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ISSN 0250 3638

DIFFERENTIAL FUNCTION FIELDS AND MODULI OF ALGEBRAIC VARIETIES

by

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PREPRINT SERIES IN MATHEMATICS
No.49/1984

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August 1984

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PREFACE

The aim of these notes is to extend to several variables the "one variable theory" developed in: M.Matsuda, First order algebraic differential equations, Lecture Notes in Mathematics, Springer 1980. Methods from Matsuda's book do not apply to several variables, so our approach is entirely different. It is based on moduli theory of algebraic varieties, the main ingredient being the algebraicity of certain moduli spaces (cf. Mumford, Popp).

We made our exposition independent of Matsuda's book. Standard terminology of differential algebra is borrowed from Chapter 1 of Kolchin's book "Differential algebra and algebraic groups". A basic reference for moduli theory will be H.Popp, Moduli theory and classification theory of algebraic varieties, Lecture Notes in Mathematics, Springer 1977.

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CONTENTS

0. Introduction.

(0.1) The present work is devoted to the following:

<u>Problem.</u> Classify all differential extensions $K \subset L$ enjoying the property that L is the function field of a smooth projective K-variety V all of whose local rings $\mathcal{O}_{V,p}$ are differential subrings of L.

Recall from [17] Chapter 1 that a differential extension means a field extension $K \subset L$ together with commuting derivations $\delta_1, \ldots, \delta_r$ on L such that $\delta_j(K) \subset K$ for all $j=1,\ldots,r$. The condition that $\mathcal{O}_{V,p}$ is a differential subring of L means that $\delta_j(\mathcal{O}_{V,p}) \subset \mathcal{O}_{V,p}$ for all j; the condition that this should happen for all $p \in V$ is the differential algebraic transcription of Fuchs' condition expressing the absence of movable singularities of algebraic differential equations (see [23] [28]). Extensions $K \subset L$ as in the statement of the problem will be called in this paper Fuchs extensions. Put $n = \text{tr.deg}_K L$. For $n=r=1$ Fuchs extensions were classified in [23](pp.13,37,91); they were called there "differential algebraic function fields of one variable with no movable singularities". Methods in [23] do not generalize to the case $n \geqslant 2$; they are based on Weierstrass normal form of elliptic curves and on Weierstrass points on curves of genus $\geqslant 2$. In the present approach we introduce a method working for general $n \geqslant 1$ and $r \geqslant 1$ and we give (under very reasonable hypothesis on the ground constant field, see (1.1) and (1.3) below) a complete classification of Fuchs extensions for $n=2$ (Section 8) and significant information in the case $n=3$ (Section 9).

(0.2) The main idea of our approach is to bring Fuchs extensions into the setting of moduli of algebraic varieties and then to use algebraicity of certain coarse moduli spaces [29] and arguments of deformation theory.

(0.3) The interest for classification of Fuchs extensions goes back to Fuchs, Poincaré [28] and Painlevé [27](see also [23]) who investigated algebraic differential equations with no movable singularities (see [6],[12] for a modern approach via foliation theory). On the other hand Fuchs extensions are deeply related with some Galois-theoretic results of Kolchin [17], [18], [19]. We would like to make two remarks:

1) Classification theorems in [23] and in our paper are not simply abstract differential algebraic analogs of Poincaré's and Painlevé's analytic results; indeed our analytic corollaries are stronger than the previous results of the same kind. In particular our concept of "algebraic solution" is stronger that the one in [12] p.215 (see (6.3)).

2) At least for $\mathrm{tr.deg.}_K L \geq 2$, Kolchin's Galois theory requires a certain normality hypothesis on the extensions $K \subset L$ (called "strong normality" [17] p.393). This hypothesis is very natural for the Galois-theoretic setting but is very restrictive from the point of view of our problem. Therefore Kolchin's Galois

theory cannot be applied to classification of Fuchs extensions. On the other hand, as suggested by [23] classification of Fuchs extensions may be used to deduce Galois-theoretic results (see (5.11)).

(0.4) To illustrate the results of our paper let's briefly discuss in what follows some of our analytic corollaries.

Let $w_1,\ldots,w_r$ be coordinates on C^r (C=the complex field). For any domain $\mathcal{A}\subset C^r$ let $\mathrm{Mer}(\mathcal{A})$ be the differential field of meromorphic functions on $\mathcal{A}$ with derivations $\delta_j=\partial/\partial w_j$. If $\mathcal{A}^*$ is a subdomain of $\mathcal{A}$ and if we have differential extensions $C\subset K\subset L\subset \mathrm{Mer}(\mathcal{A})$ and $C\subset K^*\subset L^*\subset \mathrm{Mer}(\mathcal{A}^*)$ such that $K\subset K^*$ and $L\subset L^*$ are finite extensions then we say that $K\subset L$ has a finite embedding into $K^*\subset L^*$.

Let $K\subset L$ be a differential extension of differential subfields of $\mathrm{Mer}(\mathcal{A})$. Call it a Kolchin extension if there exists a lattice $\Lambda\subset C^n$ ($n=\mathrm{tr.deg.}_K L$), a subdomain $\mathcal{A}^*$ of $\mathcal{A}$ and functions $\beta_1,\ldots,\beta_n$

$Mer(\mathcal{A}^*)$ such that

1) C^n/Λ is an abelian variety.

2) $\beta_1,\dots,\beta_r$ are primitive over K (i.e. $\delta_i\beta_j \in K$ for all i,j).

3) L is generated over K by functions of the form $\wp(\beta_1,\dots,\beta_n)$ with $\wp$ abelian functions with respect to Λ (i.e. $\wp \in Mer(C^n)$, $\wp(w+a)=\wp(w)$ for all $a \in \Lambda$ and $w \in C^n \setminus \{\text{poles of } \wp\}$).

These extensions were considered by Kolchin in [19].

(0.5) Suppose now $K \subset L$ is a Fuchs extension of differential subfields of $Mer(\mathcal{A})$ of finite transcendence degree over C. Let $n, \varkappa, q, \alpha$ be the transcendence degree, the Kodaira dimension, the irregularity and the Albanese dimension of $K \subset L$ (see(1.2)).

Here is the effect of the classification in [23]:

<u>Theorem 1</u>. Suppose $n=1$ and $\varkappa \neq -\infty$. Then $\varkappa=0$ and $K \subset L$ has a finite embedding into a Kolchin extension $K^* \subset L^*$ such that $L^*=K^*(L)$.

