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NATURAL LOCALIZATION AND NATURAL SHEAF PROPERTY

IN STANDARD H-CONES FUNCTIONS (I)

N.Boboc and Gh.Bucur

I. Natural localization in standard H-cones of functions.

Introduction. The aim of this paper is to develop a theory of localization in a standard H-cone of functions S on a set X. It is essentialy different from that developed in [1] or [4] and they coincide only in the case where the axiom of polarity is fullfiled. It turn out that this new typ of localization is more adequated even in the case of standard H-cones of superharmonic functions associated with an harmonic space on a locally compact space.

We succeeded in improving some results from the theory of standard H-cones using intensively the tool of absolutely continuous resolvent associated with a standard H-cone.

If G is a fine open set in X then we denote by $S'(G)$ the set of all positive functions f on G which are finite on a fine dense subset of G and such that there exists a sequence $(s_n)_n$ in S, $s_n \leq \infty$ such that

$$s_n - B_{s_n}^{X\setminus G} \uparrow f$$

where $B_{s_n}^{X\setminus G}$ is the balayage of s_n on $X\setminus G$.

We show that $S'(G)$ is a standard H-cone of functions on G and the map

$$G \longrightarrow S'(G)$$

is a pre-sheaf. Also we show that the fine topology on G given by $S'(G)$ coincides with the induced fine topology of X on G and a similar assertion for the natural topology when G is open.

Finally, some complementary results on balayage theory on $S'(G)$ are given.

In the second part of this paper we deal with the natural sheaf property for the pre-sheaf $G \longrightarrow S'(G)$ considered above.

1. Preliminaries and first results

In this paper S will be a standard H-cone. Many concepts and results concerning the theory of standard H-cones which will be used may be recover in [1].

So we recall that whenever a standard H-cone S is given, a natural dual H-cone denoted by S^* may be associated with S. The elements of S^* are called H-integrals on S. In every standard H-cone S we distinguish the subset S_o of all universally continuous elements. The coarsest topology on S^* for which all the functions

$$\mu \longrightarrow \mu(s), \qquad s \in S_o$$

are continuous is called natural topology on S^*.

The set S^* endowed with the natural topology is a complete separable metrisable space.

The coarsest topology on S^* for which all the functions

$$\mu \longrightarrow \mu(s) \qquad s \in S$$

are continuous is called the fine topology on S^*.

In many situations S is given as a standard H-cone of functions on a set X. In fact in this case X is a subset of S^* such that:

a) $s \leq t$ iff $s(x) \leq t(x)$ $(\forall)$ $x \in X$.

b) S separates the points of X.

c) S is a min-stable convex cone of functions on X such that $\inf(s,1) \in S$ for any $s \in S$.

Any standard H-cone may be represented as a standard H-cone of functions, namely for any weak unit u of S we consider the set X_u of all non-zero extreme points of the cap

$$K_u := \{\mu \in S^* \mid \mu(u) \leq 1\}$$

of S^* endowed with the natural topology and then we identify any element $s \in S$ with its restriction to X_u. Thus S becomes a standard H-cone of functions on X_u. This is the canonical representation of S associated with the weak unit u and it possesses the following remarkable property: the positive constant functions belong to S and any H-integral μ such that $\mu(1) < \infty$ is represented as a Borel measure on X_u.

In general if S is a standard H-cone then S may be identified with a convex subcone of S^{**}, S is solid in S^{**} with respect to the natural order and S is dense in order from below in S^{**}. If S is given as a standard H-cone of functions on a set X then

S^{**} may be identified with the standard H-cone $\bar{S}$ of all functions f on X, which are finite on a dense subset (with respect to the natural or fine topology) of X and such that inf(s,f) belongs to S for any $s \in S$. Thus, from the definition of S it follows that $\bar{S}$ contains the positive constant functions. So it is no loss of generality if we suppose that a standard H-cone of functions contains the positive constant functions.

Suppose now that S is a standard H-cone of functions on X which contains the positive constant functions. Then the function $1=1_X$ is a weak unit of S and we may consider the canonical representation of S on the set X_1. In this way X may be identified with a subset of X_1. If $\mu \in S^*$ is such that there exists a Borel measure m on X (X endowed with the natural topology), for which

$$\mu(s) = \int s\,dm \qquad (\forall) \quad s \in S,$$

we say that μ is an <u>H-measure on X</u> (with respect to S). The set X is called <u>saturated</u> if any $\mu \in S^{**}$ with $\mu(1) < \infty$ is an H-measure on X. Obviously X is saturated iff $X=X_1$. For any $s \in S$ we denote by s_1 the function on X_1 which extends natural s and by S_1 the set

$$S_1 = \left\{ s_1 \,\middle|\, s \in S \right\}.$$

The pair (S_1, X_1) is called the <u>natural extension</u> of (S,X) and X_1 is called <u>the saturated of X</u>.

If A is a subset of X and $s \in S$ we consider the reduite of s on A

$$R_s^A =: \inf \left\{ s' \in S \,\middle|\, s' \geq s \text{ on } A \right\}$$

and the balayage of s on A

$$B_s^A =: \bigwedge \{ s' \in S \mid s' \geq s \text{ on } A \} .$$

Generally we have $R_s^A = B_s^A$ on $X \setminus A$. If A is fine closed and $B_s^A = s$ on A for any $s \in S$ then A is called a basic set. Any basic set A is a G_δ subset of X with respect to the natural topology and the map

$$B^A : S \longrightarrow S \, , \qquad B^A(s) := B_s^A$$

is a balayage on S. The set X is called nearly saturated if, conversely, for any balayage B on S there exists a basic set $A \subset X$ such that $B = B^A$. In this case A is uniquely determined and it is called the base of B and it is denoted by b(B).

A subset A of X is called thin at $x \in X$ if there exists $s \in S$ such that $B_s^A(x) < s(x)$. The set of all points $x \in X$ such that A is not thin at x is called the base of A and it is denoted by b(A). We say that A is totally thin if A is thin at any point $x \in X$. Also we say that A is semipolar if it is a countable union of totally thin sets.

If X is saturated then it is nearly saturated. Generally X will be nearly saturated iff $X_1 \setminus X$ is negligible i.e. any compact subset of $X_1 \setminus X$ is semipolar (with respect to the pair (S_1, X_1)) or equivalently any $\mu \in S_o^*$ is an H-measure on X.

A subset A of X is called polar if $B_s^A = 0$ for any $s \in S$.

Suppose that S is a standard H-cone of functions on a nearly saturated set X. For any balayage B on S we denote

$$d(B) := X \setminus b(B) .$$

The set $d(B)$ is fine open and the set of all functions on $d(B)$ of the form

$$(s-Bs)\big|_{d(B)}, \quad s \in S, \quad s < \infty$$

is a standard H-cone of functions on $d(B)$ which is denoted by S_B. We denote by $\overline{S_B}$ the set of all functions f on $d(B)$ which are finite on a fine dense subset of $d(B)$ and for which $\inf(f,t) \in S_B$ for any $t \in S_B$.

For any fine open set G we have denoted [1] by $\mathcal{S}(G)$ the set of all restriction to G of the functions $f \in \overline{S_B}$ where B is the greatest balayage on S whose base lies in $X \setminus G$.

