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Alexandru DIMCA

Let $(X,0)$ be an isolated singularity of complete intersection in $\underline{C}^m$ defined by the weighted homogeneous polynomials f_i of degree d_i with respect to the positive integer weights $wt(X_j)=w_j$ for $i=1,\dots,p$ and $j=1,\dots,m$.

Let $f:(X,0)\longrightarrow(\underline{C},0)$ be a function germ induced by a weighted homogeneous polynomial of degree d with respect to the weights $\underline{w}=(w_1,\dots,w_m)$ such that $(X_o,0)=(f^{-1}(0),0)$ is again an isolated singularity of complete intersection with $n=\dim X_o=\dim X-1\geq 1$. If $\bar{X}_o$ denotes the Milnor fiber of the singularity $(X_o,0)$, then there is a natural (complex) monodromy operator $h:H^n(\bar{X}_o,\underline{C})\longrightarrow H^n(\bar{X}_o,\underline{C})$ associated to the function f [8].

In the first part of this note we show that this monodromy operator is diagonalisable and compute its characteristic polynomial

$$\Delta(\lambda)=\det(\lambda.Id-h)$$

in terms of the weights $\underline{w}$ and the degrees $\underline{d}=(d_1,\dots,d_p)$ and d.

In the special case of Brieskorn-Pham singularities this result is due to Hamm [9], not to mention the case when X is smooth, treated already by Milnor and Orlik [11] and Brieskorn [2].

Our proof depends on the relation between the monodromy operator h and the Gauss-Manin connection of the function f (as suggested by an example in Looijenga [10], p.166) and on the knowledge of the Poincaré series of $\Omega^n_{X_o}/d\Omega^{n-1}_{X_o}$ computed by Greuel and Hamm [7].

In the second part we derive some topological consequences. Namely, there are two spaces naturally associated to the singularity (X,0): its link $K=X\cap S$, where S is the unit sphere in $\mathbb{C}^m$ and the quasi-smooth weighted complete intersection Y defined by the polynomials f_i in the weighted projective space $P(\underline{w})$ [4]. We show that the results in the first section allow one to compute the (middle) Betti numbers of K and Y in terms of $\underline{w},\underline{d}$. Equivalently, we determine the rank of the intersection form of the Milnor lattice of (X,0).

We also prove that all the quasi-smooth weighted complete intersections of the same type $(\underline{w},\underline{d})$ are homeomorphic.

1. The monodromy operator

Let $\mathcal{O}_K$ denote the $\mathbb{C}$-algebra of germs of holomorphic functions at the origin of $\mathbb{C}^k$, I_X the ideal generated by $f_1,\ldots,f_p$ in $\mathcal{O}_m$.

The weights $\underline{w}$ give rise to a filtration on the $\mathcal{O}_m$-module Ω^k of germs of holomorphic k-forms at the origin of $\mathbb{C}^m$, such that a monomial form

$$\varphi=x^a dx_{i_1}\wedge\ldots\wedge dx_{i_K}$$

had degree $\deg(\varphi)=\deg(x^a)+w_{i_1}+\ldots+w_{i_K}$, where $\deg(x^a)=$ $=a_1w_1+\ldots+a_mw_m$.

This filtration induces a filtration (compatible with the derivations) on the stalk at the origin of the sheaf of holomorphic k-forms relative f: $\Omega^k_f=\Omega^k/I_X.\Omega^k+df_1\wedge\Omega^{k-1}+\ldots+df_p\wedge\Omega^{k-1}+df\wedge\Omega^{k-1}$.

It is known that

(i) $\Omega_f^n/d\Omega_f^{n-1}$ is (via f) a free $\mathcal{O}_1$-module of rank $\mu=\mu(X_o)$, the Milnor number of $(X_o,0)$ [6], [10].

(ii) $A:=\Omega_f^n/d\Omega_f^{n-1}+(f)\Omega_f^n=\Omega^n_{X_o}/d\Omega^{n-1}_{X_o}$ is a μ-dimensional vector space over $\underline{C}$ with a natural grading $A=\bigoplus_{k\geq 0} A_k$ coming from the above filtration. Moreover, the Poincaré series of A

$$P(s)=\sum_{k\geq 0}(\dim A_k)s^k$$

is computed in [7] and in our case is given by

$$P(s)=\mathrm{res}_{t=0}\,\frac{t^{-m+p}}{1+t}\left[\prod_{i=1}^{m}\frac{1+ts^{w_i}}{1-s^{w_i}}\prod_{j=1}^{p+1}\frac{1-s^{d_j}}{1+ts^{d_j}}+t\right]$$

where $d_{p+1}=d$.

(iii) $H^n(\Omega_f^{\bullet})\subset\Omega_f^n/d\Omega_f^{n-1}\subset H^n(\Omega_f^{\bullet})[u^{-1}]$

where u denotes the coordinate on $\underline{C}$ ([10], Proposition 8.24).

Our main result is the following.

