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by

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II. Natural sheaf property and axiom D_o on a standard H-cone of functions

In this part we deal with the study of the property that the map

$$G \longrightarrow S'(G)$$

given on the set of all open subsets of X is a sheaf. We call this property, natural sheaf property.

We prove that the fact that one of the maps

$$G \longrightarrow S'(G) , \qquad G \longrightarrow S(G)$$

given on the set of all fine open subsets of X is a sheaf implies that the other is also a sheaf and this hapens if an only if S satisfies axiom D.

We prove also that the fact that the map

$$G \longrightarrow S(G)$$

given on the set of all open subset of X is a sheaf is equivalent with the fact that axiom of polarity and natural sheaf property hold on X. This result allows us to assert that for any open subset G of X the standard H-cone S'(G) of functions on G (and not S(G)) must be taken as the local structure on G associated with S.

We prove that if the natural sheaf property holds on X then for any two open subsets G_1, G_2 of X such that $G_1 \cup G_2 = X$ we have

$$B^{G_1} B^{G_2} = B^{G_2} B^{G_1}$$

We call this last assertion axiom D_o on X. We show that axiom D_o strongly depends on the representation of S as standard H-cone of functions, namely, axiom D_o holds on the set X_u for any weak unit u of S iff axiom D holds on S.

Some other characterisations of axiom D are given. One of them go to the "old" meaning of this axiom introduce by Brelot in the framework of harmonic spaces. Namely, if S satisfies the natural sheaf property then axiom D holds on S iff any universally bounded element of S is the sum of a sequence of universally continuous elements of S.

3. Natural localization and axiom D_o in standard H-cone of functions

In this part S will be a standard H-cone of functions on a set X. We remember that we denote by X_1 the saturated of X and by S_1 the standard H-cone of all functions on X_1 obtained by the natural extention of each function from S to a function on X_1.

More generally if Y is a set such that

$$X \subset Y \subset X_1$$

then any element $s \in S$ may be naturally extended to a function $\bar{s}$ on Y and the set

$$S_Y := \left\{ \bar{s} : Y \longrightarrow \overline{R}_+ \mid s \in S \right\}$$

is a standard H-cone of functions on Y.

If A is a subset of Y and $s \in S$ we put

$${}^Y B^A s = \bigwedge_{S_Y} \left\{ \bar{t} \in S_Y \mid \bar{t} \geq \bar{s} \text{ on } A \right\}$$

Proposition 3.1. a) For any subset A of X and any $s \in S$ we have

$${}^Y B^A s = \overline{(B^A s)} \quad \text{or equivalently} \quad ({}^Y B^A \bar{s})\Big|_X = B^A s \; .$$

b) If G is a fine open subset of Y and $s \in S$ then

$${}^Y B^G s = \overline{(B^{G \cap X} s)} \quad \text{or equivalently} \quad ({}^Y B^G \bar{s})\Big|_X = B^{X \cap G} s \; .$$

c) If X is semi-saturated then for any subset A of Y and any $s \in S$ we have

$${}^Y B^A \bar{s} = \overline{(B^{A \cap X} s)} \quad \text{or equivalently} \quad ({}^Y B^A \bar{s})\Big|_X = B^{A \cap X} s \; .$$

Proof. a) Let A and s as in the point a) and let $t \in S$. Certainly we have

$$t \geq s \text{ on } A \iff \bar{t} \geq \bar{s} \text{ on } A$$

and therefore

$$\overline{(\bigwedge_S \{t\in S \mid t \geq s \text{ on } A\})} = \bigwedge_{S_Y} \{\bar{t}\in S_Y \mid \bar{t} \geq \bar{s} \text{ on } A\}.$$

b) If G is a fine open subset of Y then the fine closure $\overline{G\cap X}^f$ in Y of the set $G\cap X$ coincides with the fine closure in Y of G since in the contrary case we have the following relations

$$\emptyset \neq G \setminus \overline{(G\cap X)}^f \subset Y \setminus X$$

which contradicts the fact that S is a standard H-cone of functions on X.

c) If X is semi-saturated we know that any borel subset of $Y\setminus X$ is polar with respect to S_Y. It is sufficient to prove the assertion c) for a borel subset A of Y. We know that there exists a borel subset M of Y such that $A\cap X\subset M$ and such that

$$Y_B M = Y_B A\cap X$$

Since the set $A\setminus M$ is a borel part of Y contained in $Y\setminus X$ it is a polar subset of Y and therefore

$$Y_B A \leq Y_B M = Y_B A\cap X \leq Y_B A.$$

<u>Corollary 3.2</u>. If X is semi-saturated then the axiom of polarity holds on X iff the axiom of convergence holds on S.

<u>Proof</u>. It is sufficient to use the assertion c) of the preceding proposition and Theorem 5.6.3 from [3].

<u>Theorem 3.3</u>. If X is semi-saturated with respect to S then the following assertion are equivalent:

1) $S(D)=S'(D)$ for any open subset D of X,

2) $S(D)=S'(D)$ for any fine open subset D of X,

3) the axiom of polarity holds on X.

Proof. The relation 3) $\Longrightarrow$ 2) follows from the fact that for any subset A of X the map B^A is a balayage on S.

The relation 2) $\Longrightarrow$ 1) is obvious.

1) $\Longrightarrow$ 3) Let F be a closed semipolar subset F of X. Since

$$S(X\setminus F)=S'(X\setminus F)$$

we deduce that for any $s\in S$, $s<\infty$ we have

$$s-B^F s\in S(X\setminus F)$$

Noting $\beta(F)$ the essential base of F we get

$$s-B^F s\leq s-B^{\beta(F)}s$$

and therefore using ([3], proposition 5.1.2 and 5.1.4) we deduce that there exists $s'\in S$, $s'\leq s$ such that

$$s-B^F s=s'-B^{\beta(F)}s' \quad \text{on } X\setminus F$$

Since F is semipolar we get $\beta(F)=\emptyset$ and

$$s-B^F s=s'-B^{\beta(F)}s'=s' \quad \text{on } X.$$

Thus

$$B^F s\preccurlyeq s \quad (\forall)\quad s\in S$$

and therefore

$$B^F s\preccurlyeq t$$

for any $s \in S$ and any $t \in S$ such that

$$t=s \text{ on } F$$

Let now G be a fine open subset of X such that $F \subset G$. We have

$$B^G s=s \quad \text{on} \quad F \qquad (\forall)\ s \in S$$

and therefore

$$B^F s \leq B^G s\ , \qquad (\forall)\ s \in S$$

Hence for any $s \in S$ we have

$$B^G B^F s=B^F s$$

for any fine open subset G, $G \supset F$ and thus we get

$$B^F (B^F s)=B^F s \qquad (\forall)\ s \in S.$$

From this fact it follows that B^F is a balayage on X with respect to S whose base lies in F. Hence $B^F=0$ and therefore F is polar. Using Proposition 3.1 c) and ([3] Theorem 5.2.1) we deduce that for any Borel subset A of X we have

$$B^A=\vee \left\{ B^F \mid F \text{ closed}, F \subset A \right\}.$$

From this fact and from the preceding considerations we get that any Borel semipolar subset of X is polar.

Definition. If S is a standard H-cone of functions on a set X we say that S (or the pair (S,X)) satisfies axiom D_o if for any two open subsets G_1 , G_2 of X we have

$$B^{G_1}B^{G_2}=B^{G_2}B^{G_1}$$

whenever

$$G_1 \cup G_2 = X .$$

Remark. In the sequel we shall prove that if X is nearly saturated with respect to S then the H-cone S satisfies axiom D iff for any two fine open subsets G_1, G_2 of X we have

$$B^{G_1}B^{G_2}=B^{G_2}B^{G_1}$$

whenever

$$G_1 \cup G_2 = X .$$

Proposition 3.4. a) If axiom D_o holds for the pair (S,X) it holds also for the pair (S_1,X_1).

b) If axiom D_o holds for the pair (S_1,X_1) it holds for the pair (S_Y,Y) where Y is an arbitrary semi-saturated subset of X_1.

