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DAN TIMOTIN

In [9] Peller obtained a precise criterion for the Hankel operators H_φ to belong to a given Schatten-von Neumann class: namely, $H_\varphi \in \mathcal{C}_p$ $(1 \leq p < \infty)$ if and only if the antianalytic part of φ is in a certain Besov space on the unit circle (Recall that $H_\varphi : H^2 \longrightarrow L^2 \ominus H^2$ is given by the formula $H_\varphi f = (I-P_+)\varphi f$, where P_+ is the Riesz projection from L^2 onto H^2). Peller extended subsequently his result to other operator ideals ([10]; see also [11],[13]).

On the other hand, since the commutator (in $L^2(\mathbb{T})$) of the multiplication operator M_φ with the Riesz projection has the matrix representation

$$\begin{pmatrix} 0 & H_\varphi \\ H_{\bar\varphi}^* & 0 \end{pmatrix}$$

Rochberg ([12]) has suggested, as higher dimensional generalizations of Peller's result, the estimation of commutators of multiplication operators with singular integral operators of Calderón-Zygmund type. In this case the frame of R^d seems more natural. Results for such commutators and, more recently, for iterated commutators have been obtained by Janson and Wolff ([7]) Janson and Peetre ([6]). For a comprehensive survey of Hankel operators, Peller's results and subsequent generalizations, see [9].

The present note considers a further generalization, suggested by the Fourier transform of the previous case. We obtain results for integral operators defined by a Kernel of type:

(1) $\quad A(x,y)\,\hat\varphi(x-y)$

where the main condition imposed on A is its invariance under the action of a fixed discrete multiplicative subgroup G of $\mathbb{R}_+^*$:

$$(2) \quad A(gx, gy) = A(x,y), \ \forall x,y \in \mathbb{R}^d, \ g \in G$$

It seems improbable to obtain a necesary and sufficient characterization, in terms of both A and φ, of operators with kernel (1) belonging to a Schatten-von Neumann class. Our results are of the following type (see theorems 1 and 2) under certain conditions preimposed on A, the operator defined by the kernel (1) is in $\mathcal{C}_p$ if and only if $\varphi \in \dot{B}^{d/p}_{pp}$ (for the definition of the homogeneous Besov spaces used here, see [1]). This goal is actually achieved only for $1 \leq p \leq 2$. For $2 < p < \infty$, we obtain only the necessity; however , the proof is more general and simpler then that in [7] or [9]. Note that in [6], about which we recently learned, kernels of type (1), but satisfying the stronger symmetry condition

$$A(\lambda x, \mu y) = \lambda\mu A(x,y), \ \forall x,y \in \mathbb{R}^d, \ \lambda, \mu > 0$$

are mentioned as a possible further generalization of iterated commutators. Also, the use of interpolation in theorem 1 below is similar to that in [6] .

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1. Let T (A, φ) be the operator whose kernel is given by (1), and T(A) the operator with kernel A(x,y). We suppose that A is a locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying condition (2) and that φ belongs, say, to $\mathcal{S}(\mathbb{R}^d)$. We shall consider T(A) and T(A, φ) as densely defined operators on $L^2(\mathbb{R}^d)$. Suppose also

that G is generated by $g_o>1$.

The following two lemmas provide the basic estimates. The proof of the first follows a technique of Peller ([9]), while the second is a straightforward computation.

Lemma 1. Consider $E=\{(x,y)\mid 3\sqrt{d}\leq|x-y|\leq 3\sqrt{d}g_o^2\}$, and let $\chi(x,y)$ be the characteristic function of some set $E'\supset E$. Then

$$\|T(A,\varphi)\|_{\mathcal{C}_1}\leq Ca_1(A)\cdot\|\varphi\|_{\dot{B}^d_{11}}$$

where $a_1(A)=\|T(\chi A)\|_{\mathcal{C}_1}$.

(as everywhere below, C denotes a universal constant, not necessarily the same in different inequalities).

Proof. Let $\psi\in\mathcal{S}(\mathbb{R}^d)$ be such that $\operatorname{supp}\hat{\psi}\subset\{x\in\mathbb{R}^d\mid 3\sqrt{d}\leq|x|\leq 3\sqrt{d}\,g_o^2\}$ and $\sum\hat{\psi}_k=1$ on $\mathbb{R}^d\setminus\{0\}$, where $\hat{\psi}_k(x)=\psi(g_o^{-k}x)$.

