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# THE THEORY OF THE GENERAL MOTION OF A THIN PROFILE AT A SMALL INCIDENCE IN AN INVISCID INCOMPRESSIBLE FLUID

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# The theory of the general motion of a thin profile at a small incidence in an inviscid incompressible fluid

by

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## Summary

The paper gives an analytical method to study the motion of the thin aerofoil in an incompressible inviscid fluid. The only restriction imposed to the aerofoil motion is that the incidence angle be small.

The solution needs the solving of a first kind Volterra type integral equation which, in the case of the rectilinear motion of the aerofoil, coincides with the Wagner's integral equation. The theory is next applied to study the case of the incidence variation of the flat plate in rectilinear motion and to the case of the flat plate having a circular motion at a small incidence angle.

## Notations

$a$ The complex coordinate of the leading edge projection on the aerofoil velocity direction at the moment $t (a=a(t))$.

$a_k$ The coefficients in series expansion (4.3) $(a_k= a_k(t))$.

A,B The projection of the leading edge and trailing edge, respectively, on the curve $C_o$.

$b$ The complex coordinate of the trailing edge projection on the profile velocity direction at the moment $t$ $(b=b(t))$.

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| | |
|---|---|
| $C$ | The profile contour. |
| $C_o$ | The trajectory of the point $O_1$. |
| $C_L$ | The lift coefficient. |
| $C_M$ | The pitching moment coefficient. |
| $f$ | The integration constant (with respect to s) in relation (3.5). |
| $F$ | The complex velocity potential. |
| $h$ | The local lift on the profile with reversed sign. |
| $\hat{h}$ | The local force exerced by the body on the fluid. |
| $i$ | The imaginary unit in complex plane. |
| $\vec{i}, \vec{j}$ | The unit vectors of the Ox-, Oy- axes, respectively. |
| $L$ | The halfchord of the profile. |
| $\mathcal{L}$ | The overall lift on the profile. |
| $m$ | The function defined by relation (2.11). |
| $\mathcal{M}$ | The overall pitching moment. |
| $\vec{n}$ | The unit vector of the outward normal to the curve C. |
| $O_1$ | The mean position of the mid-point of the profile chord. |
| $p$ | The dimensionless pressure. |
| $q$ | The intensity of the mass sources. |
| $R$ | The dimensionless radius of the circle. |
| $s$ | The curvilinear coordinate along the curve $C_o$. |
| $t$ | The dimensionless time. |
| $\vec{v}$ | The dimensionless fluid velocity. |
| $\tilde{V}_o$ | The characteristic velocity of the motion. |
| $\vec{V}_o$ | The mean velocity of the point $O_1$ at the moment t. |
| $\vec{v}_c$ | The velocity of an arbitrary point of the aerofoil. |
| $x, y$ | The dimensionless variables in fixed reference frame. |
| $Y_{\pm}$ | The equations of the two surfaces of the aerofoil (eq.3.1). |
| $z$ | The complex variable x + iy. |
| $\gamma$ | The vorticity. |

$\delta$ The incidence of the aerofoil. ($\delta_0$ a constant incidence).

$\varepsilon$ The parameter determining the aerofoil thickness.

$\rho$ The fluid density.

$\varphi$ The velocity potential.

$\psi$ The stream function.

$\sigma$ The curvilinear abscissa along the curve C.

$\omega$ The angular velocity of the moving reference frame.

$\omega_0$ The dimensionless parameter $R^{-1}$.

$\theta$ The angular coordinate in circular motion.

The subscript 1 denotes the reference quantities with respect to the moving reference frame.

The superscript ' indicates that the quantity should be taken at the moment t'.

The superscript · denotes the derivative with respect to time for fixed z.

## 1. Introduction

We consider the motion of a thin aerofoil at low incidence in an incompressible fluid. In this case the real (viscous) flow does not separate from body. In order to obtain various aerodynamic coefficients of interest it is necessary to calculate the unsteady boundary layer, but this needs the knowledge of the pressure distribution on the aerofoil. If the Reynolds number is sufficiently large the thickness of the boundary layer is small and, in the first approximation, the inviscid region can be considered to be bounded by body itself rather than by the outer edge of the boundary layer.

The impulsive rectilinear start from the rest of a two-dimensional aerofoil at incidence was considered by H. Wagner [1]. In his theory all vortices are confined to a very thin layer which can be assimilated with a vortex sheet behind the aerofoil in accordance with Birnbaum's hypothesis [2]. The problem of the unsteady rectilinear motion of the thin profile was considered by Karman and Sears [3], Söhngen [4], C. Jacob [5]. In [6] the unsteady motion of a thick symmetrical profile at zero incidence was studied.

More recently Basu and Hancock [7] developed a numerical method to study the unsteady motion of a two-dimensional aerofoil in an inviscid incompressible fluid.

The present paper is concerned with the determination of the first inviscid approximation of the unseparated flow of a thin aerofoil which will provide the pressure distribution on the profile. The viscosity is present only by means of the Kutta-Joukowsky condition at the trailing edge of the aerofoil.