Proof. The first assertion follows from the point b) of Proposition 3.1.

b) Let Y be a semi-saturated set with respect to S_Y $(Y \subset X_1)$, let G_1, G_2 be two open subsets of X_1 such that $Y \subset G_1 \cup G_2$ and let p be a finite continuous element of S_1. Since the set

$$M := X_1 \setminus (G_1 \cup G_2)$$

is a polar subset of X_1 we find a decreasing sequence (D_n) of open subsets of X_1 such that

$$\bigwedge_n 1_B^{D_n} p=0 \quad \text{and} \quad M \subset D_n \qquad (\forall)\ n \in N$$

For any $n \in N$ we have

$$(G_1 \cup D_n) \cup G_2 = X_1 \quad ,$$

$$1_B^{G_1}(1_B^{G_2}p) \leq 1_B^{G_1 \cup D_n}(1_B^{G_2}p) =$$

$$= 1_B^{G_2}(1_B^{G_1 \cup D_n}p) \leq 1_B^{G_2}(1_B^{G_1}p) + 1_B^{D_n}p \quad .$$

and therefore

$$1_B^{G_1}(1_B^{G_2}p) \leq 1_B^{G_2}(1_B^{G_1}p)$$

The element p being arbitrary we get
$1_B^{G_1}(1_B^{G_2}) = 1_B^{G_2}(1_B^{G_1})$.

The assertion follows now using Proposition 3.1, point b).

<u>Proposition 3.5</u>. Suppose that X is semi-saturated with respect to S. Then the following assertions are equivalent:

a) For any open subset G of X and any closed subset F of X such that $G \supset F$ we have

$$B^G B^F = B^F \quad .$$

b) For any open subset G of X and any subset A of X such that $G \supset A$ we have

$$B^G B^A = B^A$$

c) For any open subset G of X and any $x \notin G$ the H-measure $(B^G)^*(\varepsilon_x)$ doesn't charge any semipolar subset of G.

Proof. a) $\Longrightarrow$ c). Let G be an open subset of X and let for any $x \in X \setminus G$, μ_x the H-measure $(B^G)^*(\varepsilon_x)$. Since any totally thin subset of G is contained in a Borel totally thin subset of G and since X is metrisable then for the proof of c) it is sufficient to show that

$$\mu_x(F)=0$$

for any closed totally thin subset F of X, $F \subset G$. In this case if p is a bounded continuous generator of S then

$$B^F p(y) < p(y) \qquad (\forall)\ y \in X.$$

Let now $(G_n)_n$ be a decreasing sequence of open subsets of G such that

$$F \subset G_m \ , \ G_{n+1} \subset G_n \ , \ \inf_n B^{G_m} p(x) = B^F p(x)$$

If we denote

$$g = \inf_n B^{G_n} p$$

then we have $g \geq B^F p$,

$$\mu_x(g) = \inf_n \mu_x(B^{G_n} p) = \inf_n B^G(B^{G_n} p)(x) = \inf_n B^{G_m} p(x) = B^F p(x).$$

Since by hypothesis

$$\mu_x(B^F p) = B^F p(x)$$

we get

$$\mu_*(g-B^F p)=0 ,$$

$$\mu_*(R^F p-B^F p)=0 ,$$

and therefore

$$\mu_*((p-B^F p)1_F)=0 ,$$

$$\mu_*(F)=0 .$$

c) $\Longrightarrow$ b) Let A and G as in b). Obviously we may suppose that there exists a closed subset F of X, F G such that

$$A \subset F .$$

In this case we consider an open subset G_o of X such that

$$F \subset G_o \subset \overline{G}_o \subset G .$$

For any $p \in S_o$ we chose a decreasing sequence $(G_n)_{n \in N}$ of open subsets of X such that

$$A \subset G_n \subset G_o \qquad (\forall)\ n \in N$$

and such that

$$\bigwedge_n B^{G_n} p = B^A p .$$

We have from ([3], Corollary 5.1.7)

$$B^A p = \bigwedge_n B^{G_n} p = \inf_n B^{G_n} p \quad \text{on } X \setminus \overline{G}_o .$$

From hypothesis we deduce that for any $x \notin G$ we have

$$B^G(B^A p)(x) = \mu_x(B^A p) = \mu_x(\inf_n B^{G_n} p) = \inf_n \mu_x(B^{G_n} p) =$$

$$=\inf_n B^G(B^{G_n}p)(x)=\inf_x B^{G_n}p(x)=B^A p(x)$$

where μ_x means the H-measure $(B^G)^*(\mathcal{E}_x)$.

<u>Theorem 3.6.</u> Let S be a standard H-cone of functions on a semi-saturated set X. Then the following assertions are equivlaent:

1. S satisfies axiom D_o.

2. For any closed subset A of X and any $x \in X \setminus A$ the H-measure $(B^A)^*(\mathcal{E}_x)$ is carried by the natural boundary of A.

3. For any open subset G of X, any $t \in S$, $t<\infty$ and any $s \in S'(G)$ such that

$$\liminf_{G \ni x \to y} s(x) \geq t(y) \qquad (\forall)\ y \in \partial G,$$

the function $s-B_t^{X \setminus G}$ belongs to $S'(G)$.

3'. The same assertion as in 3) but for any $s \in S'(G)$ of the form $s=u|_G$ where $u \in S$.

4. For any open set G of X, any $s \in S'(G)$ and any $t \in S$ such that

$$\liminf_{G \ni x \to y} s(x) \geq t(y) \qquad (\forall)\ y \in \partial G,$$

the function

$$s'(x) := \begin{cases} t(x) & \text{if } x \in X \setminus G \\ \inf(t(x), s(x)), & \text{if } x \in G \end{cases}$$

belongs to S.

4'. The same assertion as in 4, but for any $s \in S'(G)$ of the form $s=u|_G$ where $u \in S$.

5. For any G, s, t as in the preceding point 4) we have

$$s(x) \geq B_t^{X \setminus G}(x) \qquad (\forall) \quad x \in G .$$

5'. The same assertion as in 5) but for any $s \in S'(G)$ of the form $s = u|_G$ where $u \in S$.

<u>Proof</u>. 1) $\Longrightarrow$ 2) Let p be a finite continuous generator of S and let D be an open subset of X such that $\bar{D} \subset \mathring{A}$. Since $p = B^{X \setminus \bar{D}}p$ on $X \setminus \bar{D}$ then the function $B^{X \setminus \bar{D}}p$ is continuous on a neighbourhood of A and therefore, using axiom D_o, we have for any $x \in X \setminus A$,

$$B^A(B^{X \setminus \bar{D}}p)(x) = R^A(B^{X \setminus \bar{D}}p)(x) =$$

$$= \inf\left\{B^G(B^{X \setminus \bar{D}}p)(x) \mid G \text{ open}, G \supset A\right\} =$$

$$= \inf\left\{B^{X \setminus \bar{D}}(B^G p)(x) \mid G \text{ open}, G \supset A\right\} =$$

$$= \inf\left\{B^G p(x) \mid G \text{ open}, G \supset A\right\} = B^A p(x) .$$

Since for any $x \in X \setminus A$, $(B^A)^*(\mathcal{E}_x)$ is an H-measure carried by A from the above considerations we get

$$(B^A)^*(\mathcal{E}_x)(p - B^{X \setminus \bar{D}}p) = 0$$

and therefore the measure $(B^A)^*(\mathcal{E}_x)$ doesn't charge the set $\left[p > B^{X \setminus \bar{D}}p\right]$ which contains D. The open subset D being arbitrary we deduce that the measure $(B^A)^*(\mathcal{E}_x)$ is carried by ∂A.