In this way $S(G)$ is a standard H-cone of functions on G, namely a nearly-saturated representation of the standard H-cone $\overline{S_B}$.

<u>Definition</u>. Let S be a standard H-cone. Then an element $s \in S$ is called <u>universally bounded</u> if for any weak unit u of S we have $s \leq \alpha u$ for a suitable $\alpha \in R_+$.

<u>Proposition 1.1</u>. Let S be a standard H-cone and let $s \in S$. Then the following assertions are equivalent:

1) $\mu \in S^{**} \Rightarrow \mu(s) < \infty$.

2) s is universally bounded.

3) There exists $s_o \in S_o$ such that $s \leq s_o$.

<u>Proof</u>. 1) $\Rightarrow$ 2). Let u be a weak unit of S. If we suppose that $s \nleq \alpha u$ for any $\alpha \in R_+$ then for any $n \in N$ there exists $\mu_n \in S^*$ such that $\mu_n(s) > 4^n \cdot \mu_n(u)$. Obviously we may suppose that $\mu_n(u)=1$ and thus the element

$$\mu := \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_n$$

belongs to S^*. We have arrived to the contradictory relation

$$+\infty > \mu(s) \geq \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_n(s) \geq \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot 4^n = +\infty.$$

2) $\Longrightarrow$ 3) Let u be a weak unit of S and let $(s_n)_n$ be a sequence of S_o increasing to u. We shall show that there exists $\alpha > 0$ and $n \in N$ such that $s \leq \alpha s_n$. In the contrary case there exists a sequence $(\mu_n)_n$ in S^* such that $\mu_n(s_n) \leq 2^{-n}$, $\mu_n(s) = 1$, $\mu_n(u) < +\infty$.

We put, for any $n \in N$,

$$\beta_n := \sup_{1 \leq k \leq n} \mu_k(u), \quad \alpha_n := 2^{-n}(1+\beta_n)^{-1}.$$

Since $\sum_{n=1}^{+\infty} \alpha_n < +\infty$ and since $\sum_{k=1}^{+\infty} \alpha_k s_k \leq u$ it follows that

$$v := \sum_{k=1}^{+\infty} \alpha_k s_k$$

is an element of S and moreover v is a weak unit of S. Using the hypothesis, there exists $\beta > 0$ such that $s \leq \beta v$. For any $n \in N$ we have

$$1 = \mu_n(s) \leq \beta \cdot \mu_n(v) = \beta \cdot \sum_{k=1}^{n} \mu_n(\alpha_k s_k) +$$

$$+ \beta \sum_{k=n+1}^{\infty} \mu_n(\alpha_k s_k) \leq \beta \cdot 2^{-n} \cdot \sum_{k=1}^{n} \alpha_k + \beta \cdot \sum_{k=n+1}^{\infty} \alpha_k \beta_k \leq \beta \cdot 2^{-n+1}$$

which is a contradiction.

3) $\Longrightarrow$ 1) follows from the fact that any $\mu \in S^*$ is finite on S_o.

Definition. Let S be a standard-H-cone of functions on a set X. We say that X is semi-saturated if any H-integral on S dominated by an H-measure on X (with respect to S) is also an H-measure on X.

Since any H-measure on X is a σ-finite measure on X then X will be semi-saturated iff any H-integral on S dominated by a finite H-measure is also an H-measure.

Proposition 1.2. Let S be a standard H-cone of functions on a set X and let (S_1, X_1) be the natural extension of (S, X). Then the following assertions are equivalent:

1. X is semi-saturated ;
2. any compact subset of $X_1 \setminus X$ is polar ;
3. any universally bounded H-integral on S is an H-measure on X.

Proof. $1) \Rightarrow 2)$. Let F be a compact subset of $X_1 \setminus X$ and $x \in X$. We consider the H-integral on S defined by

$$s \longrightarrow {}^1B^F_{s_1}(x)$$

where s_1 is the natural extention of s to X_1 and ${}^1B^F_{s_1}$ means the balayage of s_1 on F considered in X_1. It is known that there exists an unique measure μ_x on X_1 such that

$${}^1B^F_{s_1}(x) = \mu_x(s_1) \qquad (\forall)\ s \in S$$

and this measure is carried by F. On the other hand since

$${}^1B^F_{s_1}(x) \leq s_1(x) = s(x) = \varepsilon_x(s)$$

it follows, from hypothesis, that there exists a Borel measure θ_x on X such that

$$^{1}B^{F}_{s_1}(x) = \theta_x(s) \qquad (\forall)\ s \in S.$$

Obviously θ_x may be considered as a measure on X_1 which doesn't charge F. From the preceding considerations we get $\theta_x = \mu_x = 0$. Hence

$$^{1}B^{F}_{s_1}(x) = 0 \qquad (\forall)\ x \in X, \quad (\forall)\ s \in S$$

and therefore F is polar.

2) $\Longrightarrow$ 3). Let $\mu \in S^*$ be such that it is dominated by an element $\nu \in S^*_o$. Since X_1 is saturated and $\mu(1) \leq \nu(1) < +\infty$ it follows that μ, ν are H-measures on X_1. Since $\nu \in S^*_o$ it follows that for any compact subset F of $X_1 \setminus X$ we have

$$\mu(K) \leq \int {}^{1}R^{K}_{1} d\mu \leq \inf \left\{ \int s_1 d\mu \,\middle|\, s_1 \in S_1,\ s_1 \geq 1 \text{ on } K \right\} \leq$$

$$\leq \inf \left\{ \int s_1 d\nu \,\middle|\, s_1 \in S_1, s_1 \geq 1 \text{ on } K \right\} = \int R^{K}_{1} d\nu = \int B^{K}_{1} d\nu = 0,$$

and therefore μ may be considered as a measure on X. The assertion 3) follows now using the preceding proposition.

3) $\Longrightarrow$ 1). Let $\mu \in S^*$ and let m be a finite measure on X such that

$$\mu(s) \leq \int s dm \qquad (\forall)\ s \in S.$$

Since $\mu(1) \leq \int 1 dm < \infty$, μ may be considered as a measure on X_1. We want to prove that μ doesn't charge any compact subset of $X_1 \setminus X$. First we show that any compact part F of $X_1 \setminus X$ is polar with respect to the pair (S_1, X_1). Indeed for any $\nu \in S^*_o = (S^*_1)_o$ the map ν_1 defined on S_1 by

$$\nu_1(s_1) := \nu({}^1B^F_{s_1})$$

is an H-integral on S dominated by ν and therefore it is an universally bounded H-integral on S. Using the hypothesis ν_1 is a measure on X. On the other hand it is known ([1], Prop. 4.3.12) that ν_1 is an H-measure on X_1 carried by F. Hence $\nu_1=0$ for any $\nu \in S_o^*$ and therefore ${}^1B^F_{s_1} = 0$ i.e. F is polar. Now we have, for a compact subset F of $X_1 \setminus X$;

$${}^1B^F_1 = {}^1R^F_1 \text{ on } X, \qquad {}^1B^F_1 = 0 \text{ on } X_1$$

$$\int_F 1 d\mu \leq \int {}^1R^F_1 d\mu \leq \inf \left\{ \int s_1 d\mu \,\middle|\, s_1 \in S_1, s_1 \geq 1 \text{ on } F \right\} \leq$$

$$\inf \left\{ \int s_1 dm \,\middle|\, s_1 \in S_1, s_1 \geq 1 \text{ on } F \right\} = \int {}^1R^F_1 dm = \int {}^1B^F_1 dm = 0$$

and so μ doesn't charge the compact subset F of $X_1 \setminus X$.