Since $A(x,y)\hat{\varphi}(x-y)=\sum_{k\in\mathbb{Z}}\hat{\psi}_k(x-y)A(x,y)\hat{\varphi}(x-y)$

we have

$$(3)\quad \|T(A,\varphi)\|_{\mathcal{C}_1}\leq\sum_{k\in\mathbb{Z}}\|T(A,\varphi*\psi_k)\|_{\mathcal{C}_1}$$
$$=\sum_{k\in\mathbb{Z}}T(\chi_kA,\varphi*\psi_k)\|_{\mathcal{C}_1}$$

where $\chi_k(x,y)=\chi(g_o^{-k}x,g_o^{-k}y)$

But we may write, for a kernel B,

$$T(B,\varphi)=\int T(B_\xi)\varphi(\xi)d\xi$$

where $B_\xi(x,y)=e^{ix.\xi}B(x,y)e^{-iy.\xi}$

Since

(4) $$\| T(B_\xi)\|_{\mathcal{C}_1} = \|T(B)\|_{\mathcal{C}_1}$$

it follows that

$$\| T(B,\varphi)\|_{\mathcal{C}_1} \leq \| T(B)\|_{\mathcal{C}_1} \cdot \|\varphi\|_{L^1}$$

Combining (3) and (4), we obtain:

$$\| T(A,\varphi)\|_{\mathcal{C}_1} \leq \sum_{k\in\mathbb{Z}} \|T(\chi_k A)\|_{\mathcal{C}_1} \cdot \|\varphi * \psi_k\|_{L^1}$$

But $T(\chi_k A)$ has kernel (by (2))

$$\chi_k(x,y)A(x,y)=\chi(g_o^{-k}x, g_o^{-k}y)A(x,y)=\chi(g_o^{-k}x, g_o^{-k}y)A(g_o^{-k}x, g_o^{-k}y)$$

and therefore

$$\| T(\chi_k A)\|_{\mathcal{C}_1} = g_o^{kd} \|T(\chi A)\|_{\mathcal{C}_1}$$

whence

$$\| T(A,\varphi\|_{\mathcal{C}_1} \leq \sum_{k\in\mathbb{Z}} g_o^{kd} \|T(\chi A)\|_{\mathcal{C}_1} \|\varphi * \psi_k\|_{L^1} \leq C \| T(\chi A)\|_{\mathcal{C}_1} \cdot \|\varphi\|_{\dot{B}^d_{11}}$$

Lemma 2. Define

$$a_2(A) = \sup_{3\sqrt{d} \leq |t| \leq 3\sqrt{d}\, g_o^2} \int |A(y+t,y)|^2 dy$$

Then

$$\| T(A,\varphi)\|_{\mathcal{C}_2} \leq c \cdot a_2(A) \cdot \|\varphi\|_{\dot{B}^{d/2}_{22}}$$

Proof. As in the proof of lemma 1, we have

$$A(x,y)\hat{\varphi}(x-y)= \sum_{k\in\mathbb{Z}} \hat{\psi}_k(x-y)A(x,y)\hat{\varphi}(x-y)$$

and therefore

$$\| T(A,\varphi)\|^2_{\mathcal{C}_2} \leq 3 \sum_{k\in\mathbb{Z}} \| T(\chi_k A, \varphi * \psi_k)\|^2_{\mathcal{C}_2}$$

since the supports of ψ_k and $\psi_{k'}$ are disjoint for $|k-k'|\geq 3$.
Then

$$\| T(\chi_k A, \varphi*\psi_k)\|^2_{\mathcal{C}_2} = \iint |\chi_k A(x,y)|^2 \, |\hat{\psi}_k(x-y)|^2 |\hat{\varphi}(x-y)|^2 dxdy=$$
$$= \int |\hat{\psi}_k(t)|^2 |\varphi(t)|^2 \left(\int |\chi_k A(y+t,y)|^2 dy\right) dt$$

and

$$\int |\chi_k A(y+t,y)|^2 dy = g_o^{kd} \int |\chi(y'+g_o^{-k}t,y')A(y'+g_o^{-k}t,y')|^2 dy$$
$$\leq \quad g_o^{kd}(a_2(A))^2$$

Therefore

$$\| T(A,\varphi)\|^2_2 \leq (a_2(A))^2 \sum_{k\in\mathbb{Z}} g_o^{kd} \|\psi_k * \varphi\|^2_{L^2} \leq c.(a_2(A))^2 \|\varphi\|^2_{\dot{B}^{d/2}_{22}}$$

