proof. It will be sufficient to prove that $x\in\mathcal{B}^o_{sa}(A)\Rightarrow x^2\in\mathcal{B}^o_{sa}(A)$. Let us define

$$\mathcal{M}=\{x\in\mathcal{B}^o_{sa}(A);\ x^k\in\mathcal{B}^o_{sa}(A),\ \forall k\in\mathbb{N}\}.$$

We obviously have that $\mathcal{G}_{sa}(A)^-\subset\mathcal{M}\subset\mathcal{B}^o_{sa}(A)$. If we can prove that $\mathcal{M}$ is sequentially monotone closed in $\mathcal{B}^o_{sa}(A)$, then the assertion will follow from Corollary 3 to Lemma 1. Indeed, let $(x_n)_{n\geq 0}$ be a monotone increasing sequence in $\mathcal{M}$, with $\lim_{n\to\infty} x_n=x\in\mathcal{B}^o_{sa}(A)$. We can assume that $\|x_n\|\leq 1$, $n\geq 0$. Then, for any $t\in[0,1)$, the series expansion

$$(1-tx_n)^{-1}=\sum_{k=0}^{\infty} t^k x^k$$

is norm convergent. Since $x_n\in\mathcal{M},\ \forall n\geq 0$, we infer that $(1-tx_n)^{-1}\in\mathcal{B}^o_{sa}(A),\ \forall n\geq 0$. Since

$$(1-tx_n)^{-1}\uparrow(1-tx)^{-1},$$

we infer that $(1-tx)^{-1}\in\mathcal{B}^o_{sa}(A)$. It follows that

$$\frac{1}{t^2}\left[(1-tx)^{-1}-(1+tx)\right]\in\mathcal{B}^o_{sa}(A),\ \forall t\in[0,1),$$

and, therefore, by norm convergence for $t\to 0$, we get that

$x^2 \in \mathcal{B}^o_{sa}(A)$. By induction, from

$$x^k = \lim_{t\to 0} \frac{1}{t^k}\Big[(1-tx)^{-1} - \sum_{\ell=0}^{k-1} t^\ell x^\ell\Big],$$

we infer that $x^k \in \mathcal{B}^o_{sa}(A)$, for any $k \geq 0$, and, therefore, $x \in \mathcal{M}$. The Proposition is proved.

Proposition 2. $\mathcal{B}^o(A)$ is a $\mathcal{B}_o(A)$ - bimodule.

Proof. We have to prove that $x \in \mathcal{B}^o(A)$, $y \in \mathcal{B}_o(A) \Rightarrow xy \in \mathcal{B}^o(A)$ and $yx \in \mathcal{B}^o(A)$.

From the identity

$$(1) \qquad 2xyx = (xy+yx)x + x(xy+yx) - (yx^2 + x^2 y)$$

we infer that $x, y \in \mathcal{B}^o_{sa}(A) \Rightarrow xyx \in \mathcal{B}^o_{sa}(A)$.

The identity

$$(2) \qquad i(xy-yx) = (x+i1)^* y(x+i1) - xyx - y, \qquad x, y \in A^{**}_{sa}$$

implies that

$$x \in A_{sa}, y \in (A_{sa})^m \Rightarrow i(xy-yx) \in \mathcal{S}_{sa}(A) \subset \mathcal{B}^o_{sa}(A).$$

It immediately follows that

$$(3) \qquad x \in A_{sa}, \ y \in \mathcal{S}_{sa}(A) \Rightarrow i(xy-yx) \in \mathcal{S}_{sa}(A) \subset \mathcal{B}^o_{sa}(A).$$

From identities (1) and (2) we now infer that

$$x \in A_{sa}, \ y \in \mathcal{S}_{sa}(A) \Rightarrow (y+i1)^* x(y+i1) \in \mathcal{B}^o_{sa}(A)$$

and, therefore, by a sequential monotone closure argument,

$$x \in \mathcal{B}_o(A)_{sa},\ y \in \mathcal{S}_{sa}(A) \Rightarrow (y+i1)^* x (y+i1) \in \mathcal{B}^o_{sa}(A).$$

By identities (1) and (2) again, we infer that

$$x \in \mathcal{B}_o(A)_{sa},\ y \in \mathcal{S}_{sa}(A) \Rightarrow (x+i1)^* y (x+i1) \in \mathcal{B}^o_{sa}(A)$$

and, therefore, by Corollary 3 to Lemma 1, we infer that

$$x \in \mathcal{B}_o(A)_{sa},\ y \in \mathcal{B}^o_{sa}(A) \Rightarrow (x+i1)^* y (x+i1) \in \mathcal{B}^o_{sa}(A).$$

By (1) and (2) again we finally infer that

(4) $$x \in \mathcal{B}_o(A)_{sa},\ y \in \mathcal{B}^o_{sa}(A) \Rightarrow i(xy-yx) \in \mathcal{B}^o_{sa}(A).$$

Since we have

(5) $$x, y \in \mathcal{B}^o_{sa}(A) \Rightarrow xy+yx \in \mathcal{B}^o_{sa}(A),$$

from (4) and (5) the Proposition now immediately follows.

Remark. Even in the commutative case, we have, in general, $\mathcal{B}_o(A) \neq \mathcal{B}^o(A)$. Of course, one would expect, at first glance, that $\mathcal{B}^o(A)$ be a $C^*$-algebra, as it happens with $\mathcal{B}_o(A)$. Our conjecture is that this is not the case in general (see [14] for conditions in order that a JC-algebra be the self-adjoint part of a $C^*$-algebra).

§6. Let us now define $\mathcal{B}(A)$ to be the smallest $C^*$-subalgebra of $A^{**}$, such that

a) $(A_{sa})^m \subset \mathcal{B}(A)$;

b) the self-adjoint part $\mathcal{B}(A)_{sa}$ of $\mathcal{B}(A)$ is closed in $A^{**}_{sa}$ with respect to the sequential monotone convergence.

We immediately infer that $\mathcal{B}^o(A) \subset \mathcal{B}(A)$.

Lemma 4. $\mathcal{B}(A)$ is the $C^*$-algebra whose self-adjoint part is the sequential monotone closure of the self-adjoint part of th $C^*$-algebra $\mathcal{B}_1(A)$ generated by $\mathcal{B}^o_{sa}(A)$.

Proof. We have to consider

i) the $C^*$-algebra $\mathcal{B}_1(A)$ generated by $\mathcal{B}^o_{sa}(A)$;

ii) the smallest sequentially monotone closed subset $\mathcal{B}_{sa}(A) \subset A^{**}_{sa}$, containing the self-adjoint part $\mathcal{B}_1(A)_{sa}$ of $\mathcal{B}_1(A)$; and to prove that

iii) $\mathcal{B}_{sa}(A)$ is the self-adjoint part of a $C^*$-algebra, which coincides with $\mathcal{B}(A)$.

Since we obviously have $\mathcal{B}_{sa}(A) \subset \mathcal{B}(A)$, the Lemma will immediately follow from assertion iii), and, as a consequence, we have that $\mathcal{B}_{sa}(A) = \mathcal{B}(A)_{sa}$.

It is obvious that $\mathcal{B}_1(A)$ is the norm closure in $A^{**}$ of the $*$-algebra

$$\mathcal{A} = \left\{ \sum_{k=1}^{n_1} A'_{k_1} \cdots A'_{km_1} + i \sum_{k=1}^{n_2} A''_{k_1} \cdots A''_{km_2} ;\ A'_{k\ell}, A''_{k\ell} \in \mathcal{B}^o_{sa}(A) \right\}$$

Let now $\mathcal{B}_{sa}(A) \subset A^{**}_{sa}$ be the smallest sequentially monotone closed subset of $A^{**}_{sa}$, containing the real vector subspace $\mathcal{B}_1(A)_{sa}$.