Proposition 1.3. If S is a standard H-cone of functions on a semi-saturated set X and B is a balayage on S then the set $d(B) = X \setminus b(B)$ is semi-saturated with respect to the standard H-cone of functions $S(d(B))$.

Proof. Let μ be an H-integral on $S(d(B))$ and let ν be a finite H-measure on $d(B)$ with respect to $S(d(B))$ such that $\mu \leq \nu$. We show that μ is also an H-measure on $d(B)$. We denote by μ' the map from S into $\overline{R}_+$ defined by

$$\mu'(s) := \mu(s|_{d(B)}) \qquad (\forall)\ s \in S.$$

Obviously μ' is an H-integral on S and we have

$$\mu'(s) = \mu(s|_{d(B)}) \leq \nu(s|_{d(B)}) = \int s d\nu .$$

Since $\nu(1)<\infty$ it follows that $\mu'(1)<\infty$ and therefore, X being semi-saturated with respect to S, μ' is a measure on X. For any bounded element s of S we have

$$\mu'(s-Bs)=\mu'(s)-\mu'(Bs)=\mu(s|_{d(B)})-\mu(Bs|_{d(B)})=\mu(s-Bs)$$

and therefore

$$\mu'|_{d(B)}(s-Bs)=\mu'(s-Bs)=\mu(s-Bs).$$

Any element u of S(d(B)) being the limit of an increasing sequence $(s_n-Bs_n)_n$ where s_n is bounded for any n we deduce that

$$\mu'|_{d(B)}(u)=\mu(u).$$

<u>Lemma 1.4.</u> Let S be a standard H-cone of functions on a nearly saturated set X and let $(s_n)_n$ be a sequence in S such that the function f on X defined by

$$f(x):=\liminf_{n\to\infty} s_n(x)$$

is finite on a dense subset of X. Then the lower semicontinuous regularisation $\hat{f}$ of f belongs to S, namely

$$\hat{f}=\bigvee_{n\in N}\left(\bigwedge_{k\geq n} s_k\right).$$

<u>Proof.</u> Let (S_1,X_1) be the natural extension of (S,X). For any $s\in S$ we denote by $\tilde{s}$ the extension of s to X_1. We consider the sequence $(\tilde{s}_n)_n$ in S_1 and for any $n\in N$ the function f_n (resp. g_n) on X (resp. X_1) defined by

$$f_n(x)=\inf_{k\geq n} s_k(x) \quad (\text{resp. } g_n(x)=\inf_{k\geq n} \tilde{s}_k(x)).$$

Obviously the sequence $(f_n')_n$ increases to f, the sequence $(g_n)_n$ increases to a function g on X_1 and we have

$$g_n=f_n \text{ on } X, \quad \hat{f}_n=\bigwedge_{k\geq n} s_k \;, \quad \hat{g}_n=\bigwedge_{k\geq n}\tilde{s}_k=\widetilde{\bigwedge_{k\geq n} s_k}$$

for any $n\in N$, where $\hat{f}_n$ (resp. $\hat{g}_n$) is the lower semicontinuous regularisation of f_n on X (resp. of g_n on X_1). Hence we have

$$\hat{f}_n=\hat{g}_n \quad \text{on} \quad X, \quad [\hat{f}_n<f_n]\subset[\hat{g}_n<g_n] \qquad (\forall)\quad n\in N$$

and therefore

$$\left[\sup_{n\in N}\hat{f}_n<f\right]=X\cap\left[\sup_{n\in N}\hat{g}_n<g\right] .$$

Since for any $n\in N$ the set $[\hat{g}_n<g_n]$ is a Borel semipolar subset of X_1 it follows that the set

$$M:=\left[\sup_{n\in N}\hat{g}_n<g\right]$$

is a Borel semipolar subset of X_1 .

By hypothesis the function f is finite on a dense subset of X and therefore (see [1], Theorems 3.1.5, 4.4.4) the function $\sup_{n\in N}\hat{f}_n$ (resp. $\sup_{n\in N}\hat{g}_n$) belongs to S (resp. S_1) and

$$\sup_{n\in N}\hat{f}_n=\sup_{n\in N}\hat{g}_n \quad \text{on} \quad X.$$

We show now that $\hat{f}=\sup_{n\in N}\hat{f}_n$. ~~Indeed,~~ from the above considerations we have

$$\sup_{n\in N}\hat{f}_n\leq\hat{f}\leq f$$

and the set

$$D:=\left[\sup_{n\in N} \hat{f}_n < \hat{f}\right]$$

is fine open in X and it is contained in M. We remark that $D=\emptyset$ because in the contrary case there exists a balayage B on S_1 such that $b(B)\cap X\subset D$. On the other hand since M is a Borel semipolar subset of X_1 we get that $b(B)\setminus M$ is a Borel non-semipolar subset of X_1. Hence there exists a balayage B_1 on S_1 such that

$$B_1\leq B\ ,\quad b(B_1)\subset X_1\setminus M,\quad b(B_1)\cap X=\emptyset$$

The last equality shows that the balayage B_1 is not representable on X which contradicts the hypothesis.

<u>Lemma 1.5.</u> a) Let S be a standard H-cone of functions on a set X and let A be an arbitrary subset of X. If $s,t\in S$ are such that $s+B^At\leq t+B^As$ then we have $B^As\leq s+B^At$. If moreover $s\wedge B^As=0$ then $s\leq t$ and $B^As\leq B^At$.

b) Let u be a function on $X\setminus A$ such that there exist a sequence $(s_n)_n\subset S$, $s_n<\infty$, and an element $t\in S$ such that

$$s_n-B^As_n+B^At\leq t \qquad (\forall)\quad n\in N,$$

$$(s_n-B^As_n)_n\uparrow u \quad \text{on} \quad X\setminus A.$$

Then there exists $s'\in S$ such that $B_t^A\leq s'\leq t$ and $s'=u+B^At$ on $X\setminus A$.

<u>Proof</u>. a) First we remark that X may be supposed saturated. Since for any polar subset M of A we have

$$B_s^A = B_s^{A\setminus M} , \qquad B_t^A = B_t^{A\setminus M}$$

then we may suppose also that s, t are finite on A.

