INSTITUTUL
DE
MATEMATICĂ

INSTITUTUL NAȚIONAL
PENTRU CREAȚIE
ȘTIINȚIFICĂ ȘI TEHNICĂ

ISSN 0250 3638

A DESINGULARIZATION THEOREM OF NÉRON TYPE

by

Mihai CIPU and Dorin POPESCU

PREPRINT SERIES IN MATHEMATICS

No.36/1984

BUCUREȘTI

A DESINGULARIZATION THEOREM OF NÉRON TYPE

by

Mihai CIPU and Dorin POPESCU*)

June 1984

*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd. Pacii 220, 79622 Bucharest, Romania

A DESINGULARIZATION THEOREM OF NÉRON TYPE

Mihai CIPU and Dorin POPESCU

§1. INTRODUCTION

The Implicit Function Theorem (shortly IFT) is a powerful method for studying the solvability of certain polynomial equations. In general it would be nice to reduce the solvability of some polynomial equations to the solvability of another ones for which it is possible to apply the IFT. For example Néron's p-desingularization [N] says in particular that an unramified extension of discrete valuation rings $R \longrightarrow R'$ inducing separable extensions on fraction and residue fields, is a filtered inductive limit of smooth R-algebras of finite type. More precisely, given an R-algebra of finite type B and an R-morphism $\alpha : B \longrightarrow R'$, there exist a smooth R-algebra of finite type B' and two morphisms $\beta : B \longrightarrow B'$, $\gamma : B' \longrightarrow R'$ such that the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{\ \beta\ } & B' \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\gamma} & \\ & R' & \end{array}$$

In particular Néron's result gives a method to substitute the solvability in R' of certain polynomial equations over R with the solvability of another ones for which it is possible to apply the IFT. A natural extension of Néron's result is the following:

(1.1) <u>Theorem</u> ($[P_3]$ Theorem (2.5)). Let $u : A \longrightarrow A'$ be a morphism of noetherian rings. Then the following statements are equivalent:

i) u is regular,

ii) u is a filtered inductive limit of smooth morphisms of finite type,

iii) A' is a filtered inductive limit of standard smooth A-algebras,

iv) for every A-algebra B and every A-morphism $\alpha: B \longrightarrow A'$ there exist a standard smooth A-algebra B' and two A-morphisms $\beta: B \longrightarrow B'$, $\gamma: B' \longrightarrow A'$ such that the following diagram commutes:

(1.1.1)
$$\begin{array}{ccc} B & \xrightarrow{\beta} & B' \\ & {\scriptstyle\alpha}\searrow & \swarrow{\scriptstyle\gamma} \\ & A' & \end{array}$$

The above Theorem answers positively the Question (4.2.1) from $[P_1]$.
It is the purpose of this paper to show that Theorem (1.1) still holds if in iv) we require for β to be smooth "wherever" possible - roughly speaking except above the singular (nonsmooth) locus of B over A (see Theorem (2.2) below).
This result is a positive answer to one conjecture due to M.Artin [A]. The proof of our result is just an application (see (2.5)) of Theorem (1.1) and Proposition (2.3) below (the latter being proved in §§ 3-4). Section 3 is closely related with $[P_2]$ §7, while Section 4 is a simple adaptation to our framework of [AD] page 16-17.

§2. GENERAL NÉRON DESINGULARIZATION

(2.1) Let $f=(f_1,\dots,f_m)$ be a system of polynomials in some variables $Y=(Y_1,\dots,Y_n)$ over a ring A. Given a system $g=(g_1,\dots,g_r)$, $r\leq n$ of r-polynomials from the ideal (f), we consider the ideal Δ_g generated in A[Y] by all rxr-minors of $(\frac{\partial g}{\partial Y})$. The ideal $H_f:=\sqrt{(f)+\sum_g \Delta_g((g):(f))}$, the sum being taken over all systems g of r-polynomials from (f), $1\leq r\leq n$, defines the singular (nonsmooth) locus of $B:=A[Y]/(f)$, i.e. if $q\in \text{Spec } B$, then B_q is smooth iff $q\not\supset H_f B$.