Let us first remark that $\overline{\mathcal{B}_{sa}(A)}$ is a real vector subspace of $A^{**}_{sa}$. Indeed, for $x \in \mathcal{B}_1(A)_{sa}$, the set

(1) $\{y\in\mathcal{B}_{sa}(A);\ x+y\in\mathcal{B}_{sa}(A)\}$

is a sequentially monotone closed subset of $\mathcal{B}_{sa}(A)$, containing $\mathcal{B}_1(A)_{sa}$. It follows that the set in (1) equals $\mathcal{B}_{sa}(A)$, and, therefore,

$$x\in\mathcal{B}_1(A)_{sa},\ y\in\mathcal{B}_{sa}(A)\Rightarrow x+y\in\mathcal{B}_{sa}(A).$$

A repetition of the argument shows that

$$x,y\in\mathcal{B}_{sa}(A)\Rightarrow x+y\in\mathcal{B}_{sa}(A)$$

and, similarly,

$$\lambda\in\mathbb{R},\ x\in\mathcal{B}_{sa}(A)\Rightarrow\lambda x\in\mathcal{B}_{sa}(A).$$

Lemma 1 now immediately implies that $\mathcal{B}_{sa}(A)$ is a norm closed real vector subspace of $A^{**}_{sa}$.

We shall now prove that $\mathcal{B}_{sa}(A)$ is a JC-algebra. Let us consider the set

$$\mathcal{M}_1=\{x\in\mathcal{B}_{sa}(A);\ x^k\in\mathcal{B}_{sa}(A),\ \forall\ k\in\mathbb{N}\}.$$

We obviously have that

(2) $\mathcal{B}_1(A)_{sa}\subset\mathcal{M}_1\subset\mathcal{B}_{sa}(A).$

If we manage to prove that $\mathcal{M}_1$ is sequentially monotone closed, then from (2) we would infer that $\mathcal{M}_1=\mathcal{B}_{sa}(A)$. Indeed, let $(x_n)_{n\geq 0}$ be an increasing sequence in $\mathcal{M}_1$, converging to $x\in\mathcal{B}_{sa}(A)$. We can assume that $\|x_n\|\leq 1$, $n\in\mathbb{N}$. Then for any $t\in[0,1)$ the series expansion

(3) $$(1-tx_n)^{-1} = \sum_{k=0}^{\infty} t^k x_n^k$$

is norm convergent. From the fact that $x_n \in \mathcal{M}_1$ and from (3) we infer that $(1-tx_n)^{-1} \in \mathcal{B}_{sa}(A)$, $n \in \mathbb{N}$.

Since the sequence $((1-tx_n)^{-1})_{n\geq 0}$ is monotone increasing to $(1-tx)^{-1}$, we infer that $(1-tx)^{-1} \in \mathcal{B}_{sa}(A)$, for any $t \in [0,1)$. As above, we infer that

$$x^2 = \lim_{t\to 0} \frac{1}{t^2}\left[(1-tx)^{-1} - (1+tx)\right] \in \mathcal{B}_{sa}(A)$$

and, by induction, we get that $x^k \in \mathcal{B}_{sa}(A)$, $\forall\, k \in \mathbb{N}$.

It follows that $x \in \mathcal{M}_1$ and this shows that $\mathcal{M}_1$ is sequentially monotone closed and, therefore, $\mathcal{M}_1 = \mathcal{B}_{sa}(A)$. Hence, $\mathcal{B}_{sa}(A)$ is a JC-algebra.

In order to prove that $\mathcal{B}_{sa}(A)$ is the self-adjoint part of a $C^*$-algebra it will suffice now to prove that

(4) $$x, y \in \mathcal{B}_{sa}(A) \Rightarrow i(xy-yx) \in \mathcal{B}_{sa}(A).$$

To this end we shall use again the identities

(5) $$i(xy-yx) = (x+i1)^* y (x+i1) - xyx - y, \ \forall\, x, y \in A_{sa}^{**},$$

and

(6) $$2xyx = x(xy+yx) + (xy+yx)x - (x^2y+yx^2); \ \forall\, x, y \in A^{**};$$

therefore, $x, y \in \mathcal{B}_{sa}(A) \Rightarrow xyx \in \mathcal{B}_{sa}(A)$.

Let us now remark that

(7) $$x, y \in \mathcal{B}_1(A)_{sa} \Rightarrow (x+i1)^* y (x+i1) \in \mathcal{B}_1(A)_{sa}.$$

Let then $x \in \mathcal{B}_1(A)_{sa}$ and define

$$\mathcal{M}_2 = \{y \in \mathcal{B}_{sa}(A);\ i(xy-yx) \in \mathcal{B}_{sa}(A)\}.$$

From (6) and (7) we infer that

$$\mathcal{B}_1(A)_{sa} \subset \mathcal{M}_2,$$

whereas (5) implies that $\mathcal{M}_2$ is sequentially monotone closed in $\mathcal{B}_{sa}(A)$. We infer that

$$\mathcal{M}_2 = \mathcal{B}_{sa}(A),$$

and therefore

(8) $\quad x \in \mathcal{B}_1(A)_{sa},\ y \in \mathcal{B}_{sa}(A) \Rightarrow i(xy-yx) \in \mathcal{B}_{sa}(A).$

Let now $y \in \mathcal{B}_{sa}(A)$ and define

$$\mathcal{M}_3 = \{x \in \mathcal{B}_{sa}(A);\ i(xy-yx) \in \mathcal{B}_{sa}(A)\}.$$

From (8) we infer that

$$\mathcal{B}_1(A)_{sa} \subset \mathcal{M}_3$$

and, therefore, with the help of identities (4) and (5), we get

$$x \in \mathcal{B}_1(A)_{sa},\ y \in \mathcal{B}_{sa}(A) \Rightarrow \underline{(y+i1)^* x}(y+i1) \in \mathcal{B}_{sa}(A).$$

We infer that $\mathcal{M}_3$ is sequentially monotone closed in $\mathcal{B}_{sa}(A)$; hence $\mathcal{M}_3 = \mathcal{B}_{sa}(A)$, and this shows that

$$x,y \in \mathcal{B}_{sa}(A) \Rightarrow i(xy-yx) \in \mathcal{B}_{sa}(A).$$

It follows that

$$\mathcal{B}(A) = \mathcal{B}_{sa}(A) + i\mathcal{B}_{sa}(A)$$

is a $C^*$-algebra, and the Lemma is proved.

Remarks 1. The $C^*$-algebra $\mathcal{B}(A)$ has been defined as being the smallest $C^*$-subalgebra of $A^{**}$, whose self-adjoint part is sequentially monotone closed and contains the cone $(A_{sa})^m$. It is natural therefore, to consider that the elements of $\mathcal{B}(A)$ are the Borel elements of $A^{**}$ over A.

The definition we have given to the Borel enveloping $C^*$-algebra of A, however natural, would not be very useful if $\mathcal{B}(A)$, so defined, would not possess some basic properties which bore resemblance to the $C^*$-algebra $\mathcal{B}^b(X)$ of the bounded Borel measurable complex functions on the maximal spectrum X of A, in the case that A is commutative. It is, however, easy to see that if A is commutative, then the $C^*$-algebra $\mathcal{B}(A)$, defined as above, can be identified with $\mathcal{B}^b(X)$.

2. The approach by stages we have obtained in Lemma 4 for the $C^*$-algebra $\mathcal{B}(A)$ is necessary in order to avail of the implication

$$a \in (A_{sa})^m \Rightarrow a^2 \in \mathcal{U}(A),$$

which follows from Proposition 1, and will be used below in establishing the disintegration properties of $\mathcal{B}(A)$.

§7. We shall now study the behaviour of $\mathcal{B}(A)$ with respect to the canonical irreducible disintegrations of the cyclic representations of A, as defined in ([22], §2).

For any $p \in P(A)$ let $e_p \in A^{**}$ be its support, which is a minimal projection. We have

$$Ae_p = A^{**}e_p, \quad p \in P(A),$$

and we shall consider the vector space $\prod_{p \in P(A)} (Ae_p)$ and the mapping $\tau : A^{**} \to \prod_{p \in P(A)} Ae_p$, given by $\tau(a) = (\tau_p(a))_{p \in P(A)}$, where

$$\tau_p : A^{**} \to Ae_p, \quad p \in P(A),$$

is defined by $\tau_p(a) = ae_p$, $a \in A^{**}$.

Since we have

$$e_p a e_p = p(a)e_p, \quad a \in A^{**}, \; p \in P(A),$$

if we endow $Ae_p$ with the structure of a Hilbert space, given by the scalar product

$$(ae_p | be_p)_p = p(b^* a), \quad a, b \in A^{**},$$

we infer that the irreducible representation of A, corresponding by the GNS-construction to $p \in P(A)$, is unitarily equivalent to the left regular representation $\lambda_p$ of A on the left A-module $Ae_p$. As in [22], we define $\Gamma = \tau(A)$.