The motion of the aerofoil is restricted to the only requirement of small incidence. We consider the form in distributions of the Euler equations [8], [9]; these equations hold in regular points inside the fluid as well as on the surface of the profile and on the discontinuity surfaces inside the flow (i.e. on the vortex sheets). By means of this form we obtain the vortex sheet as being a surface described by the aerofoil inside the fluid; the vortex intensity is related to the local force exerceted by the aerofoil upon the fluid particles. Thus we replaced the Birnbaum's hypothesis about the vortex sheet by the linearisation hypothesis valid in the case of the thin aerofoil at low incidence [10].

In the following the time derivative of the complex potential function will be the solution of a Dirichlet boundary value problem. That part of the solution due to the thickness of the aerofoil can be obtained similarly to the steady motion. The other part contains an undetermined function $\dot{f}(t)$ which enter all aerodynamic parameters of interest. By imposing the initial boundary condition a first kind Volterra type integral equation for determining the function $\dot{f}(t)$ follows. We have a theorem of existence and unicity of the solution and an algorithm for numerical solution of the integral equation.

The developed theory is next applied to the rectilinear motion of the flat plate which changes its incidence and then to the circular motion of the flat plate at low incidence. In the last case the influence of the circular motion and of the unsteadyness of the motion upon the lift coefficient is pointed out.

## 2. Statement of the problem

We consider the motion of an aerofoil of 2L chord length in an incompressible inviscid fluid. Let $O_1$ be the mid-point of the mean position of the profile chord at the moment t. The mean profile motion is described by means of the velocity $\vec{V}_o(t)$ of the point $O_1$ and the angular velocity $\Omega(t)$ around the point $O_1$. We suppose that the profile is thin and the incidence, defined as the angle of the velocity $\vec{V}_o(t)$ with the upward chord direction, is small. We refere the motion to a fixed system of coordinates Oxy. The equations characterising the fluid motion are

$$\operatorname{div} \vec{v} = 0 \tag{2.1}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\,\operatorname{grad})\vec{v} + \operatorname{grad} p = \hat{h}\cdot\vec{n}\;\delta_c \tag{2.2}$$

The term in the r.h.s. of relation (2.2) is the local action of the aerofoil on the fluid [8],[9]. If $\theta(x,y,t)$ denotes the characteristic function of the aerofoil domain we have

$$\operatorname{grad} \theta(x,y,t) = -\vec{n}\cdot\delta_c \tag{2.3}$$

$\vec{n}$ being the outward normal unit vector at the curve C.

We considered dimensionless variables by choosing the following reference quantities: L for x and y, $\tilde{V}_o$ for velocity, $L/\tilde{V}_o$ for time, $\rho\tilde{V}_o^2$ for pressure and local force $\hat{h}$ on the curve C.

Let $\gamma(x,y,t)$ be the vorticity

$$\gamma(x,y,t) = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \qquad (2.4)$$

The equation governing the vorticity is found by taking the curl of the momentum equation (2.2) which gives

$$\frac{\partial \gamma}{\partial t} + (\vec{v}\ \mathrm{grad})\gamma = \left(\frac{\partial \hat{h}}{\partial x} n_y - \frac{\partial \hat{h}}{\partial y} n_x\right) \delta_c \qquad (2.5)$$

If we denote by $\sigma$ the arc length along the contour C we can write

$$\vec{n} = \frac{dy}{d\sigma}\vec{i} - \frac{dx}{d\sigma}\vec{j}$$

and hence

$$\frac{D\gamma}{Dt} = -\frac{\partial \hat{h}}{\partial \sigma}\delta_c \qquad (2.6)$$

This equation can be integrated in Lagrangean coordinates $x^o, y^o$ (the position of the fluid particle at the initial moment) [11]

$$\gamma(x^o,y^o,t) = \gamma(x^o,y^o,0) - \int_0^t \left(\frac{\partial \hat{h}}{\partial \sigma}\delta_c\right)(x^o,y^o,t')dt' \qquad (2.7)$$

For fluid particles which do not touch the aerofoil we have $\delta_c(x^o,y^o,t') = 0$ for all t' and it follows the classical theorem of vorticity conservation in inviscid flows, i.e.

$$\gamma(x^o,y^o,t) = \gamma(x^o,y^o,0)$$

If the motion at the initial moment is irrotational and the flow does not separate from the body then the vorticity is confined to the aerofoil contour and to the curve $C_1$ described by fluid particles which touched the surface of the body.

The continuity equation is identically satisfied if we put

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x} \tag{2.8}$$

$\psi(x,y,t)$ being the stream function. In the irrotational region there is a potential function for velocities $\varphi(x,y,t)$ such that the complex potential function

$$F(z,t) = \varphi(x,y,t) + i\,\psi(x,y,t) \tag{2.9}$$

is an analytic function of the complex variable $z = x + iy$.