Then $H_{B/A}:=H_f B$ does not depends of the presentations chosen for B over A. An element $d\in B$ is a <u>standard element for</u> B <u>over</u> A if there exist a presentation $B\cong A[Y]/(f)$ and a system of polynomials g (like above) such that $d\in\sqrt{\Delta_g((g):(f))B}$.

When 1 is a standard element for B over A then we call B <u>standard smooth</u> over A. In fact B is standard smooth over A if B has a presentation $B\cong(A[Y]/(g))_h$, $Y=(Y_1,\dots,Y_n)$, $g=(g_1,\dots,g_r)$, $r\leq n$, h being a polynomial from Δ_g. Note that if D is an A-algebra then $H_{B\otimes_A D/D}\supset H_{B/A}\cdot(B\otimes_A D)$ by base change (see also [P_2] Note (2.3)).

(2.2) <u>Theorem</u>. Let $u:A\longrightarrow A'$ be a morphism of noetherian rings. Then the following statements are equivalent:

i) u is regular

ii) for every A-algebra B of finite type and every A-morphism $\alpha:B\longrightarrow A'$ there exist a standard smooth A-algebra B' and two A-morphisms $\beta:B\longrightarrow B'$, $\gamma:B'\longrightarrow A'$ such that

ii_1) the diagram (1.1.1) commutes, i.e. $\alpha=\gamma\beta$

ii_2) $\alpha(H_{B/A})\subset\sqrt{\gamma(H_{B'/B})}$.

(2.2.1) <u>Remark</u>. The condition ii_2) says in fact that for every prime ideal $q\subset A'$, $B'_{\gamma^{-1}q}$ is smooth over B if (thus iff) $B_{\alpha^{-1}q}$ is smooth over A.

As we shall see below the proof follows from Theorem (1.1) and the following

(2.3) <u>Proposition</u>. Let A be a noetherian ring, B an A-algebra of finite type, $\alpha: B \longrightarrow A$ an A-morphism and $I \subset \alpha(H_{B/A})$ an ideal from A. Then there exist a standard smooth A-algebra B' and two A-morphisms $\beta: B \longrightarrow B'$, $\gamma: B' \longrightarrow A$ such that

i) $I \subset \sqrt{\gamma(H_{B'/B})}$,

ii) the following diagram commutes

$$\begin{array}{ccc} B & \xrightarrow{\beta} & B' \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\gamma} & \\ & A & \end{array}$$

(2.3.1) <u>Remark</u>. The hypothesis "A noetherian" from the above proposition is superfluous when I is finitely generated and B is of finite presentation over A (like in [AD] page 13 we can reduce to the noetherian case descending the question to a certain finitely generated sub-$\mathbb{Z}$-algebra of A). Thus on normal domains our Proposition (2.3) coincides with Theorem (2.3) from [AD].

(2.4) <u>Corollary</u>. Theorem (2.2) holds when $A = A'$ and $u = 1_A$. For the proof take $I = \alpha(H_{B/A})$.

(2.5) <u>Proof of Theorem</u> (2.2). By Theorem (1.1) it is enough to show i) $\Rightarrow$ ii). Denote $C := B \otimes_A A'$ and let $\alpha': C \longrightarrow A'$ be the A'-morphism induced by $(\alpha, 1_{A'})$. By Corollary (2.4) there exist a standard smooth A'-algebra C' and two A'-morphisms $\beta': C \longrightarrow C'$, $\gamma': C' \longrightarrow A'$ such that

1) $\alpha'(H_{C/A'}) \subset \gamma'(H_{C'/A'})$

2) $\alpha' = \gamma' \beta'$

Suppose that $C' \cong C[Y]/(f)$, $Y = (Y_1, \ldots, Y_N)$, where f is a system of polynomials from C[Y]. Using Theorem (1.1) we can express A' as a filtered inductive limit of standard smooth A-algebras. Then we can choose a standard smooth A-algebra D with its limit map $\varphi: D \longrightarrow A'$ such that the coefficients of f come from $B \otimes_A D$