Here are our analytic results:

Theorem 2. Suppose $n=2$ and $\varkappa \neq -\infty$. Then $\varkappa=0$ and $K \subset L$ has a finite embedding into a Kolchin extension $K^* \subset L^*$.

Theorem 3. Suppose $n=3$. Then $K \subset L$ has a finite embedding into a Kolchin extension $K^* \subset L^*$ provided we are in one of the following cases:

1) $\varkappa = 0$.

2) $\varkappa \geqslant 1$ and $\alpha \neq 3$.

Theorem 4. Suppose $n \geqslant 1$ and $\varkappa \neq -\infty$. Then $q \neq 0$.

Theorem 5. Suppose L is the field of rational functions of an abelian variety over K. Then $K \subset L$ has a finite embedding into a Kolchin extension $K^* \subset L^*$ such that $L^* = K^*(L)$.

Theorems 1-5 will be deduced in (6.2),(8.4),(9.5).

We would like to note that the case $K=C$ in the above theorems should be viewed as the trivial case.

(0.6) The paper is organized as follows. In Chapter I we discuss a series of geometric properties of models of differential extensions which will be needed in Chapter III. The main concept is that of a Fuchs model (1.5). In Chapter II we prove some classification results for Fuchs models which have "satisfactory" moduli-theoretic properties. Main results are Theorems (5.10),(5.11),(6.1). In Chapter III we use results proved in the first two chapters to perform classification of smooth projective Fuchs models of dimension ≤ 3. Main results are Theorems: (7.1),(8.2),(9.3).

Some appendices are also included; they contain ideas which should be useful for further developements.

Chapter I

1. The geometric setting. Fuchs models.

(1.1) We shall work over an algebraically closed ground field C of characteristic zero. All rings and morphisms of rings will be over C; same with schemes. All derivations will be C-derivations. Fibre products $X \times_{\text{Spec}(C)} Y$ will be denoted by $X \times Y$. Due to applications we have in view and in order to simplify the exposition and proofs we shall suppose throughout the paper that C is the complex field; many results hold however without this hypothesis. If X is a reduced irreducible scheme we denote by R(X) its field of rational functions. If A is an integral domain, R(A) will denote its quotient field.

(1.2) If K is a filed and $\overline{K}$ its algebraic closure, a K-variety V will mean a K-algebraic scheme V such that $\overline{V} = V \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$ is irreducible reduced. If in addition V is quasi-projective we shall say V is a model of $K \subset L$ where $L=R(V)$. As well known a field extension $K \subset L$ has a model if and only if it is finitely generated and K is algebraically closed in L; in this case $L \otimes_K \overline{K}$ is a field and we can define invariants $n, \varkappa, q, \alpha$ as follows:

n = tr.deg.$_K L$

$\varkappa$ = Kodaira dimension of any smooth projective model W of $L \otimes_K \overline{K}$

q = irregularity of any W as above

α = dimension of the image of $W \longrightarrow \mathrm{Alb}(W)$ with W as above ; call it Albanese dimension.

(1.3) Terminology of differential algebra will be freely borrowed from [17], Chapter ~~1. For~~ a differential field K we denote by K_0 the field of constants of K. By the hypothesis made at (1.1) K_0 will always contain C.

We shall suppose throughout Chapters I,II,III that all all differential fields under consideration (except those explicitely written as $Mer(\mathcal{A})$) have finite transcendence degree over C. This is a very reasonable hypothesis for function-theoretic applications; on the other hand we shall decribe in Appendix A a method which sometimes permits to reduce the case $tr.deg._C K = \infty$ to the case $tr.deg._C K < \infty$.

(1.4) Let $K \subset L$ be a differential extension with model V (so here the word "model" is used in its sense from algebraic geometry (1.2) and not in its differential sense [14]). For any (not necessarily closed) point $p \in V$ let $\mathcal{O}_{V,p}$ denote the local ring of V at p, m_p the maximal ideal of $\mathcal{O}_{V,p}$ and $R(p) = \mathcal{O}_{V,p}/m_p$ the residue field. We define (see also [5]) subsets of V as follows:

$$V_C \subset V_D \subset V_F \subset V.$$

$$V_F = \{p \in V;\quad \delta_j(\mathcal{O}_{V,p}) \subset \mathcal{O}_{V,p} \quad \text{for all } j\}$$

$$V_D = \{p \in V_F;\quad \delta_j(m_p) \subset m_p \quad \text{for all } j\}$$

$$V_C = \{p \in V_D;\quad (R(p))_o = C\}$$

Note that for $p \in V_D$, $R(p)$ has a natural structure of differential field so it makes sense to speak about $(R(p))_o$. Now it is easy to see that V_F is Zariski open in V (see also [5] Lemma 3). The structure of V_D and V_C is much more complicated (see [5] [32]). Note (although we won't use this fact) that V_D is the subiacent set of a ringed space which is the basic object in "differential algebraic geometry" (see [14] [4]); V_C plays then the role of a "set of rational points of V_D" (compare with (1.7) below). For a geometric interpretation of V_D and V_C in case $K=C$ see (2.9) below.

(1.5) In notations above we say that V is a Fuchs model if $V_F=V$; clearly for any V, V_F is a Fuchs model. A differential extension $K \subset L$ will be called a Fuchs extension if it has a smooth projective Fuchs * model.

(1.6) Suppose that in (1.4) K is a differential subfield of some $Mer(\mathcal{A})$ (this is what is called in [31] the "analytic case"). Define V_A to be the set of all differential extensions $f:R(p) \longrightarrow Mer(\mathcal{B})$ with $p \in V_C$, $\mathcal{B}$ a subdomain of $\mathcal{A}$, such that the composed map $K \longrightarrow R(p) \longrightarrow Mer(\mathcal{B})$ equals the natural indusion $K \subset Mer(\mathcal{A}) \subset Mer(\mathcal{B})$. Note that V_A does not identify with a subset of V.