Lemmas 1 and 2 may be applied to a variety of concrete situations (that is, conditions on A) to obtain criteria for $T(A,\varphi)$ to be trace class or Hilbert-Schmidt. More interesting is the possibility to interpolate between these two results.Our main application is the following theorem:

Theorem 1. Suppose A is ~~C^∞ on~~ $\mathbb{R}^{2d}\setminus\{0\}$ satisfies (2), and, together with its derivatives of order $\leq N-1$, vanishes on the diagonal $\Delta=\{x=y\}$. Assume also $N\geq d/p$ and $1\leq p\leq 2$. Then there is a constant C_p, depeding on A, such that

$$\|T(A,\varphi)\|_{\mathcal{C}_p} \leq C_p \cdot \|\varphi\|_{\dot{B}^{d/p}_{pp}}$$

<u>Proof</u>. Any C^∞ function which vanishes of order N on the diagonal may be written as a sum of terms of the form

$$\frac{(x_{i_1}-y_{i_1})\ldots(x_{i_N}-y_{i_N})}{(|x|^2+|y|^2)^{N/2}} \cdot \tilde{A}(x,y)$$

where A is C^∞ and satisfies (2). Also, by polarization, we may consider only kernels of the form

$$\mu(x,y)^N(|x|^2+|y|^2)^{-N/2}\tilde{A}(x,y)\hat{\varphi}(x-y)$$

where μ is a linear function on $\mathbb{R}^{2N}$, vanishing on Δ. Denote by $\tilde{\mu}(x,y)=\mu(x,y)(|x|^2+|y|^2)^{-1/2}$. Then $\tilde{\mu}$ is $\mathbb{R}^*_+$- homogeneous (therefore G-homogeneous), and

(5) $$|\tilde{\mu}(x,y)| \leq C.\frac{|x-y|}{(|x|^2+|y|^2)^{1/2}}$$

Consider the family of operators $T(A'_\lambda,\varphi)$, corresponding to the kernels

$$A'_\lambda(x,y)=\tilde{A}(x,y)|\tilde{\mu}(x,y)|^\lambda \operatorname{sign}(\tilde{\mu}(x,y))^N$$

We will interpolate between $\operatorname{Re}\lambda=Np$ and $\operatorname{Re}\lambda=Np/2$.

For $\operatorname{Re}\lambda=Np$, in order to apply lemma 1, we have to choose first a set $E'\supset E$. For $m\in\mathbb{Z}^d$, we will denote by $Q_m\subset\mathbb{R}^d$ the cu be $\prod_{i=1}^{d}[m_i, m_i+1]$. Let $\Xi\subset\mathbb{Z}^d\times\mathbb{Z}^d$ be defined by

$$\Xi = \{(m,n)\in\mathbb{Z}^d\times\mathbb{Z}^d \mid 2\sqrt{d}\leq|m-n|\leq(3g_0^2+1)\sqrt{d}\}$$

and $E' = \bigcup_{(m,n)\in\Xi} Q_m \times Q_n$. Then $0 \notin E'$, and $E \subset E'$. Let χ be the characteristic function of E'. We have

$$(6) \quad a_1(A'_\lambda) = \| T(\chi A'_\lambda) \|_{\mathcal{C}_1} \leq \sum_{(m,n)\in\Xi} \| P_{Q_m} T(\chi A'_\lambda) P_{Q_n} \|_{\mathcal{C}_1}$$

Now, $P_{Q_m} T(\chi A'_\lambda) P_{Q_n}$, considered as an operator from $L^2(Q_n)$ to $L^2(Q_m)$, has kernel of class $\mathcal{C}^d$ (since $Np > d$). Moreover, for $|\alpha| \leq d$, using (5) and the G-homogeneity of $\tilde{A}$ and $\tilde{\mu}$, we obtain that $|D^\alpha A'_\lambda(x,y)|$ is majorized by a sum of terms of the form

$$C(\lambda) |x-y|^{Np-k} (|x|^2 + |y|^2)^{-1/2(Np+|\alpha|-k)} .$$

where $0 \leq k \leq |\alpha|$ and $C(\lambda)$ is polynomial in $|\operatorname{Im} \lambda|$.