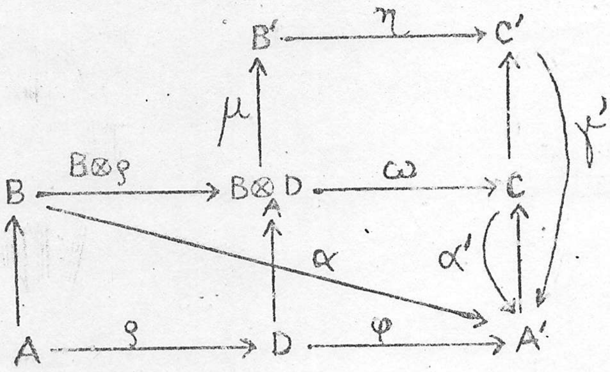
by $\omega := B\otimes_A \varphi$. Take $B' := (B\otimes_A D[Y]/(\tilde{f})$, where $\tilde{f}$ is the image of f by $\eta := \omega\otimes_{B\otimes D} B'$. $\eta : B' \longrightarrow C'$. Let $d\in B'$ be an element. Then $\eta(d)$ is a standard smooth element for C' over C if there exists a system of polynomials g from (f) such that $\eta(d)\in \Delta_g((g):(f))\cdot B'$. Choosing D such that the coefficients of g belong to $\omega(B\otimes_A D)$ we can suppose that d is also a standard element for B' over $B\otimes_A D$. A similar remark can be made if $\eta(d)$ is a standard element for C' over A'. Thus we can arrange D such that

3) B' is a standard smooth D-algebra,

4) $\mu(H_{B\otimes_A D/D}) \subset H_{B'/B\otimes_A D}$,

where μ denotes the structural algebra map $B\otimes_A D \longrightarrow B'$.

In the following commutative diagram all the squares are cocartesian



Put $\gamma := \gamma'\eta$ and let β be the composed map $B \longrightarrow B\otimes_A D \xrightarrow{\mu} B'$. Clearly we have

$$\gamma\beta = \gamma'\circ\eta\,\mu(B\otimes\rho) = \alpha'\omega(B\otimes\rho) = \alpha$$

and so ii_1). Since D is standard smooth over A we get B' standard smooth over A too (see 3)). As $B\otimes\rho$ is smooth and $H_{B\otimes_A D/D} \supset (B\otimes\rho)(H_{B/A})$ we get also ii_2) (see (2.1)).□

The Proposition (2.3) follows from the following two Lemmas.

(2.6) <u>Lemma</u>. Proposition (2.3) holds if $IB \subset H_{B/A}$.

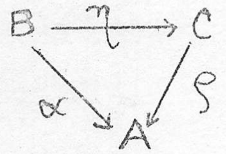
(2.7) <u>Lemma</u>. Let A be a ring, B an A-algebra of finite presentation, $x \in B$ an element and $\alpha : B \longrightarrow A$ an A-morphism. Suppose that $\alpha(x) \in \sqrt{\alpha(H_{B/A})A}$. Then there exist an A-algebra C of finite presentation and two A-morphisms $\eta : B \longrightarrow C$, $\varrho : C \longrightarrow A$ such that

1) $\eta(x) \in H_{C/A}$

2) $\eta(H_{B/A}) \subset H_{C/B}$

3) the following diagram commutes

(2.7.1)



Indeed, applying Lemma (2.7) for $x = d1_B$, d being a certain element from I, we can reduce our question (changing B by C) to the case when $x \in H_{B/A}$. Since I is finitely generated we arrive after several applications of Lemma (2.7) to the case when $IB \subset H_{B/A}$. Now it is enough to use Lemma (2.6).

The proof of Lemma (2.6) is given in §§3-4. Lemma (2.7) is a particular form of Lemma (2.4) from [P_2]. For the proof it is enough to choose a system of generators $b = (b_1, \ldots, b_t)$ for $H_{B/A}$, a positive integer n and some elements $z = (z_i)$ from A such that

$$\alpha(x^n) = \sum_{i=1}^{t} \alpha(b_i) z_i$$

Then $C_i := B[Z]/(x^n - \sum_{i=1}^{t} b_i Z_i)$, $\varrho : C \longrightarrow A$, $Z \leadsto z$ and the canonical map $\eta : B \longrightarrow C$ are the wanted ones.

§3. <u>AN ELEMENTARY DESINGULARIZATION</u>

The aim of this section is to prove the following Proposition (its proof follows faithfully the Desingularization Prin ciple from [P_2]).

