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NORMAL DEGENERATIONS OF RATIONAL AND RULED SURFACES

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NORMAL DEGENERATIONS OF RATIONAL AND RULED SURFACES

Lucian Bădescu

This note is a preliminary report of an attempt to understand the structure of the normal projective degenerations either of the rational surfaces with the second Betti number ≤ 10, or of the arbitrary ruled non-rational surfaces (see theorems (16) and (18) below). At the end of the paper some elementary examples of degenerations are given and some open questions are discussed.

(1) Let $f:X \longrightarrow T$ be a projective flat morphism of algebraic varieties over the complex field $\mathbb{C}$. Throughout this paper we shall assume that:

- X is a 3-fold,
- T is a smooth connected affine curve with $\omega_T = O_T$ (where $\omega_T = \Omega^1_{T/\mathbb{C}}$),
- there is a distinguished point $o \in T$ such that the fibre $X_o = f^{-1}(o)$ is a normal surface, and
- for every $t \in T$, $t \neq o$, the fibre $X_t = f^{-1}(t)$ is a smooth surface.

Since X_o is normal, X is also normal, and if $x \in X$ is a singular point of X then x is also a singular point of X_o (and in particular, X has only isolated singularities). In general we shall follow the standard terminology and notations.

(2) <u>Definition.</u> If Y is a normal projective surface we define:

- the irregularity $q(Y)$ of Y by $q(Y) = h^1(Y, O_Y)$,
- the geometric genus $p_g(Y)$ of Y by $p_g(Y) = h^2(Y, O_Y)$, which by duality on Y coincides to $h^0(Y, \omega_Y)$, where ω_Y is the Grothendieck dualizing sheaf of Y,
- for every $n \geqslant 1$, the n-genus $p_n(Y)$ of Y by $p_n(Y) = h^0(Y, \omega_Y^{(n)})$, where $\omega_Y^{(n)}$ denotes the double dual $(\omega_Y^{\otimes n})^{vv}$ of $\omega_Y^{\otimes n}$ (in particular, $p_1(Y) = p_g(Y)$). If $n < 0$ we define $p_n(Y)$ in the same way, with $\omega_Y^{(n)} = \underline{Hom}_Y(\omega_Y^{(-n)}, O_Y)$.

(3) <u>Proposition.</u> <u>In the situation of (1)</u> ~~we have~~ $p_g(X_o) = p_g(X_t)$ <u>and</u> $q(X_o) = q(X_t)$ <u>for every</u> $t \in T$.

<u>Proof.</u> Since the function $t \longrightarrow h^1(X_t, O_{X_t})$ is upper-semi-continuous, it is constant for $t \in T$ general. Shrinking T a little bit, we can assume that

this function is constant on $T-\{o\}$. Denote by q this constant value. Then by the base-change theorem (see e.g. [13], theorem 12.11, page 290), $R^1f_*(O_X)/T-\{o\}$ is locally free of rank q. By the relative duality (see e.g [16]) we have a canonical isomorphism

$$R^1f_*(\omega_X) \cong \underline{\mathrm{Hom}}_T(R^1f_*(O_X),O_T),$$

from which we deduce that $R^1f_*(\omega_X)$ is locally free of rank q on T.

On the other hand, since $H^3(X_t,\omega_{X_t}) = o$ for every $t\in T$ and $R^3f_*(\omega_X) = o$, the base change theorem (loc.cit.) shows that the canonical map

$$R^2f_*(\omega_X)\otimes k(t) \longrightarrow H^2(X_t,\omega_{X_t})$$

is an isomorphism for every $t\in T$. Again by the relative duality we have a canonical isomorphism

$$R^2f_*(\omega_X) \cong \underline{\mathrm{Hom}}_T(f_*(O_X),O_T),$$

and since $f_*(O_X) = O_T$, we get $R^2f_*(\omega_X) \cong O_T$. Applying once again the base change theorem we infer that the canonical map

$$R^1f_*(\omega_X)\otimes k(t) \longrightarrow H^1(X_t,\omega_{X_t})$$

is an isomorphism for every $t\in T$. Recalling that $R^1f_*(\omega_X)$ is locally free of rank q, this proves (via duality on X_o) the assertion about $q(X_o)$. The other assertion of proposition (3) follows from the first and from the invariance of the Euler-Poincaré characteristic. Q.E.D.

(4) Remark. In characteristic zero this result is well known (see [11], éxposé 236, corollaire 3.6, where a proof based on the theory of the Picard schemes is given). The above proof, included for the convenience of the reader, is more elementary (because it is based on the relative duality theory in its elementary form as presented in [16]), and works in the case of surfaces in positive characteristic as well.

(5) Proposition. In the situation of (1), assume moreover that $p_g(X_t) = q(X_t) = o$ for every $t \neq o$. Then $\rho(X_o)\leq\rho(X_t)$ for every $t\neq o$, where $\rho(Z)$ denotes the rank of the Néron-Severi group $NS(Z) = Pic(Z)/Pic^o(Z)$ of a variety Z (see [17]).

Proof. Using the exponential sequence we deduce that $\rho(X_t) = b_2(X_t)$, so $\rho(X_t)$

is independent of $t \neq o$. Let $a = \rho(X_o)$ and $L_1,\dots,L_a \in \mathrm{Pic}(X_o)$ be such that they define a base of $NS(X_o)\otimes_{\mathbb{Z}}\mathbb{Q}$ (i.e. a maximal number of linearly independent elements of $NS(X_o)$). Note that by proposition (3) $q(X_o) = p_g(X_o) = o$, and in particular, $NS(X_o) = \mathrm{Pic}(X_o)$. Using Artin's approximation theory (see [1]) and the fact that $h^2(O_{X_o}) = o$, we get the existence of some $L'_1,\dots,L'_a \in \mathrm{Pic}(X)$ such that $L'_i/X_o \cong L_i$, $i = 1,\dots,a$ (eventually by replacing T with an appropriate étale neighbourhood of (T,o), see e.g. [3] , proposition 4 and its proof for details). Proposition (5) will follow if we show that for $t \neq o$ the restrictions L'_i/X_t $(i = 1,\dots,a)$ define linearly independent elements of $\mathrm{Pic}(X_t) = NS(X_t)$. Assume the contrary, i.e. there exist some integers $n_1,\dots,n_a$, not all zero, such that $L'/X_t = O_{X_t}$, with $L' = L_1'^{\,n_1}\otimes\dots\otimes L_a'^{\,n_a}$. Since $h^1(X_t,O_{X_t}) = o$ we can apply [12] (4.6.5) and deduce that L' is trivial in a neighbourhood of the form $f^{-1}(T')$ of X_t, with $T' \subseteq T-\{o\}$. Without loss of generality we can assume that $L'/X-X_o \cong O_X/X-X_o$. Since X_o is an irreducible divisor on X there is an integer m such that $L' \cong O_X(mX_o)$. Then we get $L_1^{n_1}\otimes\dots\otimes L_a^{n_a} \cong L'/X_o \cong O_X(mX_o)/X_o \cong O_{X_o}$ (the normal bundle of X_o in X is trivial), which is a contradiction because $L_1,\dots,L_a$ were supposed to be linearly independent in $\mathrm{Pic}(X_o)$. Q.E.D.