Let $\Omega_1 = \Omega_1(E_o(A))$ be the set ~~of all~~ maximal orthogonal probability Radon measures $\mu$ on $E_o(A)$, such that the barycenter $b(\mu) \in E(A)$. We shall denote by $\tilde{\mu}$ the probability measure induced on P(A) by $\mu$, and defined on the $\sigma$-algebra of the Borel measurable subsets of P(A), with respect to the maximal orthogonal topology of P(A). (We refer to [1], [17], [18], [19], [20], [21] and [22], for all questions regarding the construction and the proofs of the properties of the induced measures $\tilde{\mu}$). We shall denote $\tilde{\Omega}_1=\tilde{\Omega}_1(P(A))=\{\tilde{\mu};\mu\in\Omega_1(E_o(A))\}$.

Once $\tilde{\mu}$ (i.e., $\mu\in\Omega_1(E_o(A))$) has been chosen in $\tilde{\Omega}_1$, we can define a (non-separated and non-complete) scalar product on $\Gamma$, by

$$(\tau(a)\mid\tau(b))=\int_{P(A)} p(b^*a)d\tilde{\mu}(p),$$

and we can consider the $L^2$-completion of $\Gamma$ with respect to $\tilde{\mu}$, denoted $\Gamma^2(\tilde{\mu})$, as in ([17], p.154).

If we denote $f=b(\mu)$, then the mapping

$$V:\Gamma\to H_f,$$

given by

$$V(\tau(a))=\pi_f(a)\xi_f^o\ ,\ a\in A,$$

is correctly defined, isometric, and it has a dense range in $H_f$; hence, V can be uniquely extended as a unitary linear mapping onto $H_f$. If we denote by W this extension, then we have

$$W_\mu[(\lambda_p(a)\xi_p)_p]=\pi_f(a)\ W[(\xi_p)_p],\ (\xi_p)\in\Gamma^2(\tilde{\mu}).$$

(note that the elements of $\Gamma^2(\tilde{\mu})$ are (square integrable) vector fields); and also

$$\int_{P(A)}\|\lambda_p(a)\xi_p\|_p^2 d\tilde{\mu}(p)=\|\pi_f(a)W_\mu[(\xi_p)_p]\|^2,$$
$$a\in A,\ (\xi_p)_p\in\Gamma^2(\tilde{\mu}).$$

Moreover, since $\mu$ is orthogonal, the vector space $\Gamma^2(\tilde{\mu})$ is an $\mathcal{L}^\infty(P(A), \mathcal{B}(P(A);\Omega); \mathbb{C})$-module.

If we denote by $\tilde{\Gamma}^2(\tilde{\mu})$ the (complete and separated) Hilbert space obtained from $\Gamma^2(\tilde{\mu})$ by identifying two (strongly square integrable) vector fields which coincide $\tilde{\mu}$-a.e., then we have the canonical mapping $Q_{\tilde{\mu}}:\Gamma^2(\tilde{\mu})\to\tilde{\Gamma}^2(\tilde{\mu})$, and W factorizes through $Q_{\tilde{\mu}}$, i.e., there exists a uniquely determined Hilbert space isomorphism

$$\tilde{W}_{\tilde{\mu}}:\tilde{\Gamma}^2(\tilde{\mu})\to H_f ,$$

such that $\tilde{W}_{\tilde{\mu}}Q_{\tilde{\mu}}=W_{\tilde{\mu}}$.

Let us now consider a (bounded) field $(a_p)_{p\in P(A)}$ of operators $a_p\in\mathcal{L}(Ae_p)$, $p\in P(A)$.

We shall say that $(a_p)_{p\in P(A)}$ is a $\tilde{\mu}$ -<u>integrable</u> field of operators, if

$$(\xi_p)_{p\in P(A)}\in\Gamma^2(\tilde{\mu})\Rightarrow(a_p\xi_p)_{p\in P(A)}\in\Gamma^2(\tilde{\mu});$$

$(a_p)_{p\in P(A)}$ will be called a <u>universally integrable</u> field of operators, if it is $\tilde{\mu}$-integrable, for any $\tilde{\mu}\in\tilde{\Omega}_1$.

It is obvious that any $\tilde{\mu}$-integrable field of operators $(a_p)_{p\in P(A)}$ factors through $Q_\mu$; i.e., it exists a uniquely determined operator $\tilde{a}_{\tilde{\mu}}\in\mathcal{L}(\tilde{\Gamma}^2(\tilde{\mu}))$, such that

$$Q_\mu[(a_p\xi_p)_p]=\tilde{a}_\mu Q_\mu[(\xi_p)_p], \quad (\xi_p)_p\in\Gamma^2(\tilde{\mu}).$$

The operators in $\mathcal{L}(H_f)$ of the form

$$\tilde{W}_{\tilde{\mu}}\tilde{a}_{\tilde{\mu}}\tilde{W}_{\tilde{\mu}}^{-1}$$

corresponding to $\tilde{\mu}$-integrable fields of operators $(a_p)_p$, are said to be the decomposable operators in $\mathcal{L}(H_f)$ (with respect to the chosen irreducible disintegration, which still depends on the choice of the maximal orthogonal measure $\mu\in\Omega_1$, such that $b(\mu)=f$).

We shall say that an element $a\in A^{**}$ is universally disintegrable if the field $(\lambda_p(a))_{p\in P(A)}$ of operators $\lambda_p(a)$, given by the (extended) left regular representations $\lambda_p: A^{**}\to\mathcal{L}(Ae_p)$, $p\in P(A)$, is universally integrable, and if for the corresponding decomposable operator $\widetilde{W}_{\mu}\tilde{\lambda}(a)_{\tilde{\mu}}\widetilde{W}_{\tilde{\mu}}^{-1}$ in $\mathcal{L}(H_f)$ we have

$$\widetilde{W}_{\tilde{\mu}}\tilde{\lambda}(a)_{\tilde{\mu}}\widetilde{W}_{\tilde{\mu}}^{-1}=\tilde{\pi}_f(a),$$

where $\tilde{\pi}_f$ is the normal extension of $\pi_f$ to $A^{**}$, for any $\mu\in\Omega_1$, such that $b(\mu)=f$.

Lemma 5. i) If $a,b\in A^{**}$ are universally disintegrable, then $a+b$ and $ab$ are universally disintegrable, and $\alpha a$ is universally disintegrable, for any $\alpha\in\mathbb{C}$.

ii) The norm limit of a sequence of universally disintegrable elements of $A^{**}$ is universally disintegrable.

iii) If $(a_n)_{n\in\mathbb{N}}$ is a bounded monotone sequence of universally disintegrable elements in $A^{**}_{sa}$, then its limit is a universally disintegrable element.

Proof. i) For any $(\xi_p)_{p\in P(A)}\in\Gamma^2(\tilde{\mu})$ we have that

$$(a\xi_p)_p\in\Gamma^2(\tilde{\mu}) \text{ and } (b\xi_p)_{p\in P(A)}\in\Gamma^2(\tilde{\mu});$$

it follows that

$$((a+b)\xi_p)_p\in\Gamma^2(\tilde{\mu}) \text{ and } ((ab)\xi_p)_p=(a(b\xi_p))_p\in\Gamma^2(\tilde{\mu});$$

and also $((\alpha a)\xi_p)_p=(\alpha(a\xi_p))_p\in\Gamma^2(\tilde{\mu})$.