Let $C_o$ be the curve described by the point $O_1$. In the case of the thin profile and at low incidence we can consider the vorticity line as being the curve $C_o$. Likewise we replace the aerofoil by a line of singularities (point sources and point vortices) on its projection AB on the curve $C_o$. The vortices intensity results in the form

$$\gamma(x,y,t) = -\,\delta_{c_o} \cdot \int_0^t \frac{\partial h}{\partial s}\,dt' \tag{2.10}$$

where h is difference between the local body force on the upper side and lower side of the aerofoil surface, and s is the curvilinear coordinate along the curve $C_o$. Similarly, the intensity $q(x,y,t)$ of the point sources on the arc $\widehat{AB}$ will be taken in the form

$$q(x,y,t) = -\,\delta_{c_o} \cdot \int_0^t \frac{\partial m}{\partial s}\,dt' \tag{2.11}$$

The complex potential function corresponding to the above mentioned distribution of singularities is

$$F(z,t) = \frac{1}{2\pi i} \iint \left[\gamma(\xi,\eta) + iq(\xi,\eta)\right] \ln(z-\zeta)\,d\xi\,d\eta$$

where $\zeta = \xi + i\eta$. In our case we obtain after a by parts integration

$$F(z,t) = -\frac{1}{2\pi i}\int_0^t dt' \int_{\widehat{A'B'}} \frac{h(\zeta,t')+im(\zeta,t')}{\zeta - z}\, d\zeta + F(z,0) \qquad (2.12)$$

where $\widehat{A'B'}$ is the arc of the curve $C_0$ occupied by the aerofoil projection at the moment t'. If the dot indicates the derivative with respect to time for fixed z we get

$$\dot{F}(z,t) = -\frac{1}{2\pi i}\int_{\widehat{AB}} \frac{h(\zeta,t)+im(\zeta,t)}{\zeta - z}\, d\zeta \qquad (2.13)$$

and hence $\dot{F}(z,t)$ is a holomorphyc function, outside the arc $\widehat{AB}$, which vanishes at infinity.

## 3. The boundary conditions

We consider a moving system of coordinates $x_1O_1y_1$ with the origin $O_1$ and $O_1x_1$-axis directed oposite to the vector $\vec{V}_o(t)$ (Fig.1)

Fig. 1

Let

$$y_1 = \delta x_1 + \varepsilon Y_{\pm}(x_1,t), \quad |x_1| \leq 1 \qquad (3.1)$$

be the equations of the two aerofoil surfaces where $\delta$ is the incidence.

The boundary condition states that the fluid slides along the bording curve C. This slip-condition can be written as

$$\vec{v} \cdot \vec{n} = \vec{v}_c \cdot \vec{n} \qquad (3.2)$$

The velocity of an arbitrary point of the aerofoil is

$$\vec{v}_c = \{-V_o - (\Omega+\dot{\delta})y_1\}\vec{i}_1 + \{V_o^{(1)} + (\Omega+\dot{\delta})x_1\}\vec{j}_1 \qquad (3.3)$$

where $V_o^{(1)}$ is the vertical velocity of the profile with respect to the moving system. We have

$$\vec{n} = \frac{dy_1}{ds}\vec{i}_1 - \frac{dx_1}{ds}\vec{j}_1$$

where the variation of the arc length $\sigma$ along the curve C was approximated by means of the variation of the curvilinear abscissae s on $C_o$. The condition (3.2) becomes

$$\frac{\partial\Psi}{\partial y}\frac{dy}{ds} + \frac{\partial\Psi}{\partial x}\frac{dx}{ds} = -\left\{V_o + (\Omega+\dot{\delta})y_1\right\}\frac{dy_1}{ds} - -\left\{V_o^{(1)} + (\Omega+\dot{\delta})x_1\right\}\frac{dx_1}{ds} \tag{3.4}$$

and hence

$$\Psi(s,t) = -\left\{V_o\, y_1 + V_o^{(1)}x_1 + (\Omega+\dot{\delta})\frac{x_1^2+y_1^2}{2}\right\} + f(t) \tag{3.5}$$

on the arc $\widehat{AB}$. In relation (3.5) f(t) is an yet undetermined function and s and t are independent variables. To obtain the variation of the variable $x_1$ with respect to time we write

$$s = \int_0^t V_o(t')dt' - x_1 \tag{3.6}$$

the first term in r.h.s. being the curvilinear abscissa of the point $O_1$. Hence we obtain

$$\dot{x}_1 = V_o(t) \tag{3.7}$$

We write

$$\dot{\Psi}(s,t) = \dot{f}(t) + \dot{\Psi}_{\pm}^{(o)}(x_1,t) \quad \text{on} \quad \widehat{AB} \tag{3.8}$$

the functions $\dot{\Psi}_{\pm}^{(o)}(x_1,t)$ resulting by differentiation with respect to time of the r.h.s. of relation (3.5). We shall use the boundary conditions in the form (3.8) and lately in the form (3.5) only for $x_1 = -1$; this is equivalent with boundary condition (3.5) on the arc $\widehat{AB}$.

## 4. The solution of the boundary value problem

The determination of the function $\dot{F}(z,t)$ requires to solve a Dirichlet problem for the domain outside the $\widehat{AB}$ arc with boundary conditions (3.8). In solving this boundary-value problem we shall replace the arc $\widehat{AB}$ by the segment $I = [-1,1]$ of the $O_1x_1$-axis. Let $z_1 = x_1 + iy_1$ be the complex variable associated with the moving axis. We have

$$z = \frac{b-a}{2} z_1 + \frac{b+a}{2} \tag{4.1}$$

where $a = a(t)$ and $b = b(t)$ are the complex coordinates with respect to the fixed reference frame of the two endpoints of the segment I.