Now we want to get informations about the behaviour of the higher plurigenera in an algebraic family. First we note that if X_o is smooth, a result of Iitaka (see [15]) implies that $p_n(X_o) = p_n(X_t)$ for every $n \geq 1$ and $t \in T$. The algebraic part of Iitaka's result has been subsequently generalized to the case where X_o is Gorenstein by Wilson (see [26]; however, he has to assume that the Kodaira dimension of the general fibre of f is not one). The case where X_o is normal but not Gorenstein seems to be very complicated, and the results we are able to prove are very partial.

(6) Lemma. $\underline{\mathrm{Hom}}_X(\omega_X^{(n)},\omega_X) \cong \omega_X^{(1-n)}$ and $\underline{\mathrm{Hom}}_{X_t}(\omega_X^{(n)}/X_t,\omega_{X_t}) = \omega_{X_t}^{(1-n)}$ for every $t \in T$ and $n \in \mathbb{Z}$.

Proof. Since ω_X and $\omega_X^{(n)}$ are reflexive sheaves of rank one, the first isomorphism follows from [7], while the second one follows in the same way remarking moreover that the double dual of $\omega_X^{(n)}/X_t$ is just $\omega_{X_t}^{(n)}$. Q.E.D.

(7) For every $t \in T$ one has a canonical map

$$\psi_t : f_*(\underline{\mathrm{Hom}}_X(\omega_X^{(n)},\omega_X))\otimes k(t) \longrightarrow \underline{\mathrm{Hom}}_{X_t}(\omega_X^{(n)}/X_t,\omega_{X_t}),$$

in the following way. First we have

$$\mathrm{Hom}_{X_t}(\omega_X^{(n)}/X_t, \omega_{X_t}) = H^o(X_t, \underline{\mathrm{Hom}}_{X_t}(\omega_X^{(n)}/X_t, \omega_{X_t}) = \text{(by lemma (6))}$$
$$= H^o(X_t, \omega_{X_t}^{(1-n)}).$$

Using again lemma (6) we have $\underline{\mathrm{Hom}}_X(\omega_X^{(n)}, \omega_X) \cong \omega_X^{(1-n)}$. Therefore we have to define a map

$$\psi_t : f_*(\omega_X^{(1-n)}) \otimes k(t) \longrightarrow H^o(X_t, \omega_{X_t}^{(1-n)}),$$

and the latter is by definition just the composition of the canonical base-change map

$$f_*(\omega_X^{(1-n)}) \otimes k(t) \longrightarrow H^o(X_t, \omega_X^{(1-n)}/X_t)$$

with the map which is induced by the canonical homomorphism into the bidual

$$\omega_X^{(1-n)}/X_t \longrightarrow (\omega_X^{(1-n)}/X_t)^{vv} = \omega_{X_t}^{(1-n)}.$$

(8) If F is a coherent sheaf on T and $t \in T$ is a point, we have the following canonical map (defined in an obvious way):

$$\lambda_t(F) : \mathrm{Hom}_T(F, O_T) \otimes k(t) = F^v \otimes k(t) \longrightarrow \mathrm{Hom}_{k(t)}(F \otimes k(t), k(t)) = (F \otimes k(t))^v.$$

In general $\lambda_t(F)$ is injective; moreover $\lambda_t(F)$ is an isomorphism if and only if F is locally free near t. Indeed, since T is a smooth curve we can write F = F'$\oplus$F", with F' locally free near t and Supp(F")$\subseteq \{t\}$. Then we have $\lambda_t(F) = \lambda_t(F') \oplus \lambda_t(F")$, and $\lambda_t(F')$ is an isomorphism and $\lambda_t(F") = o$.

(9) Since $\omega_T = O_T$, the dualizing sheaf ω_f relative to the morphism f (see [16]) coincides to ω_X, and therefore the relative duality (with respect to f) yields

$$f_*(\underline{\mathrm{Hom}}_X(\omega_X^{(n)}, \omega_X)) \cong \underline{\mathrm{Hom}}_T(R^2 f_*(\omega_X^{(n)}), O_T)$$

or else (via lemma (6)):

(9') $$f_*(\omega_X^{(1-n)}) \cong \underline{\mathrm{Hom}}_T(R^2 f_*(\omega_X^{(n)}), O_T).$$

In particular, $f_*(\omega_X^{(1-n)})$ is locally free.