On the other hand, we have

$$\tilde{W}\tilde{\lambda}(a+b)\tilde{W}^{-1}=\tilde{W}(\tilde{\lambda}(a)+\tilde{\lambda}(b))\tilde{W}^{-1}=$$
$$=\tilde{W}\tilde{\lambda}(a)\tilde{W}^{-1}+\tilde{W}\tilde{\lambda}(b)\tilde{W}^{-1}=\tilde{\pi}_f(a)+\tilde{\pi}_f(b)=\tilde{\pi}_f(a+b)$$

and

$$\tilde{W}\tilde{\lambda}(ab)\tilde{W}^{-1}=\tilde{W}\tilde{\lambda}(a)\tilde{\lambda}(b)\tilde{W}^{-1}=$$
$$=\tilde{W}\tilde{\lambda}(a)\tilde{W}^{-1}\tilde{W}\tilde{\lambda}(b)\tilde{W}^{-1}=\tilde{\pi}_f(a)\tilde{\pi}_f(b)=\tilde{\pi}_f(ab),$$
$$\tilde{W}\tilde{\lambda}(\alpha a)\tilde{W}^{-1}=\alpha\tilde{W}\tilde{\lambda}(a)\tilde{W}^{-1}=\alpha\tilde{\pi}_f(a)=\tilde{\pi}_f(\alpha a).$$

Assertion i) is proved.

ii) Let $a_n\in A^{**}$, $n\in\mathbb{N}$, be universally disintegrable and assume that $\lim_{n\to\infty} a_n=a$ in the uniform topology of $A^{**}$. Then

$$(\xi_p)_p\in\Gamma^2(\tilde{\mu}) \Rightarrow (a_n\xi_p)_p\in\Gamma^2(\tilde{\mu}),\ \forall n\in\mathbb{N}.$$

We then have $\lim_{n\to\infty}(a_n\xi_p)_p=(a\xi_p)_p$ in the (seminorm) topology of $\Gamma^2(\tilde{\mu})$, and, therefore, we have that $(a\xi_p)_p\in\Gamma^2(\tilde{\mu})$.

It is obvious that $\lim_{n\to\infty}\tilde{\lambda}(a_n)_{\tilde{\mu}}=\tilde{\lambda}(a)_{\tilde{\mu}}$ and also $\lim_{n\to\infty}\tilde{\pi}_f(a_n)=\tilde{\pi}_f(a)$, whence we immediately infer that

$$\tilde{W}\tilde{\lambda}(a)\tilde{W}^{-1}=\tilde{\pi}_f(a).$$

iii) Assume that $(a_n)_{n\geq 0}$ is an increasing sequence in $A^{**}_{sa}$, converging to $a\in A^{**}_{sa}$, and that $a_n$ is universally disintegrable, for any $n\in\mathbb{N}$. By Vigier's Theorem, we have

$$\lim_{n\to} \tilde{\pi}_u(a_n) = \tilde{\pi}_u(a), \text{ strongly in } H_u,$$

and, therefore, from the equalities

$$\|(a_n b e_p)_p - (a_m b e_p)_p\|^2 = \int_{P(A)} p(b^*(a_n - a_m)^*(a_n - a_m)b)\, d\tilde{\mu}(p) =$$
$$= \|\tilde{\pi}_f(a_n - a_m)\pi_f(b)\xi_f^o\|^2 = f(b^*(a_n - a_m)^2 b),$$

which hold for $b \in A$, $m,n \in \mathbb{N}$, we infer that $(abe_p)_p \in \Gamma^2(\tilde{\mu})$ for any $b \in A$. Since $\Gamma$ is dense in $\Gamma^2(\tilde{\mu})$, we obtain the implication

$$(\xi_p)_p \in \Gamma^2(\tilde{\mu}) \Rightarrow (a\xi_p) \in \Gamma^2(\tilde{\mu}).$$

On the other hand, from the equalities

$$\|(a_m b e_p)_p\|^2 = \int_{P(A)} p(b^* a_m^2 b)\, d\tilde{\mu}(p) = \|\tilde{\pi}_f(a_m)\pi_f(b)\xi_f^o\|^2 =$$
$$= f(b^* a_m^2 b), \quad b \in A,\ m \in \mathbb{N},$$

we infer that

$$\|(abe_p)_p\|^2 = \int_{P(A)} p(b^* a^2 b)\, d\tilde{\mu}(p) = \|\tilde{\pi}_f(a)\pi_f(b)\xi_f^o\|^2 =$$
$$= f(b^* a^2 b), \quad b \in A,$$

whence we get that

$$\tilde{W}_{\tilde{\mu}} \tilde{\lambda}(a)_{\tilde{\mu}} \tilde{W}_{\tilde{\mu}}^{-1} = \tilde{\pi}_r(a);$$

i.e., a is universally disintegrable. The Lemma is proved.

<u>Remark</u>. We were not able to decide whether $a^*$ is universally disintegrable whenever a is so.

the formula

$$\int_{P(A)} p(c^*ab)\,d\tilde{\mu}(p)=f(c^*ab)$$

holds for any $b,c\in A$. In particular, the boundary barycentric calculus holds for $\rho(a)$.

Proof. Since $(be_p)_{p\in P(A)}\in\Gamma^2(\tilde{\mu})$, we have $(abe_p)_{p\in P(A)}\in\Gamma^2(\tilde{\mu})$, and for any $b,c\in A$ we have

$$\int_{P(A)} p(c^*ab)\,d\tilde{\mu}(p)=\int_{P(A)}(\lambda_p(a)be_p|ce_p)_p\,d\tilde{\mu}(p)=$$

$$=(\lambda(a)(be_p)_p|(ce_p)_p)=(W_{\tilde{\mu}}\lambda(a)(be_p)_p|W_{\tilde{\mu}}(ce_p)_p)=$$

$$=(\tilde{\pi}_f(a)\theta_f(b)|\theta_f(c))=f(c^*ab).$$

On the other hand, we have $(e_p)_{p\in P(A)}\in\Gamma^2(\tilde{\mu})$ and $W[(e_p)_{p\in P(A)}]=\xi_f^o$ (see [17], Proposition 4.5 and Theorem 4.3). We immediately infer that

$$\int_{P(A)} p(a)\,d\tilde{\mu}(p)=f(a).$$

The Lemma is proved.

Let $\mathcal{D}(A)\subset A^{**}$ be the set of all universally disintegrable elements in $A^{**}$; and define $\mathcal{D}_o(A)=\mathcal{D}(A)\cap\mathcal{D}(A)^*$, where $\mathcal{D}(A)^*=\{a^*\in A^{**};\ a\in\mathcal{D}(A)\}$. Then we have

Proposition 3. $\mathcal{D}_o(A)$ is a $C^*$-algebra whose selfadjoint part is sequentially monotone closed.

Remark. It is proper to call the elements of $\mathcal{D}_o(A)$ the regular universally disintegrable elements in $A^{**}$ (see [11] ).

Proposition 4. The atomic representation is faithful on $\mathfrak{D}_o(A)$.

Lemma 7. We have $b^*\mathcal{U}(A)b\subset\mathcal{U}(a)$, for any $b\in A$.

Proof. Let $a\in\mathcal{U}(A)$, $f\in E(A)$ and $\varepsilon>0$ be given, and define $g=b.f.b^*$, which obviously belongs to $f(b^*b)E(A)$. We can find $h,k\in(A_{sa})^m$, such that

$$-k\leq a\leq h \quad \text{and} \quad g(h+k)<\varepsilon.$$

We then have that $b^*hb$, $b^*kb\in(A_{sa})^m$ and

$$-b^*kb\leq b^*ab\leq b^*hb \quad \text{and} \quad f(b^*hb+b^*kb)<\varepsilon.$$

The Lemma is proved.

Corollary. For any $a\in\mathcal{U}(A)$ and $b,c\in A$ the complex function

$$P(A)\ni p\longmapsto p(c^*ab)$$

is $\tilde{\mu}$-integrable, for any $\tilde{\mu}\in\tilde{\mathcal{M}}_1$, and

$$(*) \qquad \int_{P(A)} p(c^*ab)\,d\tilde{\mu}(p)=f(c^*ab),$$

where $f=b(\mu)$.

Proof. We have

$$p(c^*ab)=\frac{1}{4}\big[p((b+c)^*a(b+c))-p((b-c)^*a(b-c))+$$
$$(1) \qquad +ip((b+ic)^*a(b+ic))-ip((b-ic)^*a(b-ic))\big],$$

whence the $\tilde{\mu}$-measurability immediately follows, if we take into account Lemma 7 and ([20], Theorem 3;[22], Theorem 1.8). Formula (*) now follows from (1), if we take into account the fact that for the elements of $\mathcal{U}(A)$ the boundary barycentric calculus holds (see [20], Theorem 3 and [22], Theorem 1.8). The Corollary is proved.

Lemma 8. Any $a\in\mathcal{B}^o_{sa}(A)$ is universally disintegrable.