2 $\Longrightarrow$ 3. We suppose that t is finite, continuous and we denote, for any $n \in N$,

$$G_n = \left\{x \in G \mid s(x) + \frac{1}{n} > t(x)\right\}, \quad D_n = \overline{G_n \cup (X \setminus G)}$$

Obviously G_n, D_n are open subsets of X and

$$\partial D_n \subset G, \quad \partial G \subset G_n$$

We consider an open set G_o such that $\overline{G}_o \subset G$ and a sequence $(V_n)_n$ of open sets such that

$$\overline{V}_{n+1} \subset V_n, \quad \overline{V}_n \subset D_n, \quad \bigcap_n V_n = X \setminus G$$

$$\overline{G}_o \cap \overline{V}_n = \emptyset, \quad \bigwedge_n (B^{V_n} t) = B^{X \setminus G} t$$

Using proposition 2.3 [1] we have for any $u \in S$, $u < \infty$

$$B^{\overline{V}_n} u = \hat{B}^{\overline{V}_n \cap G}(u - B^{X \setminus G} u) + B^{X \setminus G} u \quad \text{on } G.$$

For any $x \in G_o$, let μ_x be the measure on ∂V_n such that

$$\mu_x(p) = B^{\overline{V}_n} p(x) \qquad (\forall)\ p \in S.$$

Using proposition 3.5 we deduce

$$B^{\overline{V}_n}(B^{X \setminus G} u)(x) = B^{X \setminus G} u(x)$$

and therefore

$$\hat{B}^{\overline{V}_n \cap G}(u - B^{X \setminus G} u)(x) = \mu_x(u - B^{X \setminus G} u).$$

Hence

$$\hat{B}^{\overline{V}_n \cap G}(p|_G)(x) = \mu_x(p|_G) = B^{\overline{V}_n} p(x) \qquad (\forall)\ p \in S.$$

Since $s + \frac{1}{n} > t$ on $\overline{V_n \cap G}$ and using ([1] Theorems 2.1, 2.4) we get

$$\left(s + \frac{1}{n} - B^{\overline{V}_n} t\right)\Big|_{G_o} = \left(s + \frac{1}{n} - \hat{B}^{\overline{V}_n \cap G} t\right)\Big|_{G_o} \in [S'(G)]'(G_o) \subset S'(G_o)$$

and therefore

$$(s-B^{X\setminus G_t}t)\big|_{G_o} \in S'(G_o) .$$

From the remark at the Proposition 2.2 we deduce

$$s-B^{X\setminus G_t}t \in S'(G)$$

If $t \in S$ is arbitrary, $t<\infty$ then we take a sequence $(t_n)_n$ in S_o, increasing to t. We have

$$(s-B^{X\setminus G}t)=\inf_n (s-B^{X\setminus G}t_n)=\inf_n \widehat{(s-B^{X\setminus G}t_n)}$$

i.e. $\quad s-B^{X\setminus G}t \in S'(G)$.

3) $\Longrightarrow$ 4) and 3') $\Longrightarrow$ 4'). Suppose t finite and continuous. We consider a sequence $(V_n)_n$ of open sets such that

$$\bigcap_{n\in N} V_n = X\setminus G, \quad \bigwedge_n B^{V_n}t = B^{X\setminus G}t, \quad \overline{V}_{n+1} \subset V_n \qquad (\forall)\quad n\in N$$

From hypothesis we have

$$(s-B^{X\setminus G}t)\big|_{X\setminus\overline{V}_{n+1}} \in S(X\setminus\overline{V}_{n+1}) \qquad (\forall)\ n\in N$$

and therefore there exists $u \in S$, $u<\infty$ such that

$$(s-B^{X\setminus G}t)\wedge(t-B^{V_n}t)=u-B^{V_n}u \quad \text{on} \quad d(B^{V_n})$$

Hence, we have $u-B^{V_n}u+B^{V_n}t \in S$ and therefore the function

$$s_n(x):=\begin{cases}\inf\,(t(x),\, s(x)-B^{X\setminus G}t(x)+B^{V_n}t(x)) & \text{if } x\in d(B^{V_n})\\ B^{V_n}t(x) & \text{if } x\notin d(B^{V_n})\end{cases}$$

belongs to S. Passing to the limit we get

$$\lim_n s_n(x) = \begin{cases} \inf\,(t(x), s(x)) & \text{if } x \in G \\ t(x) & \text{if } x \in X \setminus G. \end{cases}$$

Since the function on X

$$x \longrightarrow \lim_n s_n(x)$$

is a finite lower semicontinuous fonction, from ([1] Lemma 1.3), we deduce that it belongs to S. If t is arbitrary we consider a sequence $(t_n)_n$ in S_o increasing to t and for any $n \in N$ we denote

$$s_n(x) := \begin{cases} \inf\,(t_n(x), s(x)) & \text{if } x \in G \\ t_n(x) & \text{if } x \notin G \end{cases}$$

From the above considerations $s_n \in S$ for any n and therefore, the sequence $(s_n)_n$ being increasing to the function s′ on X defined by

$$s'(x) = \begin{cases} \inf(t(x), s(x)) & \text{if } x \in G \\ t(x) & \text{if } x \notin G \end{cases}$$

we deduce $s' \in S$.

The relations 3) ⟹ 3′), 4) ⟺ 4′), 5) ⟹ 5′), 4) ⟹ 5) and 4′) ⟹ 5′) are obviously.

5″) ⟹ 1). We show that $B^{G_1} B^{G_2} s \geq B^{G_1} s \wedge B^{G_2} s$ for any $s \in S$. Indeed, the inequality is obvious on G_1. Let $u \in S$ be such that $u \geq B^{G_2} s$ on G_2. By hypothesis the function

$$t(x) := \begin{cases} \inf(u(x), s(x)) & \text{if } x \in G_2 \\ s(x) & \text{if } x \in X \setminus G_2 \end{cases} .$$

belongs to S and $t \geq B^{G_1}s$. Hence

$$u(x) \geq B^{G_1}s(x) \qquad (\forall)\ x \in G_2 ,$$

$$B^{G_1}(B^{G_2}s)(x) \geq B^{G_1}s(x) \qquad (\forall)\ x \in G_2 ,$$

$$B^{G_1}(B^{G_2}s) \geq B^{G_1}s \wedge B^{G_2}s,\ B^{G_1}(B^{G_2}s) = B^{G_1}s \wedge B^{G_2}s$$

and therefore $B^{G_1}B^{G_2} = B^{G_2}B^{G_1}$.

Corollary 3.7. If X is semi-saturated with respect to S and if S satisfies axiom D_0 then for any closed subset F and any open subset G of X, $F \subset G$ we have

$$B^G(B^F) = B^F .$$

Proof. The assertion follows from Theorem 3.5 (1) $\Longrightarrow$ (2) using Proposition 3.5.

Theorem 3.8. If X is semi-saturated with respect to S and if S satisfies axiom D_0 then for any open subset G of X the standard H-cone of functions $S'(G)$ on G satisfies also axiom D_0.

Proof. From ([1], Proposition 2.5) we deduce that G is semi-saturated with respect to the standard H-cone of functions $S'(G)$. Hence it is sufficient to show that one of the assertions 1) — 5) from Theorem 3.6 holds for $S'(G)$. We shall prove that $S'(G)$ verifies the assertions 4). Let U be an open subset of G and let $t \in S'(G)$, $s \in [S'(G)]'(U)$ be such that

$$\liminf_{U \ni x \to y} s(x) \geq t(y) \qquad (\forall)\ y \in \partial_G U$$

where $\partial_G U$ means the boundary of U with respect to G. We consider the function f on G defined by

$$f(x) = \begin{cases} \inf(s(x), t(x)) & \text{if } x \in U \\ t(x) & \text{if } x \in G \setminus U \end{cases}$$

and we want to show that $f \in S'(G)$.