(10) Consider the base-change map

$$\varphi_t : R^2 f_*(\omega_X^{(n)}) \otimes k(t) \longrightarrow H^2(X_t, \omega_X^{(n)}/X_t).$$

Since the relative dimension of the morphism f is 2, the base-change theorem implies that φ_t is always an isomorphism. We claim that the maps ψ_t are always injective, and moreover ψ_t is an isomorphism if and only if $R^2 f_*(\omega_X^{(n)})$ is locally free near the point t. This comes from the foregoing discussion and the following commutative diagram

$$
\begin{array}{ccc}
f_*(\omega_X^{(1-n)})\otimes k(t) & \xrightarrow{\ \psi_t\ } & \mathrm{Hom}_{X_t}(\omega_X^{(n)}/X_t, \omega_{X_t}) \\
\text{(by (9'))}\ \big\downarrow \wr & & \wr\big\downarrow\ \text{(by duality on } X_t) \\
\underline{\mathrm{Hom}}_T(R^2f_*(\omega_X^{(n)}), \mathcal{O}_T)\otimes k(t) & & \\
\lambda_t \big\downarrow & & \\
\mathrm{Hom}_{k(t)}(R^2f_*(\omega_X^{(n)})\otimes k(t), k(t)) & \xleftarrow[\sim]{\ \varphi_t^\vee\ } & \mathrm{Hom}_{k(t)}(H^2(X_t, \omega_X^{(n)}/X_t), k(t))
\end{array}
$$

(11) If $n \leqslant o$ and $t \neq o$ then ψ_t is always an isomorphism. Indeed, by a result of Iitaka (already mentioned above) the function $t \longrightarrow h^2(X_t, \omega^n_{X_t}) = h^o(X_t, \omega^{1-n}_{X_t}) = p_{1-n}(X_t)$ is constant on $T-\{o\}$. Then by the base-change theorem the sheaf $R^2f_*(\omega_X^{(n)})$ is locally free on $T-\{o\}$ of rank $p_{1-n}(X_t)$, and by (9'), $f_*(\omega_X^{(1-n)})$ is locally free of rank $p_{1-n}(X_t)$ everywhere (i.e. including the point o). Since ψ_o is injective we get the inequality

$$p_{1-n}(X_t) \leqslant p_{1-n}(X_o) \quad \text{for every } t\in T \text{ and } n \leqslant o.$$

(12) If $n \geqslant 1$ the sheaf $R^2f_*(\omega_X^{(n)})$ can also be assumed to be locally free in $T-\{o\}$ (by shrinking eventually T) of rank $p_{1-n}(X_t)$. Therefore, as in case $n \leqslant o$, we also get the inequality:

$$p_{1-n}(X_t) \leqslant p_{1-n}(X_o) \quad \text{for every } n \geqslant 1 \text{ and } t\in T \text{ general.}$$

Summing up the discussion of (6) - (12) we get:

(13) Proposition. i) For every $n \in \mathbb{Z}$ the sheaf $f_*(\omega_X^{(1-n)})$ is locally free of rank equal to the rank of $R^2f_*(\omega_X^{(n)})$ (and also equal to $p_{1-n}(X_t)$ for every $t \neq o$ if $n \leqslant o$, or for $t \in T$ general if $n \geqslant 1$).

ii) The natural map (see (7))

$$\psi_t : f_*(\omega_X^{(1-n)})\otimes k(t) \longrightarrow H^o(X_t, \omega_{X_t}^{(1-n)})$$

is injective for every $t\in T$ and $n \in \mathbb{Z}$. Moreover, ψ_t is an isomorphism if and only if the sheaf $R^2f_*(\omega_X^{(n)})$ is locally free near t.

iii) $p_n(X_t) \leqslant p_n(X_o)$ for every $n \geqslant 2$ and $t \neq o$, and also for every $n < o$ and $t \in T$ general. Moreover, $p_n(X_t) = p_n(X_o)$ if and only if ψ_o is an isomorphism.

... can state the first result concerning the behaviour of the plurigenera

(14) Proposition. *In the situation of (1), assume that X_t is a rational surface with $K_t^2 \geqslant o$ for some $t \neq o$, where K_t is a canonical divisor on X_t. Then $p_n(X_o) = o$ for every $n > 1$.*

Proof. We distinguish two cases:

1) $K_t^2 > o$. First we remark that $X_{t'}$ is rational for every $t' \neq o$ (see [15]). Therefore the particular point t does not play any special role in our considerations because the self-intersection number $K_{t'}^2$ is constant in $T-\{o\}$. So, if $t \neq o$ and $K_t^2 > o$, the Riemann-Roch theorem gives

$$p_{1-n}(X_t) \geqslant \frac{n(n-1)}{2} K_t^2 + 1.$$

Applying proposition (13) iii) we get

$$p_{1-n}(X_o) \geqslant p_{1-n}(X_t) \geqslant \frac{n(n-1)}{2} K_t^2 + 1, \quad \text{for every } n \geqslant 1.$$

Let U_o be the smooth locus of X_o. The above inequalities yield $h^o(U_o, \omega_{U_o}^{1-n}) \geqslant r(n)$, with $r(n) = \frac{n(n-1)}{2} K_t^2 + 1 > 1$. Assume that the conclusion of our proposition fails, i.e. there is a $m > 1$ such that $p_m(X_o) = h^o(U_o, \omega_{U_o}^m) > o$. Using the fact that $H^o(U_o, O_{U_o}) = \mathbb{C}$ and the above inequality (with $n = 1+m$), we get an isomorphism $\omega_{U_o}^m \cong O_{U_o}$, or else $\omega_{X_o}^{(m)} \cong O_{X_o}$ (because $\omega_{X_o}^{(m)}/U_o = \omega_{U_o}^m$ and $\omega_{X_o}^{(m)}$ is a reflexive O_{X_o}-module of rank one). From this we deduce that $\omega_{X_o}^{(ma)} \cong O_{X_o}$ for every $a \in \mathbb{Z}$, and in particular, $h^o(X_o, \omega_{X_o}^{(-ma)}) = 1$ for every $a \geqslant 1$, which contradicts the inequality $h^o(X_o, \omega_{X_o}^{(-ma)}) \geqslant r(1+ma) \longrightarrow \infty$ as $a \longrightarrow \infty$.

2) $K_t^2 = o$. By Riemann-Roch we have $\chi(X_t, O_{X_t}(nK_t)) = 1$ for every $n \geqslant 1$, and taking $n = 2$ we get $p_{-1}(X_t) = h^2(X_t, O_{X_t}(2K_t)) > o$ for every $t \neq o$. By proposition (13) iii), $p_{-1}(X_o) \geqslant p_{-1}(X_t) > o$ (for $t \in T$ general), or else $h^o(X_o, \omega_{X_o}^{(-1)}) > o$. Assume by contradiction that $p_m(X_o) > o$ for some $m > 1$. Then we claim that $\omega_{X_o} \cong O_{X_o}$. In fact, let $o \neq s \in H^o(U_o, \omega_{U_o}^{-1})$ be a non-zero section; then $o \neq s^m \in H^o(U_o, \omega_{U_o}^{-m})$, and hence the last complex vector space is not zero. Since $H^o(U_o, O_{U_o}) = \mathbb{C}$ and $H^o(U_o, \omega_{U_o}^m) = H^o(X_o, \omega_{X_o}^{(m)}) \neq o$, we infer that $\omega_{U_o}^{-m} \cong O_{U_o}$, and in particular, $s^m(y) \neq o$ for every $y \in U_o$. But this in turn implies that s itself is a nowhere vanishing section, i.e. $\omega_{U_o} \cong O_{U_o}$, and the latter isomorphism is equivalent

to the claim. From this point one can proceed exactly as in [26], page 29, in order to get the conclusion. Q.E.D.