Proof. We have

$$\mathcal{B}^o_{sa}(A)\subset\mathcal{U}(A)$$

(see §4). From Proposition 1 we infer that $a^2\in\mathcal{B}^o_{sa}(A)$. Let us now consider the vector space $\Gamma_a$ given by $\Gamma_a=\{((ab+c)e_p)_p;b,c\in A\}$. We obviously have that

$$\Gamma_A\subset\Gamma_a\subset\prod_{p\in P(A)}Ae_p$$

and

$$p((ab_2+c_2)^*(ab_1+c_1))=p(b_2^*a^2b_1)+p(b_2^*ac_1)+ +p(c_2^*ab_1)+p(c_2^*c_1);$$

hence, the function

$$P(A)\ni p\mapsto(\xi_p|\eta_p)_p\in\mathbb{C}$$

is $\tilde{\mu}$-integrable, for any $(\xi_p)_p,(\eta_p)_p\in\Gamma_a$. We can, therefore, consider the $L^2$-completion of $\Gamma_a$ with respect to $\tilde{\mu}$ (see [17], §4), which we shall denote by $\Gamma_a^2(\tilde{\mu})$. We can define correctly a linear mapping

$$V_a:\Gamma_a\to H_f$$

by

$$V_a[((ab+c)e_p)_p]=\widetilde{\pi}_f(a)\theta_f(b)+\theta_f(c), \quad b,c\in A,$$

where $\widetilde{\mu}\in\widetilde{\Omega}_1$ , and $f=b(\mu)$.

Indeed, if $p((ab+c)^*(ab+c))=0$, $p\in P(A)$, then, by the preceding Corollary, we have

$$f((ab+c)^*(ab+c))=\int_{P(A)} p((ab+c)^*(ab+c))\,d\widetilde{\mu}(p)=0,$$

and this implies that $\widetilde{\pi}_f(a)\theta_f(b)+\theta_f(c)=0$.
It is easy to see that $V_a$ is an isometric linear mapping; therefore, it extends uniquely to an isometric linear surjective mapping $W_a:\Gamma^2_a(\widetilde{\mu})\to H_f$. We infer that

$$\Gamma^2(\widetilde{\mu})=\Gamma^2_a(\widetilde{\mu}),$$

and, therefore, for any $b\in A$, there exists a strongly integrable vector field $(\xi_p)_{p\in P(A)}\in\Gamma^2(\widetilde{\mu})$, such that $abe_p=\xi_p$, $\widetilde{\mu}$-a.e. We infer that

$$(\xi_p)_{p\in P(A)}\in\Gamma^2(\widetilde{\mu})\Rightarrow(a\xi_p)_{p\in P(A)}\in\Gamma^2(\widetilde{\mu}).$$

On the other hand, from the formula

$$\int_{P(A)} p(b^*a^2b)\,d\widetilde{\mu}(p)=f(b^*a^2b)=\|\widetilde{\pi}_f(a)\theta_f(b)\|^2,$$

which holds for any $b\in A$, we infer that

$$\widetilde{W}_{\widetilde{\mu}}\widetilde{\lambda}(a)_{\widetilde{\mu}}\widetilde{W}^{-1}_{\widetilde{\mu}}=\widetilde{\pi}_f(a),$$

and the Lemma is proved.

The following Theorem is the main result of the paper.

Theorem 1. Any $a\in\mathcal{B}(A)$ is universally disintegrable.

Proof. From Lemma 8 and part i) of Lemma 5 we infer that any element $a\in\mathcal{A}$ (where $\mathcal{A}$ is the $*$-algebra defined in the proof of Lemma 4) is universally disintegrable. From part ii) of Lemma 5 we then infer that any element $a\in\mathcal{B}_1(A)$ is universally disintegrable. Let now

$$\mathcal{M}_4=\{a\in\mathcal{B}(A)_{sa};\ a \text{ is univ.disint.}\}.$$

From the above argument, we have

$$\mathcal{B}_1(A)_{sa}\subset\mathcal{M}_4\subset\mathcal{B}(A)_{sa}.$$

From part iii) of Lemma 5 we infer that $\mathcal{M}_4$ is a sequentially monotone closed subset of $A^{**}_{sa}$. Lemma 4 now implies that $\mathcal{M}_4=\mathcal{B}(A)_{sa}$, whereas part i) of Lemma 5 now ends the proof.

Let $z_o=\sup\{e_p;\ p\in P(A)\}$ be the supremum of all minimal projections of $A^{**}$. Then $z_o$ is a central projection and the mapping $A^{**}\ni a\mapsto az_o$ is the (space-free) atomic representation.

Corollary 1. The atomic representation is isometric on $\mathcal{B}(A)$.

Proof. The assertion immediately follows from Lemma 6.

Remark. This result answers positively to a question raised in ([15], Ch.II, §10.15, Remark (2), p.434).

§8. In this section we shall study the functor $A \mapsto \mathcal{B}(A)$ with respect to surjective morphisms $\pi: A \to B$ of $C^*$-algebras.

Let $E$ be a Hausdorff locally convex topological real vector space, and $K \subset E$ a (non-empty) compact convex subset. We recall that a convex subset $F \subset K$ is said to be a <u>face</u> of $K$ if

$$x \in F,\ x = \alpha x' + (1-\alpha)x'',\ x', x'' \in K,\ \alpha \in (0,1) \Rightarrow x', x'' \in F.$$

The face $F$ is said to be <u>supplemented</u> if there exists a face $F' \subset K$, such that $F \cap F' = \emptyset$ and for any $x \in K \setminus (F \cup F')$ there exist unique $\alpha \in [0,1]$, $y \in F$ and $y' \in F'$ such that $x = \alpha y + (1-\alpha)y'$ (and, of course, $K = co(F \cup F')$). The face $F'$ is said to be a <u>supplement</u> of $F$. As shown by Perdrizet, if $F$ has a supplement $F'$, then this is unique (see [9]).

The following Lemma is due to Perdrizet (see [9]; [15], Ch.II, Lemma 9.1.8; [16], Ch.III, Lemma 6.26, p.170).

<u>Lemma 9.</u> <u>Let $F \subset K$ be a compact supplemented face. Then any bounded lower (respectively, upper) semi-continuous affine function $\varphi: F \to \mathbb{R}$ has a bounded lower (respectively, upper) semicontinuous affine extension $\psi: K \to \mathbb{R}$. If $\varphi$ is positive on $F$, then $\psi$ can be chosen to be positive on $K$; and, in any case, $\|\psi\| = \|\varphi\|$.</u>

<u>Proof.</u> Assume that $\varphi \geq 0$ on $F$; and that $\varphi$ is upper semicontinuous. Let $F'$ be a supplement of $F$, and consider the compact convex set

$$M = co((K \times \{0\}) \cup \{(x,t);\ x \in F,\ 0 \leq t \leq \varphi(x)\}) \subset K \times \mathbb{R}.$$

Then by $\psi(x) = \sup\{t;\ (x,t) \in M\}$, $x \in K$, we define a positive bounded upper semicontinuous affine function on $K$, which extends $\varphi$. We obviously have $\|\varphi\| = \|\psi\|$ . The general case reduces to this one by the transformations $\varphi \mapsto \varphi + \|\varphi\|$ and $\varphi \mapsto \|\varphi\| - \varphi$.

Lemma 10. Let A be any $C^*$-algebra and $I \subset A$ a closed two-sided ideal. Then the sets

$$F_I = \{f \in E_o(A);\ f|I = 0\}$$

and

$$F_I' = \{f \in E_o(A);\ \|f|I\| = 1\}$$

are supplementary faces of $E_o(A)$, whereas F is compact.

Proof. This is another formulation of ([5], Proposition 2.11.7 ; see, also, [16], Ch.III, Proposition 6.27, p.171).

The following result is essentially due to Combes, who proved it for $C^*$-algebras possessing the unit element (see [4]; [15], Ch.II, §9.2, Proposition 9.2.16).

Lemma 11. Let A, B be $C^*$-algebras and $\pi : A \to B$ a surjective morphism of $C^*$-algebras. Then we have $\pi^{**}(((A_{sa})^m)^-) = ((B_{sa})^m)^-$ and $\pi^{**}(((A_{sa})^m)^-_+) = ((B_{sa})^m)^-_+$ .