(1.7) Now if we cover V_F in (1.6) with affine open subsets V_i, each of the V_i's may be written as $Spec(K\{y_1,\ldots,y_N\}/P)$ where $P=\{F_1,\ldots,F_M\}$ is a differential prime ideal in the ring of differential polynomials $K\{y_1,\ldots,y_N\}$. So to each V_i there is associated a system S_i of differential algebraic e-quations $F_1=F_2=\ldots=F_M=0$ with coefficients in K and V may be viewed as obtained by "glueing" these systems S_i. This glueing process is very classical and implicitely contained in Fuchs's "condition at infinity" [28]. To is immediate to see that if $p \in V_i$ then giving an element $f:R(p) \longrightarrow Mer(\mathcal{B})$ of V_A is equivalent to giving an "analytic zero" in Ritt's sense [31] p.166 of the system S_i. This interpretation shows in particular that V_A is always non-empty (use Ritt's theorem of zeroes [31] p.176)We also see that it is important to know the structure of the extensions $K \subset f(R(p))$ since these extensions are generated by the "components" of the analytic solutions of the systems S_i. After all we see that describing V_A means in fact to "integrate the systems S_i". Note that in the "abstract case" (i.e. when K is not necessarily a differential subfield of some $Mer(\mathcal{A})$) V_C should be viewed as the substitute for V_A.

(1.8) The birational classification problem we are dealing with in this paper is: describe all Fuchs extensions. It will turn out that it is more convenient to treat its biregular analog: describe all smooth projective Fuchs models. Furthermore one is interested by considerations of algebraic differential equations (1.7) to describe the sets V_A (or V_C) for such models. Finally, as we shall see in Chapter III, to obtain the structure of V_C one needs to understand the structure of W_D's for $\dim(W) < \dim(V)$.

So after all it will be convenient to make a systematic study of the geometry of V_F, V_D, V_C.

(1.9) Some more notational conventions. For any scheme X over a scheme S put $T_{X/S} = \underline{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{O}_X) =$ = sheaf of $\mathcal{O}_S$-derivations from $\mathcal{O}_X$ into itself. For any K-variety X and any K-rational point $y \in X$ denote by T_yX the Zariski tangent space of X at y. Note that if $K \subset L$ is a differential extension and V is a model then V is a Fuchs model if and only if $\delta_1, \ldots, \delta_r \in H^0(V, T_{V/C})$.

2. First properties.

We begin with some "well known" facts of differential algebra which we summarize in the following two lemmas:

(2.1) Lemma. Let K be a differential field and $K \subset L$ an algebraic extension. Then the derivations $\delta_1, \ldots, \delta_r$ on K uniquely extend to derivations on L. Furthermore if $K_0 = C$ then $L_0 = C$.

The proof is easy and we omit it.

(2.2) Lemma. Let A be a noetherian ring and δ a derivation on A. Then:

1) $\delta(nil(A)) \subset nil(A)$, where $nil(A)$ is the nilpotent radical of A.

2) For any minimal prime P of A, $\delta(P) \subset P$.

3) If A is integral and we still denote by δ the induced derivation on $R(A)$ then $\delta(A^{nor}) \subset A^{nor}$ where

A^{nor} is the normalization of A in $R(A)$.

4) If A is a C-algebra of finite type and $P \in \operatorname{Spec}(A)$ is such that A_P is not regular but A_Q is regular for any prime $Q \subset P$, $Q \neq P$ then $\delta(P) \subset P$.

Proof. 1) is [13] Lemma 1.8; 2) is proved for instance in [30]; 3) and 4) were proved by Seidemberg in [33] and [32] respectively.

(2.3) Lemma. Let $K \subset L$ be a differential extension with model V. Then:

1) If V is normal, $V \setminus V_F$ has pure codimension one in V.

2) If V is a Fuchs model and $p \in V_D$, then the closure W of p in V (with its reduced structure) is a Fuchs model of $K^* \subset R(p)$, where K^* is the algebraic closure of K in $R(p)$.

3) If $Z_1, Z_2, \ldots, Z_m$ are irreducible reduced subschemes of V whose generic points belong to V_D then

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all generic points of the irreducible components of $Z_1 \cap Z_2 \cap \ldots \cap Z_m$ belong to V_D.

4) If V is a Fuchs model then so is V^{nor}.

Proof. 1) and 2) are easy and we omit proofs. 3) follows from (2.2),2). Finally 4) follows from (2.2),3).

(2.4) Lemma. Consider differential extensions $K \subset L \subset L^*$ and $K \subset K^* \subset L^*$ and let V and V^* be models of $K \subset L$ and $K^* \subset L^*$ respectively. Suppose we have a morphism $f: V^* \longrightarrow V$ extending $\mathrm{Spec}(K^*) \longrightarrow \mathrm{Spec}(K)$ and whose residual extension at the generic points is $L \subset L^*$. Then:

1) If f is flat then $f(V^*_F) \subset V_F$.

Suppose in addition that V and V^* are Fuchs models. Then:

2) $f(V^*_D) \subset V_D$.

3) For any $p \in V_D$ the generic points of the irreducible components of $\overline{f^{-1}(p)}$ belong to V^*_D.

4) If $p^* \in V^*$, $p=f(p^*)$ and the extension $R(p) \longrightarrow R(p^*)$ is algebraic then $p^* \in V^*_C$ if and only if $p \in V_C$.

5) Suppose K is a differential subfield of some $Mer(\mathcal{A})$. Suppose furthermore that $g:R(p) \longrightarrow Mer(\mathcal{B})$ is an element of V_A, $p^* \in f^{-1}(p)$ and $R(p) \longrightarrow R(p^*)$ is finite. Then there is an element $h:R(p^*) \longrightarrow Mer(\mathcal{B}^*)$ of V^*_A with $\mathcal{B}^* \subset \mathcal{B}$ such that the composed morphism $R(p) \longrightarrow R(p^*) \xrightarrow{h} Mer(\mathcal{B}^*)$ equals to $R(p) \xrightarrow{g} Mer(\mathcal{B}) \subset \subset Mer(\mathcal{B}^*)$.

Proof. 1) follows from the equality $\mathcal{O}_{V^*,p^*} \cap L = \mathcal{O}_{V,p}$ where $p^* \in V^*$ and $p=f(p^*)$. 2) is trivial. 3) follows from (2.2),2). Statement 4) follows from (2.1). Finally 5) follows from Ritt's theorem of zeroes.