By means of the transform (4.1) the function $\dot{F}(z,t)$ becomes a holomorphyc function $\dot{F}(z_1,t)$ of variable $z_1$ in the domaine outside I. We have

$$\dot{F}(z_1,t) = \frac{1}{2\pi} \int_{-1}^{+1} \frac{\dot{\psi}_+^{(o)}(\xi,t) - \dot{\psi}_-^{(o)}(\xi,t)}{\xi - z_1} d\xi +$$

$$+ if(t)\left(1-\sqrt{\frac{z_1-1}{z_1+1}}\right) + \frac{1}{2\pi i}\sqrt{\frac{z_1-1}{z_1+1}} \int_{-1}^{1} \sqrt{\frac{1+\xi}{1-\xi}} \frac{\dot{\psi}_+^{(o)}(\xi,t) + \dot{\psi}_-^{(o)}(\xi,t)}{\xi - z_1} d\xi \tag{4.2}$$

where the Kutta-Joukowsky condition was applied át the trailing edge of the profile. The solution has singularity on the leading edge of the aerofoil. To remove this singularity one can apply the technique for rending uniformly valid approximate solutions of boundary value problems for domaines external to thin regions [12], [13].

For numerical computations we write the symmetrical part of the condition (3.8) in the form

$$\frac{\dot{\psi}_+^{(o)}(\xi,t)+\dot{\psi}_-^{(o)}(\xi,t)}{2} = \dot{f}(t) + \sum_{k=o}^{\infty} a_k(t)\, T_k(x_1) \qquad |x_1|<1 \tag{4.3}$$

where $T_k(x_1)$ are the Chebyshev polynomials. We have

$$\dot{F}(z_1,t) = i\big(\dot{f}(t) + a_o(t)\big)\Big(1 - \sqrt{\frac{z_1-1}{z_1+1}}\Big) + i\sum_{k=1}^{\infty} a_k(t)\,\big(z_1-\sqrt{z_1^2-1}\big)^k + \frac{1}{2\pi}\int_{-1}^{+1}\frac{\dot{\psi}_+^{(o)}(\xi,t)-\dot{\psi}_-^{(o)}(\xi,t)}{\xi - z_1}\,d\xi \tag{4.4}$$

The square root determination in the above formulae are choosen such that we have

$$\sqrt{\frac{z_1-1}{z_1+1}} = 1 + O\Big(\frac{1}{z_1}\Big), \quad z_1 - \sqrt{z_1^2-1} = O\Big(\frac{1}{z_1}\Big).$$

at infinity.

## 5. Aerodynamic coefficients of the profile

The Plemelj relations applied to the relation (2.13) give

$$\dot{F}(x_1 \pm io,t) = \mp\frac{1}{2}\{h(x_1,t) + im(x_1,t)\} - \frac{1}{2\pi i}\int_{-1}^{+1}\frac{h(\xi,t)+im(\xi,t)}{\xi - x_1}\,d\xi \tag{5.1}$$

the integral being the Cauchy principal value. Hence we obtain

$$m(x_1,t) = \dot{\psi}_-^{(o)}(x_1,t) - \dot{\psi}_+^{(o)}(x_1,t) \tag{5.2}$$

$$h(x_1,t) = -2\{\dot{f}(t) + a_o(t)\}\sqrt{\frac{1-x_1}{1+x_1}} - 2\sum_{k=1}^{\infty} a_k(t)\,\sin(k\arccos(x_1)) \tag{5.3}$$

The relation (5.2) determines directly the source intensity and relation (5.3) gives the local force of the aerofoil on the fluid. The local lift on the profile will be given by function $h(x_1,t)$ with reversed sign. This gives

$$C_L \equiv \frac{\mathcal{L}}{\rho \tilde{V}_0^2 L} = - \int_{-1}^{+1} h(x_1,t)dx_1 = 2 \left\{ \mathring{f}(t) + a_o(t) + \frac{a_1(t)}{2} \right\} \qquad (5.4)$$

for the lift coefficient and

$$C_M \equiv \frac{\mathcal{M}}{2\rho \tilde{V}_0^2 L^2} = - \frac{1}{2} \int_{-1}^{+1} (x_1+1)h(x_1,t)dx_1 =$$

$$= \frac{\pi}{2} \left\{ \mathring{f}(t) + a_o(t) + a_1(t) + \frac{a_2(t)}{2} \right\} \qquad (5.5)$$

for the pitching moment coefficient about the leading edge. The relations (5.4), (5.5) are generalisations of those known in steady aerodynamics.

To obtain the pressure on the aerofoil we shall use the Bernoulli equation in the form

$$p = p_\infty - \frac{\partial \varphi}{\partial t} - \frac{1}{2} \vec{v}^2 \qquad (5.6)$$

In linearised theory the last term will be neglected and relation (4.4) gives

$$p(x_1, \pm 0, t) = p_\infty - \frac{1}{2\pi} \int_{-1}^{+1} \frac{\dot{\psi}_+^{(o)}(\xi,t) - \dot{\psi}_-^{(o)}(\xi,t)}{\xi - x} d\xi \mp$$

$$\mp \left\{ \left(\mathring{f}(t) + a_o(t)\right) \sqrt{\frac{1-x_1}{1+x_1}} + \sum_{k=1}^{\infty} a_k(t) \sin(k \arccos(x_1)) \right\} \qquad (5.7)$$

It is to be noticed that all aerodynamic quantities of interest contain the unknown function $\mathring{f}(t)$.