(15) Remark. The proof of proposition (14) is inspired from the proof of proposition 3.3 in Wilson [26], which deals with the case where the special fibre X_o is moreover Gorenstein (but without any restriction on K_t^2). We do not see how the proof of Wilson can be directly extended also to the case $K_t^2 < o$ and X_o normal (and non-Gorenstein). However, we suspect that proposition (14) remains still valid in case $K_t^2 < o$ (and X_t rational). For example, if the general fibre X_t is rational with $K_t^2 < o$, but, moreover, $p_{-n}(X_t)$ behaves like n or n^2 when $n \longrightarrow \infty$ (i.e. the anti-Kodaira dimension $\kappa^{-1}(X_t)$ of the general fibre X_t is $\geqslant 1$, in the terminology of [24]), then we still have $p_n(X_o) = o$ for every $n \geqslant 1$. This fact can be proved using the same argument as in case 1) of the proof of proposition (14). On the other hand, there are many examples of rational surfaces S with $K_S^2 < o$ and $\kappa^{-1}(S) = 2$; take for example any surface S obtained by blowing up n points of the surface $F_e = P(O_{P^1} \oplus O_{P^1}(-e))$, with $9 \leqslant n \leqslant e+4$ (see [24], 8.1.2).

(16) Theorem. <u>In the situation of (1), assume that X_t is a rational surface with $K_t^2 \geqslant o$ (or equivalently $b_2(X_t) \leqslant 10$, by Noether's formula) for one (and hence for all) point $t \neq o$. Then the following statements hold:</u>

i) $p_n(X_o) = o$ for every $n \geqslant 1$ and $\rho(X_o) \leqslant b_2(X_t) = 10 - K_t^2$.

ii) If $u: X'_o \longrightarrow X_o$ is the minimal desingularization of X_o, then X'_o is a ruled surface and $\dim_{\mathbb{C}} R^1 u_*(O_{X'_o}) = q$, where q is the irregularity of X'_o. In particular, if X'_o is rational, then all the singularities of X_o are rational.

iii) If X'_o is ruled non-rational, then X_o has precisely one non-rational singularity x, and (possibly) finitely many rational singularities. The irreducible components of the fibre $u^{-1}(x)$ are: a section of the canonical ruled fibration $\pi : X'_o \longrightarrow B$ (with B a smooth curve of genus q) plus (possibly) some components of the degenerated fibres of π. The exceptional fibre of u over every rational singularity of X_o is contained in a degenerated fibre of π.

iv) $b_2(X'_o) - b_2(X_o)$ is equal to the number of the irreducible components of all exceptional fibers of u.

Proof. Part i) follows from propositions (14) and (5). To prove part ii) observe that if D_n is the sum of all n-dualizing divisors of the singularities of X_o (see [4] for the definition of the dualizing divisors of a two-dimensional singularity and their properties), we have:

$$o = H^o(X_o, \omega_{X_o}^{(n)}) \cong H^o(U_o, \omega_{X_o}^{(n)}) \cong H^o(U'_o, \omega_{X'_o}^{(n)}) \cong H^o(X'_o, \omega_{X'_o}^{n} \otimes O_{X'_o}(D_n)),$$

where $U'_o = u^{-1}(U_o)$ and U_o the smooth locus of X_o (the last isomorphism comes from the very definition of D_n, see loc. cit.).

Therefore the fact that $p_n(X_o) = o$ for every $n \geqslant 1$, translates into:

$$(*) \qquad |nK + D_n| = \emptyset \quad \text{for every } n \geqslant 1, \text{ with } K = K_{X'_o}$$

Since $D_n \geqslant o$ we have in particular that $p_n(X'_o) = o$ for every $n \geqslant 1$, and consequently X'_o is ruled by Enriques' criterion of ruledness (see [6]). We note that the fact that X'_o is ruled is also a consequence of [20], page 77 (even without any restriction concerning K_t^2). But (later) we'll need the stronger information contained in (*)!

On the other hand, the Leray spectral sequence of the morphism u yields the exact sequence

$$o = H^1(O_{X_o}) \longrightarrow H^1(O_{X'_o}) \longrightarrow R^1u_*(O_{X'_o}) \longrightarrow H^2(O_{X_o}) = o,$$

from which we derive the formula $\dim_{\mathbb{C}} R^1u_*(O_{X'_o}) = q$. This proves ii).

Assume now $q > o$. The proof of part iii) is quite similar to the proof of a part of theorem 2 in [2]. We'll divide it into three steps.

<u>Step 1.</u> Every singularity of X_o whose exceptional fibre is contained in a degenerated fibre of π, is rational.

Step 1 is just lemma 7 in [2] (and follows using standard arguments).

<u>Step 2.</u> There is precisely one non-rational singularity x of X_o, and this singularity has geometric genus q.

For, the formula $\dim_{\mathbb{C}} R^1u_*(O_{X'_o}) = q$ shows that X_o has at least one non-rational singularity x. By step 1 the fibre $u^{-1}(x)$ contains at least one irreducible component E_o which is not contained in any fibre of π. Then E_o dominates B, and hence $p_a(E_o) \geqslant p_a(B) = q$, where $p_a(C)$ denotes the arithmetic genus of a curve C. On the other hand, since $p_a(E_o) \leqslant \dim_{\mathbb{C}} R^1u_*(O_{X'_o}) = q$, we get that $p_a(E_o) = q$ and the geometric genus of (X_o, x) is q. This last fact together with the above formula imply that the other singularities of X_o are rational. <u>Moreover,</u> the exceptional fibre of every rational singularity of X_o is contained in a degenerated fibre of π.