Proof. The dual mapping $\pi^* : B^* \to A^*$ restricts to an injective continuous mapping $\pi^* : E_o(B) \to E_o(A)$ and $\pi^*(E_o(B)) = F_{\ker\pi}$ , with the above notation. Since $\pi^{**} : A^{**} \to B^{**}$ is normal, we have

$$\pi^{**}((A_{sa})^m) \subset (B_{sa})^m,$$

and, therefore, since $\pi^{**}$ is norm continuous, we have

$$\pi^{**}(((A_{sa})^m)^-) \subset ((B_{sa})^m)^-.$$

Let now $b \in ((B_{sa})^m)^-$. Then $\rho_B(b)$ is a bounded lower semicontinuous function of $E_o(B)$, such that $\rho_B(b)(0)=0$. We infer that there exists a (unique) bounded lower semicontinuous function $\varphi : F_{ker} \to \mathbb{R}$, such that $\varphi(0)=0$ and

$$\varphi \circ \pi^* = \rho_B(b).$$

By Lemma 9, there exists a bounded lower semicontinuous function $\psi : E_o(A) \to \mathbb{R}$, such that $\psi | F_{ker\pi} = \varphi$, $\|\psi\| = \|\varphi\|$ ; and, $\psi \geq 0$ if $\varphi \geq 0$. From ([8], Proposition 3.11.5) we infer that there exists a unique $a \in ((A_{sa})^m)^-$, such that $\rho_A(a) = \psi$, and $\|a\| = \|\psi\| = \|\varphi\| = \|b\|$ ; moreover, $a \geq 0$ if $b \geq 0$. It is obvious that $\pi^{**}(a) = b$, and the Lemma is proved.

The following result is due to Pedersen (see [15], Ch.II, §6, Lemma 6.1.7).

Lemma 12. Let $\pi : A \to B$ be a $C^*$-homomorphism of $C^*$-algebras and $J \subset A$ a JC-subalgebra. Then $\pi(J) \subset B$ is a JC-subalgebra. For any $x_o, x_1, x_2 \in J$, such that $x_o \leq x_2$ and $\pi(x_o) \leq \pi(x_1) \leq \pi(x_2)$, there exists an $x \in J$, such that $x_o \leq x \leq x_2$ and $\pi(x) = \pi(x_1)$.

We recall the by a $C^*$-homomorphism is meant here any selfadjoint (hermitean) linear mapping, such that

$$\pi(\tfrac{1}{2}(ab+ba)) = \tfrac{1}{2}(\pi(a)\pi(b) + \pi(b)\pi(a)),$$

It is obvious that any $*$-homomorphism of $C^*$-algebras is a $C^*$-homomorphism.
For the proof of this Lemma we also refer to a paper by Størmer (see [14]).

The following result is essentially due to Combes, who proved it for $C^*$-algebras having the unit element (see [4]; [15], Ch.II, §9.2, Proposition 9.2.16).

Lemma 13. $\pi^{**}(\mathcal{S}_{sa}(A)^-)=\mathcal{S}_{sa}(B)^-$.

Proof. For any $c\in\mathcal{S}_{sa}(A)$ there exist $a,b\in(A_{sa})^m$, such that $c=a-b$ and, therefore, $\pi^{**}(c)=\pi^{**}(a)-\pi^{**}(b)\in\pi^{**}(((A_{sa})^m)^-)-$ $-\pi^{**}(((A_{sa})^m)^-)=((B_{sa})^m)^- -((B_{sa})^m)^-\subset\mathcal{S}_{sa}(B)^-$. This shows that $\pi^{**}(\mathcal{S}_{sa}(A)^-)\subset\mathcal{S}_{sa}(B)^-$.

For any $c'\in\mathcal{S}_{sa}(B)$ there exist $a',b'\in(B_{sa})^m$, such that $c'=a'-b'$, and we can find $a,b\in((A_{sa})^m)^-$, such that $a'=\pi^{**}(a)$, $b'=\pi^{**}(b)$. We infer that $c'\in\pi^{**}(((A_{sa})^m)^-)-\pi^{**}(((A_{sa})^m)^-)\subset$ $\subset\pi^{**}(\mathcal{S}_{sa}(A)^-)$. Therefore, we have that

$$(*)\qquad \mathcal{S}_{sa}(B)\subset\pi^{**}(\mathcal{S}_{sa}(A)^-)\subset\mathcal{S}_{sa}(B)^-.$$

Since $\mathcal{S}_{sa}(A)^-$ is a JC-algebra, whereas $\pi^{**}:A^{**}\longrightarrow B^{**}$ is a $C^*$-homomorphism, by Lemma 12, we infer that $\pi^{**}(\mathcal{S}_{sa}(A)^-)$ is a JC-algebra. From $(*)$ we infer then the required equality. The Lemma is proved.

Lemma 14. $\pi^{**}(\mathcal{B}^o_{sa}(A))=\mathcal{B}^o_{sa}(B)$.

Proof. Let us define

$$\mathcal{M}=\{a\in\mathcal{B}^o_{sa}(A);\ \pi^{**}(a)\in\mathcal{B}^o_{sa}(B)\}$$

We have

$$\mathcal{S}_{sa}(A)^{-} \subset \mathcal{M}$$

and $\mathcal{M}$ obviously is sequentially monotone closed in $\mathcal{B}^{o}_{sa}(A)$. It follows that

$$\mathcal{M} = \mathcal{B}^{o}_{sa}(A),$$

and, therefore,

(1) $$\mathcal{S}_{sa}(B)^{-} \subset \pi^{**}(\mathcal{B}^{o}_{sa}(A)) \subset \mathcal{B}^{o}_{sa}(B).$$

We shall now prove that $\pi^{**}(\mathcal{B}^{o}_{sa}(A))$ is sequentially monotone closed in $\mathcal{B}^{o}_{sa}(B)$, with the help of Lemma 12. Indeed, let $(b_n)_{n\geq 0}$ be an increasing bounded sequence in $\pi^{**}(\mathcal{B}^{o}_{sa}(A))$. Then there exists a $k \in \mathbb{R}$, such that

$$b_o \leq b_n \leq b_{n+1} \leq k1, \quad n \in \mathbb{N}.$$

Let $a_o \in \mathcal{B}^{o}_{sa}(A)$ be such that $\pi^{**}(a_o) = b_o$. By Lemma 12 we can find an $a_1 \in \mathcal{B}^{o}_{sa}(A)$, such that $a_o \leq a_1 \leq k1$, and $\pi^{**}(a_1) = b_1$. Inductively, we can find an increasing sequence $(a_n)_{n\geq 0}$ in $\mathcal{B}^{o}_{sa}(A)$, such that

$$a_o \leq a_1 \leq a_n \leq a_{n+1} \leq k1 \quad \text{and} \quad \pi^{**}(a_n) = b_n,$$

for any $n \in \mathbb{N}$. If $a = \lim_{n\to\infty} a_n$ in $\mathcal{B}^{o}_{sa}(A)$, then $\pi^{**}(a) = b$, by virtue of the normality of $\pi^{**}$. It follows that $b \in \pi^{**}(\mathcal{B}^{o}_{sa}(A))$, and, therefore the latter set is sequentially monotone closed in $\mathcal{B}^{o}_{sa}(B)$. From (1) we now infer that $\pi^{**}(\mathcal{B}^{o}_{sa}(A)) = \mathcal{B}^{o}_{sa}(B)$, and the Lemma is proved.

Remark. This Lemma is due to Combes (see [4]; [15], Ch. II, §10, Proposition 10.8(2)).

We shall now denote by $\mathcal{A}(A)$, respectively $\mathcal{A}(B)$, the $*$-algebras, corresponding to A, respectively B, as in the proof of Lemma 4.

Lemma 15. $\pi^{**}(\mathcal{A}(A))=\mathcal{A}(B)$.

Proof. Immediate consequence of Lemma 14 and of the definition of $\mathcal{A}$.

Lemma 16. $\pi^{**}(\mathcal{B}_1(A))=\mathcal{B}_1(B)$.

Proof. Since $\pi^{**}$ is norm continuous, we have

$$(*) \qquad \mathcal{A}(B)\subset\pi^{**}(\mathcal{B}_1(A))\subset\mathcal{B}_1(B),$$

Since $\pi^{**}$ is a $*$-homomorphism of $C^*$-algebras, $\pi^{**}(\mathcal{B}_1(A))$ is a $C^*$-algebra, and the required equality now immediately follows from $(*)$.