(2.5) Let V be a projective K-variety (no differential structure is assumed here). Call C-presentation of V any pair (f,j) with $f:X \longrightarrow S$ a flat projective morphism of C-varities and $j:R(S) \longrightarrow K$ a field extension such that V is K-isomorphic to $X \times_S \mathrm{Spec}(K)$. If (f',j') is another C-presentation $f':X' \longrightarrow S'$, $j':R(S') \longrightarrow K$ we say that (f',j') dominates (f,j) if there exists a dominant morphism $S' \longrightarrow S$ such that the composed extension $R(S) \longrightarrow R(S') \xrightarrow{j'} K$ equals j and we have $X' \cong X \times_S S'$ over S'.

Note that any projective K-variety V has a C-presentation. Indeed write $V = \mathrm{Proj}(K[T]/P)$ where P is a prime homogenous ideal in $K[T]$, $T=(T_0,\ldots,T_N)$ and let A be a finitely generated C-algebra contained in K such that $A[T]$ contains a system of generators of P. Let $f_1:X_1 \longrightarrow S_1$, $X_1 = \mathrm{Proj}(A[T]/P \cap A[T])$, $S_1 = \mathrm{Spec}(A)$. There exists a Zariski open set S in S_1 such that $f_1^{-1}(S) = X$ is flat over S; it is easy to see that $f:X \longrightarrow S$ and the inclusion $R(S) \subset K$ give a C-presentation of V. If V is smooth, then one can choose (f,j) such that in addition f is smooth (use [10] p.275). Note also that if $tr.deg._C K < \infty$ then we can choose (f,j) such that j is an algebraic extension.

(2.6) Lemma. Let $K \subset L$ be a differential extension, V a projective model and (f,j) a C-presentation of V. Then there exists a C-presentation (f',j') of V dominating (f,j), $f':X' \longrightarrow S'$ such that $R(S')$ is a differential subfield of K and $R(X')$ is a differential subfield of L (such a presentation will be called a differential presentation). If in addition V is a Fuchs model we may choose (f',j') such that X' and S' are Fuchs models.

Proof. Of course we may suppose $S=Spec(A)$. Let $Spec(B)$ be an open subset of X; suppose $A=C[a_1,\dots,a_m]$ $B=A[b_1,\dots,b_n]$. Since $\delta_j b_i \in L=R(B \otimes_A K)$, there exist elements $c_1,\dots,c_p \in K$ such that $\delta_j b_i \in R(B \otimes_A A[c_1,\dots,c_p])$ for all i,j. Now it is easy to see that the field

$R(B \otimes_A C\{a_1,\ldots,a_m,c_1,\ldots,c_p\})$, which by flatness of $A \subset B$ embeds in L, is a differential subfield of L. Suppose we know that $C\langle a_1,\ldots,a_m,c_1,\ldots,c_p\rangle/C$ has a model S_1 (this will follow from Lemma(2.7) below). Then let S' be the Zariski open set where the rational map $S_1 \dashrightarrow S$ is defined; it is clear that $f':X'= X \times_S S' \longrightarrow S'$ and $j':R(S')=R(S_1) \longrightarrow K$ give the desired C-presentation.

Now suppose in addition that V is a Fuchs model. By (2.4),1) $X' \times_{S'} \mathrm{Spec}(R(S'))$ is a Fuchs model so by properness of f', $f'(X' \setminus X'_F)$ will be a proper closed subset of S'. Put $S^*=S'_F \setminus f'(X' \setminus X'_F)$ and then $X^*=X' \times_{S'} S^* \longrightarrow S^*$ will give the desired C-presentation with X^* and S^* Fuchs models.

(2.7) <u>Lemma</u>. Let $K \subset L=K\langle x_1,\ldots,x_n\rangle$ be a differential extension of finite transcendence degree. Then $K \subset L$ is finitely generated as a (non-differential) field extension.

Proof. This lemma follows from the theory developed in [17], Chapter 2, but we give here for convenience a quite short proof. Clearly we may suppose $n=1$ so $L=K\langle x\rangle$. Let Θ be the set of operators $\theta=\delta_1^{a_1}\ldots\delta_r^{a_r}$. If $\eta=\delta_1^{b_1}\ldots\delta_r^{b_r}$ we write $\theta<\eta$ if either $a_1+\ldots+a_r<$ $<b_1+\ldots+b_r$ or there exists t such that $a_i=b_i$ for $i<t$ and $a_t<b_t$. We write $\theta\leq\eta$ if either $\theta<\eta$ or $\theta=\eta$. We write $\theta\subseteq\eta$ if $a_i\leq b_i$ for all i. For any $\theta\in\Theta$ put $L^\theta=K(\eta x;\ \eta<\theta)$ and construct inductively with respect to the order $\leq$ subsets Σ^θ of Θ in the following way: $\Sigma^0=\phi$ and if η is the successor of θ put $\Sigma^\eta=\Sigma^\theta$ if ηx is algebraic over L^θ and $\Sigma^\eta=\Sigma^\theta\cup\{\eta\}$ if ηx is transcendent over L^θ. Put $\Sigma=\bigcup_\theta\Sigma^\theta$, $\Lambda=\Theta\setminus\Sigma$ and let Λ_{min} be the set of minimal elements of Λ with respect to the order $\subseteq$. Now Λ_{min} is a finite set $\{\theta_1,\ldots,\theta_q\}$. Define $M=K(\theta x,\ \theta\in\Sigma)$; by construction M is purely transcendental over K so by our hypothesis M is finitely generated over K (in

(2.9) Lemma. Let L be a differential field and X a smooth Fuchs model of $C \subset L$.

1) Let Y be an irreducible reduced subscheme of X and $p \in X$ the generic point of Y. Then the following are equivalent:

a) $p \in X_D$.

b) There exists an open Zariski subset $Y_o \subset Y_{reg}$ such that for all closed points $y \in Y_o$ we have $\delta_1(y), \ldots, \delta_r(y) \in T_y Y$, where $\delta_j(y)$ is the image of $\delta_j \in H^o(X, T_{X/C})$ in $T_y X$ (one says [12] that Y is an integral subvariety of X for $\delta_1, \ldots, \delta_r$).

c) Same as b) with $Y_o = Y_{reg}$.

2) Let $f \in L \setminus C$ and look at f as a rational function $f: U \longrightarrow C$ for U a Zariski open subset in X. Suppose that for any closed point $x \in U$ there is an integral subvariety Y of X for $\delta_1, \ldots, \delta_r$ such that $x \in Y_{reg} \subset f^{-1}(f(x))$. Then $\delta_j f = 0$ for all j.