In this case, using Proposition 3.2.4 from [1] we have

$$B^A s = \bigwedge \{ B^G s \mid G \text{ fine open, } G \supset A \} ,$$

$$B^A t = \bigwedge \{ B_t^G \mid G \text{ fine open, } G \supset A \} .$$

On the other hand S being a standard H-cone there exists a decreasing sequence $(G_n)_n$ of fine open subsets of X such that $A \subset G_n$ for any $n \in N$ and such that

$$B^A s = \bigwedge_n B^{G_n} s = \widehat{\lim_{n\to\infty} B^{G_n} s} ,$$

$$B^A t = \bigwedge_n B^{G_n} t = \widehat{\lim_{n\to\infty} B^{G_n} t} .$$

Since for any $n \in N$ the map $u \longrightarrow B^{G_n} u$ is a balayage on S we deduce, using ([1], Proposition 5.1.2 and Theorem 5.1.5) that in the H-cone S_{B_n} (where $B_n = B^{G_n}$) we have

$$(s - B^{G_n} s) \wedge (t - B^{G_n} t) \leq t - B^{G_n} t ,$$

$$u_n := (s - B^{G_n} s) \wedge (t - B^{G_n} t) + B^{G_n} t \in S ,$$

$$u_n \leq t \qquad (\forall) \quad n \in N.$$

Hence, for any $n \in N$, we have

$$u_n + B^{G_n} s = (s + B^{G_n} t) \wedge (t + B^{G_n} s) .$$

Using the fact that the sequences $(B^{G_n} s)_n$, $(B^{G_n} t)_n$ are decreasing and passing to the limit in the last equality we get

$$\liminf_{n\to\infty} u_n + \lim_{n\to\infty} B^{G_n} s = \inf(s+\lim_{n\to\infty} B^{G_n} t, t+\lim_{n\to\infty} B^{G_n} s) .$$

From the preceding lemma we deduce

$$\widehat{\liminf_{n\to\infty} u_n} \in S ,$$

$$\widehat{\liminf_{n\to\infty} u_n} + B^A s = \widehat{\liminf_{n\to\infty} u_n + \lim_{n\to\infty} B^{G_n}} s =$$

$$= \inf(\widehat{s+\lim_{n\to\infty} B^{G_n}} s, \widehat{t+\lim_{n\to\infty} B^{G_n}} t) = (s+B^A_t) \wedge (t+B^A s) = s+B^A t .$$

Hence $B^A s \lesssim s+B^A t$. If $s \wedge B^A s=0$ then, obviously, we deduce $B^A s \lesssim B^A t$, $s \leq t$.

b) Since the sequence $(s_n - B^A s_n)_n$ is increasing on $X \setminus A$ we deduce that the sequence $(s_n - B^A s_n)_n$ increases on X. From the preceding point we get that the sequence $(s'_n)_n$, where

$$s'_n := s_n - B^A s_n + B^A t \qquad (\forall) \quad n \in N$$

is an increasing sequence of S dominated by the element $t \in S$. Hence the function

$$x \longrightarrow s'(x) := \sup_{n \in N} s'_n(x)$$

belongs to S and it satisfies the required conditions.

2. Localization in a standard H-cone of functions.

In this section S will be a standard H-cone of functions on a nearly saturated set X. A sub-Markovian resolvent $\mathcal{V} = (V_\alpha)_{\alpha>0}$ of kernels on X is called associated with S if its initial kernel V is such that $Vf \in S$ for any positive bounded Borel function f on X and such that $V1$ is a bounded, continuous generator of S.

Theorem 2.1. Let S be a standard H-cone of functions on a nearly saturated set X and let G be a fine open subset of X. Let further $\mathcal{V}=(V_\alpha)_{\alpha>0}$ be a sub-Markovian resolvent of kernels on X associated with S. Then we have:

a) There exists a sub-Markovian resolvent of kernels $\mathcal{W}=(W_\alpha)_{\alpha>0}$ on G (considered as a measurable subspace of X) such that its initial kernel W is defined by

$$Wf=(V\bar{f}-B^{X\setminus G}(V\bar{f}))|_G$$

where for any positive, bounded Borel function f on G, $\bar{f}$ denotes a positive, bounded Borel extention of f to X;

b) The convex cone $\mathcal{E}_{\mathcal{W}}$ of all excessive functions on G with respect to the resolvent $\mathcal{W}$ is a standard H-cone of functions on G such that G is nearly saturated with respect to this cone. Moreover we have: for any $s,t\in S$, $t<\infty$, $s\geq t$ on $X\setminus G$ the restriction to G of the function $s-B^{X\setminus G}t$ belongs to $\mathcal{E}_{\mathcal{W}}$; any element f of $\mathcal{E}_{\mathcal{W}}$ which is dominated by the element $(t-B^{X\setminus G}t)|_G$ is of the form $(v-B^{X\setminus G}t)|_G$ where $v\in S$ and $B^{X\setminus G}t\leq v\leq t$.

c) The fine topology on G with respect to $\mathcal{E}_{\mathcal{W}}$ coincides with the trace on G of the fine topology of X with respect to S; the natural topology on G with respect to $\mathcal{E}_{\mathcal{W}}$ is finer then the trace on G of the natural topology of X with respect to S and they coincide on any open subset of X contained in G.

Proof. We denote by μ the Borel measure on X defined by

$$\mu(A):=\sum_{n=1}^{\infty}\frac{1}{2^n}V(1_A)(x_n)$$

where the set $\{x_n \mid n\in N\}$ is naturally dense in X. From hypothesis it follows that V is absolutely continuous with respect to μ and moreover, we have

$$\mu(A)=0 \Longleftrightarrow V(1_A)=0 .$$

Obviously any closed Borel subset A such that $\mu(A)>0$ is not totally thin because the element $V(1_A)$ is nearly continuous and therefore there exists a balayage on S whose base lies in A. Hence any semipolar subset of X is μ-negligible and therefore, using the fact that for any fine closed subset F of X the set $F\setminus b(F)$ is semipolar, any fine open subset of X is μ-measurable. We deduce also that the kernel V may be naturally extended to the measurable functions with respect to the fine topology, preserving the complete maximum principle. Since for any μ-negligible subset A of X we have

$$B_S^{X\setminus A}s = s \quad (\forall)\ s\in S$$

we deduce that any non empty fine open subset of X is not μ-negligible.

a) Obviously for any μ-measurable positive bounded function g on X vanishing on G we have

$$Vg-B^{X\setminus G}(Vg) = 0 .$$

Hence, for any μ-measurable positive bounded function f on G, the function on G

$$\overset{\circ}{W}f = V\bar{f} - B^{X\setminus G}(V\bar{f})$$

where $\bar{f}$ is an arbitrary positive bounded ~~μ-me~~asurable extention of f to X, does not depend on this extention. So, from now on, we denote by $\bar{f}$ the canonical extention of f to X defined by

$$\bar{f}(x)=\begin{cases} f(x) & \text{if } x\in G,\\ 0 & \text{if } x\in X\setminus G.\end{cases}$$

We show that the kernel W satisfies the complete maximum principle and for any $s,t\in S$, $t<\infty$, $t\leq s$ the restriction to G of the function $s-B^{X\setminus G}t$ is a W-dominant function.