## 6. Integral equation for the function $\mathring{f}(t)$

To determine the function $F(z,t)$ we must integrate the function $\dot{F}$ with respect to time for fixed z. By using relations (4.1), (4.4) we have

$$\dot{F}(z,t) = i\left(\dot{f}(t) + a_o(t)\right)\left(1 - \sqrt{\frac{z-b}{z-a}}\right) +$$

$$+ i\sum_{k=1}^{\infty} a_k(t)\left\{\frac{2}{b-a}\left(z - \frac{a+b}{2} - \sqrt{(z-a)(z-b)}\right)\right\}^k \qquad (6.1)$$

$$+ \frac{1}{2\pi}\int_{-1}^{1}\frac{\dot{\psi}_+^{(o)}(\xi,t) - \dot{\psi}_-^{(o)}(\xi,t)}{\xi - [2z-(b+a)]/(b-a)}\,d\xi$$

Hence

$$F(z,t) = i\int_0^t \left(\dot{f}(t') - a_o(t')\right)\left\{1 - \sqrt{\frac{z-b'}{z-a'}}\right\}dt' +$$

$$+ i\sum_{k=1}^{\infty}\int_0^t a_k(t')\left\{\frac{2}{b'-a'}\left(z - \frac{a'+b'}{2} - \sqrt{(z-a')(z-b')}\right)\right\}^k dt' + \qquad (6.2)$$

$$+ \frac{1}{2\pi}\int_0^t dt'\int_{-1}^{1}\frac{\dot{\psi}_+^{(o)}(\xi,t') - \dot{\psi}_-^{(o)}(\xi,t')}{\xi - [2z-(b'+a')]/(b'-a')}\,d\xi + F(z,o).$$

Here $a' = a(t')$, $b' = b(t')$.

To obtain the function $F(z,t)$ in the form (6.2) we used the boundary condition (3.8). In order for the very boundary condition (3.5) to be verified we need in addition to impose this condition to be verified at a point, say $z = a$.

We have for the symmetrical part of the relation (3.5)

$$\int_0^t \left[\dot{f}(t') + a_o(t')\right]\left[1 - \mathrm{Re}\left\{\sqrt{\frac{a-b'}{a-a'}}\right\}\right]dt' +$$

$$+ \sum_{k=1}^{\infty}\int_0^t a_k(t')\,\mathrm{Re}\left\{\frac{2}{b'-a'}\left(a - \frac{a'+b'}{2} - \sqrt{(a-a')(a-b')}\right)\right\}^k dt' + \qquad (6.3)$$

$$+ \psi(a,o) = f(t) - \left\{-V_o\delta - V_o^{(1)} + 0.5(\Omega + \dot{\delta}) + \right.$$

$$\left. + 0.5\,V_o\left[Y_+(-1,t) + Y_-(-1,t)\right]\right\}$$

Hence we obtain

$$\int_0^t \dot{f}(t')\,\mathrm{Re}\left\{\sqrt{\frac{a-b'}{a-a'}}\right\}dt' = g(t) \qquad (6.4)$$

where

$$g(t) = \int_0^t a_o(t')\left[1 - \mathrm{Re}\left\{\sqrt{\frac{a-b'}{a-a'}}\right\}\right]dt' +$$

$$+\sum_{k=1}^{\infty}\int_0^t a_k(t')\ \mathrm{Re}\left\{\frac{2}{b'-a'}\left(a - \frac{a'+b'}{2} - \sqrt{(a-a')(a-b')}\right)\right\}^k dt' - \qquad (6.5)$$

$$- V_o\delta - V_o^{(1)} + 0.5(\Omega+\dot{\delta}) + 0.5\ V_o[Y_+(-1,t)+Y_-(-1,t)] + \Psi(a,o)\ .$$

The relation 6.4) is a first kind Volterra integral equation whose kernel has a weak singularity for t' = t. The existence and uniqueness of solutions of the integral equation (6.4) is guaranteed under the following conditions.

Theorem. The integral equation (6.4) has an unique continuous solution $\dot{f}(t)$ for $t \in [0, T]$ if

1) $\dot{a}(t)$, $\dot{b}(t)$ are continuous for $t \in [0,T]$

2) $a(t) \neq a(t')$ for $t \neq t'$

3) $\hat{g}(t) = \frac{d}{dt}\int_0^t \frac{g(t')}{\sqrt{t-t'}}\,dt'$ is continuous for $t \in [0,T]$ .

The theorem follows from the theorem given in [14]pp.80-82. The condition 2) requires that the profile doesn't intersect the line of vortices.

Analytical solution of the equation (6.4) can be obtained only in very particular cases. Therefore, in order to obtain the function $\dot{f}(t)$ we must recourse to numerical computations. A numerical method for generalised Abel integral equation was developed in [15]. The equation (6.4) can be transformed such that we use this method. However for large time intervals (as is our case) the resulting algorithm is slow. In order to avoid this difficulty we considered some modifications to the product integration method which give a faster algorithm.