Proof. 1) Clearly c) implies b). Suppose b) holds.

and let's prove that $p \in X_D$. Choose any $y_o \in Y_o$, put $A = \mathcal{O}_{X,y_o}$, $B = \mathcal{O}^{an}_{X,y_o}$ and choose holomorphic coordinates $w_1,\ldots,w_n$ around y_o such that Y is given around y_o by $w_1 = \ldots = w_r = 0$. Let $P \in \mathrm{Spec}(A)$ correspond to Y (hence $PB = (w_1,\ldots,w_r)B$). Write $\delta_j = a_{1j}(w)\partial/\partial w_1 + \ldots + a_{nj}(w)\partial/\partial w_n$ with $a_{ij}(w)$ holomorphic functions $\in B$. Now b) implies $a_{1j},\ldots,a_{rj} \in (w_1,\ldots,w_r)B$ hence $\delta_j(P) \subset A \cap (w_1,\ldots,w_r)B = P$ and we are done. Finally the implication a) $\Rightarrow$ c) may be done in the same way.

2) Let $x_o \in U$ such that there exist holomorphic coordinates $w_1,\ldots,w_n$ on U around x_o such that f identifies with the projection $(w_1,\ldots,w_n) \mapsto w_1$. Now if a_{ij} are as in the proof of 1) we have to prove that $a_{1j} = 0$ for all j, i.e. that $\delta_j(x) \in T_x f^{-1}f(x)$ for all x near x_o. But by our assumption plus the equivalence in 1) we get $\delta_j(x) \in T_x Y_{reg} \subset T_x f^{-1}f(x)$ and we are done.

3. Coverings, blowing-ups, automorphisms, discriminants.

(3.1) Proposition. Let $K \subset L$ be a differential extension with Fuchs model V and let $f:W \longrightarrow V$ be an étale morphism with W irreducible reduced. Then W is a Fuchs model of $K^* \subset R(W)$ where K^* is the algebraic closure of K in $R(W)$

Proof. By [24] p.26 for any $y \in W$ there exist open neighbourhoods $\mathrm{Spec}(B)$ and $\mathrm{Spec}(A)$ of y and $f(y)$ such that $f(\mathrm{Spec}(B)) \subset \mathrm{Spec}(A)$ and $B = (A[T]/(F))_b$ with $F \in A[T]$ monic, $b \in A[T]/(F)$ and $F'(T)$ a unit in B. Let t be the image of T in B. From $F(t)=0$ we get $F_j(t)+F'(t)(\delta_j t)=0$ where $F_j(T)$ is obtained from $F(T)$ applying δ_j to each coefficient. Since $\delta_j(A) \subset A$ we get $\delta_j t \in B$ hence $\delta_j(B) \subset B$ and we are done.

(3.2) Proposition. Let $K \subset L$ be a differential extension, V a normal Fuchs model, $\mathcal{L}\in Pic(V)$, $s\in$

$\in H^0(V,\mathcal{L}^k)$, $V'=Spec(\mathcal{O}_V\oplus\mathcal{L}^{-1}\oplus\cdots\oplus\mathcal{L}^{-k+1})\longrightarrow V$

be the cyclic covering defined by s and V'' any component of $(V'_{red})^{nor}$. Suppose that the generic points of the irreducible components of the zero locus Z(s) of s belong to V_D. Then V'' is a Fuchs model of $K^*\subset R(V'')$ where K^* is the algebraic closure of K in R(V'').

Proof. The problem is local so we may suppose V= =Spec(A), $s\in A$, V'=Spec(B), $B=A[t]=A[T]/(T^k-s)$. First we claim that $s^{-1}\delta_j s\in A$. Indeed for a fixed $p\in\in Spec(A)$ of height one, denote by x the parameter of A_p and write $s=ux^n$, u invertible in A_p. By hypothesis if $n\geqslant 1$, $x^{-1}\delta_j x\in A_p$. So for any $n\geqslant 0$, $s^{-1}\delta_j s=u^{-1}\delta_j u+nx^{-1}\delta_j x\in A_p$. By normality of A we get $s^{-1}\delta_j s\in A$. Now $\delta_1,\dots,\delta_r$ extend to $A[T]$ by the rule $\delta_j T=k^{-1}(s^{-1}\delta_j s)T$. One immediately checks that $\delta_j(T^k-s)\in(T^k-s)A[T]$: $sA[T]=(T^k-s)A[T]$ hence δ_j

descend to B. Now we may conclude by (2.2),1),2),3).

(3.3) Proposition. Let $K \subset L$ be a differential extension with Fuchs model V. Let Z be an irreducible reduced closed subscheme in V, let $p \in V$ be the generic point of Z and $W \longrightarrow V$ the blowing up of V with respect to the ideal sheaf of Z.

1) If $p \in V_D$ then W is a Fuchs model.

2) If W is a Fuchs model and $p \in V_{reg}$ is a point of codimension $\geqslant 2$ then $p \in V_D$.

Proof. 1) We may suppose $V = Spec(A)$, $Z = Spec(A/p)$, $p \in Spec(A)$, $p = (x_1, \ldots, x_t)$. W will be covered by open subsets $Spec(B_k)$, $B_k = A[x_1/x_k, \ldots, x_t/x_k] \subset R(A)$. By hypothesis we have $\delta_j x_i = \sum_{s=1}^{t} a_{ijs} x_s$ with $a_{ijs} \in A$. We get

$$\delta_j(x_i/x_k) = \sum_{s=1}^{t} (a_{ijs}(x_s/x_k) - a_{kjs}(x_s/x_k)(x_i/x_k)) \in B_k$$

so $\delta_j(B_k) \subset B_k$ and we are done.

2) Put $T = Spec(\mathcal{O}_{V,p})$; then $W \times_V T$ is the blowing

up of T at its closed point m_p. Let $y_1,\dots,y_s$ be a regular system of parameters for $\mathcal{O}_{V,p}$. Clearly the coordinate rings of the affine pieces of $W\times_V T$ are closed under δ_j's so in particular

$$(\delta_i y_j)/y_k - (y_j\delta_i y_k)/y_k^2 = \delta_i(y_j/y_k) = F(y_1/y_k,\dots,y_s/y_k)$$

where F is polynomial with coefficients in $\mathcal{O}_{V,p}$. Multiplying the relation above with a suitable power of y_k and using the fact that the graded ring $G_{m_p}(\mathcal{O}_{V,p})$ is a polynomial ring over $R(p)$ in the indeterminates $\hat{y}_1,\dots,\hat{y}_s$ where $\hat{y}_i = y_i \mod m_p^2$, we get that the image of $\delta_i y_j$ in $R(p)$ is zero so $\delta_i(m_p)\subset m_p$.