Let f be a positive, bounded Borel measurable function on G, let $\bar{f}$ be the canonical extention of f to X and suppose that

$$V\bar{f}-B^{X\setminus G}(V\bar{f})\leq s-B^{X\setminus G}t \text{ on } [\bar{f}>0].$$

We choose a decreasing sequence $(\Gamma_n)_n$ of fine open subsets of X such that $\Gamma_n\supset X\setminus G$ and such that

$$\bigwedge_{n\in N} B^{\Gamma_n}(V\bar{f})=B^{X\setminus G}(V\bar{f}),\quad \bigwedge_{n\in N} B^{\Gamma_n}t=B^{X\setminus G}t.$$

Since for any $n,p\in N$ we have

$$(s-B^{\Gamma_{n+p}}t)\big|_{d(B^{\Gamma_n})}\in\overline{S}_{B^{\Gamma_n}}$$

we deduce that

$$(s-B^{X\setminus G}t)\big|_{d(B^{\Gamma_n})}\in\overline{S}_{B^{\Gamma_n}}\qquad(\forall)\ n\in N.$$

Also we have

$$V\bar{f}-B^{\Gamma_n}(V\bar{f})\leq s-B^{X\setminus G}t \quad\text{on } [\bar{f}>0]\cap d(B^{\Gamma_n}).$$

Since the function on $d(B^{\Gamma_n})$ defined by

$$u:=\inf(V\bar{f}-B^{\Gamma_n}(V\bar{f}),s-B^{X\setminus G}t)$$

belongs to $\overline{S}_{B^{\Gamma_n}}$ and it is dominated by $V\bar{f}-B^{\Gamma_n}(V\bar{f})$, we deduce that there exists $u_n \in S$ such that

$$u = u_n - B^{\Gamma_n} u_n \leq V\bar{f} - B^{\Gamma_n}(V\bar{f}).$$

From ([1], Propozition 5.1.2) and from the inequality

$$V\bar{f} - B^{\Gamma_n}(V\bar{f}) \leq u \quad \text{on} \quad d(B^{\Gamma_n}) \cap [\bar{f} > 0]$$

it follows that the element

$$w_n := u + B^{\Gamma_n}(V\bar{f})$$

belongs to S, $w_n = V\bar{f}$ on $X \setminus d(B^{\Gamma_n})$,

$$V\bar{f} \leq w_n \quad \text{on} \quad [\bar{f} > 0]$$

and therefore

$$V\bar{f} \leq w_n \quad \text{on } X$$

$$V\bar{f} - B^{\Gamma_n}(V\bar{f}) \leq u_n - B^{\Gamma_n} u_n \leq s - B^{X \setminus G} t \quad \text{on} \quad G.$$

The assertion a) follows now using Hunt's approximation theorem ([1], Theorem 1.2.1).

b) Obviously if we denote by λ the restriction of μ to G it follows immediately that the kernel W is absolutely continuous with respect to λ. Moreover if $W(1_A)=0$ then $\lambda(A)=0$. Indeed, if $\lambda(A) > 0$ then there exists A' a natural closed subset of A such that $\lambda(A') > 0$. Since $V(1_{A'}) = B^{X \setminus G}(V(1_{A'}))$ on G we get

$$V(1_{A'}) \leq B^{X \setminus G}(V(1_{A'})) \quad \text{on } A',$$

$$V(1_{A'}) \leq B^{X \setminus G}(V(1_{A'})) \quad \text{on } X,$$

and therefore we deduce that the balayage B on S associated with the nearly continuous element $V(1_{A'})$ as in ([1], Proposition 4.3.13), is equal to zero, because $b(B) \subset A' \cap (X \setminus G) = \emptyset$. Hence $V(1_{A'}) = 0$, $\mu(A') = 0$ which contradicts the relation $\lambda(A') > 0$. Since W is absolutely continuous with respect to λ we deduce that $\mathcal{E}_W$ is a standard H-cone. We show now that any W-dominant function f on G which is fine lower semicontinuous and finite on a dense subset of G belongs to $\mathcal{E}_W$. For this it will be sufficient to prove that for any $s \in S$, $s < \infty$ the function

$$g := \inf(f, (s - B^{X \setminus G} s)|_G)$$

belongs to $\mathcal{E}_W$. Using the considerations made in a) we deduce that g is W-dominant. Let $\hat{g}$ be the greatest element of $\mathcal{E}_W$ such that $\hat{g} \leq g$. It is known that $W(1_{[g > \hat{g}]}) = 0$ and therefore $\mu([g > \hat{g}]) = 0$. To finish the proof it will be sufficient to show that $\hat{g}$ is fine continuous. In fact we show that for any $h \in \mathcal{E}_W$ such that $h \leq (s - B^{X \setminus G} s)|_G$ for some element $s \in S$, $s < \infty$ we have

$$h = (v - B^{X \setminus G} s)|_G \text{ where } v \in S, \ v \leq s.$$

Indeed, let $(f_n)_n$ be a sequence of positive, bounded, Borel functions on G such that the sequence $(Wf_n)_n$ increases to h. Since for any $n \in N$ we have

$$(V\bar{f}_n - B^{X \setminus G}(V\bar{f}_n))|_G = Wf_n \leq h \leq s - B^{X \setminus G} s \quad \text{on } G,$$

$$V\bar{f}_n - B^{X \setminus G}(V\bar{f}_n) \leq s - B^{X \setminus G} s \quad \text{on } X.$$

From Lemma 1.5 it follows that, for any n, the function v_n on X defined by

$$v_n := V\tilde{f}_n - B^{X\setminus G}(V\tilde{f}_n) + B^{X\setminus G}s$$

belongs to S and moreover $v_n \leq s < \infty$. Obviously the sequence $(v_n)_n$ increases to an element $v \in S$, $v \leq s$. Also we have

$$h + B^{X\setminus G}s = v \quad \text{on} \quad G, \quad h = (v - B^{X\setminus G}s)|_G .$$

From the preceding considerations it follows that for any $s, t \in \mathcal{E}_{\mathcal{W}}$ the function

$$G \ni x \longrightarrow \inf(s(x), t(x))$$

belongs to $\mathcal{E}_{\mathcal{W}}$. Also for any $s, t \in S$, $s \geq t$, $t < \infty$ the function $(s - B^{X\setminus G}t)|_G$ belongs to $\mathcal{E}_{\mathcal{W}}$. Particularly the function $s|_G$ belongs to $\mathcal{E}_{\mathcal{W}}$ for any $s \in S$ and $\mathcal{E}_{\mathcal{W}}$ is a standard H-cone of functions on G.