Since any $t \in S'(G)$ is the limit of an increasing sequence of elements of the form $v - B^{X \setminus G}v$ where v is a bounded continuous element of S (see [1], Theorem 2.1) it is sufficient to show that the preceding assertion holds for t of the form

$$t = v - B^{X \setminus G}v$$

where v is bounded and continuous. Let now v′ be an element of S such that $v' \geq v$ on $X \setminus G$. We have

$$\liminf_{U \ni x \to y} (s+v')(x) \geq v(y) \qquad (\forall)\ y \in \partial U$$

where ∂U is the boundary of U in X. Using the assertion 4) for the standard H-cone S we deduce that the function $f_{v'}$ on X defined by

$$f_{v'}(x) = \begin{cases} \inf((s+v')(x), v(x)) & \text{if } x \in U \\ v(x) & \text{if } x \in X \setminus U \end{cases}$$

belongs to S and

$$B^{X \setminus G}_{f_{v'}} = B^{X \setminus G}_{v} ,$$

$$f_{v'}-B^{X\setminus G}f_{v'}=f_{v'}-B^{X\setminus G}v$$

Since on the set G we have

$$\inf\left\{v' \mid v'\in S,\ v'\geq v \text{ on } X\setminus G\right\}=B^{X\setminus G}v$$

$$f=:\inf_{v'}(f_{v'}-B^{X\setminus G}v)=\begin{cases}\inf\ (s(x),t(x)) & \text{if } x\in U\\ t(x) & \text{if } x\in G\setminus U\end{cases}$$

and since f is lower semicontinuous we get that f belongs to $S'(G)$.

The following Lemma will allows us to improve Theorem 2.4 from [1].

Lemma 3.9. If the standard H-cone of functions S on a semi-saturated set X satisfies axiom D_0 then for any two subset A_1, A_2 of X and for any $s\in S$ we have

$$B^{A_1}s\vee B^{A_2}s=B^{A_1\cup A_2}s\quad.$$

Proof. Obviously it is sufficient to suppose that $s\in S_0$. In this case we may assume also that A_1, A_2 are G_δ-sets. Since X is semi-saturated then from Proposition 3.1 c) and from ([3], Theorem 5-2-1) we deduce that for any Borel subset A of X we have

$$B^{A}s=\bigvee\left\{B^{F}s \mid F \text{ closed},\ F\subset A\right\}.$$

Using this fact we see that it is sufficient to show that

$$B^{A_1\cap F}s\vee B^{A_2\cap F}s=B^{F}s$$

for any closed set F, $F \subset A_1 \cup A_2$. Replacing $A_1 \cap F$ by

$$F \setminus (A_2 \cap F)$$

we may suppose that $A_1 \cap F'$ is an F_σ-set. Since

$$B^{\bigcup_n C_n} s = \bigvee_n B^{C_n} s \qquad (\forall)\ s \in S$$

for any increasing sequence $(C_n)'_n$ of subsets of X then we may suppose that $A_1 \cap F$ is closed. Repeating the preceding procedure we may also suppose that A_1 and A_2 are closed.

Let now $s_1, s_2 \in S$ be such that

$$s_1 \geq s \text{ on } A_1 \ , \quad s_2 \geq s \text{ on } A_2$$

and let f be the function on $X \setminus (A_1 \cup A_2)$ defined by

$$f = B^{A_1} s \vee B^{A_2} s + (s_1 - B^{A_1} s) + (s_2 - B^{A_2} s).$$

From ([1] Theorem 2-1 and Proposition 2.2) it follows that

$$f \in S'(X \setminus (A_1 \cup A_2))$$

On the other hand for any $i = 1, 2$, and for any

$$y \in \partial(X \setminus (A_1 \cup A_2)) \cap \partial F_i \ ,$$

we have

$$\liminf_{X \setminus (A_1 \cup A_2) \ni x \to y} f(x) \geq \liminf_{X \setminus (A_1 \cup A_2) \ni x \to y} s_i(x) \geq s(y).$$

Using Theorem 3-5 we deduce that the function t on X defined by

$$t(x)=\begin{cases} \inf\,(f(x),s(x)) & \text{if } x\in X\setminus(A_1\cup A_2) \\ s(x) & \text{if } x\in A_1\cup A_2 \end{cases}$$

belongs to S and moreover

$$t\geq B^{A_1\cup A_2}s \quad \text{on} \quad X.$$

Hence

$$B^{A_1}s\vee B^{A_2}s+s_1+s_2\geq B^{A_1\cup A_2}s+B^{A_1}s+B^{A_2}s$$

and therefore, the elements s_1, s_2 being arbitrary,

$$B^{A_1}s\vee B^{A_2}s \quad B^{A_1\cup A_2}s.$$

Remark. For the case where S is the cone of all superharmonic functions of on armonic space the above Lemma was proved in [4].

Theorem 3-10. If the standard H-cone of functions S on a semi-saturated set X satisfies axiom D_c then for any two open subsets G_1, G_2 of X, $G_1\subset G_2$ we have

$$\left[S'(G_2)\right]'(G_1)=S'(G_1) \quad .$$

Proof. Theorem 2-4 from [1] asserts that we have

$$\left[S'(G_2)\right]'(G_1)\subset S'(G_1)$$

As for the converse inclusion it is sufficient to show that for any $s\in S$, $s<\infty$ we have

$$s-B^{X\setminus G_1}s=(s-B^{X\setminus G_2}s)-\hat{B}^{(G_2\setminus G_1)}(s-B^{X\setminus G_2}s) \text{ on } G_1$$

or equivalently

$$B^{X\setminus G_1}s=B^{X\setminus G_2}s+\hat{B}^{(G_2\setminus G_1)}(s-B^{X\setminus G_2}s) \quad \text{on} \quad G_1$$

The inequality

$$B^{X\setminus G_1}s\geq B^{X\setminus G_2}s+\hat{B}^{(G_2\setminus G_1)}(s-B^{X\setminus G_2}s) \quad \text{on} \quad G_1$$

follows from ([1], Proposition 2-3). Let now t be an element of $S'(G_2)$ such that

$$t\geq s-B^{X\setminus G_2}s \quad \text{on} \quad G_2\setminus G_1 \quad ,$$

$$t\leq s-B^{X\setminus G_2}s \quad \text{on} \quad G_2$$

From ([1], Lemma 1.4 b)) we deduce that there exists $s'\in S$ such that

$$B^{X\setminus G_2}s\leq s'\leq s \text{ and } s'(x)=t(x)+B^{X\setminus G_2}s(x) \quad (\forall) \quad x\in G_2$$

Hence

$$s'\geq B^{X\ G_2}s \text{ and } s'(x)\geq s(x) \qquad (\forall) \quad x\in G_2\setminus G_1$$

Let now $(F_n)_{n\in N}$ be an increasing sequence of closed subsets of X such that $\bigcup_n F_n=G_2$.

Using the previous lemma we deduce

$$s' \geq B^{X\setminus G_2}s \vee B^{G_2\setminus G_1}s = \bigvee_{n\in N}(B^{X\setminus G_2}s \vee B^{F_n\setminus G_1}s) =$$

$$= \bigvee_{n\in N} B^{(X\setminus G_2)\cup(F_n\setminus G_1)}s = B^{X\setminus G_1}s$$

The element t being arbitrary we obtain

$$B^{X\setminus G_2}s + B^{(G_2\setminus G_1)'}(s - B^{X\setminus G_2}s) \geq B^{X\setminus G_1}s \text{ on } G_1 .$$

and therefore

$$B^{X\setminus G_2}s + B^{(G_2\setminus G_1)}(s - B^{X\setminus G_2}s) = B^{X\setminus G_1}s \quad \text{on } G_1 .$$

We remember that a standard H-cone satisfy <u>axiom D</u> if for any two balayages B_1 , B_2 on S such that $B_1 \vee B_2 = I$ we have

$$B_1 B_2 = B_2 B_1 .$$

We give now new characterizations of axiom D in three different ways.

<u>Theorem 3-11</u>. If S is a standard H-cone of function on X then the following assertion are equivalent.