## 7. Unsteady rectilinear motion of the flat plate

In order to verify the above given theory we consider the case of the flat plate which moves with constant velocity $\widetilde{V}_o$ and changes the incidence by the rule

$$\delta(t) = \begin{cases} 0 & \text{for } t < 0 \\ \delta_o(-3+2t/t_o)\cdot t^2/t_o^2 & \text{for } 0 \leq t \leq t_o \\ -\delta_o & \text{for } t > t_o \end{cases} \qquad (7.1)$$

For $t \leq 0$ the function $F(z,t)$ is vanishing identically. For $t > 0$ we have

$$\begin{aligned} a(t) &= -1 - t\,, & b(t) &= 1 - t \\ a_o(t) &= -(\delta + 0.25\ddot{\delta}), & a_1(t) &= -2\dot{\delta} \\ a_2(t) &= -0.25\ddot{\delta}\,, & a_j &= 0\,,\ j \geq 3 \end{aligned} \qquad (7.2)$$

and the integral equation (6.4) becomes

$$\int_0^t \dot{f}(t')\sqrt{\frac{2+t-t'}{t-t'}}\,dt' = g(t) \qquad (7.3)$$

$$\begin{aligned} g(t) = &\int_0^t a_o(t')\left(1 - \sqrt{\frac{2+t-t'}{t-t'}}\right)dt' + \\ &+ \sum_{j=1}^{2} \int_0^t a_j(t')\left\{t'-t+\sqrt{(t-t')(2+t-t')}\right\}^j dt' - \delta(t) + 0.5\,\dot{\delta}(t) \end{aligned} \qquad (7.4)$$

The integral equation (7.3) is just the Wagner's integral equation for the unsteady rectilinear motion of the thin profile. For numerical computations we took $t_o = 3$ and the solution of the integral equation was obtained up to $T = 100$. The overall lift coefficient

$$C_L' = C_L / (2\pi\delta_o) \qquad (7.5)$$

was plotted in Fig.2 and 3.

Fig. 2          Fig. 3

It is to be remarked that first $C'_L$ increases up to the value 0.82 and then decreases due to the second derivative $\ddot{\delta}(t)$. When the rotation of the plate ceases $C'_L$ reaches the value 0.67 and then increases monotonically to the steady-state value $C'_L = 1$. The convergence to this final value is slow. Thus, for $t = 100$, which corresponds to the time necesary for the aerofoil to cover a distance equal to 50 chord length, the difference of this lift coefficient and its value from the steady case is about 0.01.

## 8. The motion of the flat plate on a circle

Let us suppose that the point $O_1$ has a circular motion given by relations

$$x = - R \sin\omega t$$
$$y = R(1-\cos \omega t) \qquad (8.1)$$

The plate starts from rest at $t = 0$ and its incidence $\delta(t) = -\delta_o$ is constant during the motion. We have

$$a(t) = iR\,(1 - e^{-i\omega t}) - e^{-i\omega t}$$
$$b(t) = iR\,(1 - e^{-i\omega t}) + e^{-i\omega t} \qquad (8.2)$$

======
Fig. 4
======

In order to avoid the singularity of the solution at the moment $t = 0$ we took for $\omega$ a time dependence of the form $(\dot{7}.1)$ on a very short time interval $t_o$. We have $\omega_o R = 1$. (In dimensional variables $\tilde{\omega}_o\tilde{R} = \tilde{V}_o$, $\tilde{V}_o$ being the velocity of the point $O_1$ and $\tilde{R}$ the circle's radius). (Fig.4). Consequently we have

$$a_o(t) = -\delta + 0.25\,\omega_o\dot{\omega}\ , \quad a_1(t) = \omega_o\omega$$
$$a_2(t) = 0.25\,\omega_o\omega\ , \quad a_j(t) = 0 \quad j \geqslant 3 \qquad (8.3)$$

There are two dimensionless parameters of the problem $\delta_o$ and $\omega_o$. We have

$$C_L = 2\pi\{\delta_o\, C_L^{(1)}(\theta,\omega_o) + 0.5\omega_o\, C_L^{(2)}(\theta,\omega_o)\},\ \theta = \omega_o t \quad (8.4)$$

The coefficients $C_L^{(1)}$, $C_L^{(2)}$ were computed for $\omega_o = 0.05$ and $\omega_o = 0.1$. Both of them oscillate around the value 1 but the amplitude of the second coefficient is smaller such that for $\delta_o \neq 0$ we can put $C_L^{(2)} = 1$. In Fig.5 we plotted the variation of the coefficient $C_L^{(1)}$ with respect to $\theta$.

Fig. 5

In order to estimate the influence of the two terms in (8.4) to the lift coefficient we take $\delta_o = 0.2$, $\omega_o = 0.05$. If we would consider the motion of the plate as being uniform with velocity $\tilde{V}_o$ the maximum error in $C_L$ with respect to the value given by (8.4) would be 18%. The error done by neglecting the term $\Omega$ in boundary condition (3.5) is 8%. The rest of 10% is the influence of the free vortices.