(3.4) <u>Proposition.</u> Let K be algebraically closed, $K\subset L$ a differential extension and V a Fuchs model of dimension $n\leq 2$. Then there exists a projective birational morphism $W\longrightarrow V$ with W a smooth Fuchs model.

Proof. Case $n=1$ follows from (2.3),4). Suppose n=2. By [22] there exists a sequence of morphisms

$$W=V_n \xrightarrow{f_n} V_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_o=V$$

such that each f_k is the blowing up the singular locus of $(V_{k-1})^{nor}$. We conclude by induction using (2.3),4), (2.8) and (3.3),1).

(3.5) It would be important to dispose of an analog of (3.4) for general n. It would follow in this case that given a smooth projective Fuchs model V over an algebraically closed K, then any residual extension $K \subset R(p)$ $(p \in V_D)$ is a Fuchs extension. Anyway the above statement holds for $\dim(V) \leq 3$ (by (2.3),2) and (3.4)).

(3.6) <u>Proposition</u>. Let $K \subset L$ be a differential extension with model V a simple abelian variety and K algebraically closed. If the group of differential K-automorphisms of L is infinite, then V is a Fuchs model.

Proof. Consider the reduced divisor E having support $V \setminus V_F$. By [20] p.81 and 85, E is either zero or ample; suppose E is ample. Any element λ of the group $\mathcal{G}$ of differential K-automorphisms of L induces a biregular automorphism of V which clearly invariates V_F, hence $\lambda^* E = E$. Consider the exact sequence

$$0 \longrightarrow T \longrightarrow \mathrm{Aut}(V) \longrightarrow \mathrm{Aut}_o(V) \longrightarrow 1$$

where T is the group of translations, $\mathrm{Aut}(V)$ is the group of all biregular maps from V to V and $\mathrm{Aut}_o(V)$ is the group of invertiblre homomorphisms from V to V. Let $\mathrm{Aut}(V,E) = \{\sigma \in \mathrm{Aut}(V);\ \sigma^* E \sim E\}$ (here $\sim$ denotes the linear equivalence). We get an exact sequence

$$0 \longrightarrow T \cap \mathrm{Aut}(V,E) \longrightarrow \mathrm{Aut}(V,E) \longrightarrow G \longrightarrow 1$$

with $G \subset \mathrm{Aut}_o(V)$. Now $T \cap \mathrm{Aut}(V,E)$ is finite [20] p.85. Furthermore $\mathrm{Aut}_o(V)$ is countable [20] p.184, so G is at most countable. But $\mathrm{Aut}(V,E)$ is in one-to one correspondence with the K-rational points of an algebraic subgroup of some PGL(n). Since K is uncountable Aut(V,E) must be finite. But $\mathcal{G} \subset$ Aut(V,E), contradiction, and the proposition is proved.

To state the following proposition recall that if $f:V \longrightarrow W$ is a proper morphism of K-varieties (assume for convenience K algebraically closed) and if $V_1 \subset V$ is the open set where f is flat([10] p.266), $W_o = W \setminus f(V \setminus V_1)$, $V_o = f^{-1}(W_o)$ then the set D of points $y \in W$ such that $f^{-1}(y)$ is not smooth is closed in W_o [10] p.275 . The closed subset $(W \setminus W_o) \cup \bar{D}$ of W where $\bar{D}$ is the closure of D in W will be called the discriminant of f. Note that if V and W are smooth then any component of $W \setminus W_o$ has codimension $\geqslant 2$ in W (use [10] p.256 and 276).

(3.7) Proposition. Let $K \subset L$ be a differential extension with smooth projective Fuchs model V and K algebraically closed. Let $f:V \longrightarrow W$ be a dominant morphism of projective K-varieties.

1) If $f_{*}\mathcal{O}_V = \mathcal{O}_W$ then $R(W)$ is a differential subfield of L and W is a Fuchs model.

2) If $R(W)$ is a differential subfield of L and W is a Fuchs model then the generic points of the codimension one components of the discriminant of f belong to W_D.

Proof.1) Consider the canonical morphism $f^{*}\Omega_{W/C} \to \Omega_{V/C}$. Taking $Hom_V(-, \mathcal{O}_V)$ we get a morphism

$$H^0(T_{V/C}) \longrightarrow Hom_V(f^{*}\Omega_{W/C}, \mathcal{O}_V) = Hom_W(\Omega_{W/C}, f_{*}\mathcal{O}_V) = H^0(T_{W/C})$$

which already proves 1) by the remark made at (1.9).

Let's prove 2). Using (2.6) we may construct projective morphisms of C-varieties $F:X \longrightarrow Z$, $H:Z \longrightarrow S$, $G:X \longrightarrow S$ with $G=H\circ F$ and an embedding $j:R(S) \longrightarrow K$ such that:

a) (G,j) and (H,j) are differential presentations of V and W respectively, and j is algebraic.

b) S,X,Z are Fuchs models.

c) $X \times_S \operatorname{Spec}(K) \longrightarrow Z \times_S \operatorname{Spec}(K)$ identifies with $i_*($

d) G is smooth and X, S are smooth.