1. axiom D holds on S,

2. for any two fine open subsets G_1 , G_2 of X such that $X = G_1 \cup G_2$ we have

$$B^{G_1} B^{G_2} = B^{G_2} B^{G_1}$$

3. for any two fine open subsets G_1 , G_2 of X such that $X = G_1 \cup G_2$ and such that there exists four bounded elements s_1 , s_1' , s_2, s_2' in S which satisfy the relations

$$s_1 \leq s_1' \;,\; s_2 \leq s_2' \;;\; G_1=[s_1 < s_1'] ,\; G_2=[s_2 < s_2']$$

we have

$$B^{G_1}B^{G_2}=B^{G_2}B^{G_1}$$

Proof. The relations 1) $\Rightarrow$ 2) $\Rightarrow$ 3) are obvious. 3) $\Rightarrow$ 1).

3) $\Rightarrow$ 1. Let B_1, B_2 be two balayages on S such that $B_1 \vee B_2 = I$ and let p be a bounded generator of S. For any $n \in N$ we put

$$G_n := \left[p < (1+\tfrac{1}{n}) B_1 p \right] ,\; G_n' := \left[p + B_1 p < (1+\tfrac{1}{n}) B_2 (p+B_1 p) \right]$$

Obviously the sequences of fine open subsets $(G_n)_n$ and $(G_n')_n$ are decreasing and we have:

$$b(B_1)=\bigcap_n G_n \;,\; b(B_2)=\bigcap_k G_k' \;,\; G_n \cup G_k' = X \qquad (\forall)\; n,k \in N,$$

$$B^{G_n} p \leq (1+\tfrac{1}{n}) B_1 p \qquad (\forall)\; n \in N$$

$$B^{G_k'}(p+B_1 p) \leq (1+\tfrac{1}{k}) B_2 (p+B_1 p) \qquad (\forall)\; n \in N,$$

$$B^{G_k'} B_1 p \leq (1+\tfrac{1}{k}) B_2 B_1 p + \tfrac{1}{k} B_2 p \qquad (\forall)\; k \in N,$$

$$B_1 B_2 p \leq B^{G_n} B^{G_k'} p = B^{G_k'} B^{G_n} p \leq (1+\tfrac{1}{n}) B^{G_k'} B_1 p \leq (1+\tfrac{1}{n})(1+\tfrac{1}{k}) B_2 B_1 p + (1+\tfrac{1}{n}) \cdot \tfrac{1}{k} B_2 p$$

$$(\forall)\; n,k \in N,$$

Passing to the limit we deduce

$$B_1 B_2 p \leq B_2 B_1 p \;,\; B_1 B_2 p = B_2 B_1 p$$

From the last relation one can easily shaw that

$$B_1B_2=B_2B_1$$

Theorem 3.12. If X is nearly saturated with respect to S then the following assertions are equivalent:

1) axiom D holds on S,

2) for any balayage B on S and for any $x \in X\setminus b(B)$, $B^*(\mathcal{E}_x)$ is an H-measure carried by the fine boundary of $b(B)$.

3) for any fine closed subset F of X and for any $x \in X\setminus F$, $(B^F)^*(\mathcal{E}_x)$ is an H-measure on X carried by the fine boundary of F.

4) for any fine open subset G of X and for any $x \in X\setminus G$, $(B^G)^*(\mathcal{E}_x)$ is an H-measure on X carried by the fine boundary of G.

4') the same assertion as in the preceding point but only for all fine subset G of X of the form

$$G=[s<t] \quad , s,t \in S,$$

5) for any subset A of X and for any $x \in X\setminus A$, $(B^A)^*(\mathcal{E}_x)$ is an H-measure on X carried by the fine boundary of A.

Proof. The relations $5) \Rightarrow 3) \Rightarrow 2)$ and $5) \Rightarrow 4) \Rightarrow 4')$ are obvious. The relation $1) \Leftrightarrow 2)$ is nothing else than the assertion $1) \Leftrightarrow 2)$ from ([3], Theorem 5.6.10).

$1) \Rightarrow 5)$. Using ([3], Theorems 5.6.8, 5.4.6) we deduce that for any subset A of X the map B^A is a balayage on S whose base is $b(F)$ where F is the fine closure of F. For $x \in X\setminus b(F)$ the assertion follows now from the relation $1) \Rightarrow 2)$ and from the fact that the fine boundary of F is contained in the fine boundary of A. If $x \in (X\setminus A)\cap b(F)$ then obviously x belongs to the fine boundary of A and on the other hand we have

$$(B^{A})^{*}(\varepsilon_{x})=(B^{b(F)})^{*}(\varepsilon_{x})=\varepsilon_{x}.$$

4') $\Longrightarrow$ 1). Using the preceding theorem it is sufficient to shaw that for any two fine open subsets G_1 , G_2 of X such that $X=G_1\cup G_2$ and such that there exist s_1, s_1', s_2, s_2' bounded functions in S for which

$$G_1=[s_1<s_1'], \qquad G_2=[s_2<s_2']$$

we have

$$B^{G_1}B^{G_2}=B^{G_1}B^{G_2}$$

Let now G_1 , G_2 be two fine open subsets of X with the above property. We shaw that for any $p\in S$, p bounded we have

$$B^{G_1}(B^{G_2}p)=B^{G_1}p \quad B^{G_2}p .$$

Obviously this equality holds on G_1 . If $x\in X\setminus G_1$ then $(B^{G_1})^{*}(\varepsilon_x)$ is an H-measure on X carried by the fine boundary $\partial^{f}G_1$ of the set G_1. Since

$$\partial^{f}G_1\subset G_2.$$

we deduce

$$B^{G_1}(B^{G_2}p)(x)=(B^{G_1})^{*}(\varepsilon_x)(B^{G_2}p)=(B^{G_1})^{*}(\varepsilon_x)(p)=B^{G_1}p(x)\geq(B^{G_1}p\wedge B^{G_2}p)(x)$$

Hence

$$B^{G_1}(B^{G_2}p)=B^{G_1}p\wedge B^{G_2}p$$

Similarly we have

$$B^{G_2}(B^{G_1}p)=B^{G_1}p\wedge B^{G_2}p$$

and therefore, p being arbitrary, we get

$$B^{G_1}B^{G_2}=B^{G_2}B^{G_1} .$$

Theorem 3.13. If X is nearly saturated with respect to S the following assertions are equivalent:

1) axiom D holds on S,

2) for any fine open subset G of X such that $X\setminus G$ is a basic set, for any $s,t\in S$ such that

$$\text{fine}\liminf_{G\ni x\to y} s(x)\geq t(y)$$

for any point y of the fine boundary of G we have

$$B^{X\setminus G}t\preceq s \quad (\text{"}\preceq\text{" with respect to } S(G)),$$

3) for any s, t as in the preceding point 2) the function s' on X defined by

$$s'(x)=\begin{cases}\inf(s(x),t(x)) & \text{if } x\in G\\ t(x) & \text{if } x\in X\setminus G\end{cases}$$

belongs to S,

4) for any s, t as in the preceding point 2) we have

$$B^{X\setminus G}t\leq s \quad \text{on } G.$$

2'), 3') respectively 4') the same assertions as in 2), 3) respectively 4) but for any fine open subset G of X.

Proof. Obviously 2') ⟹ 2), 3') ⟹ 3), 4') ⟹ 4). The relation 1) ⟹ 2) follows from ([3], Theorem 5.6.1) and the proof of the relations 2) ⟹ 3) ⟹ 4) ⟹ 1) is similar with the proof of the relation 1) ⟹ 3) ⟹ 4) ⟹ 5) ⟹ 1) in the above quated theorem.

Hence 1) $\Longleftrightarrow$ 2) $\Longleftrightarrow$ 3) $\Longleftrightarrow$ 4). The relation 2) $\Longrightarrow$ 2'), 3) $\Longrightarrow$ 3'), 4) $\Longrightarrow$ 4') may be obtained remarking that for any fine open set G there exists a basic set F such that

$$G \subset X \setminus F \subset G \cup \partial^f G$$

(where $\partial^f G$ is the fine boundary of G) and moreover if S satisfies axiom D we have

$$S(X \setminus F)\Big|_G = S(G).$$

<u>Theorem 3.14</u>. If X is nearly saturated with respect to S the following assertions are equivalent.