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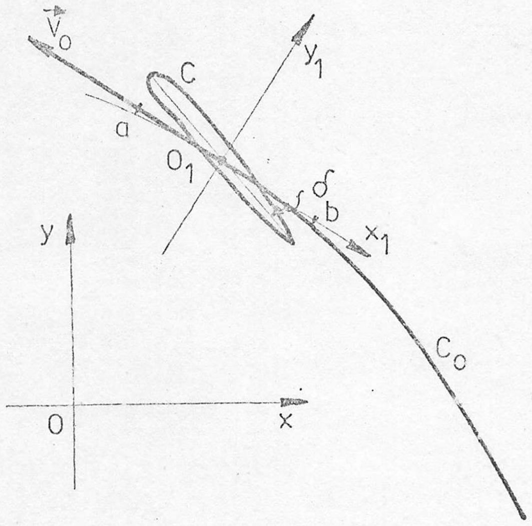
$\vec{V}_0$
C
$y_1$
a
$O_1$
δ
b
$x_1$
y
$C_0$
0
x

Fig. 1

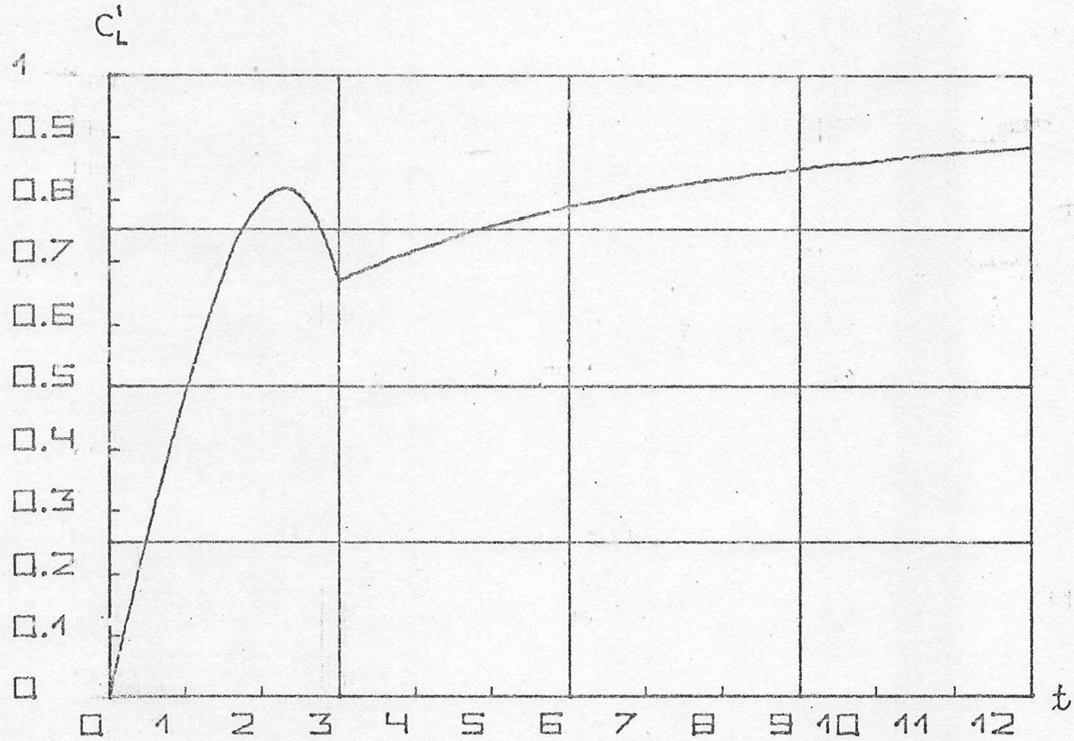
$C_L'$
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
t
0 1 2 3 4 5 6 7 8 9 10 11 12