(3.1) Proposition. Let A be a noetherian ring, B a finite type A-algebra; d an element of A and $\alpha: B \longrightarrow A$ an A-morphism. Suppose that B_d is a smooth A-algebra (i.e. $dB \subset H_{B/A}$). Then there exist a standard smooth A-algebra B' and two A-morphisms $\beta: B \longrightarrow B'$, $\gamma: B' \longrightarrow A$ such that

i) $\beta(d) \in H_{B'/B}$

ii) $\gamma\beta = \alpha$

Proof. First note that we can reduce the question to the case when d is a standard element for B over A using the following Lemma (in fact a particular case of Lemma (3.4) from [P_2] inspirated by [E]).

(3.2) Lemma. Let A be a noetherian ring, B an A-algebra of finite type and $\alpha: B \longrightarrow A$ an A-morphism. Then there exist a finite type A-algebra C and two A-morphisms $\eta: B \longrightarrow C$, $\varrho: C \longrightarrow A$ such that:

1) there exists a presentation of C over A for which all elements of $\eta(H_{B/A})$ are standard,

2) $\eta(H_{B/A}) \subset H_{C/B}$

3) the diagram (2.7.1) commutes.

Indeed applying Lemma (3.2) we get C, η, ϱ and changing B by C and α by ϱ we can suppose that d is a standard element of B over A.

Thus there exists a presentation $B = A[y] \cong A[Y]/p$, $Y = (Y_1, \ldots, Y_N)$, $y = (y_1, \ldots, y_N) \in B^N$, the isomorphism being given by $Y \rightsquigarrow y$ and some polynomials $f = (f_1, \ldots, f_r)$ from p such that $d^n \in (\Delta_f((f):p))_{(y)}$ for a certain positive integer n. Replacing d by d^n we reduce the question to the case

$$d \in (\Delta_f((f):p))(y) \subset \Delta_f(y) \cap ((f):p)(y).$$

Choose a polynomial $P \in A[Y]$, some rxr-minors $\{M_i\}_{1 \leq i \leq t}$ of the jacobian matrix $J := (\frac{\partial f}{\partial Y})$ and some nonzero polynomials $L_i \in A[Y]$ such that

$$d = P(y) = \sum_{i=1}^{t} (L_i M_i)(y). \tag{3.1.1}$$

Next, by the noetherianity, we can find a natural number $e \geqslant 2$ with the property

$$\mathrm{Ann}_A d^e = \mathrm{Ann}_A d^{e+1} \tag{3.1.2}$$

Since α induces an isomorphism $B/\alpha^{-1}(d^{2e+1}A) \cong A/d^{2e+1}A$ there exist $y' \in A^N$, $y'' \in d^e A^N$ such that

$$\alpha(y) = y' + d^{e+1} y'' \tag{3.1.3}$$

Therefore, by Taylor's formula, we get

$$\begin{aligned} d &\equiv P(y') \\ d &\equiv (\sum_{i=1}^{t} L_i M_i)(y') \end{aligned} \qquad \mathrm{mod}\ d^{2e+1}$$

Thus we can find two elements $s, s' \in 1 + d^{2e}A$ such that

$$\begin{aligned} s'd &= P(y') \\ sd &= \sum_{i=1}^{t} (L_i M_i)(y'). \end{aligned} \tag{3.1.4}$$

Adding some rows of the unit NxN-matrix I_N to J, we obtai a NxN-matrix H_i over $A[Y]$ with det $H_i = M_i$. Also there exist some matrices $G_i' \in \mathcal{M}(N, A[Y])$ for which

$$H_i G_i' = G_i' H_i = M_i I_N \ , \ i=1,\dots,t$$

and so $G_i := L_i G_i'$ are such that

$$(3.1.5) \qquad H_i G_i = G_i H_i = (L_i M_i) I_N \ , \quad i=1,\dots,t$$

Moreover, by (3.1.4) we get

$$(3.1.6) \qquad \sum_{i=1}^{t} (H_i G_i)(y') = \sum_{i=1}^{t} (G_i H_i)(y') = sdI_N$$