If $s, t \in S$ and $s \geq t$ on $X\setminus G$ then for any $u \in S_o$, $u \leq t$ and any $n \in N$ the set

$$D := \left[u < s + \frac{1}{n}\right]$$

is open and it contains the set $X\setminus G$. Obviously we have

$$s + \frac{1}{n} \geq B^D u , \quad B^{X\setminus G}(B^D u) = B^{X\setminus G}u$$

and from the preceding considerations we deduce

$$\frac{1}{n} + s - B^{X\setminus G}u \in \mathcal{E}_{\mathcal{W}} \qquad (\forall) \quad n \in N$$

Since $n \in N$ and $u \in S_o$, $u \leq t$ are arbitrary we get

$$s - B^{X\setminus G}t \in \mathcal{E}_{\mathcal{W}} .$$

We show now that G is nearly saturated with respect to $\mathcal{E}_{\mathcal{W}}$. For this let θ be an universally continuous element of $\mathcal{E}_{\mathcal{W}}^{*}$. We consider the real function $\bar{\theta}$ on S given by

$$\bar{\theta}(s) = \theta(s|_G)$$

Obviously $\bar{\theta}$ is an H-integral on S. We show that for any decreasing sequence $(s_n)_n$ from S we have

$$\inf_n \bar{\theta}(s_n) = \bar{\theta}(\bigwedge_n s_n) .$$

Indeed, since for any element $u \in \mathcal{E}_{\mathcal{W}}$ and any element $s \in S$, $s < \infty$ the function

$$G \ni x \to \inf(u(x), s(x) - B^{X \setminus G} s(x))$$

belongs to $\mathcal{E}_{\mathcal{W}}$ it follows that any element u of $\mathcal{E}_{\mathcal{W}}$ is fine continuous on G and therefore

$$(\bigwedge_n s_n)|_G = \bigwedge_{\mathcal{E}_{\mathcal{W}}} (s_n|_G), \ \bar{\theta}(\bigwedge_n s_n) = \theta((\bigwedge_n s_n)|_G) = \inf_n \theta(s_n|_G) = \inf_n \bar{\theta}(s_n)$$

Since X is nearly saturated with respect to S then $\bar{\theta}$ is an H-measure on X which doesn't charge any semipolar subset of X. Particularly the subset

$$\left[X \setminus \beta(X \setminus G)\right] \setminus G = (X \setminus G) \setminus \beta(X \setminus G)$$

is semipolar, hence G is $\bar{\theta}$-measurable and

$$\bar{\theta}(X \setminus \beta(X \setminus G)) = \bar{\theta}(G) .$$

For any positive, bounded, Borel function f on G we have

$$\theta(Wf)=\theta((V\bar{f}-B^{X\setminus G}V\bar{f})|_G)=\int(V\bar{f}-B^{X\setminus G}V\bar{f})d\bar{\theta}=$$

$$=\int_{X\setminus\beta(X\setminus G)}(V\bar{f}-B^{X\setminus G}V\bar{f})d\bar{\theta}=\int_G(V\bar{f}-B^{X\setminus G}V\bar{f})d\bar{\theta}=\bar{\theta}|_G(Wf)$$

and therefore $\hat{\theta}|_G$ is a measure on G which represents θ.

c) From the preceding considerations we deduce that the fine topology on G with respect to $\mathcal{E}_W$ coincides with the trace on G of the fine topology of X with respect to S. Let now $s\in S$ and $\alpha\in R_+$ be such that $s<\alpha$. We have

$$\alpha=(\alpha-B^{X\setminus G}s)+B^{X\setminus G}s \quad \text{on } G.$$

Since the elements $(\alpha-B^{X\setminus G}s)|_G$, $(B^{X\setminus G}s)|_G$ belong to $\mathcal{E}_W$ we deduce that the function $(B^{X\setminus G}s)|_G$ is continuous with respect to the natural topology on G given by $\mathcal{E}_W$. Using [1] Proposition 4.4.5 and Theorem 4.4.6 there exists a Borel function ψ on G such that $0<\psi\leq 1$ and such that for any positive, bounded, Borel function g on G the function $W(\psi\cdot g)$ is 1-continuous in $\mathcal{E}_W$.

Let ψ' be the function on X equal ψ on G and equal 1 on $X\setminus G$. For any positive, bounded, Borel function f on X we have

$$V(\psi' f)|_G=W(\psi f|_G)+(B^{X\setminus G}V(\psi' f))|_G$$

and therefore $V(\psi' f)|_G$ is continuous with respect to the natural topology on G associated with $\mathcal{E}_W$. Hence this topology is finer then the trace on G of the natural topology of X with respect to S. Moreover if D is an open subset of X, $D\subset G$ then from [1] Proposition 5.6.14 it follows that $(B^{X\setminus G}V(\psi' f))|_D$ is continuous with respect to the trace on D of the natural topolgy of X. Using now the equality

$$W(\psi f|_G)|_D = V(\psi' f)|_D - B^{X\setminus G}(V(\psi' f))|_D$$

we get that the trace on D of the natural topology on G with respect to $\mathcal{E}_{\mathcal{W}}$ coincides with the trace on D of the natural topology of X with respect to S.

Proposition 2.2. Let S, X, G, $\mathcal{V}$ and $\mathcal{W}$ be as in the preceding theorem and let f be a positive function on G which is finite on a fine dense subset. Then the following assertions are equivalent:

1) For any fine open subset D of X such that $\overline{D}^{\text{fine}} \subset G$ we have $f|_D \in S(D)$.

2) For any fine open subset D of X such that $\overline{D} \subset G$ we have $f|_D \in S(D)$.

3) For any balayage B on S such that $d(B) \subset G$ we have $f|_{d(B)} \in \overline{S}_B$

S_B.

4) For any fine open subset D of X such that $D \subset G$ and such that $B^{X\setminus D}$ is a balayage we have $f|_D \in S\ (D)$.

5) There exists a sequence $(s_n)_n$ in S such that $s_n < +\infty$ $(\forall)$ $n \in N$ and such that the sequence $(s_n - B^{X\setminus G}s_n)_n$ increases to f on G.

6) There exists a sequence $(s_n - B^{X\setminus G}t_n)_n$ increasing to f where $s_n, t_n \in S$, $t_n < \infty$, $t_n \leq s_n$ for any $n \in N$.

7) f is excessive with respect to $\mathcal{W}$.

Proof. 6) $\Rightarrow$ 7) follows from points b) and c) of the preceding theorem; 7) $\Rightarrow$ 5) follows from Hunt's approximation theorem; 5) $\Rightarrow$ 6) and 4) $\Rightarrow$ 3) $\Rightarrow$ 1) $\Rightarrow$ 2 are trivial.

2) $\Rightarrow$ 7). Since for any $x \in G$ there exists a fine neighbourhood G' of x such that the natural closure $\overline{G'}$ of G' in X is contained in G we deduce that f is fine lower semi-continuous.

It is now sufficient (see the proof of point b) from the preceding theorem) to show that the function f is $\mathcal{W}$-dominant. Let for this g be a positive, bounded, Borel function on X such that

$$V_g - B^{X\setminus G}(V_g) \leq f \quad \text{on} \quad G \cap [g > 0].$$

Since V_g is continuous we deduce that there exist a decreasing sequence $(\Gamma_n)_{n \in N}$ of open subsets of X such that $X \setminus G \subset \Gamma_n$ $(\forall)$ $n \in N$ and

$$\bigwedge_{n \in N} B^{\Gamma_n} V_g = B^{X\setminus G} V_g .$$

Obviously the fine open set $D_n := d(B^{\Gamma_n})$ is such that $\overline{D}_n \subset G$. From hypothesis we have

$$f|_{D_n} \in S(D_n) = \overline{S_{B^{\Gamma_n}}}$$

and therefore

$$\overline{S_{B^{\Gamma_n}}} \ni \inf(f, V_g - B^{\Gamma_n} V_g) \leq V_g - B^{\Gamma_n} V_g \; ;$$

$$\inf(f, V_g - B^{\Gamma_n} V_g) = u - B^{\Gamma_n} u \; , \text{ where } u \in S, \; u \leq V_g < \infty$$

$$u - B^{\Gamma_n} u + B^{\Gamma_n} V_g \in S,$$

$$u - B^{\Gamma_n} u + B^{\Gamma_n} V_g \geq V_g \quad \text{on} \quad [g > 0] \cap d(B^{\Gamma_n})$$

$$u - B^{\Gamma_n} u + B^{\Gamma_n} V_g \geq V_g \quad \text{on } [g > 0],$$

$$f + \bigwedge_n B^{\Gamma_n} V_g \geq V_g \quad , \quad f \geq V_g - B^{X\setminus G} V_g \quad .$$