Since, for $(x,y) \in E'$, $|x-y|$ is bounded below and above, $|x|^2 + |y|^2$ is bounded below and, moreover, of the same order as $|m|^2 + |n|^2$ if $(x,y) \in Q_m \times Q_n$, we obtain, on $Q_m \times Q_n$, the estimate

$$|D^\alpha A'_\lambda(x,y)| \leq C(\lambda) . (|m|^2 + |n|^2)^{-\frac{Np}{2}} \quad \text{for } 0 \leq |\alpha| \leq d.$$

It follows, by [4], XI, 9, that $P_{Q_m} T(\chi A'_\lambda) P_{Q_n}$ is trace class, and

$$\| P_{Q_m} T(\chi A'_\lambda) P_{Q_n} \| \leq C(\lambda) (|m|^2 + |n|^2)$$

whence (by (6))

$$a_1(A'_\lambda) \leq C(\lambda) . \sum_{(m,n)\in\Xi} (|m|^2 + |n|^2)^{-\frac{Np}{2}}$$

and therefore, since $Np > d$, $a_1(A'_\lambda)$ is bounded, polynomially in $|\operatorname{Im} \lambda|$.

For $\mathrm{Re}\, z = \frac{Np}{2}$, we have to estimate (lemma 2)

$\int |A'_\lambda(y+t,y)|^2 dy$, for $3\sqrt{d} \leq |t| \leq 3\sqrt{d}\, y_0^2$. But, again by (5), for $\mathrm{Re}\lambda = \frac{Np}{2}$,

$$|A'_\lambda(y+t,y)| \leq c \frac{|t|^{\frac{Np}{2}}}{(|y|^2+|y+t|^2)^{\frac{Np}{4}}} \leq C(1+|y|^2)^{-\frac{Np}{4}}$$

and, therefore

$$\int |A'(y+t,y)|^2 dy \leq C\int (1+|y|^2)^{-\frac{Np}{2}} dy$$

is bounded independently of t and $|\mathrm{Im}\,\lambda|$.

To end the proof, consider, for $\frac{Np}{2} \leq \mathrm{Re}\,\lambda \leq Np$, the analytic family of operators $\mathcal{T}_\lambda$ which associate to the function φ the operator $T(A'_\lambda, \varphi)$. Then, for $\mathrm{Re}\lambda = \frac{Np}{2}$, $\mathcal{T}_\lambda$ maps $\dot{B}^{d/2}_{22}$ into $\mathcal{C}_2$ (with uniformly bounded norm in $\mathrm{Im}\lambda$) and, for $\mathrm{Re}\,\lambda = Np$, it map $\dot{B}^{d}_{11}$ into $\mathcal{C}_1$ (with bound increasing polynomially in $|\mathrm{Im}\,\lambda|$). The desired conclusion follows by interpolation.

Theorem 1 may be extended to cover other cases of interest. It is obvious from the proof that me may take $A \in C^m(\mathbb{R}^{2d} \setminus \{0\})$, where $m = \max\{d, N-1\}$. Moreover, it is sufficient only to suppose $A(x,.)$ differentiable for $y \neq 0$ and $A(.,y)$ differentiable for $x \neq 0$, since the result of [4] quoted in the proof uses only derivatives with respect to one of the variables, while the condition $(m,n) \in \mathcal{E}$ implies that, for m,n fixed we have either $x \neq 0$ on Q_m or $y \neq 0$ on Q_n. This applies, for instance, to the case of iterated commutators (see [6]), which give rise to kernels of the form $A(x,y) = \Pi(\hat{K}_j(x) - \hat{K}_j(y))$, where $\hat{K}_j$ are $\mathbb{R}^*_+$- homogeneous. Actually, for iterated commutators the proof is simpler: there is no need of estimates on the derivatives (nor of quoting [4]), since $P_{Q_m} T(\chi A'_\lambda) P_{Q_n}$ is then of finite rank.

2. The reverse problem can be treated in more generality (see theorem 2 below). We rely again on two simple lemmas.

Lemma 3. Let A be some locally integrable kernel, and $\alpha, \alpha' \in C_0^\infty(\mathbb{R}^d)$. Define the function a by its Fourier transfor

(7) $$\hat{a}(u)=\alpha'(u)\int A(x+u,x)\alpha(x)\,dx$$

Suppose P, P' are the projections onto $L^2(\operatorname{supp}\alpha)$, $L^2(\operatorname{supp}\alpha+\operatorname{supp}\alpha')$, respectively. Then

$$\| a \|_{L^1} \leq C \| P'T(A)P \|_{\mathscr{C}_1}$$

where c depends on the multiplier norm of α' and on the uniform norm of α.