We can now repeat step by step the above arguments, in order to prove that

$$\pi^{**}(\mathcal{B}_{sa}(A))=\mathcal{B}_{sa}(B).$$

In this we manner, we get the following

Theorem 2. For any surjective $*$-homomorphism $\pi: A \to B$ we have $\pi^{**}(\mathcal{B}(A))=\mathcal{B}(B)$.

§ 9. In this section we shall introduce the "concrete" Borel enveloping $C^*$-algebras of the "concrete" $C^*$-algebras $A \subset \mathcal{L}(H)$, where $H$ is any Hilbert space; we shall assume that $A$ is a non-degenerate $C^*$-subalgebra of $\mathcal{L}(H)$.

Let $A''$ be the bicommutant of $A$ in $\mathcal{L}(H)$, and $\pi : A^{**} \to A''$ the normal extension of the identical representation $A \to \mathcal{L}(H)$.

We shall denote by $(A_{sa})^m_H$ the set of all elements in $A''_{sa}$, which are suprema (hence wO- or so- limits) of increasing nets in $A_{sa}$.

<u>Lemma 17</u>. $(A_{sa})^m_H = \pi((A_{sa})^m)$.

<u>Proof</u>. Obvious, since $\pi$ is a normal $*$-homomorphism.

We shall denote $\mathcal{S}_{sa}(A;H) = (A_{sa})^m_H - (A_{sa})^m_H$. Of course, $\mathcal{S}_{sa}(A;H)$ is a real vector subspace of $A''_{sa}$. From $1 \in (A_{sa})^m_H$ we infer that $1 \in \mathcal{S}_{sa}(A;H)$. Let $\mathcal{S}^-_{sa}(A;H)$ be the norm closure of $\mathcal{S}_{sa}(A;H)$ in $A''_{sa}$. We have the following result due to Combes ([4]; [15], Ch.II, §9.2, Proposition 9.2.18).

<u>Lemma 18</u>. $\pi(\mathcal{S}^-_{sa}(A)) = \mathcal{S}^-_{sa}(A;H)$.

<u>Proof</u>. We obviously have that

$$\mathcal{S}_{sa}(A:H) \subset \pi(\mathcal{S}^-_{sa}(A)) \subset \mathcal{S}^-_{sa}(A;H),$$

since $\pi$ is norm continuous. Since $\pi$ is a $C^*$-homomorphism, $\pi(\mathcal{S}^-_{sa}(A))$ is a (norm closed) JC-algebra, and the equality obtains (see Lemma 12).

Corollary 1. $\mathcal{S}_{sa}^{-}(A;H)$ is a JC-algebra.

Remark. The preceding Corollary, which is due to Combes, is proved in ([15], Ch.II, §9.2, Remark 9.2.19) by another way, under the assumption that $1\in A$ (see, also, [4]).

We shall now consider the sequentially monotone closure $\mathcal{B}_{sa}^{o}(A;H)$ of $\mathcal{S}_{sa}^{-}(A;H)$ in $A_{sa}''$.

The following result is due to Combes ([4]; [15], Ch.II, §10, Proposition 10.8(2)).

Lemma 19. $\pi(\mathcal{B}_{sa}^{o}(A))=\mathcal{B}_{sa}^{o}(A;H)$.

Proof. We obviously have that

$$\mathcal{S}_{sa}^{-}(A;H)\subset\pi(\mathcal{B}_{sa}^{o}(A)).$$

Let us define

$$\mathcal{M}=\{a\in\mathcal{B}_{sa}^{o}(A);\pi(a)\in\mathcal{B}_{sa}^{o}(A;H)\}.$$

Then $\mathcal{M}$ is sequentially monotone closed and

$$\mathcal{S}_{sa}^{-}(A)\subset\mathcal{M}\subset\mathcal{B}_{sa}^{o}(A).$$

We infer that $\mathcal{M}=\mathcal{B}_{sa}^{o}(A)$ and, therefore,

$$\mathcal{S}_{sa}^{-}(A;H)\subset\pi(\mathcal{B}_{sa}^{o}(A))\subset\mathcal{B}_{sa}^{o}(A;H).$$

The proof now proceeds similarly to that of Lemma 14.

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We shall denote by $\mathcal{A}(A;H)$ the complex $*$-algebra generated in $\mathcal{L}(H)$ by $\mathcal{B}^{o}_{sa}(A;H)$. We obviously have

Lemma 20. $\pi(\mathcal{A})=\mathcal{A}(A;H)$.

and, if we denote by $\mathcal{B}_1(A;H)$ the norm closure of $\mathcal{A}(A;H)$ in $\mathcal{L}(H)$, we have the following

Corollary 1. $\pi(\mathcal{B}_1(A))=\mathcal{B}_1(A;H)$.

We can now consider the sequentially monotone closure $\mathcal{B}_{sa}(A;H)$ of the self-adjoint part $\mathcal{B}_1(A;H)_{sa}$ of $\mathcal{B}_1(A;H)$ in $A''_{sa}$.

We can now state the following

Theorem 3. i) $\mathcal{B}_{sa}(A;H)$ is the self-adjoint part of the smallest $C^*$-algebra $\mathcal{B}(A;H)$, which has the properties:

a) $(A_{sa})^m_H \subset \mathcal{B}(A;H) \subset A''$;

b) the self-adjoint part $\mathcal{B}(A;H)_{sa}$ is sequentially monotone closed in $A''_{sa}$.

ii) $\pi(\mathcal{B}(A))=\mathcal{B}(A;H)$.

Proof. Similar to that of Lemma 4.

§10. In this section we shall consider surjective $*$-homomorphisms of concrete $C^*$-algebras.

The following Theorem is an extension of some results of Combes (see [15], Ch.II, §9.2, Proposition 9.2.18 and §10, Proposition 10.8(2); [4]).

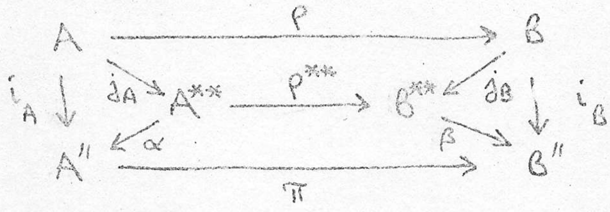
Theorem 4. Let $A\subset\mathcal{L}(H)$ and $B\subset\mathcal{L}(K)$ be concrete non-degenerate $C^*$-algebras and let $\pi: A''\to B''$ be a surjective normal $*$-homomorphism, such that $\pi(A)=B$. Then we have

a) $\pi(((A_{sa})^m_H)^-)=((B_{sa})^m_K)^-$.

b) $\pi(\mathcal{S}^{-}_{sa}(A;H))=\mathcal{S}^{-}_{sa}(B;K)$;

c) $\pi(\mathcal{B}(A;H))=\mathcal{B}(B;K)$.

Proof. We have the following commutative diagram




where $i_A$ and $i_B$ are the inclusions, $\alpha$ and $\beta$ are their normal extensions, $\rho=\pi|A$, whereas $j_A$ and $j_B$ are the canonical mappings. The details of the proof are now left to the reader.

Remark. The preceding Theorem, part a) and b) are due to Combes for $C^*$-algebras possessing the unit element. Part c) appears here for the first time.

§11. Let $f\in E(A)$ and let $\mu\in\Omega_1$, be a maximal orthogonal Radon probability measure; let $\tilde{\mu}\in\tilde{\Omega}_1$ be the corresponding boundary measure. We shall consider the associated cyclic representation $\pi_f:A\to\mathcal{L}(H_f)$ by the GNS - construction, and the canonical irreducible disintegration, as described in section 7.

From Theorem 1 and Theorem 3, c) we immediately infer the following

Theorem 5. Any operator $a\in\mathcal{B}(\pi_f(A);H_f)$ is decomposable with respect to any canonical irreducible disintegration of $\pi_f$.

Remark. This Theorem is an extension of ([22], Proposition 2.1), where the case of the Baire operators were considered.