5) $\Rightarrow$ 3). Let B be a balayage on S such that $d(B) \subset G$. We show that for any $s \in S$, $s < \infty$ we have $s - B^{X\setminus G} s \in \overline{S_B}$. We consider for

this a decreasing sequence $(\Gamma_n)_n$ of fine open sets of X such that $\Gamma_n \supset X \setminus G$ and such that

$$\bigwedge_n B^{\Gamma_n} s = B^{X \setminus G} s \quad .$$

We denote $B_n := B^{\Gamma_n \cap b(B)}$ and we remark that B_n is a balayage on S and we have

$$\Gamma_n \cap b(B) \supset X \setminus G, \; B \geq B^{\Gamma_n} \geq B_n \qquad (\forall)\; n \in N,$$

$$\bigwedge_n B_n s = B^{X \setminus G} s \;,$$

$$d(B) \subset d(B_n) \subset G \qquad (\forall)\; n \in N,$$

$$(s - B_n s)|_{d(B)} \in \overline{S_B} \qquad (\forall)\; n \in N,$$

$$(s - B^{X \setminus G} s)|_{d(B)} = \sup_n (s - B_n s)|_{d(B)} \in \overline{S_B} \; .$$

2) $\Rightarrow$ 4). Let D be a fine open subset of G such that $B^{X \setminus D}$ is a balayage on S. We denote by φ the restriction of f to D. Using the hypothesis it follows that for any fine open subset D_1 of X, $\overline{D_1} \subset D$ we have

$$\varphi|_{D_1} \in S(D_1)$$

From the above considerations we deduce that there exists a sequence $(s_n)_n$ in S, $s_n < \infty$, such that the sequence $(s_n - B^{X \setminus D} s_n)_n$ increases to φ.

Obviously

$$s_n - B^{X \setminus D} s_n \in \overline{S_{B^{X \setminus D}}} \qquad (\forall)\; n \in N$$

$$s_n - B^{X\setminus D} s_n \in S(D) \qquad (\forall)\ n \in N$$

$$\varphi \in S(D)$$

<u>Definition.</u> Let S be a standard H-cone of functions on a nearly saturated set X. For any fine open set G of X we denote by $S'(G)$ the set of all positive functions f on G such that

a) f is finite on a fine dense subset of G

b) For any fine open subset D of X such that $\overline{D}^{fine} \subset G$ we have $f|_D \in S(D)$.

From the above definition and from theorem 2.1 and proposition 2.2 it follows that for any fine open subset G of X the set $S'(G)$ is a standard H-cone of functions on G such that the fine topology on G with respect to $S'(G)$ coincides with the trace on G of the fine topology of X with respect S and the natural topology on G given by $(S'(G))_o$ is finer than the trace on G of the natural topology of X given by S_o and they coincide if G is open.

Also we have

a) if $B^{X\setminus G}$ is a balayage then $S'(G)=S(G)$

b) if G_1, G_2 are two fine open subset of X such that $G_1 \subset G_2$ then

$$f \in S'(G_2) \Longrightarrow f|_{G_1} \in S'(G_1)$$

<u>Remark.</u> If G is an open subset of X and f is a positive function on G which is finite on a dense subset of G then the following assertions are equivalent.

1) $f \in S'(G)$.

2) for any open subset G_o of X such that

$\overline{G}_o \subset G$ we have $f|_{G_o} \in S'(G_o)$

3) for any open subset G_o of X such that $\overline{G}_o \subset G$ we have $f|_{G_o} \in S(G_o)$.

Indeed, from the above proposition we have 1) $\Rightarrow$ 3) $\Rightarrow$ 2).

For the relation 2) $\Rightarrow$ 1) let D be a fine open subset of G such that $\overline{D} \subset G$ and let G_o be an open subset of X such that

$$\overline{D} \subset G_o \;, \qquad \overline{G}_o \subset G \;.$$

Since $f|_{G_o} \in S'(G_o)$, using again the above proposition, we get

$$f|_D = (f|_{G_o})|_D \in S(D)$$

and therefore, the fine open subset D being arbitrary, we have $f \in S'(G)$.

Proposition 2.3. Let G be a fine open subset of X, let $s \in S$ be a finite element of S and let M be a subset of G. Then we have

$$B^{M \cup (X \setminus G)} s - B^{X \setminus G} s \geq \hat{B}^M (s - B^{X \setminus G} s) \quad \text{on} \quad G$$

where for any $f \in S'(G)$, $\hat{B}^M_f$ means the balayage of f on M in the standard H-cone of functions $S'(G)$. Moreover if M is such that the fine closure of M in X contains any point $x \in X \setminus G$ for which $X \setminus G$ is thin at x then

$$B^{M \cup (X \setminus G)} s - B^{X \setminus G} s = \hat{B}^M (s - B^{X \setminus G} s) \quad \text{on } G \;.$$

Proof. Since s is finite and S is a standard H-cone there exists a decreasing sequence of fine open subsets (G_n) of X such that

$$M\cup(X\setminus G)\subset G_n \qquad (\forall)\ n\in N,$$

$$B^{M\cup(X\setminus G)}s=\widehat{\inf_{n\in N} B^{G_n}s}\ .$$

From the relations

$$B^{G_n}s-B^{X\setminus G}s=B^{G_n}s-B^{X\setminus G}(B^{G_n}s)=s-B^{X\setminus G}s \quad \text{on } M, \quad (\forall)\ n\in N,$$

we get

$$B^{G_n}s-B^{X\setminus G}s\geq \hat{B}^M(s-B^{X\setminus G}s),\ \inf_{n\in N} B^{G_n}s-B^{X\setminus G}s\geq \hat{B}^M(s-B^{X\setminus G}s) \quad \text{on } G.$$

Since $\hat{B}^M(s-B^{X\setminus G}s)$ is lower semicontinuous on G with respect to the fine topology of X, from the last inequality and the preceding considerations we get

$$B^{M\cup(X\setminus G)}s-B^{X\setminus G}s\geq \hat{B}^M(s-B^{X\setminus G}s) \quad \text{on } G.$$

Suppose now that the fine closure of M in X contains any point $x\in X\setminus G$ for which $X\setminus G$ is thin at x. Let $u\in S'(G)$ be such that

$$u\geq s-B^{X\setminus G}s \quad \text{on } M, \quad u\leq s-B^{X\setminus G}s \quad \text{on } G.$$