Denoting $z^{(i)} := H_i(y')y'' \in d^e A^N$, it results from (3.1.6) and (3.1.3)

$$(3.1.7) \qquad s\alpha(y) = sy' + d^e \sum_{i=1}^{t} G_i(y')z^{(i)}$$

Because the first r rows of the matrix H_i are the rows of J, for every $i=1,\dots,t$ we have

$$(3.1.8) \qquad z_k^{(i)} = z_k^{(1)} \ , \quad k=1,\dots,r.$$

Let $Z = (Z_1^{(1)},\dots,Z_r^{(1)}, \{Z_j^{(i)}\}_{\substack{1\le i\le t \\ r<j\le N}})$, $Z' = (Z_1',\dots,Z_N')$ be some variables; also we put $Z_j^{(i)} := Z_j^{(1)}$ when $j=1,\dots,r$. We define the following polynomials from $A[Y,Z,Z']$, $j=1,\dots,N$

$$(3.1.9) \qquad h_j = sY_j - sy_j' - d^{e+1} Z_j' - d^e \sum_{i=1}^{t} G_i(y')Z_j^{(i)}$$

By (3.1.7) α extends to a morphism of A-algebras

$\varphi: A[Y,Z,Z']/_{(\beta,h)} \longrightarrow A$, $h=(h_1,\dots,h_N)$ given by $Y \rightsquigarrow \alpha(y)$, $Z \rightsquigarrow z$, $Z' \rightsquigarrow 0$.

Let $m=\max\{\deg P, \max_{j=1\dots r} \deg f_j\}$. Then $s^m f_j, s^m P$ can be expressed as polynomials $\hat{f}_j, \hat{P}$ in sY. Taking into account (3.1.4) we get by Taylor's formula

$$(3.1.10) \qquad s^m P = \hat{P}(sY) \equiv \hat{P}(sy') = s^m P(y') \equiv P(y') \mod (d^e, h)$$

$$s^m f = \hat{f}(sY) \equiv \hat{f}(sy') + d^e \frac{\partial \hat{f}}{\partial(sY)}(sy')[dZ' + \sum_{i=1}^{t} G_i(y')Z^{(i)}] + d^{2e}Q' \mod h$$

for some polynomials $Q'=(Q'_j)_{1\leq j\leq r}$ from $A[Z,Z']$ containing only monomials of degree at least two.

Since $\frac{\partial \hat{f}}{\partial(sY)}(sy') = s^{m-1}J(y')$, the last relation can be rewritten

$$(3.1.11) \quad s^m f \equiv s^m f(y') + s^{m-1} d^e \sum_{i=1}^{t} (JG_i)(y')Z^{(i)} + d^{e+1}Q \mod h$$

where $Q := s^{m-1}J(y')Z' + d^{e-1}Q'$.

From (3.1.5) we get $JG_i = L_i M_i E$, where $E := (I_r|0)$, so from above it follows:

$$EZ^{(i)} = (Z_j^{(i)})_{j=1,\dots,r} = (Z_j^{(1)})_{j=1,\dots,r} = EZ^{(1)}, \quad i=1\dots t$$

Accordingly to (3.1.6) it results

$$\sum_{i=1}^{t} (JG_i)(y')Z^{(i)} = \sum_{i=1}^{t} (L_i M_i)(y'))EZ^{(i)} =$$

$$= \sum_{i=1}^{t} (L_i M_i)(y')EZ^{(1)} = sdEZ^{(1)}$$

and so (3.1.11) becomes

$$(3.1.12)\qquad s^m Y \equiv s^m f(y') + s^m d^{e+1} Z^{(1)} + d^{e+1} Q \mod h.$$

Mapping (3.1.12) by φ we get

$$s^m f(y') \equiv 0 \mod d^{2e+1}$$

Since $s \in 1 + d^{2e}A$, it induces a nonzero divisor in $A/d^{2e+1}A$ and so we get $f(y') \in d^{2e+1}A$, let us say $f(y') = d^{2e+1}c$ for a certain $c \in A^r$. Denote $g := s^m d^e c + s^m Z^{(1)} + Q \in A[Z,Z']^r$.