<u>Step 3.</u> Let E_o be the component of $u^{-1}(x)$ from step 2, and $E_1, \dots, E_n$ all the other components of $u^{-1}(x)$. Then E_o is a section of π, and E_i $(i = 1, \dots, n)$ is contained in a degenerated fibre of π.

First we see that every of the components $E_1, \ldots, E_n$ is contained in a (degenerated) fibre of π. For, if E_1 (say) is not contained in any such fibre, the argument of step 2 shows that $p_a(E_1) = q$. On the other hand, since the fibre $u^{-1}(x)$ is connected, one can find s+1 distinct components $E_{i_o} = E_o, \ldots, E_{i_s} = E_1$ of $u^{-1}(x)$ such that E_{i_t} meets $E_{i_{t+1}}$ for every $t = o, 1, \ldots, s-1$. Then an easy induction shows that:

$$p_a(E_{i_o} + \ldots + E_{i_s}) \geqslant p_a(E_{i_o}) + p_a(E_{i_s}) = p_a(E_o) + p_a(E_1) = 2q.$$

Since the arithmetic genus of every curve with support in $u^{-1}(x)$ is less than or equal to the geometric genus of (X_o, x) $(= \dim_{\mathbb{C}} R^1 u_*(O_{X_o'})_x)$ (which is an easy consequence of Zariski's holomorphic functions theorem), we get the desired contradiction.

Therefore E_o is the only component of $u^{-1}(x)$ which is not contained in any fibre of π. In order to finish the proof of step 3 we have only to show that E_o is a section of π. If $q \geqslant 2$, this fact is obvious because E_o dominates B, E_o and B have the same arithmetic genus (= q), and via the Hurwitz's formula. Assume therefore q = 1, i.e. E_o is an elliptic curve. Then E_o is just the so-called "minimally elliptic cycle" of the singularity (X_o, x) in the sense of Laufer (see [18]). On the other hand, the 1-dualizing divisor of every two-dimensional singularity of geometric genus one coincides to the minimally elliptic cycle of that singularity (see [4], proposition 3.5), and in particular, $D_1 \geqslant E_o$. Since we also have $D_n \geqslant nD_1$, we get $D_n \geqslant nE_o$ for every $n \geqslant 1$. Recalling the equalities (*) we infer that $|n(K+E_o)| = \emptyset$ for every $n \geqslant 1$, which - via [22], lemma 12 - implies that E_o is a section of π. This proves step 3 and thereby part iii) of the theorem.

Finally, by i), ii) and iii) we get that $p_g(X_o') = p_g(X_o)$ (= o) and that every irreducible component of the exceptional fibres of u is smooth and these components meet transverally and no three in a point. Therefore we can apply [8], corollary 3 (3) (iii) and [9], lemma 9 to deduce part iv) of theorem (16). Q.E.D.

(17) Remarks. a) In the assumptions of theorem (16) we have $b_2(X_o) = \rho(X_o)$. This follows from the exponential sequence of X_o, a GAGA-type result and the equalities $p_g(X_o) = q(X_o) = o$.

b) If the minimal desingularization X_o' of X_o is ruled non-rational, part iii) of theorem (16) shows that X_o has precisely one non-rational singularity x; this singularity is however pararational in the sense of [9], as one can easily see.

c) The assumption "$K_t^2 \geqslant 0$" was necessary to apply proposition (14) in order to deduce that $p_n(X_0) = 0$ for every $n \geqslant 1$, which in turn was used to show that E_0 is a section of the ruled fibration $\pi : X'_0 \longrightarrow B$ when B is an elliptic curve. Therefore theorem (16) is valid without any restriction about K_t^2 (but X_t rational) as soon as one knows how to prove proposition (14) in case $K_t^2 < 0$. As we have remarked in (15), proposition (14) is valid if $K_t^2 < 0$ but the anti-Kodaira dimension of the general fibre of t is greater than or equal to one.

d) If we assume furthermore that X_0 is Gorenstein, then more precise information about the structure of X_0 can be obtained (see theorem (22) below).

e) Assume that $K_t^2 > 0$. Then $p_{-n}(X_t) \geqslant \frac{n(n-1)}{2} K_t^2 + 1$, and hence $\kappa^{-1}(X_t) = 2$. By proposition (13) iii) above and lemma 1.6 in [24] we get $\kappa^{-1}(X'_0) = \kappa^{-1}(X_0) = 2$. Therefore it makes sense to speak about the anticanonical model of X'_0 in the sense of [24] (if X'_0 is rational) and [2] (if X'_0 is ruled non-rational). Recall that the anticanonical model Y_0 of X'_0 is a normal projective surface which is obtained from X'_0 by blowing down all irreducible curves E of X'_0 with the property that $P.E = 0$, where $-K_{X'_0} = P + N$ is the Zariski decomposition of an anticanonical divisor of X'_0 (loc. cit.), with P a numerically effective $\mathbb{Q}$-divisor and either $N = 0$, or if $N > 0$, the intersection matrix of Supp(N) is negative definite; moreover, $P.E = 0$ for every $E \subseteq \mathrm{Supp}(N)$. Then X_0 and Y_0 are related in the following way: there exists a commutative diagram of the form

$$\begin{array}{ccc} X'_0 & \xrightarrow{\ v\ } & Y_0 \\ & {\scriptstyle u}\searrow \quad \nearrow{\scriptstyle w} & \\ & X_0 & \end{array}$$

where v is the canonical blowing-down morphism. To see this it will be sufficient to prove that every irreducible component of the exceptional fibres of u is contracted by v to a point. Let E be such a component. If $p_a(E) > 0$, then the genus' formula together with the fact that $E^2 < 0$ shows that $-K.E < 0$ (with $K = K_{X'_0}$), or else $P.E + N.E < 0$. Since $P.E \geqslant 0$, we have $N.E < 0$; therefore E is a component of the effective divisor N, and consequently $P.E = 0$. In other words $v(E)$ is a point. Assume now that $p_a(E) = 0$; then $E^2 \leqslant -2$ because X'_0 is the minimal desingularization of X_0. By the genus'formula $E^2 + K.E = -2$. Thus $-K.E \leqslant 0$, or else $P.E + N.E \leqslant 0$. This inequality again implies that $P.E = 0$, as required.