Theorem 1. The complex monodromy operator h is diagonalisable and its eigenvalues are d-roots of the unity. The multiplicity of the root $e^{2\pi ik/d}$ is

$$\sum_{j\equiv k\,(\mathrm{mod}\ d)}\dim A_j=d^{-1}\sum_{s^d=1}P(s)s^{-k}\ .$$

Proof. Chose a homogeneous ~~basis~~ $\varphi_1,\ldots,\varphi_\mu$ for A. Then by (i) they form a basis of $\Omega_f^n/d\Omega_f^{n-1}$ over $\mathcal{O}_1$.

The vector field $\eta=u\frac{d}{du}$ on $(\underline{C},0)$ can be lifted to the vector field $\xi = d^{-1} \sum_{K=1,m} w_K x_K \frac{\partial}{\partial x_K}$ on $(X,0)$.

The 1-parameter flow generated by ξ is obviously

$F_t(x) = (e^{w_1 t/d} x_1, \ldots, e^{w_n t/d} x_m)$.

The Lie derivative L_ξ is easy to compute for a homogeneous form φ

$$L_\xi(\varphi) = \lim_{t \to 0} \frac{F_t^*(\varphi) - \varphi}{t} = \deg(\varphi) d^{-1} \varphi.$$

Using this and (iii), it follows that a (multivalued) horizontal section of $R^n f_* \underline{C} \times \mathcal{O}_{\underline{C}}$ over $\underline{C} \setminus \{0\}$ is given by $u \longmapsto u^{-\deg(\varphi)/d} \varphi$.

Taking $\varphi = \varphi_1, \ldots, \varphi_\mu$ we get a frame in each fiber. Thus, if we put $u = \rho e^{2\pi i \theta}$ and let θ go from 0 to 1, then we find that the monodromy operator h multiplies φ_K with $e^{2\pi i \deg(\varphi_K)/d}$. This ends the proof of the Theorem. $\square$

<u>Example 2.</u> Consider the simple space curve singularity $X_o = U_7 : g_1 = x^2 + yz = 0$, $g_2 = xy + z^3 = 0$ corresponding to $\underline{w} = (4,5,3)$ and $\underline{d} = (8,9)$ [5].

Then a direct computation using the formula for P(s) given in (ii) shows that

$$P(s) = s^{14} + s^{13} + s^{11} + s^{10} + s^9 + s^8 + s^7.$$

Let $\Delta_i(\lambda)$ be the characteristic polynomial of the monodromy operator of the function germ $g_i : (\{g_j = 0\}, 0) \longrightarrow (\underline{C}, 0)$ for $i \neq j$. Then Theorem 1 gives us

$$\Delta_1(\lambda) = (\lambda^8 - 1)(\lambda + 1)^{-1}, \quad \Delta_2(\lambda) = (\lambda^9 - 1)(\lambda^2 + \lambda + 1)^{-1}.$$

2. The Betti numbers of K and Y

Recall from the introduction the definition of the spaces K and Y associated to the singularity $(X,0)$. Let K_o and Y_o be the similar spaces associated to the singularity $(X_o,0)$.

Note first that K is a smooth compact oriented $(2n+1)$-dimensional manifold which is $(n-1)$-connected [8]. In particular, we have to determine only the middle Betti numbers $b_n(K)=b_{n+1}(K)$. On the other hand, it is known that

$$b_n(K)=\mu(X)-\text{rank } S$$

where S is the intersection form of the Milnor lattice of $(X,0)$ [10]. Hence we will get a procedure to compute rank S in terms of $(\underline{w},\underline{d})$. One of the applications of the computation of rank S is the estimation of the number of singularities which may occur on a fiber in a deformation of $(X,0)$ [3].

As to the projective variety Y, it is a V-variety and hence a 2n-dimensional $\underline{Q}$-manifold [4].

The action of S^1 on S given by

$$t.(x_1,\ldots,x_m)=(t^{w_1}x_1,\ldots,t^{w_m}x_m)$$

leaves K invariant and $K/S^1=Y$. For a point $y=[x]=(x_1:\ldots:x_m)\in Y$ we define

$$w(y)=\text{g.c.d.}\{w_i;\ x_i\neq 0\}.$$

It follows easily that the isotropy group S^1_x of a point $x\in K$ is precisely the group of $w([x])$-roots of the unit. In particular, if $w(y)$ is constant for $y\in Y$, then Y is in a natural way a smooth manifold ([1], p.72). We will say in this case that Y is <u>strongly smooth</u>. Note that Y can be a smooth algebraic variety without being strongly smooth!

First we show that the topology of a quasi-smooth complete intersection depends only on its type.

<u>Proposition 3</u>. Two quasi-smooth complete intersections Y_1 and Y_2 of the same type $(\underline{w},\underline{d})$ are homeomorphic. Moreover, if one of them is strongly smooth then so is the other and they are diffeomorphic.

<u>Proof</u>. Let $P(\underline{w},d)$ be the vector space of homogeneous polynomials of degree d with respect to $\underline{w}$ and $P = P(\underline{w},d_1)\times\ldots\times P(\underline{w},d_p)$.