1) axiom D holds on S,

2) the map $G \longrightarrow S(G)$ defined on the set of all fine open subsets of X is a sheaf,

2') the map $G \longrightarrow S'(G)$ defined on the set of all fine open subsets of X is a sheaf.

<u>Proof</u>. The relation 1) $\Longleftrightarrow$ 2) is the assertion from ([3], Theorem 5.6.12).

The relation 2) $\Longrightarrow$ 2') follows from the fact that if S satisfies axiom D then axiom of polarity holds on X (See [3] Proposition 5.6.8) and therefore for any fine open subset G of X we have

$$S'(G) = S(G)$$

To finish the proof it is sufficient now, using Thoerem 3.11 to show that for any two fine open subset G_1 , G_2 of X such that $X = G_1 \cup G_2$ and any $s \in S$ we have

$$B^{G_1} B^{G_2} s = B^{G_1} s \quad B^{G_2} s$$

Obviously we have

$$B^{G_1}B^{G_2}s \leq B^{G_1}s \wedge B^{G_2}s \quad \text{on } X$$

$$B^{G_1}B^{G_2}s \geq B^{G_1}s \wedge B^{G_2}s \quad \text{on } G_1$$

Let $t \in S$ such that $t \geq B^{G_2}$, on G_1. From the hypothesis we deduce that the function on X defined by

$$t'(x) = \begin{cases} \inf(t(x), s(x)) & \text{if } x \in G_2 \\ s(x) & \text{if } x \in G_1 \end{cases}$$

belongs to S. Hence

$$t' \geq B^{G_1}s \text{ on } X \ , \ t \geq t' \geq B^{G_1}s \wedge B^{G_2}s \text{ on } G_2$$

The element t being arbitrary we deduce

$$B^{G_1}B^{G_2}s \geq B^{G_1}s \wedge B^{G_2}s \ .$$

The following result shows how much axiom D_o on a standard H-cone S depends on the representation of S as standard H-cone of functions. More precisely if we suppose that S is a standard H-cone of functions on a semi-saturated set X such that axiom D_o holds on X then, taking another weak unit u of S and the representation of S on the saturated space X_u, axiom D_o may have no place on X_u.

<u>Theorem 3.15.</u> The standard H-cone S satisfies axiom D iff for any weak unit $u \in S$ the ~~axiom D_o~~ holds for the standard H-cone of functions S_u on the set X_u.

<u>Proof.</u> If the H-cone S satisfies axiom D then, obviously, for any weak unit u of S axiom D_o holds on X_u.

We suppose now that for any weak unit u of S axiom D_o holds on X_u. Let u be a fixed weak unit of S and let G_1, G_2 be two fine open subset of X_u such that $G_1 \cup G_2 = X_u$ and such that there exist $s_1, s_1', s_2, s_2' \in S$ bounded functions on X_u from S for which

$$G_1 = [s_1 < s_1']^* \cap X_u , \quad G_2 = [s_2 < s_2']^* \cap X_u$$

where for any $s, t \in S$ the set $[s < t]^*$ means the subset of S^* defined by

$$[s < t]^* = \{ \mu \in S^* | \mu(s) < \mu(t) \} .$$

Let now v be the weak unit on S defined by

$$v = u + s_1 + s_2 + s_1' + s_2' .$$

In the representation (S, X_v) the element v is equal 1 on X_v and the elements u, s_1, s_1', s_2, s_2' become continuous functions on X_v. Since

$$u \leq v \leq \alpha u$$

for a suitable positive number α we deduce that the sets G_1', G_2' defined by

$$G_1' = [s_1 < s_1']^* \cap X_v , \quad G_2' = [s_2 < s_2']^* \cap X_v$$

are open subsets of X_v and

$$G_1' \cup G_2' = X_v .$$

Hence, using axiom D_o on X_v we have

$$B^{G_1'}B^{G_2'}=B^{G_2'}B^{G_1'}$$

But from the preceding considerations the maps $B^{G_1'}$, $B^{G_2'}$, B^{G_1}, B^{G_2} may be considered balayages on the standard H-cone S and we have

$$B^{G_1'}=B^{G_1}, \quad B^{G_2'}=B^{G_2} .$$

Hence

$$B^{G_1}B^{G_2}=B^{G_2}B^{G_1}$$

The open subsets G_1, G_2 of X_u being arbitrary we deduce, using Theorem 3.10, 3) that axiomD holds on S.

4. Natural potentials and natural sheaf property

Definition. Let S be a standard H-cone of functions on a nearly saturated set X. An element $p \in S$ is called natural potential (on X) if for any sequence G_n of open subsets of X such that $\bigcup_n G_n=X$ we have

$$\bigwedge_{(l_1,l_2,\ldots l_n)\in\mathcal{F}} B^{X\setminus G_{l_1}}B^{X\setminus G_{l_2}}\ldots B^{X\setminus G_{l_n}}p=0$$

where $\mathcal{F}$ is the set of all finite systems of natural numbers.

Proposition 4.1. a) The set of all natural potentials of S is a solid convex subcone of S with respect to the natural order and for any sequence $(s_n)_n$ of natural potentials of S the function $\sum_n s_n$ is also a natural potential if it belongs to S.

b) If G is an open subset of X, p is a finite element of S and $f:=p-B^{X\setminus G}p$ then the restriction of f at G is a natural potential in $S'(G)$ iff for any sequence $(G_n)_n$ of open subset of X such that $\overline{G}_n \subset G$ for any n and such that $\bigcup_n G_n = G$ we have

$$B^{X\setminus G}p = \bigwedge_{(\iota_1,\iota_2,\ldots\iota_n)\in\mathcal{F}} B^{X\setminus G_{\iota_1}} B^{X\setminus G_{\iota_2}} \ldots B^{X\setminus G_{\iota_n}} p \quad ;$$

proof. The assertion a) may be drown from the fact that the map B^A is additive on S for any $A \subset X$.

b) Let $(G_n)_n$ be a sequence of open sets of X such that

$$\bigcup_n G_n = G \ , \ \overline{G}_n \subset G \text{ for any } n \in N.$$

Applying now ([1], proposition 2.3) we get for any $(\iota_1,\ldots,\iota_n) \in \mathcal{F}$

$$\hat{B}^{G\setminus G_{\iota_1}} \hat{B}^{G\setminus G_{\iota_2}} \ldots \hat{B}^{G\setminus G_{\iota_n}} (p - B^{X\setminus G}p) = B^{X\setminus G_{\iota_1}} B^{X\setminus G_{\iota_2}} \ldots B^{X\setminus C_{\iota_n}} p - B^{X\setminus G}p$$

on G, where, for any subset $M \subset G$, $\hat{B}^M$ means the balayage on M relative to the H-cone of functions $S'(G)$.

Hence we have

$$\bigwedge_{(\iota_1,\iota_2,\ldots\iota_n)\in\mathcal{F}} \hat{B}^{G\setminus G_{\iota_1}} \hat{B}^{G\setminus G_{\iota_2}} \ldots \hat{B}^{G\setminus G_{\iota_n}} (p - B^{X\setminus G}p) = \bigwedge_{(\iota_1,\iota_2,\ldots\iota_n)\in\mathcal{F}} B^{X\setminus G_{\iota_1}} \ldots B^{X\setminus G_{\iota_n}} p - B^{X\setminus G}p$$

From the above formula and using the fact that any open subset D of X may be writen as a countable union

$$D = \bigcup_{n\in N} D_n$$

of open subset $D_n \subset X$ such that $\overline{D}_n \subset D$ for any $n \in N$ we deduce the assertion b).

Proposition 4.2. Let S be a standard H-cone of functions on a nearly saturated set X. If there exists a natural potential in S which is also an weak unit (or equivalently if any $s\in S_o$ is a natural potential) then X is semi-saturated.