(18) <u>Theorem</u>. <u>In the situation of (1), assume that X_t is a ruled non-rational surface for one (and hence for all) $t \neq 0$. Then the following statements hold:</u>

i) $p_n(X_o) = o$ for every $n > o$.

ii) If $q(X_t) = q$ and $u: X'_o \longrightarrow X_o$ is the minimal desingularization of X_o, then X'_o is a ruled surface of irregularity q.

iii) X_o has at most rational singularities and their exceptional fibres are contained in the fibres of the ruled fibration $\pi: X'_o \longrightarrow B$.

Proof. The fact that X'_o is ruled follows (as in case of rational surfaces) from [20], page 77. However, one can also give the following direct argument. The Leray spectral sequence of the morphism u yields the formula

$$\chi(O_{X_o}) - \chi(O_{X'_o}) = \dim_{\mathbb{C}} R^1 u_*(O_{X'_o}),$$

and since $\chi(O_{X_o}) = \chi(O_{X_t}) = 1-q \leqslant o$, we get $\chi(O_{X'_o}) \leqslant o$. If $\chi(O_{X'_o}) < o$, the classification of surfaces (see [6]) implies that X'_o is ruled. Consider therefore the case $\chi(O_{X'_o}) = o$, which occurs iff $q = 1$ and X_o has at most rational singularities. To prove that X'_o is ruled also in this case, consider the Albanese fibration $Alb^1(X/T)$ (cf. [5], proof of the theorem; see [11] for the definition and the basic properties of $Alb^1(X/T)$), which fits in the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\;h\;} & Alb^1(X/T) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & T & \end{array}$$

Since $Alb^1(X/T)$ is compatible with the base-change, $g^{-1}(t)$ is isomorphic to the Albanese variety $Alb(X_t)$ of the surface X_t. Since the general fibre of the morphism $h_t: X_t \longrightarrow Alb(X_t)$ (induced by h) is P^1 for every $t \neq o$, the general fibre of h is also P^1. Therefore the general fibre of $h_o: X_o \longrightarrow Alb(X_o)$ is P^1, because it is at any rate smooth. It follows that the general fibre of the composition $h_o \circ u : X'_o \longrightarrow Alb(X_o)$ is P^1, and then the Noether-Tsen criterion of ruledness (see [6]) implies that X'_o is ruled, as required.

As soon as we know that X'_o is ruled, part iii) follows easily. In fact we have

$$\dim_{\mathbb{C}} R^1 u_*(O_{X'_o}) = \chi(O_{X_o}) - \chi(O_{X'_o}) \leqslant -\chi(O_{X'_o}) = q'-1, \text{ where } q' = q(X'_o).$$

In particular, X'_o is not rational, i.e. $q' > o$. If X_o would have a non-rational singularity x, it should exist an irreducible component E of the fibre $u^{-1}(x)$ which is not contained in the fibres of the ruled fibration π (see step 1 in the proof of theorem (16), or lemma 7 in [2]). Then necessarily $p_a(E) \geqslant q'$, and hence $\dim_{\mathbb{C}} R^1 u_*(O_{X'_o}) \geqslant p_a(E) \geqslant q'$, which contradicts the above inequality. Therefore X_o

has only rational singularities, $q = q'$ and all the exceptional fibres of u are contained in the degenerated fibres of π.

It remains to prove i). Since X_o has only rational singularities and the divisor class group of a rational singularity is finite (see e.g. [19]), there is a positive integer $a \geqslant 1$ such that $\omega_{X_o}^{(a)}$ is invertible. If D_a is the sum of the a-dualizing divisors of the singularities of X_o, we have (see [4]):

$$u^*(\omega_{X_o}^{(na)}) \cong \omega_{X'_o}^{na} \otimes O_{X'_o}(nD_a) \quad \text{for every } n \geqslant 1,$$

and $Supp(D_a)$ is contained in the fibres of π. On the other hand, since X'_o is ruled, $\omega_{X'_o} = O_{X'_o}(-2C) \otimes O_{X'_o}(D)$, with D a divisor whose support is contained in the fibres of π, and C a section of π. Comparing these two equalities we get

$$u^*(\omega_{X_o}^{(na)}) \cong O_{X'_o}(-2C+nD_a+naD).$$

Since $Supp(nD_a+naD)$ is contained in the fibres of π and C is a section of π, we easily get $|-2C+(nD_a+naD)| = \emptyset$ for every $n \geqslant 1$, or else $p_{na}(X_o) = o$ for every $n \geqslant 1$. These last equalities already imply i). Q.E.D.

(19) <u>Construction of certain normal degenerations of surfaces.</u> The method of constructing normal degenerations of surfaces we are going to describe is classical and known in the modern literature as the "sweeping out of the cone with hyperplane sections" method (see [21], page 46).

Start with a smooth projective surface F, a very ample line bundle L on F and a smooth curve Y belonging to the complete linear system $|L|$. We shall assume that $|L|$ yields an arithmetically Cohen-Macaulay embedding $i = i_L : F \hookrightarrow P^N$ (with $N = \dim|L|$). Then there is a hyperplane H of P^N such that $Y = F \cap H$. Let $C(F,i)$ be the projective cone over F in P^{N+1} (i.e. with respect to the embedding i), and H' the hyperplane "at infinity" of P^{N+1}. Then H is a 2-codimensional linear subspace of P^{N+1}, which generates the pencil $\{H_t\}_{t \in P^1}$ of all the hyperplanes of P^{N+1} containing H. We may assume that the parametrization is taken in such a way that H_o is the hyperplane of this pencil passing through the vertex of $C(F,i)$ and $H_\infty = H'$. Then for every $t \in P^1 - \{o\}$, $H_t \cap C(F,i)$ is isomorphic to F, while $H_o \cap C(F,i)$ is just the cone $C(Y,i')$ over Y with respect to the embedding $i' : Y \hookrightarrow H = P^{N-1}$. Since we assumed that F is arithmetically Cohen-Macaulay in P^N, Y is arithmetically normal in P^{N-1}, and hence the cone $C(Y,i')$ is normal.

In this way we got a family $f : X \longrightarrow T = A^1 = P^1 - \{\infty\}$ such that $X_t \cong F$ for $t \neq o$, and $X_o \cong C(Y,i')$, and moreover, satisfying all the assumptions of (1).