From the definition of $S'(G)$ there exists a sequence $(s_n)_n$ of finite elements of S such that the sequence $(s_n-B^{X\setminus G}s_n)_n$ increases to u on G. Since $u\leq s-B^{X\setminus G}s$, and using Lemma 1.5, we deduce that for any $n\in N$ the function

$$t_n:=s_n-B^{X\setminus G}s_n+B^{X\setminus G}s$$

belongs to S and $B^{X\setminus G}s\leq t_n\leq s$. The sequence $(t_n)_n\in S$ being increasing and dominated we deduce that the function

$$t:=\sup_n t_n=\bigvee_n t_n$$

belong to S and moreover we have

$$t = u + B^{X \setminus G} s \quad \text{on } G, \quad t \geq s \text{ on } M, \quad t \geq B^{X \setminus G} s$$

Since $t \geq s$ on the fine closure $\overline{M}^{\text{fine}}$ of M and

$$\overline{M}^{\text{fine}} \supset (X \setminus G) \setminus b(X \setminus G)$$

we get

$$t \geq s \quad \text{on } M \cup (X \setminus G), \quad t \geq B^{M \cup (X \setminus G)} s .$$

Hence we have

$$u \geq B^{M \cup (X \setminus G)} s - B^{X \setminus G} s \quad \text{on} \quad G$$

and therefore, u being arbitrary

$$\hat{B}^M (s - B^{X \setminus G} s) \geq B^{M \cup (X \setminus G)} s - B^{X \setminus G} s \quad \text{on} \quad G.$$

Theorem 2.4. For any fine open sets G_1, G_2 of X such that $G_1 \subset G_2$ we have

$$[S'(G_2)]'(G_1) \subset S'(G_1) .$$

Proof. Let $f \in [S'(G_2)]'(G_1)$ and let B be a balayage on S such that $d(B) \subset G_1$. From Proposition 2.2 it is sufficient to show that

$$f|_{d(B)} \in \overline{S}_B = S(d(B)) .$$

Now, since $f|_{d(B)}$ belongs to $[S'(G_2)]'(d(B))$ the proof is finished if we show that for any finite element s of S we have

$$(s - B^{X \setminus G_2} s) - \hat{B}^{G_2 \setminus d(B)} (s - B^{X \setminus G_2} s) = s - Bs \text{ on } d(B)$$

or equivalently

$$B s=B^{(X\setminus G_2)\cup(G_2\setminus d(B))} s=\hat{B}^{G_2\setminus d(B)}(s-B^{X\setminus G_2}s)+B^{X\setminus G_2}s \text{ on } d(B) .$$

Since $X\setminus d(B)$ is not thin at any point of $X\setminus d(B)$ and $X\setminus d(B)=$ $=(X\setminus G_2)\cup(G_2\setminus d(B))$ we deduce that the fine closure in X of the set $G_2\setminus d(B)$ contains any point x of $X\setminus G_2$ for which $X\setminus G_2$ is thin at x and therefore the last equivalent relation follows from the preceding proposition.

<u>Proposition 2.5.</u> If X is semi-saturated with respect to S then any fine open subset G of X is semi-saturated with respect to the standard H-cone $S'(G)$ of functions on G.

<u>Proof</u>. Let μ be an universally bounded element of $(S'(G))^*$ and let μ_o be an universally continuous H-integral with respect $S'(G)$ such that $\mu\leq\mu_o$. From Theorem 2.1, b) and from the fact that X is nearly saturated we deduce that μ_o is an H-measure on G which does not charge any semipolar subset of G and therefore μ_o may be considered as an H-measure on X which does not charge any semipolar subset of X. Let s be a finite generator of S and let $(D_n)_n$ be a decreasing sequence of fine open subset of X such that

$$X\setminus G\subset D_n \quad (\forall)\ n\in N,$$

$$\bigwedge_n B^{D_n}s=B^{X\setminus G}s .$$

From the above considerations ~~we have~~:

$$\inf \mu_o(B^{D_n}s)=\mu_o(\bigwedge_n B^{D_n}s)=\mu_o(B^{X\setminus G}s),$$

$$(B^{D_n}s-B^{X\setminus G}s)|_G=(B^{D_n}s-B^{X\setminus G}(B^{D_n}s))|_G\in S'(G) ,$$

We denote by θ the map on S given by

$$\theta(t) := \mu(t|_G) \quad (\forall) \quad t \in S.$$

Obviously θ is an H-integral on S dominated by the H-measure μ_o and therefore is an H-measure on X.

Since $\mu \leq \mu_o$ in $(S'(G))^*$ we get

$$\theta(B^{D_n}s - B^{X\setminus G}s) = \mu((B^{D_n}s - B^{X\setminus G}s)|_G) \leq$$

$$\leq \mu_o((B^{D_n}s - B^{X\setminus G}s)|_G) = \int (B^{D_n}s - B^{X\setminus G}s)\, d\mu_o ,$$

$$\inf_n \theta (B^{D_n}s - B^{X\setminus G}s) = 0 ,$$

$$\theta (\inf_n B^{D_n}s - B^{X\setminus G}s) = 0$$

and therefore the set

$$M := [\inf_n B^{D_n}s > B^{X\setminus G}s]$$

is θ-negligible. From this fact and from the relations

$$(X\setminus G \subset \bigcap_n \bar{D}_n^{fine} , \quad (\bigcap_n \bar{D}_n^{fine}) \setminus (X\setminus G) \subset M ,$$

we deduce that G is θ-measurable and

$$\theta|_G = \theta|_{X\setminus \bigcap_n \bar{D}_n^{fine}} .$$

Further, we have,

$$\mu((s - B^{X\setminus G}s)|_G) = \theta(s - B^{X\setminus G}s) = \theta(s - B^{D_n}s) + \theta(B^{D_n}s - B^{X\setminus G}s) ,$$

$$\mu((s-B^{X\setminus G}s)|_G)=\sup_n \theta(s-B^{D_n}s)=\theta(s-\inf_n B^{D_n}s)=$$

$$=\int(s-\inf_n B^{D_n}s)\,d\theta|_{X\setminus\bigcap_n \bar{D}_n}\text{-fine}=\int(s-\inf_n B^{D_n}s)\,d\theta|_G=$$

$$=\int(s-B^{X\setminus G}s)\,d\theta|_G .$$

Since for any finite element $u\in S$ we have

$$\mu((u-B^{X\setminus G}u)|_G)=\theta(u-B^{X\setminus G}u)\geq\int(u-B^{X\setminus G}u)\,d\theta|_G ,$$

we deduce that if $u\in S$, $u\lessapprox s$ then

$$\mu((u-B^{X\setminus G}u)|_G)=\int(u-B^{X\setminus G}u)\,d\theta|_G ,$$

and therefore the same equality holds for any finite element u of S because any such element is the limit of an increasing sequence $(u_n)_n$, $u_n\lessapprox\alpha_n s$, $\alpha_n>0$. If $f\in S'(G)$ then there exists an increasing sequence of the form $(s_n-B^{X\setminus G}s_n)_n$ with

$s_n\in S, s_n<\infty$ such that

$f=\lim_n (s_n-B^{X\setminus G}s_n)|_G .$

and thus

$$\mu(f)=\lim_{n\to\infty}\mu((s_n-B^{X\setminus G}s_n)|_G)=\lim_{n\to\infty}\int(s_n-B^{X\setminus G}s_n)\,d\theta|_G=\int f\,d\theta|_G$$

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