Let $p \in W$ be as in 2). If $p \in W_{sing}$ then we are done by (2.8). Suppose $p \in W_{reg}$. Let q be the image of p under the projection $W \longrightarrow Z$; then $\mathcal{O}_{Z,q}$ is regular. Let Y be the closure of q in Z; we have $Y \cap Z_{reg} \neq \emptyset$. Since the set of points $y \in Z_{reg}$ above which F is not flat has codimension ≥ 2 in Z_{reg}, there exists a Zariski open subset $Z_o \subset Z_{reg}$ above which F has fibres of pure dimension $d = \dim(X) - \dim(Z)$ and such that $Z_o \cap Y = Y_o \neq \emptyset$ and $Z_o \longrightarrow S$ is smooth (use [10] p.272). Put $X_o = F^{-1}(Z_o)$ and let $\Delta_o \subset X_o$ be the Zariski closed subset of closed points $x \in X_o$ where the tangent map $T_xF : T_xX \longrightarrow T_{F(x)}Z$ is not surjective (since the fibres have the right dimension, this is precisely the set of all $x \in X$ which are singular points of the scheme-theoretic fibre $F^{-1}F(x)$). Since $V \times_W \operatorname{Spec}(R(p))$ is not smooth it follows that $X \times_Z \operatorname{Spec}(R(Y))$ is not smooth hence $Y_o \subset F(\Delta_o)$. Since Y_o has codimension one in Z_o and $F(\Delta_o) \neq Z_o$, Y_o is dominated by some irreducible component of Δ_o; call it Δ_1. There exists a locally closed subvariety U of Δ_1 such that $U \longrightarrow Y_o$ is étale and for any point y in the image of $U \longrightarrow Y_o$ the morphism $Y \longrightarrow S$ is smooth at y. Since j is algebraic it will be sufficient(by (2.4),3))to see that $q \in Z_D$ hence (by (2.9)) that for any closed point $x \in U$ we have $(T_xF)(T_xX) \subset T_yY$ where $y=F(x)$. Now $X_x=G^{-1}G(x)$ is a smooth variety, $Z_x=H^{-1}H(y)$ is smooth at y and the fibre Y_x of $Y \longrightarrow S$ through y is smooth at y. Let $F_x:X_x \longrightarrow Z_x$ the morphism induced by F and U_x the fibre of $U \longrightarrow S$ through x.

Claim 1. $Im(T_xF_x:T_xX_x \longrightarrow T_yZ_x)=T_yY_x$.

Indeed T_yY_x is contained in $Im(T_xF_x)$ because $U_x \longrightarrow Y_x$ is étale.On the other hand T_yY_x has codimension one in T_yZ_x and T_xF_x is not surjective (x being singular on $F^{-1}F(x)$) hence the claim follows.

Claim 2. $T_xX = T_xU + T_xX_x$

Indeed computing dimensions of the vector spaces involved we see that it is sufficient to show $\dim(T_xU \cap T_xX_x) \leq$ $\leq \dim(Z_x)-1=t-1$. But if we could find linearly independent vectors $e_1,\dots,e_t$ in $T_xU \cap T_xX_x$ then since T_xF is injective on T_xU we would get linearly independent vectors $T_xF(e_i)$, $1 \leq i \leq t$ inside T_xZ_x which would imply that T_xF_x is surjective, contradiction. The two claims put together close the proof of the Proposition.

Chapter II

4. Models with C-moduli and models defined over C.

We begin with some non-differential preliminaries

(4.1) Let $f:X \longrightarrow S$ be a projective morphism of C-varieties. For any closed point $y \in S(C)$ put $X_y = f^{-1}(y)$ and $S_y = \{z \in S(C);\ X_y \text{ and } X_z \text{ are C-isomorphic}\}$. Since the functor $\Psi: \{\text{noetherian S-schemes}\} \longrightarrow \{\text{sets}\}$ $\Psi(T) = \mathrm{Isom}_T(X \times_S T, X_y \times T)$ is represented by a countable disjoint union of C-algebraic schemes [7] it follows by Chevalley's constructibility theorem that S_y is a union of at most countably many locally closed subsets of S.

(4.2) In notations above we say that a morphism $\varphi: S \longrightarrow M$ with M a C-algebraic space of finite type (see [15]) is a moduli map for f if for all $y \in S(C)$, S_y is a union of at most countably many fibres of φ.

Note that if $T \longrightarrow S$ is any morphism of C-varieties and if $X \times_S T$ is a variety then $T \to S \to M$ is a moduli map for $X \times_S T \longrightarrow T$.

(4.3) Let V be a smooth projective K-variety. We make two definitions:

1) We say V has C-moduli if there exists a C-presentation (f,j) of V (see(2.5)) such that f has a moduli map (see(4.2)).

2) We say V is defined over C if V is K-isomorphic to $Z \times \mathrm{Spec}(K)$ for some smooth projective C-variety Z.

Clearly, if V is defined over C then V has C-moduli. It is also easy to see that if $\mathrm{tr.deg.}_C K=1$ then any smooth projective K-variety has C-moduli.

As a corollary of the work of Popp [23] and Mumford [25] we prove the following:

(4.4) Theorem. Let V be a smooth projective K-variety of one of the following types:

1) a variety with ample canonical sheaf $\omega_{V/K}$.

2) a surface with $\varkappa=2$ and $\omega^{m}_{V/K}$ spanned by global sections, for $m \gg 0$.

3) a variety with $q=0$ and $\varkappa \neq -\infty$.

4) an abelian variety.

5) a hypersurface in projective space $\mathbb{P}^N_K$.

Then V has C-moduli.

Proof. Consider first that case when V is of type i, $1 \leq i \leq 4$ and take a C-presentation (f,j), $f:X \longrightarrow S$. If $i=4$ choose (f,j) such that in addition f has a section. Shrinking S we may suppose f is smooth and for any $y \in S(C)$, X_y is of type i; for $i=1$ this follows from [8], Proposition 4.6.7; if $i=2,3$ this follows from semicontinuity theorem and Grauert theorem [10] p.288; If $i=4$ we get more than that, namely $X \longrightarrow S$ is an abelian scheme [25] p.124. In cases $i=3,4$ let's fix a polarization of X/S. By the work of Popp [29] pp.45-53 there exists a C-algebraic space of finite type M which is coarse moduli space for varieties of type $i=1,2$ (respectively for polarized varieties of type $i=3,4$); of course we fixed a Hilbert polynomial in cases $i=1,3,4$ and we fixed invariants K^2 and χ in case $i=2$. Then the morphism $S \longrightarrow M$ defined by the family f is a moduli map in the sense of (4.2); indeed this is clear for $i=1,2$; for $i=3,4$ this follows from the fact that there are at most countably many polarizations on a fixed smooth projective C-variety.

Suppose now V is a hypersurface of degree d in $\mathbb{P}^N_K$. Cases $d=1,2$ are trivial. The case $d \geq N+1$ follows from cases $i=1,3$. So we may suppose $3 \leq d \leq N$. If $V=\mathrm{Proj}(K[T]/(F))$, $T=(T_0,\ldots,T_N)$ then one can take a C-presentation of the form $f:X \longrightarrow S$ with $S=\mathrm{Spec}(A)$, $A \subset K$ and $X=\mathrm{Proj}(A[T]/(F))$; we may also suppose f smooth. Consider the composed morphism $\varphi: S \longrightarrow H \longrightarrow M$ where $S \longrightarrow H$ is the natural morphism to the open set H of $|\mathcal{O}_{\mathbb{P}^N_C}(d)|$ of all smooth hypersurfaces in $\mathbb{P}^N_C$ of degree d and $H \longrightarrow M=H/PGL(N)$ is the geometric quotient map (it exists by [25]p. 79). It is easy to see that $\varphi: S \longrightarrow M$ is a moduli map for f (use the fact that the anticanonical bundle is a multiple of the hyperplane section).