This family is in fact an embedded family of surfaces in P^N, i.e. f fits into a commutative diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{\ j\ } & T\times P^N \\ & {\scriptstyle f}\searrow \qquad \swarrow{\scriptstyle pr_1} & \\ & T & \end{array}$$

with j a closed immersion.

(2o) Normal degenerations of P^2. In the construction described in (19) take $F = P^2$, $L = O_{P^2}(s)$ with $s \geqslant 1$, and Y a smooth plane curve of degree s. Therefore we get an embedded family $f:X \longrightarrow T = A^1$ of surfaces of degree s^2 in P^N (with $N = \frac{s(s+3)}{2}$), such that $X_t = P^2$ for every $t \neq o$, and $X_o = C(Y,v_s)$, where v_s is the restriction to Y of the s-fold Veronese embedding $P^2 \hookrightarrow P^N$. If $s = 1$, X_o is of course (isomorphic to) P^2 itself.

We suspect that these cones are (up to isomorphism) the only normal degenerations of P^2. For the moment we are only able to prove the following result:

(21) Theorem. In the situation of (1), assume that X_t is (isomorphic to) P^2 for one (and hence for all) $t \neq o$. Then the conclusions i) - iv) of theorem (16) hold, $K^{-1}(X'_o) = 2$, and X_o is the anticanonical model of X'_o (in the sense of [24] if X'_o is rational, and [2] if X' is ruled non-rational). Moreover, there is a positive integer $a \geqslant 1$ such that $\omega_{X_o}^{(-a)}$ is an ample invertible sheaf, or equivalently (see [2]) the anticanonical ring of X'_o, $\bigoplus_{n=0}^{\infty} H^o(X'_o, O_{X'_o}(-nK_{X'_o}))$ is a finitely generated $\mathbb{C}$-algebra. If X_o is Gorenstein, then X_o is isomorphic either to P^2, or to an elliptic cone $C(Y,v_3)$ of degree 9 in P^9 (with Y a smooth cubic plane curve).

Proof. The first part of the theorem follows from theorem (16) and remark (17) e), observing that $\rho(X_o) = b_2(X_o) = 1$. Let us prove now that there is a positive integer $a > o$ such that $\omega_{X_o}^{(a)}$ is invertible. Let L be a f-ample line bundle on X. Then $L_t = L \otimes O_{X_t}$ is of the form $O_{P^2}(a)$ with $a > o$, and the base-change theorems imply that a is independent of $t \neq o$. Consider then the sheaf $F = \omega_X^{(a)} \otimes L^3$, which is reflexive of rank one and f-flat. By the definition of F we have $F_t \cong O_{X_t}$ for every $t \neq o$. Since $H^1(O_{X_t}) = o$ for every t, by EGA III (4.6.5) we infer that, shrinking T a little bit, $F/X-X_o \cong O_X/X-X_o$. Let $x_1,\ldots,x_s$ be the singular points of X_o, and set $U = X-\{x_1,\ldots,x_s\}$ and $U_o = U \cap X_o$. Then U is smooth, U_o is an effective irredu- divisor on U, F/U is invertible and $F/U-U_o \cong O_U/U-U_o$. Therefore there is an integer

$m \in \mathbb{Z}$ such that $F/U \cong O_U(mU_o)$. Since $O_X(mX_o)/U \cong O_U(mU_o)$, $O_X(mX_o)$ is invertible, F is reflexive and X-U is a finite set of points, F is isomorphic to $O_X(mX_o)$, and in particular; F is invertible, or else $\omega_X^{(a)}$ is also invertible. It follows that $\omega_X^{(a)}/X_o$ is invertible, and recalling that $\omega_{X_o}^{(a)}$ is the double dual of $\omega_X^{(a)}/X_o$, we infer that $\omega_{X_o}^{(a)}$ is invertible and coincides to $\omega_X^{(a)}/X_o$. Moreover, since $\omega_X^{(a)}/X_t$ is not ample for every $t \neq o$, EGA III (4.7.1) implies that $\omega_{X_o}^{(a)}$ cannot be ample. On the other hand, since $\rho(X_o) = 1$, $\omega_{X_o}^{(-a)}$ is necessarily ample. By corollary 8 in [2] one can deduce once again that X_o is the anticanonical model of X'_o, and, moreover, that the anticanonical ring of X'_o is a finitely generated $\mathbb{C}$-algebra.

Assume now that X_o is Gorenstein . First of all we show that X_o cannot have any rational singularity. If this is not true, every rational singularity of X_o is a rational double point because X_o is Gorenstein. Then we can apply a result of Brieskorn-Tjurina (see e. g. [25] §3) and get a simultaneous resolution of all rational double points of X_o, i.e. we can find a complex neighbourhood T' of o in T and a commutative diagram of complex spaces:

$$\begin{array}{ccc} V & \xrightarrow{\quad \psi \quad} & f^{-1}(T') \\ {\scriptstyle g}\downarrow & & \downarrow {\scriptstyle f' = f/f^{-1}(T')} \\ T'' & \xrightarrow{\quad \varphi \quad} & T' \end{array}$$

such that g is proper and flat, φ is finite and surjective, ψ is proper and surjective, and $\psi_t : g^{-1}(t) = X'_t \longrightarrow f'^{-1}(\varphi(t)) = X_{\varphi(t)}$ is the minimal desingularization of all rational double points of $X_{\varphi(t)}$ (if any) for every $t \in T''$. In particular, this means that X'_t is a projective surface for every $t \in T''$, and if W_t is the locus of all rational double points of $X_{\varphi(t)}$, then the restriction $X'_t - \psi_t^{-1}(W_t) \longrightarrow X_{\varphi(t)} - W_t$ is an isomorphism.

Returning to our special situation we get that for every $t \in T''$ such that $\varphi(t) \neq$ $\neq o$, X'_t is isomorphic to P^2, while for every $o' \in T''$ such that $\varphi(o') = o$, $X'_{o'}$ is the minimal desingularization of all rational double points of X_o. We obtain a contradiction applying (a slightly modified version of) proposition (5) together with the observation that $\rho(X'_{o'}) > \rho(X_o) = 1$, because then $X'_{o'}$ cannot fit in the same family with P^2.