The set

$$B=\left\{(x,f)\in(\underline{C}^m\setminus\{0\})\times P;\ f=(f_1,\ldots,f_p),\ \operatorname{rk}\left(\frac{\partial f_i}{\partial x_j}(x)\right)<p\right\}$$

where $i=1,\ldots,p;\ j=1,\ldots,m$ is an algebraic subset in $(\underline{C}^m\setminus\{0\})\times P$. Let $U=P\setminus pr_2(B)$ and note that U is a Zariski open subset in P. Hence either $U=\emptyset$ or U is a dense connected subset, which is what we assume from now on.

The set

$$Z=\left\{(x,f)\in S\times U;\ \ f(x)=0\right\}$$

is a smooth manifold and the map induced by the second projection $\pi: Z\to U$ is a proper submersion. There is a S^1-action on Z coming from the action on S defined above.

Next we need the following.

<u>Lemma 4</u> (Equivariant Ehresmann fibration theorem).

Let $p:E \longrightarrow B$ be a proper submersion. If G is a compact Lie group acting on E such that all the orbits are contained in the fibers of p, then p is a locally trivial G-fibration.

[This means: for any $b \in B$ there is an open set $U \subset B$ with $b \in U$ and an equivariant diffeomorphism $f:p^{-1}(U) \longrightarrow U \times F$, where $F=p^{-1}(b)$ and G acts on $U \times F$ by the formula $g.(x,y)=(x,gy)$, such that $pr_1 \circ f=p$]

Proof. The usual proof of Ehresmann fibration theorem applies if we show that any vector field η on B can be lifted to an equivariant vector field ξ on E $\left(\text{i.e. } d_x L_g(\xi(x))=\xi(L_g(x)) \text{ for any } x \in E,\ g \in G, \text{ where } L_g(x)=g.x\right)$.
Let ξ_o be any lifting of η. Then

$$\xi(x)=\int_{g\in G}(d_x L_g)^{-1}(\xi_o(L_g(x))\,dg$$

where dg is a normalized invariant Haar measure on G, is an equivariant lifting of η. □

From this lemma we obtain that the fibers of π are equivariantly diffeomorphic and this ends the proof of proposition 3. □

Corollary 5. If a two-dimensional quasi-smooth complete intersection is nonsingular, then any other quasi-smooth complete intersection of the same type is also nonsingular.

Proof. Use the fact that the local fundamental group is a topological invariant and that the singular points on a normal surface are precisely those with nontrivial local fundamental group [12]. □

Now we give the basic result for the computation of the Betti numbers of K and Y. Let P^n be the usual projective n-space.

Proposition 6. (i) One has $b_k(Y)=b_k(P^n)$ for $k\neq n$ and $b_n(Y)=b_n(K)+b_n(P^n)$.

(ii) If Y is strongly smooth, then all the integer homology groups of Y are torsion free.

(iii) For $n\geq 2$ one has

$$b_n(K)+b_{n-1}(K_o)=\dim\ker(h-Id).$$

Proof. The Smith-Gysin exact sequence in homology with $\underline{C}$-coefficients [1] associated to the action of S^1 on K give the result (i).

When Y is strongly smooth we can use the Gysin sequence with $\underline{Z}$-coefficients and Poincaré duality over $\underline{Z}$ to get (ii).

Comparing the Smith-Gysin exact sequences associated to the S^1 actions on K and K_o, we find out that the morphism $H_n(K_o)\longrightarrow H_n(K)$ induced by inclusion is trivial for $n\geq 2$. The exact sequence of the pair (K,K_o) then gives

$$b_{n+1}(K)+b_n(K_o)=\dim H_{n+1}(K,K_o)$$

Finally, the exact sequence (1.8) in [8] shows that

$$\dim H_{n+1}(K,K_o)=\dim\ker(h-Id). \qquad \square$$

Since $\dim\ker(h-Id)$ is equal to the multiplicity of 1 as a root of $\Delta(\lambda)$, this number can be computed using Theorem 1. Then one can compute $b_n(K)$, $b_n(Y)$ by descending induction on $n=\dim Y$ as follows.

When $n=0$, K is a disjoint union of circles (and Y a finite set of points), one for each irreducible branch of the curve X. The number of branches of X is computable in terms of the type $(\underline{w},\underline{d})$ as shown by Giusti [5], Chap. II.

When $n=1$, Y is a smooth curve and there is a simple formula for its geometric genus $p_g(Y)$ in terms of $(\underline{w},\underline{d})$ [4](3.4.4). Hence $b_1(K)=b_1(Y)=2p_g(Y)$ is known in this case.

For $n>1$, there exists a weighted homogeneous function f of degree d, where d is any common multiple of $(w_1,\ldots,w_m)$ such that $X_o=X\cap f^{-1}(0)$ is an isolated singularity of complete intersection (see for instance [5], (2.4), Chap. II). Then, using (iii) we can compute $b_n(K)$ from the previously computed number $b_{n-1}(K_o)$.

When the defining equations f_i of the variety Y can be chosen such that the weighted complete intersections

$$Y_k : f_1(x)=\ldots=f_k(x)=0$$

are quasi-smooth for $k=1,\ldots,p$, then one can use (and sometimes is simpler) increasing induction on n to compute $b_n(K)$.

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