Proof. We may suppose $A\in L^2((\operatorname{supp}\alpha+\operatorname{supp}\alpha')\times\operatorname{supp}\alpha)$. Also, a depending linearly on A, it is sufficient to prove the lemma for T(A) having rank one. Then take $A(x,y)=f(x)g(y)$, $f\in L^2(\operatorname{supp}\alpha+\operatorname{supp}\alpha')$, $g\in L^2(\operatorname{supp}\alpha)$. In this case,

$$\hat{a}(u)=\alpha'(u)\int f(x+u)g(u)\alpha(x)\,dx$$

Therefore, $\hat{a}=\alpha'(f*\widetilde{\alpha g})$, and

$$\|a\|_1 \leq C\cdot\|\check{f}\cdot\check{\widetilde{\alpha g}}\|_1 \leq C\cdot\|\check{f}\|_2\cdot\|\check{\widetilde{\alpha g}}\|_2 = C\cdot\|f\|_2\cdot\|\alpha g\|_2 \leq$$
$$\leq C\cdot\|f\|_2\cdot\|g\|_2 = C\cdot\|P'T(A)P\|_{\mathscr{C}_1}.$$

Lemma 4. Under the hypothesis of Lemma 3

$$\| a \|_{L^\infty} \leq C\, R^d \| P'T(A)P \|$$

where C depends on α and α' as in Lemma 3, and $\operatorname{supp}\alpha$, $\operatorname{supp}\alpha+\operatorname{supp}\alpha'\subset\{x\in\mathbb{R}^d \mid |x|\leq R\}$.

Proof. Denote $b(u)=\int A(x+u,x)\alpha(x)dx$.

Obviously $\|a\|_{L^\infty} \leq C\cdot\|b\|_{L^\infty}$, where c is the multiplier norm of α'.

Let $\xi \in \mathbb{R}^d$. Then

$$\hat{b}(\xi) = \int e^{-i\xi\cdot u}\left(\int A(x+u,x)\alpha(x)dx\right)du =$$

$$= \iint A(s,x)\alpha(x)e^{-i\xi\cdot(s-x)}dx\,ds =$$

$$= \int_{|s|\leq R} e^{-i\xi\cdot s}\left(\int_{|x|\leq R} A(s,x)\alpha(x)e^{i\xi\cdot x}dx\right)ds =$$

$$= \langle \chi_R(\cdot)e^{-i\xi\cdot}, (P'T(A)P)(\alpha(\cdot)e^{i\xi\cdot})\rangle \leq$$

$$\leq R^d\cdot\|P'T(A)P\|$$

Corollary 1. Under the same hypothesis,

$$\|a\|_{L^p} \leq C \quad R^{d(1-\frac{1}{p})}\|P'T(A)P\|_{\mathcal{C}_p}$$

Proof. By interpolation.

Theorem 2. Let A be ~~continuous~~ C^∞ on $\mathbb{R}^{2d}\setminus\{0\}$, satisfying (2), and suppose that

(8) for any $u\in\mathbb{R}^d\setminus\{0\}$, there exists $x\in\mathbb{R}^d$, such that $A(x+u,x)\neq 0$.

Then

$$\|\varphi\|_{\dot{B}^{d/p}_{pp}} \leq c(p,A)\cdot\|T(A,\varphi)\|_{\mathcal{C}_p} \qquad \text{for } 1\leq p\leq\infty$$

Proof. Let $F=\{u\in\mathbb{R}^d \mid 1\leq|u|\leq\varrho_0\}$ contained in $\{\varrho_0^{-1}<|u|<\varrho_0^{2}\}$. Let $\{\Omega_j\}$ be a finite open cover (in $\mathbb{R}^d$) of F, D_j open sets in $\mathbb{R}^d$, $\varkappa_j\in\mathbb{C}$, $|\varkappa_j|=1$, such that $\mathrm{Re}(\varkappa_j A(x+u,x))>0$ for $u\in\Omega_j$, $x\in D_j$. Take $\Omega'_j \subset\subset \Omega_j$, such that $\{\Omega'_j\}$ is still an open cover of E, and choose $\alpha_j, \alpha'_j \in C_0^\infty(\mathbb{R}^d)$ positive functions, such that

(i) $\alpha'_j(u)>0$ for $u\in\Omega'_j$.

(ii) $\mathrm{supp}\,\alpha'_j\subset\Omega_j$ and $\mathrm{supp}\,\alpha_j\subset D_j$

(the possibility of this construction follows from (8)).