Therefore X_o is either smooth (in which case it is clearly isomorphic to P^2), or has only non-rational singularities. In the latter case X_o is Gorenstein and $\omega_{X_o}^{-1}$ is ample (because $\omega_{X_o}^{-1}$ is invertible and we have shown that there is an $a \geqslant 1$

such that $\omega_{X_o}^{-a}$ is ample). But the classification of these surfaces is known (see [10] or [14]). Since X_o has at least one non-rational singularity, from this classification we read that X_o is an elliptic cone. (One could deduce that X_o is an elliptic cone also directly, by using only the information given by theorem (16) together with the argument of the proof of theorem (22) below.) The degree of this cone can be easily calculated:

$$d = \omega_{X_o}^{-1}.\omega_{X_o}^{-1} = \omega_{X_t}^{-1}.\omega_{X_t}^{-1} = 0_{P^2}(3).0_{P^2}(3) = 9.$$

The proof of theorem (21) is complete. Q.E.D.

(22) Theorem. In the situation of (1), assume that X_o is Gorenstein and the general fibre X_t is a rational surface. Then X_o is either rational (i.e. X'_o is rational) with at most rational double points as singularities, or X'_o is ruled of irregularity one and X_o has precisely one simple elliptic singularity x and (possibly) finitely many rational double points. In the latter case the support of the fibre $u^{-1}(x)$ is a section of the ruled fibration π, and the fibres of u over the rational double points of X_o are contained in the degenerated fibres of π.

Proof. If X_o is Gorenstein we have $p_n(X_o) = o$ for every $n \geqslant 1$ (even if $K_t^2 < o$, see Wilson [26]). Then theorem (22) follows from the classification of Gorenstein surfaces with vanishing plurigenera (see [9], theorem 14, or [23]). Therefore theorem (22) is a simple consequence of some known results and is independent of the theory developed in this paper. However, this theorem was the starting point in our investigation of normal degenerations of rational surfaces (more precisely, we got interested in such kind of problems by trying to answer a question raised by F. Catanese concerning the normal degenerations of $P^1 \times P^1$).

On the other hand, theorem (22) can be also deduced using only Wilson's result quoted above, theorem (16) and a few standard arguments concerning the Gorenstein non-rational singularities, as follows. Everything is clear except the facts that, if X'_o is non-rational and x is the only non-rational singularity of X_o, then the irregularity of X'_o is one and the fibre $u^{-1}(x)$ is irreducible.

To prove these two facts, let D be the 1-dualizing divisor of (X_o,x). Since (X_o,x) is Gorenstein D can be simply defined ~~by the~~ formula (see [4])

$$u^*(\omega_{X_o}) \cong \omega_{X'_o} \otimes 0_{X'_o}(D).$$

Since ω_{X_o} is invertible it follows easily that $D > o$, $\mathrm{Supp}(D) = \mathrm{Supp}(u^{-1}(x))$ and $\omega_D \cong 0_D$. By Wilson's result we can apply theorem (16) even if $K_t^2 < o$. Let E_o

be the section of π given by theorem (16). Everything will be proved if we show that $D = E_o$ (the irreducibility of $u^{-1}(x)$ being clear, while the fact that E_o is an elliptic curve coming from the equality $\omega_D = O_D$). In any case we have $D \geqslant E_o$. Set $Y = D-E_o$. If $Y > o$ we have the exact sequence (compare with [14], proposition 1.3)

$$o \longrightarrow O_Y(-E_o) \longrightarrow O_D \longrightarrow O_{E_o} \longrightarrow o,$$

which yields the exact sequence of cohomology

$$H^o(O_D) \longrightarrow H^o(O_{E_o}) = \mathbb{C} \longrightarrow H^1(O_Y(-E_o)) \longrightarrow H^1(O_D) \longrightarrow H^1(O_{E_o}) \to o.$$

Since the first map is surjective and the last one an isomorphism (since the geometric genus of (X_o,x) is $p_a(E_o)$), we get $H^1(O_Y(-E_o)) = o$.

On the other hand, $-E_o = Y - D$ together with the definition of D and the fact that ω_{X_o} is invertible, imply that $O_Y(-E_o) \cong \omega_{X'_o} \otimes O_{X'_o}(Y) \otimes O_Y$, and therefore $O_Y(-E_o) \cong \omega_Y$. Therefore $H^1(O_Y(-E_o)) = H^1(\omega_Y)$, and by duality on Y, $H^1(O_Y(-E_o)) \cong$ $\cong H^o(O_Y) \neq o$, a contradiction. Therefore $Y = o$, or else $E_o = D$. Q.E.D.

(23) It is well known that every smooth deformation of $P^1 \times P^1$ is isomorphic to the surface $F_{2e} = F(O_{P^1} \oplus O_{P^1}(-2e))$ for some $e \geqslant o$. Theorem (22) together with a discussion which is quite similar to the last part of the proof of theorem (21) yield:

(24) Corollary. The only normal Gorenstein degenerations of $P^1 \times P^1$ are (up to isomorphism) the following: F_{2e} $(e \geqslant o)$, the quadratic cone in P^3, or an elliptic cone of degree 8 in P^8.

(25) Some open questions. a) In the situation of (1), is it true that $p_n(X_o) = p_n(X_t)$ for every $n \geqslant 1$ and $t \neq o$?

The answer to this question is yes in the following cases: X_o smooth (even if one deals with analytic deformations, see Iitaka [15]), or X_o Gorenstein and the Kodaira dimension of X_t is $\neq 1$ (see Wilson [26]), or X_o normal and either $n = 1$, or X_t ruled and $n \geqslant 1$ (but $K_t^2 \geqslant o$ if X_t is rational, see theorems (16) and (18) above).

b) Is it true that the only degenerations of P^2 are the cones from (2o) ?

c) The same question as b) but for $P^1 \times P^1$.

The answer to questions b) and c) is yes if X_o is Gorenstein (theorem (21) and corollary (24)). Question c) was raised by F. Catanese (private discussion).

Independently of the problem of degenerations of surfaces, one can formulate:

d) Give a classification of all normal projective surfaces Y with $p_n(Y) = o$ for every $n \geqslant 1$ (and eventually $q(Y) = o$) generalizing the situation from the Gorenstein case (see [9] or [23]).

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