§12. As an example, let us consider the case $A=\mathcal{K}(H)$, the

$C^*$-algebra of all compact (linear) operators on an arbitrary Hilbert space H. It is easy to see that, in this case, $\mathcal{B}(A)=\mathcal{L}(H)$, whereas $\mathcal{B}_o(A)$ is the $C^*$-algebra of all operators in $\mathcal{L}(H)$ having separable ranges; it follows that $\mathcal{D}_o(A)=\mathcal{L}(H)$.
This case has a relevance to Naimark's Problem (see[7];[8], p.225;[13], p.236;[17], §7;[21]). As it is well known, the spectrum of $A=\mathcal{K}(H)$ has only one point;i.e.,all irreducible representations of $\mathcal{K}(H)$) are unitarily equivalent (see [5], Corollary 4.1.5). The problem is to establish the converse to this Theorem which is known to be true in the separable case (see [12];[17], Theorem 7.5). Let A be any $C^*$-algebra and let $z_o \in A^{**}$ be the (central) projection which is the supremum of all minimal projections in $A^{**}$. Then Corollary 1 to Theorem 1 can be also stated in the following equivalent manner:

<u>Corollary 1'.</u> <u>The $*$-homomorphism $\mathcal{B}(A)\ni a \mapsto az_o$ is injective.</u>

<u>Remark</u>. The mapping $A^{**}\ni a \mapsto az_o$ is injective if, and only if, $z_o=1$. This obviously happens for $A=\mathcal{K}(H)$.

We recall that a cardinal number m is said to be (continously) <u>measurable</u> if on (any) set M, such that card M=m, there exists a probability measure $P:\mathcal{P}(M)\to[0,1]$, defined on the class $\mathcal{P}(M)$ of <u>all</u> the subsets of M, and such that

$$P(\{x\}) = 0, \quad \forall\ x\in M.$$

(The cardinal number m is said to be <u>measurable</u> if, moreover, the range of P is equal to $\{0,1\}$).

$\mathcal{E} \subset A^{**}$ of all minimal projections. We have then a bijective mapping

$$P(A) \ni p \mapsto e_p \in \mathcal{E},$$

which associates to any $p \in P(A)$ its support $e_p$ in $A^{**}$. For $p_1, p_2 \in P(A)$ the corresponding irreducible representations $\pi_{p_1}, \pi_{p_2}$ of A are unitarily equivalent if, and only if, the central covers of $e_{p_1}$ and $e_{p_2}$ are equal:

$$c(e_{p_1}) = c(e_{p_2}).$$

Let then $(p_i)_{i \in I}$ be a complete set of representatives for this equivalence, and define

$$z_i = c(e_{p_i}), \quad i \in I.$$

Then $z_i$, $i \in I$, are mutually orthogonal minimal central projections in $A^{**}$ and

$$\sum_{i \in I} z_i = z_o$$

If we denote $H_i = A^{**} e_{p_i}$, $i \in I$, then for the left regular representation $\lambda_i : A^{**} \to \mathcal{L}(H_i)$ we have $\ker \lambda_i = A^{**}(1 - z_i)$, $i \in I$; i.e., $A^{**} z_i$ can be identified with $\mathcal{L}(H_i)$.

For any $i \in I$, let us choose an orthonormal basis in $H_i$ consisting of partial isometries $w_{ij} \in A^{**} e_{p_i}$, $j \in I_i$, such that $w_{ij}^* w_{ij} = e_{p_i}$, $w_{ij} w_{ij}^* = e_{ij} \in A^{**} z_i$, for any $j \in I_i$ and any $i \in I$. Then $e_{ij}$ are minimal projections in $A^{**}$, such that

$$\sum_{j \in I_i} e_{ij} = z_i, \qquad i \in I.$$

It follows that

$$\sum_{i\in I}\sum_{j\in I_i} e_{ij} = z_o,$$

and, therefore, for any $p\in P(A)$, we have

$$\sum_{i\in I}\sum_{j\in I_i} e_{ij}(p) = 1.$$

Let $H_a = \bigoplus_{i\in I} H_i$. Then the direct sum $\lambda_a = \bigoplus_{i\in I}\lambda_i$ of the left regular representations can be identified with the reduced atomic representation

$$\lambda_a : A^{**} \to \mathcal{L}(H_a);$$

we have $\lambda_a(x)[(\xi_i)_{i\in I}] = (x\xi_i)_{i\in I}$, $x\in A^{**}$, $(\xi_i)_{i\in I}\in H_a$, and

$$\ker \lambda_a = A^{**}(1-z_o).$$

Let $d(A)$ be the Hilbert dimension of $H_a$ and let $J$ be the disjoint union of the sets $I_i$, $i\in I$. Then

$$d(A) = \text{card } J.$$

We recall that a $C^*$-algebra A is said to be <u>elementary</u> if it is isomorphic to a $\mathcal{K}(H)$, for a suitable Hilbert space H; A is said to be <u>scattered</u> if any $f\in E(A)$ is of the form $f=\sum_{i=0}^{\infty}\alpha_i p_i$, where $\alpha_i \geq 0$, $i\in \mathbb{N}$, $\sum_{i=0}^{\infty}\alpha_i = 1$ and $p_i\in P(A)$, $i\in\mathbb{N}$ (see[3], p.66).

The following Theorem gives a partial solution to Naimark's Problem.

Theorem 6. If $\mathcal{D}_o(A)z_o=A^{**}z_o$ and if $d(A)$ is not measurable then $z_o=1$; hence $\mathcal{D}_o(A)=A^{**}$. If, moreover, card $I=1$, then $A$ is elementary.

Proof. a) Assume, by way of contradiction, that $z_o \neq 1$. Then there exists an $f_o \in E(A)$, such that $f_o(z_o)=0$. Let $\mu_o$ be any maximal orthogonal Radon probability measure on $E_c(A)$, such that $b(\mu_o)=f_o$, and let $\tilde{\mu}_o$ be the boundary measure induced on $P(A)$ (see [22]).

For any $M \subset J$ let us define

$$e_M = \sum_{i \in I} \sum_{j \in M \cap I_i} e_{ij} .$$

Then $e_M$ is a projection in $A^{**}$, for any $M \in \mathcal{P}(J)$, and the mapping

$$\mathcal{P}(J) \ni M \longmapsto e_M \in A^{**}z_o$$

is a spectral measure. By hypothesis, for any $M \in \mathcal{P}(J)$ there exists a uniquely determined projection $d_M \in \mathcal{D}_o(A)$, such that

$$e_M = z_o d_M .$$

(see Proposition 4). We infer that we have

$$e_M(p)=d_M(p), \quad p \in P(A),\ M \in \mathcal{P}(J).$$

Since the boundary barycentric calculus holds for any $d \in \mathcal{D}_o(A)$ (see Lemma 6), we have

$$\int_{P(A)} e_M(p)\, d\tilde{\mu}_o(p) = \int_{P(A)} d_M(p)\, d\tilde{\mu}_o(p) = f_o(d_M),$$

for any $M \in \mathcal{P}(J)$.

We have the following properties

i) If $(M_n)_{n\geq 0}$ is any increasing sequence in $\mathcal{P}(J)$, then

$$e_{\bigcup\limits_{n\geq 0} M_n} = \sup\{e_{M_n};\ n\geq 0\}$$

and

$$d_{\bigcup\limits_{n\geq 0} M_n} = \sup\{d_{M_n};\ n\geq 0\},$$

because the mapping $\mathcal{D}_o(A)\ni d\mapsto dz_o\in\mathcal{L}(H_a)$ is a sequentially monotone isomorphism.

ii) $j\in J \Rightarrow e_{\{j\}} = d_{\{j\}} = e_{ij}$, if $j\in I_i$.

iii) $d_J = 1$.

It follows that the mapping $\nu: \mathcal{P}(J)\to[0,1]$, given by $\nu(M) = f_o(d_M)$, $M\in\mathcal{P}(J)$, is a probability measure on $\mathcal{P}(J)$, such that $\nu(\{j\})=0$, $\forall\, j\in J$. Since card $J=d(A)$, we arrived at a contradiction.

b) Assume now that, moreover, card $I=1$. Since $z_o=1$, we infer that $A^{**}=\mathcal{L}(H_a)$ and, therefore, $A$ is scattered. We infer that any (cyclic) representation of $A$ is of type I; hence, $A$ is of type I, as a $C^*$-algebra. It follows that $A\supset\mathcal{K}(H_a)$, since the atomic representation is irreducible, and, therefore, $A=\mathcal{K}(H_a)$, since $A$ is simple. The Theorem is proved.

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