<u>Proof</u>. Let (S_1, X_1) be the natural extention of (S,Y) and let F be a compact subset of $X_1 \setminus X$. We show that F is polar .

Let p be a natural potential of S such that p is an weak unit of S and let $\bar{p}$ be the natural extention of p to the set X_1.

Let now F_n be a decreasing sequence of closed subset of X_1 such that

$$\bigcap_{i\in N} F_i = F \;,\; F_{n+1} \subset \mathring{F}_n \qquad (\forall)\ n\in N.$$

We put for any $n\in N$,

$$G_n := X \setminus F_n$$

Obviously we have $X = \bigcup_n G_n$. Since X is nearly saturated, using Proposition 3.1, b), we have

$$({}^1B^{\mathring{F}_n}\bar{p})\big|_X = B^{\mathring{F}_n\cap X}p \qquad (\forall)\ n\in N$$

and therefore

$$({}^1B^{F_{n+1}}\bar{p})\big|_X \le ({}^1B^{\mathring{F}_n}\bar{p})\big|_X = B^{\mathring{F}_n\cap X}p \le B^{F_n\cap X}p \le B^{X\setminus G_n}p \,,$$

$$({}^1B^{F}\bar{p})_X \le \bigwedge_{n\in N} B^{X\setminus G_n}p = 0 \,.$$

Hence F is polar.

<u>Definition</u>. Let S be a standard H-cone of functions on a set X. We say that S satisfies <u>the natural global section property</u> (N.G.S. - property) on X if any function $f:X \to \overline{R}_+$ such that for any $x \in X$ there exists an open neighbourhood G of x for which

$$f|_G \in S'(G)$$

we have $f \in S$.

We say that S satisfies the <u>natural sheaf property</u> if the map

$$G \longrightarrow S'(G)$$

defined on the set of all open subset G of X is a sheaf.

It is easy to see that S satisfies the natural sheaf property iff for any open subset G of X, $S'(G)$ satisfies N.G.S property on G.

<u>Proposition 4.3</u>. If X is nearly saturated with respect to S and S satisfies N.G.S.-property on X, then we have:

a) S satisfies axiom D_o on X;

b) X is semi-saturated with respect to S;

c) any element $u \in S_o$ is a natural potential.

<u>Proof</u>. a) Let G_1, G_2 be two open subsets of X such that $G_1 \cup G_2 = X$. It will be sufficient to shaw that

$$B^{G_1}(B^{G_2}s) = B^{G_1}s \wedge B^{G_2}s \qquad (\forall)\ s \in S.$$

If $s \in S$, obviously we have

$$B^{G_1}(B^{G_2}s) \leq B^{G_1}s \wedge B^{G_2}s \text{ on } X \text{ and } B^{G_1}(B^{G_2}s) = B^{G_2}s \text{ on } G_1 .$$

Let $t \in S$ be such that $t \geq B^{G_2} s$ on $\bar{G_1}$.

Since $B^{G_2} s = s$ on G_2 we deduce, using the hypothesis, that the function

$$t'(x) = \begin{cases} \inf (t(x), s(x)) & \text{if } x \in G_2 \\ s(x) & \text{if } x \in G_1 \end{cases}$$

belongs to S. Moreover, we have

$$t' \geq B^{G_1} s \text{ on } X, \quad t \geq t' \geq B^{G_1} s \text{ on } G_2 .$$

The element t being arbitrary we get

$$B^{G_1}(B^{G_2} s) \geq B^{G_1} s \geq B^{G_1} s \wedge B^{G_2} s \text{ on } G_2$$

From the preceding considerations we have

$$B^{G_1}(B^{G_2} s) = B^{G_1} s \wedge B^{G_2} s .$$

b) From the preceding point and using proposition 3.4 a) we deduce that the pair (S_1, X_1) satisfies axiom D_0 . Let K be a compact subset of $X_1 \setminus X$. We shaw that K is polar. Let $(V_n)_n$ be a fundamental system of open neighbourhoods of K such that

$$\overline{V_{n+1}} \subset V_n \qquad (\forall)\ n \in N.$$

Let now $p \in S_0$ and let $\bar{p}$ be its natural extention to X_1. Obviously we have

$$^1B^k \bar{p} = \bigwedge_{n \in N} {^1B}^{V_n} \bar{p} = \bigwedge_{n \in N} {^1B}^{\overline{V}_n} \bar{p} ,$$

and

$$^1B^k \bar{p}(x) = \inf_{n \in N} {^1B}^{V_n} \bar{p}(x) = \inf_{n \in N} {^1B}^{\overline{V}_n} \bar{p}(x) \qquad (\forall)\ x \in X_1 \setminus K$$

Since for any $n, m \in N$, $n < m$ we have

$$(p - B^{\overline{V}_n \cap X} p)_{X \setminus \overline{V}_m} \in S'(X \setminus \overline{V}_n)$$

it follows, using Proposition 3.1, a),

$$(\bar{p} - {}^{1}B^{K}\bar{p})|_{X \setminus \overline{V}_n} = \sup_m (\bar{p} - {}^{1}B^{\overline{V}_m}\bar{p})|_{X \setminus \overline{V}_n} =$$

$$= \sup_m (\bar{p} - {}^{1}B^{V_m}\bar{p})|_{X \setminus \overline{V}_n} = \sup_m (p - B^{V_m \cap X} p)|_{X \setminus \overline{V}_n} =$$

$$= \sup_m (p - B^{\overline{V}_m \cap X} p)|_{X \setminus \overline{V}_n} \in S'(X \setminus \overline{V}_n).$$

Using the fact that (S, X) satisfies N.G.S-property we have

$$(\bar{p} - {}^{1}B^{k}\bar{p})|_X \in S, \qquad {}^{1}B^{k}\bar{p} \lessapprox p .$$

and therefore the element ${}^{1}B^{k}p$ is universally continuous. Since the pair (S_1, X_1) satisfies axiom D_0, from Corollary 3.6, we deduce

$${}^{1}B^{V_n}({}^{1}B^{K}\bar{p}) = {}^{1}B^{K}\bar{p} \qquad (\forall)\ n \in N ,$$

$${}^{1}B^{K}({}^{1}B^{K}\bar{p}) = \bigwedge_n {}^{1}B^{V_n}({}^{1}B^{K}\bar{p}) = {}^{1}B^{K}\bar{p} .$$

From the fact that ${}^{1}B^{K}\bar{p}$ is universally continuous and from the last equality we obtain, using ([2], Theorem 2.4), that the carrier of the element ${}^{1}B^{K}\bar{p}$ is contained in K which is a semipolar subset of X_1. Hence

$$1_B K\bar{p}=0$$

The element $p\in S_o$ being arbitrary we deduce that K is polar.

c) Let $u\in S_o$ and let $(G_n)_n$ be a sequence of open subsets of X_1 such that

$$X\subset \bigcup_n G_n$$

and let, for any $n\in N$, $F_n:=X_1\setminus G_n$. Using the fact that X is semi-saturated and Proposition 3.1, c) we get

$$({}^1B^{F_n}\bar{s})\Big|_X = B^{F_n\cap X} s \qquad (\forall)\ n\in N,\ (\forall)\ s\in S$$

We denote

$$v=\bigwedge_{(\ell_1,\ldots,\ell_n)\in\mathcal{F}} B^{F_{\ell_1}\cap X} B^{F_{\ell_2}\cap X}\ldots B^{F_{\ell_n}\cap X} u=$$

$$=\Big(\bigwedge_{(\ell_1,\ell_2,\ldots,\ell_n)\in\mathcal{F}} {}^1B^{F_{\ell_1}}\,{}^1B^{F_{\ell_2}}\ldots {}^1B^{F_{\ell_n}}\bar{u}\Big)\Big|_X$$

where $(\ell_1,\ell_2,\ldots,\ell_n)$ runs in set $\mathcal{F}$ of all finite systems of natural numbers.