By (3.1.12) we get

$$(3.1.13)\qquad d^{e+1} g \equiv s^m f \mod h.$$

Take $\hat{B}' := A[Y,Z,Z']/_{(p,h,g)}$, and let $\hat{\beta}$ be the composed map $B \longrightarrow B[Z,Z']/(h,g) \cong \hat{B}'$. By (3.1.13) we get

$$d^{e+1}\varphi(g+(p,h)) = \varphi(d^{e+1}g+(p,h)) = \varphi(s^m f+(p,h)) = 0$$

and so $\varphi(g+(p,h)) \in \operatorname{Ann}_A d^{e+1}$.

On the other hand it follows

$$s^m d^e c = g(0,0) \equiv g(z,0) = \varphi(g+(p,h)) \mod zA$$

and thus $\varphi(g+(p,h)) \in d^e A$. By (3.1.2) we deduce that $d^e A \cap \operatorname{Ann}_A d^e A = (0)$, and so

$$\varphi\ (g+(p,h)) = 0$$

Therefore φ induces an A-morphism $\hat{\gamma} : \hat{B}' \longrightarrow A$.

Note that we have $\left(\frac{\partial g}{\partial Y}\right)=0$, $\left(\frac{\partial h}{\partial Y}\right)=sI_N$, $\left(\frac{\partial g_i}{\partial Z_j^{(i)}}\right)_{1\leq i,j\leq r}\equiv s^m I_r$ mod $d^{e-1}A[Z,Z']$. Consequently $\Delta_{(g,h)}$ contains a power of s. Thus

$$(3.1.14)\qquad \Delta_{(g,h)}\hat{B}'\cap(1+d\hat{B}')\neq\emptyset.$$

By (3.1.13) it follows $s^m\in((d^{e+1}g,h):(f))$ and so we have

$$s^m p\in((d^{e+1}g,h):(f))((f):p)\subset((d^{e+1}g,h):p)$$

From (3.1.10) and (3.1.4) we get

$$s^m p\equiv s'd\equiv d \text{ mod } (d^e,h).$$

Choose a polynomial $F\in A[Y,Z,Z']$ such that

$$s^m p\equiv d(1+d^{e-1}F) \quad \text{mod } h$$

and so $d(1+d^{e-1}F)p\subset(d^{e+1}g,h)$.

In the algebra $\tilde{B}:=A_s[Y,Z,Z']/(h)$ the last relation becomes

$$(3.1.15)\qquad (1+d^{e-1}F)p\,\tilde{B}\subset(d^e g)\tilde{B}+\mathrm{Ann}_{\tilde{B}}d$$

On the other hand, like in (3.1.10) it holds

$$p\equiv p(y') \text{ mod } (d^e,h)A_s[Y,Z,Z']$$

and from (3.1.3) it follows $p(y')\subset d^eA$. Consequently we get

$$p\subset(h,d^e,p(y'))A_s[Y,Z,Z']\subset(h,d^e)A_s[Y,Z,Z']$$

(3.1.15) it follows

$$(3.1.16)\qquad (1+d^{e-1}F)p\tilde{B}\subset(d^eg)\tilde{B}+\mathrm{Ann}_{\tilde{B}}d\tilde{B}\cap d^e\tilde{B}$$

But $\tilde{B}$ is a flat A-algebra, being smooth over A, so we obtain from (3.1.2)

$$\mathrm{Ann}_{\tilde{B}}\,d^e\tilde{B}=(\mathrm{Ann}_A d^e)\tilde{B}=(\mathrm{Ann}_A d^{e+1})\tilde{B}=\mathrm{Ann}_{\tilde{B}}d^{e+1}\tilde{B}$$

and thus (3.1.16) means in fact

$$(1+d^{e-1}F)p\tilde{B}\subset(d^eg)\tilde{B}$$

Take a positive integer n such that $s^n(1+d^{e-1}F)p\subset(d^eg,h)$. Since $s\in 1+d^{2e}A$ we have obtained

$$\sqrt{((g,h)\div p)\hat{B}'}\cap(1+d\hat{B}')\neq\emptyset.$$

From (3.1.14) it results

$$(3.1.17)\qquad H_{\hat{B}'/A}+d\hat{B}'=\hat{B}'$$

Note that $(\frac{\partial h}{\partial Z'})=-d^{e+1}I_N$ and so $dB[Z,Z']\subset\sqrt{\Delta_h(y,Z,Z')}$. By (3.1.13) we get