Fig. 2

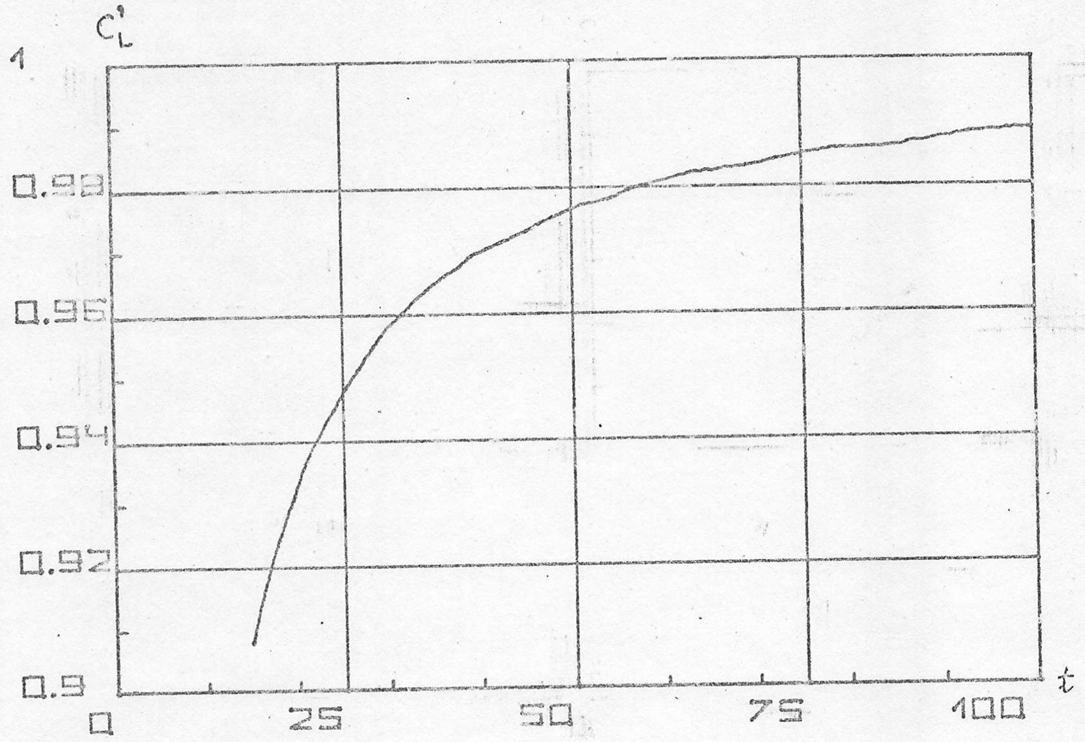


Fig. 3

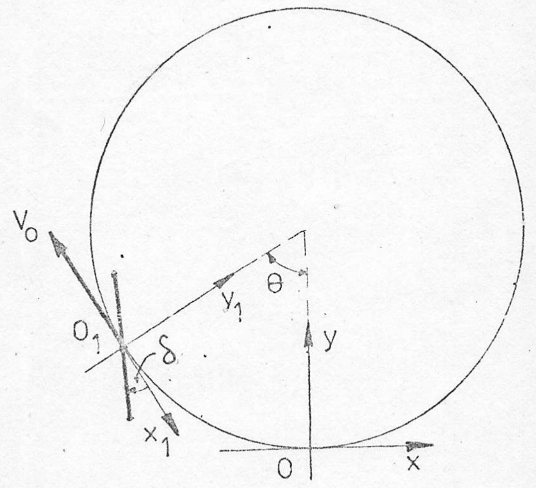


Fig. 4

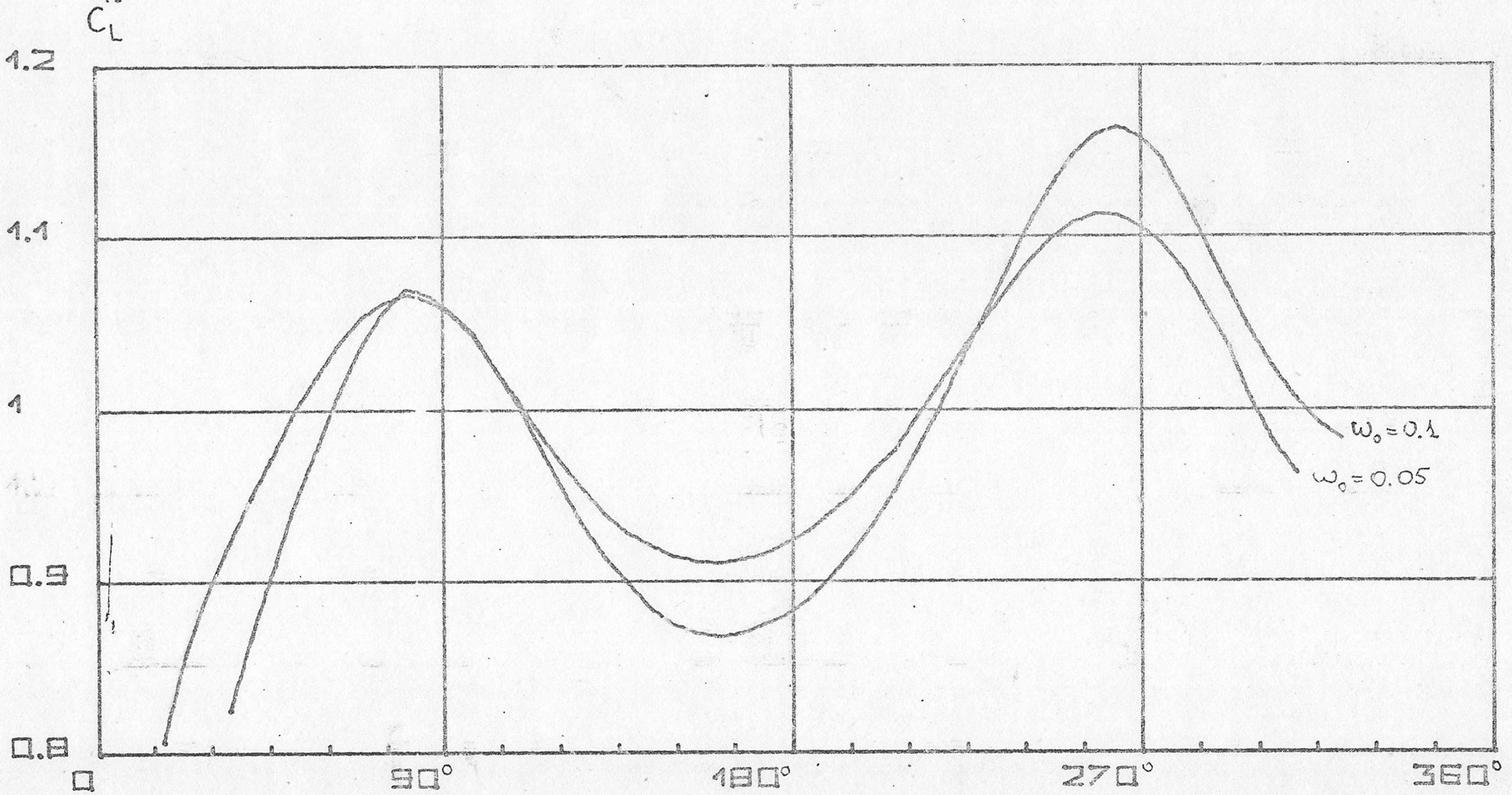


Fig. 5