Obviously, for any $n\in N$ we have

$$(u-v)\Big|_{G_n}\in S'(G_n)$$

and therefore $u-v\in S$. Hence $v\in S_o$.

From Theorem 3.5 we deduce that for any $m\in N$ and any $x\in G_m\cap X$, $(B^{F_m\cap X})^*(\varepsilon_x)$ is an H-measure carried by the boundary (in X) of the set $G_m\cap X$ and therefore

$$B^{F_m\cap X}(v)(x)=(B^{F_m\cap X})^*(\varepsilon_x)(v)=$$

$$=(B^{F_m \cap X})^*(\mathcal{E}_x)\Big[\inf_{(\ell_1,\ell_2,\dots,\ell_n)\in\mathcal{F}} (B^{F_{\ell_1}\cap X} B^{F_{\ell_2}\cap X}\dots B^{F_{\ell_n}\cap X} u\Big] =$$

$$=\inf_{(\ell_1,\ell_2,\dots,\ell_n)\in\mathcal{F}} (B^{F_m \cap X})^*(\mathcal{E}_x)(B^{F_{\ell_1}\cap X} B^{F_{\ell_2}\cap X}\dots B^{F_{\ell_n}\cap X} u) =$$

$$=\inf_{(\ell_1,\dots\ell_n)\in\mathcal{F}} B^{F_m\cap X} B^{F_{\ell_1}\cap X} B^{F_{\ell_2}\cap X}\dots B^{F_{\ell_n}\cap X} u(x)=v(x).$$

Hence, the carrier (in X) of the element $v\in S_o$ is contained in $F_m\cap X$ for any $m\in N$, i.e. $v=0$ and therefore u is a natural potential.

Theorem 4.4. Let X be semi-saturated with respect to S. Then the following assertions are equivalent:

1) the pair (S,X) satisfies N.G.S.-property,

2) the pair (S,X) satisfies axiom D_o and any element $u\in S_o$ is a natural potential on X.

Proof. The assertion $1)\Longrightarrow 2)$ follows from Proposition 4.3.

$2)\Longrightarrow 1)$ Let $f:X\to \overline{R}_+$ such that for any point $x\in X$ there exists an open neighbourhood G_x of x such that

$$f|_{G_x}\in S'(G_x) .$$

We show that if $p,q\in S$ are such that

$$\inf (f+q,p)\in S,$$

and G is an open subset of X such that

$$f|_G\in S'(G)$$

then for any open subset G_1 of X such that $\overline{G}_1\subset G$ we have

$$\inf(f+B^{X\setminus G_1}q, p) \in S .$$

Indeed, for any such open subset G_1 there exists an other open subset G_2 of X such that

$$\overline{G}_1 \subset G_2 \subset \overline{G}_2 \subset G .$$

Since

$$(f+B^{X\setminus G_1}q)\Big|_{G_2} \in S'(G_2) \quad \text{and}$$

$$f+B^{X\setminus G_1}q = f+q \quad \text{on} \quad X\setminus\overline{G}_1$$

we deduce, using Theorem 3.5, that the function

$$t(x) := \begin{cases} \inf(f(x)+B^{X\setminus G_1}q(x), p(x)) & \text{if } x \in G_2 \\ \inf(f(x)+q(x),\ p(x)) & \text{if } x \in X\setminus G_2 \end{cases}$$

belongs to S. But obviously

$$t(x) = \inf\ (f(x)+B^{X\setminus G_1}q(x), p(x)) \quad (\forall)\ x \in X .$$

Let now for any $x \in X$, G_x be an open neighbourhood of x such that

$$f\Big|_{G_x} \in S'(G_x)$$

and let $(V_n)_n$ be a sequence of open subsets of X such that $\bigcup_n V_n = X$ and for any $n \in N$ there exists $x_n \in X$ such that $\overline{V}_n \subset G_{x_n}$. If $p \in S_o$ we have

$$\inf(f+p, p) \in S$$

and therefore for any finite system $(\ell_1, \ell_2, \ldots, \ell_n)$ of natural number the following relation holds:

$$\inf(f+B^{X\setminus V_{\ell_1}}B^{X\setminus V_{\ell_2}}\ldots B^{X\setminus V_{\ell_n}}p,p)\in S.$$

The element p being a natural potential we get

$$\inf(f,p)\in S$$

and therefore, p being arbitrary

$$f\in S.$$

<u>Theorem 4.5</u>. Let X be semi-saturated with respect to S. The following assertions are equivalent

1) the pair (S,X) satisfies natural sheaf property,

2) the pair (S,X) satisfies axiom D_o and for any open set G of X there exists a strictly positive potential on G.

<u>Proof</u>. 1) $\Longrightarrow$ 2). From 1) it follows that N.G.S.-property holds for the pair (S'(G),G), for any open subset G of X. From the preceding proposition axiom D_o holds for the pair (S'(G),G) and any universally continuous element of S'(G) is a natural potential on G (with repect to S'(G)). The assertion 2) follows now using Proposition 4.1 and the fact that the function

$$(p-B^{X\setminus G}p)\restriction_G$$

is a nearly continuous element of S'(G) for any $p\in S_o$.

2) $\Longrightarrow$ 1). If axiom D_o holds for the pair (S,X) it holds also for the pair (S'(G),G) for any open subset G of X (See Theorem 3.7). From hypothesis and from proposition 4.1 we deduce that for any $p\in S_o$ the function

$$p-B^{X\setminus G}p$$

is a natural potential on G (with respect to $S'(G)$). Hence any universally continuous element of $S'(G)$ is a natural potential on G. The assertion 1) follwos now from the preceding proposition.

Theorem 4.6. If X is semi-saturated with respect to S then the following assertions are equivalent:

1) S satisfies the fine-sheaf property,

2) S satisfies the natural sheaf property and the axiom of nearly continuity,

3) S satisfies both axiom D_o and axiom of nearly continuity.

Proof. Using ([3], Proposition 5.6.8) and Theorem 3.13 we get 1) ⟹ 2). The relation 2) ⟹ 3) follows from Proposition 4.3

3) ⟹ 1) Let p be an universally continuous elements of S and let B_1, B_2 be two balayages on X such that $B_1 \vee B_2 = I$. If $b(B_1)$ (resp. $b(B_2)$) denotes the base of B_1 (resp. B_2) then we have

$$B^{b_1(B) \cup b_2(B)} = I, \quad b(B_1) \cup b(B_2) = X$$

From hypothesis the function $B_2 p$ is a nearly continuous element of S and therefore

$$B_1(B_2 p) = B^{b(B_1)}(B_2 p) = \bigwedge \{ B^G(B_2 p) \mid G \text{ open}, G \supset b(B_1) \}$$

Since $p \in S_o$ there exists a decreasing sequence $(D_n)_n$ of open set such that

$$b(B_2)=\bigcap_n D_n$$

$$B_2p=\inf_n B^{D_n}p$$

If G is an open subset of X such that $b(B_1)\subset G$ and x is an arbitrary point of X there exists a measure μ_x on the fine closure of G such that

$$\mu_x(s)=B^G s(x) \qquad (\forall)\ s\in S.$$

Particularly we have

$$B^G(B_2p)(x)=\mu_x(B_2p)=\mu_x(\inf_n B^{D_n}p)=$$

$$=\inf_n \mu_x(B^{D_n}p)=\inf_n B^G(B^{D_n}p)$$

Dince $G\cup D_n=X$ for any $n\in N$ we deduce

$$B^G B^{D_n}=B^{D_n}B^G ,$$

$$B^G(B_2p)(x)=\inf B^G B^{D_n}p(x)=\inf B^{D_n}B^G p(x)\geq$$

$$\geq B_2 B^G p(x)\geq B_2 B_1 p(x)$$

and therefore

$$B^G(B_2p)\geq B_2B_1p$$

The open subset G, $G\supset b(B_1)$ being arbitrary we have

$$B_1B_2p\geq B_2B_1p \quad , \quad B_1B_2=B_2B_1 .$$

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