$$d^{e+1}g\equiv 0 \bmod h\ A[Y,Z,Z']/_{(p)}\qquad\text{hence}$$

$$d^{e+1}B[Z,Z']\subset(hB[Z,Z']:(g))$$

Thus $d\hat{B}'\subset H_{\hat{B}'/B}$ and therefore

$$d\hat{B}'\subset H_{B/A}\hat{B}'\cap H_{\hat{B}'/B}\subseteq H_{\hat{B}'/A}$$

From (3.1.17) we conclude that $\hat{B}'$ is a smooth A-algebra and applying Lemma (3.2) we find a standard smooth A-algebra B' and two A-morphisms $\delta:\hat{B}'\longrightarrow B'$, $\gamma:B'\longrightarrow A$ such that $\hat{\gamma}=\gamma\delta$ and δ is smooth.

Let β be the composed map $B\xrightarrow{\hat{\beta}}\hat{B}'\xrightarrow{\delta}B'$. Clearly (B',β,γ) are the wanted ones.

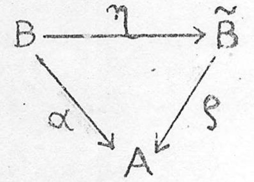
§4. Proof of Lemma (2.6)

Let $d_1,\ldots,d_t$ be a minimal system of generators of I and $\tilde{d}_i:=d_i 1_B$, $1\leq i\leq t$. We use induction on t. If $t=1$ the result is a consequence of Proposition (3.1).

Now suppose the Lemma holds for $I':=(d_1,\ldots,d_{t-1})$ by induction hypothesis. Then there exist a standard smooth A-algebra $\tilde{B}$ and two A-morphisms $\eta:B\longrightarrow\tilde{B}$, $\varrho:\tilde{B}\longrightarrow A$ such that

1) $H_{\tilde{B}/B}\supset\eta(I'B)$,

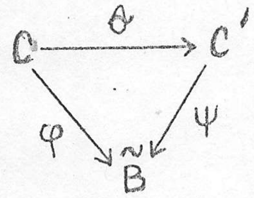
2) the following diagram commutes



Denote $C:=\tilde{B}\otimes_A B$ and let $\varphi:C\longrightarrow\tilde{B}$ be the B-morphism induced by $(1,\eta)$. Note that $H_{C/\tilde{B}}\supset H_{B/A}C$ (see (2.1)). Applying Proposition (3.1) for C,φ, $1\otimes\tilde{d}_t$ there exist a standard smooth $\tilde{B}$-algebra C' and two A-morphisms $\theta:C\longrightarrow C'$, $\psi:C'\longrightarrow\tilde{B}$ such that

3) $H_{C'/C}\supset\theta(1\otimes\tilde{d}_t)$,

4) the following diagram commutes



Take $B':=C'$, $\gamma:=\varsigma\psi$ and let β be the composed map $B\xrightarrow{\delta}C\xrightarrow{\theta}C'$, where the first map δ is given by $b\rightsquigarrow 1\otimes b$. Since C' is standard smooth over $\widetilde{B}$ and $\widetilde{B}$ is standard smooth over A we get B' standard smooth over A too.

By 2), 4) it follows

$$5)\ \gamma\beta=\varsigma\psi\theta\delta=\varsigma\varphi\delta=\varsigma\eta=\alpha$$

and so ii). For i) it is enough to show that B'_{d_i} is a smooth A-algebra for $i=1,\dots,t$. If $i=t$ then C'_{d_t} is smooth over C by 3) and so B'_{d_t} is smooth over A, δ being smooth because $\widetilde{B}$ over A is so. If $i<t$ then $\widetilde{B}_{d_i}$ over B is smooth by 1). Since we have

$$6)\ \eta=\varphi\delta=\psi\beta$$

we get $B_{d_i}\otimes\beta$ smooth by the following Lemma applied for $B_{d_i}\longrightarrow B'_{d_i}\longrightarrow\widetilde{B}_{d_i}$.

(4.1) Lemma. Let $B\xrightarrow{\beta}B'\xrightarrow{\psi}\widetilde{B}$ be two A-morphisms between smooth finite type A-algebras. Suppose that ψ is surjective and the composed map $\psi\beta$ is smooth. Then β is smooth too.