Define now functions b_{jk} by

$$\hat{b}_{jk}(u)=\hat{\varphi}(u)\alpha'_j(g_o^{-k}u)\int A(x+u,x)\alpha_j(g_o^{-k}x)dx$$

If P'_{jk}, P_{jk} are the corresponding projections, then we have, by Corollary 1,

$$\text{(9)} \qquad \|b_{jk}\|_{L^p}\leq C g_o^{kd(1-\frac{1}{p})}\|P'_{jk}T(A,\varphi)P_{jk}\|_{\mathcal{C}_p}$$

Note that C depends on the multiplier norm of α'_j and the uniform norm of α_j, and is therefore independent of k.

Denote

$$\theta_{jk}(u)=\alpha'_j(g_o^{-k}u)\int A(x+u,x)\alpha_j(g_o^{-k}x)dx$$

A change of variables yields $\theta_{jk}(u)=g_o^{kd}\,\theta_{jo}(g_o^{-k}u)$

Now, define $\psi,\psi_k\in\mathcal{S}$ by $\hat{\psi}=\sum_j r_j\theta_{jo}$, and $\hat{\psi}_k(u)=\psi(g_o^{-k}u)$.

From (9) , we obtain

$$g_o^{kd}\cdot\|\varphi*\psi_k\|_{L^p}\leq\sum_j\|b_{jk}\|_{L^p}\leq C\cdot g_o^{kd(1-\frac{1}{p})}\Big(\sum_j\|P'_{jk}T(A,\varphi)P_{jk}\|_{\mathcal{C}_p}\Big)$$

whence

$$g_o^{\frac{kd}{p}}\cdot\|\varphi*\psi_k\|_{L^p}\leq C\cdot\Big(\sum_j\|P'_{jk}T(A,\varphi)P_{jk}\|^p_{\mathcal{C}_p}\Big)^{1/p}$$

Therefore

$$\text{(10)} \qquad \sum_{k\in\mathbb{Z}} g_o^{kd}\|\varphi*\psi_k\|^p_{L^p}\leq C\cdot\sum_{k\in\mathbb{Z}}\sum_j\|P'_{jk}T(A,\varphi)P_{jk}\|^p_{\mathcal{C}_p}\leq C'\|T(A,\varphi)\|^p_{\mathcal{C}_p}$$

since P'_{jk_1}, P'_{jk_2} and P_{jk_1}, P_{jk_2} are disjoint for $|k_1-k_2|\geq 3$.

But the construction of α_j, α'_j gives Re $\sum_j \theta_{j_0}(u) > 0$ for $u \in F$, therefore Re $\hat{\gamma} > 0$ on F (and $\hat{\varphi}$ is positive, compactly supported on $\mathbb{R}^d \setminus \{0\}$). Therefore, the left member of (10) can be used to estimate the Besov norms; we get

$$\|\varphi\|_{\dot{B}_{11}^{d/p}} \leq c \| T(A,\varphi)\|_{\mathcal{C}_p}$$

which is the statement of the theorem.

3. It is well known (for the case of commutators; see [3], [5], [14]) that the good "end point" at infinity is not $\dot{B}^{\,o}_{\infty\infty}$ but BMO. From results of Coifman and Meyer ([2]) a similar theorem holds in our context.

<u>Theorem 3.</u> Suppose A is C^∞ outside the origin, satisfies (2), and, moreover, $A(x,x)=0$. Then

$$\| T(A,\varphi)\| \leq c \cdot \|\varphi\|_{BMO}$$

<u>Remark.</u> The condition $A(x,x)=0$ should be compared to the conditions in the statement of theorem 1.

<u>Proof.</u> Let $f \in L^2(\mathbb{R}^d)$, and $g = T(A,\varphi) f$. Then

$$\hat{g}(\xi) = \int e^{ix\cdot\xi} \left(A(x,y)\hat{\varphi}(x-y) f(y)\, dy\right) dx =$$

$$= \int e^{i(\alpha+y)\cdot\xi} A(\alpha+y, y)\, \hat{\varphi}(\alpha) f(y)\, dy\, d\alpha =$$

$$= \int e^{i(\alpha+y)\cdot\xi} \sigma(\alpha, y)\, \hat{\varphi}(\alpha) f(y)\, dy\, d\alpha$$

where $\sigma(\alpha,y)=A(\alpha+y,y)$.

The conditions on A imply that σ fulfills the hypothesis of proposition 2 , ch.VI of [2]; the conclusion is that

$$\|\hat{g}\|_2 \leq C \cdot \|\varphi\|_{BMO} \cdot \|f\|_2$$

whence it follows that

$$\|T(A,\varphi)\| \leq C \cdot \|\varphi\|_{BMO}$$

R E F E R E N C E S

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