Proof. It is enough to show that the map $B_{\beta^{-1}q}\longrightarrow B'_q$ induced by β is smooth for all $q\in\operatorname{Spec}B'$. Thus we can reduce to the case when $A,B,B',\widetilde{B}$ are all local rings and $A\to B,\psi,\beta$ are local morphisms. Since ψ is essentially of finite type it is enough to show that ψ is formally smooth, which follows from the below Lemma.

(4.2) Lemma. Let $A\xrightarrow{u}B\xrightarrow{v}C\xrightarrow{w}D$ be three local morphisms between noetherian local rings. Suppose that u, vu, wvu, wv are formally smooth and w is surjective. Then v is formally smooth too.

Proof. Let k be the residue field of A. Using EGA (19.7.1

we must prove that v is flat and $\bar{v}:=k\otimes_A v$ is formally smooth. If $\bar{v}$ is flat then by [M](20.G) we get v flat too. Thus we can reduce to the case A=k.

Let $\underline{m},\underline{n}$ be the maximal ideals of B,C respectively, $K:=C/\underline{n}$, $\underline{a}\subset C$ an ideal such that $D\cong C/\underline{a}$ and x a regular system of parameters from B. Since wv is formally smooth, the system (wv)(x) induces in $\underline{n}/\underline{n}^2+\underline{a}$ (so in $\underline{n}/\underline{n}^2$) a linearly K-independent system. Then v(x) is a part of a regular system of parameters from C, hence v is flat by [M](36.B) and $\bar{C}:=C/v(x)C$ is regular. It is enough to apply now the following elementary Lemma.

(4.2.1) <u>Lemma</u>. Let $(C,\underline{n})$ be a regular local algebra over a field K and $\underline{a}\subset C$ an ideal such that $D:=C/\underline{a}$ is formally smooth over K. Then C is formally smooth over K too.

<u>Proof</u>. Since the following sequence

$$0\to(\underline{a}+\underline{n}^2)/\underline{n}^2\longrightarrow\underline{n}/\underline{n}^2\longrightarrow\underline{n}/(\underline{a}+\underline{n}^2)\longrightarrow 0$$

is exact, we can find in $\underline{a}$ a system of elements $y=(y_1,\dots,y_t)$, t=dim C - dim D = ht $\underline{a}$ which is a part of a regular system of parameters from C. Thus $yC=\underline{a}$, $\underline{a}$ being necessarily prime. Let K' be a finite field p-extension of K. Then $D':=K'\otimes_K D$ is still a regular local ring. Let z be a system of elements from $C':=K'\otimes_K C$ inducing a regular system of parameters in D'. Then the system $y\cup z$ has t+dim D' - elements and generate the maximal ideal from C'. Thus it is a regular system of parameters in C', because dim C'=dim C = t+dim D = t+dim D'. In particular C is geometrically regular over K.

References

A Artin, M.: Algebraic structure of power series rings, Contemp.Math.13, Amer.Math.Soc.,Providence (1982), 223-227

AD Artin, M. and Denef, J.: Smoothing of a ring homomorphism of a section, Arithmetic and Geometry, vol.II, Birkhäuser, Boston (1983), 5-32

E Elkik, R.: Solutions d'équations à coefficients dans un anneau hensélien,Ann.Sc.Ec.Norm.Sup 4^e série, t.6 (1973), 533-604

EGA Grothendieck, A. and Dieudonné, J.: Éléments de géométrie algébrique IV, Part 1, Publ.Math.Inst.Hautes Études Sci.20(1964)

M Matsumura, H.: Commutative algebra, Benjamin, New-York,1980

N Néron, A.: Modèles minimaux des variétés abéliennes sur les corps locaux et globaux, Publ.Math.Inst.Hautes Études Sci. 21(1964)

P_1 Popescu,D.:Global form of Néron's p-desingularization and approximation, Proceedings "Week of Algebraic Geometry", Bucharest, June 30-July 6, 1980, Teubner Texte, Band 40, Leipzig, 1981

P_2 Popescu,D.:General Néron desingularization and approximation I, INCREST Preprint 44(1983), Bucharest

P_3 Popescu,D.:General Neron desingularization and approximation II (revised form), to appear.