(4.5) We would like to note that one can prove V

has C-moduli for various other types of varieties V such as : double coverings of projective spaces (use deformation of the branch locus), intersections of two quadrics in $\mathbb{P}^{2N+1}$ (use deformation of intermediate jacobians and Torelli for such varieties), certain ruled varieties (use deformation of vector bundles, see also (8.5)), a.s.o.

Now comes the key argument of our approach:

(4.6) Theorem. Let $K \subset L$ be a differential extension with smooth projective Fuchs model V and K algebraically closed. Suppose $K_0 = C$ and V has C-moduli. Then V is defined over C.

Proof. Let (f',j') be a C-presentation of V such that f has a moduli map. We may suppose of course that f' is smooth and has connected fibres. By (2.6), (f',j') is dominated by a differential C-presentation (f,j), $f: X \longrightarrow S$ with X and S smooth Fuchs models. By (4.2), f will still have a moduli map $\varphi: S \longrightarrow M$ with M a C-algebraic space of finite type.

Claim. We may suppose in addition that M is an affine smooth C-variety, S is affine and $\varphi : S \longrightarrow M$ is smooth and dominant.

Indeed by [29] p.21 there is an étale covering $N \dashrightarrow M$ with N an algebraic scheme over C. Let S_o be a connected affine scheme which is an open subspace of $S \times_M N$ [15] p.25 and put $X_o = X \times_S S_o$. Then $S_o \longrightarrow S$ and $X_o \longrightarrow X$ are étale hence X_o and S_o are smooth C-varieties and are Fuchs models by (3.1). Since K is algebraically closed, the embedding $j: R(S) \longrightarrow K$ extends to an embedding $j_o: R(S_o) \longrightarrow K$ hence (f_o, j_o), $f_o: X_o \longrightarrow S_o$ will still be a differential C-presentation. Furthermore the morphism $S_o \dashrightarrow N$ is a moduli map for f_o because $N \dashrightarrow M$ has finite fibres. So replacing f by f_o and M by N we reduced ourselves to the case when M is a C-algebraic scheme. Now the claim follows easily.

Two cases may occur.

Case 1) M is a point.

Let $g: U \longrightarrow S$ be the object representing the functor Ψ from (4.1); $U = \coprod_{n=1} U_n$ with U_n C-algebraic schemes. Since in this case g is surjective, one gets by Baire's theorem that there is at least a component U_n dominating S. Choose an integral subscheme S_1 in U_n generically finite over S. It follows that $X_1 = X \times_S S_1 \simeq X_y \times S_1$ and since $R(S) \longrightarrow K$ extends to an embedding $R(S_1) \longrightarrow K$ we get that $V \simeq X \times_S \operatorname{Spec}(K) \simeq$ $\simeq X_1 \times_{S_1} \operatorname{Spec}(K) \simeq X_y \times \operatorname{Spec}(K)$ and we are done.

Case 2) $\dim(M) \geqslant 1$.

Consider the exact sequences of $\mathcal{O}_S$-modules

$$T_{S/M} \xrightarrow{a} T_{S/C} \xrightarrow{b} f^{*}T_{M/C}$$

$$f_{*}(T_{X/C}) \xrightarrow{c} T_{S/C} \xrightarrow{d} R^1 f_{*}(T_{X/S})$$

Replacing S by some Zariski open subset we may suppose that all $\mathcal{O}_S$-modules appearing are free, all kernels and cokernels of a, b, c, d are free and that f_* and $R^1 f_*$ commute with base change $\operatorname{Spec}(R(y)) \longrightarrow S$ for $y \in S(C)$. Let η be the generic point of S and $a_\eta, b_\eta, c_\eta, d_\eta$ the corresponding morphisms of $R(S)$-vector spaces.

Claim. $\ker(d_\eta) \subset \ker(b_\eta)$.

Suppose we proved the claim. Then because S and X are Fuchs models, $\delta_1, \ldots, \delta_r$ induce vector fields $v_1, \ldots, v_r \in H^0(S, f_* T_{X/C})$ and $u_1, \ldots, u_r \in H^0(S, T_{S/C})$. Let $v_{i\eta}$ and $u_{i\eta}$ be the corresponding elements in $(f_*(T_{X/S}))_\eta$ and $(T_{S/C})_\eta$. Then $c_\eta(v_{i\eta}) = u_{i\eta}$ hence $d_\eta(u_{i\eta})=0$ hence by our claim $b_\eta(u_{i\eta})=0$ consequently $u_{i\eta} \in \mathrm{Im}(a_\eta)$, in other words $\delta_1, \ldots, \delta_r$ vanish on $R(M)$. But this contradicts the fact that $K_0=C$. So the proof will be concluded if we prove the claim. Suppose on the contrary that the claim fails. Then after replacing S by some open Zariski subset of it we may find a line bundle $Q \subset \ker(d)$ such that $Q \cap \ker(b)=0$. There exists a closed point $y \in S(C)$ such that $(Q \otimes R(y)) \cap \ker(T_y\varphi : T_yS \longrightarrow T_{\varphi(y)}M)=0$. Now any line bundle in $T_{S/C}$ is integrable hence by Frobenius [12] p.200 there exists a complex neighbourhood $\mathcal{S}$ of y in S and a smooth analytic closed subspace Δ of $\mathcal{S}$ of dimension one such that $y \in \Delta$ and such that the analytic tangent bundle T_Δ equals to the restriction $Q|\Delta$. Now for any $z \in \Delta$ the Kodaira-Spencer map $T_z\Delta \longrightarrow H^1(X_z, T_{X_z}/C)$ is the zero map because $Q \subset \ker(c)$ so by [16],6.2 the family $X \times_S \Delta \longrightarrow \Delta$ is analytically trivial, in particular $\Delta \subset S_y = \varphi^{-1}(M_y)$ with $M_y \subset M$ at most countable. On the other hand, from the way we chose y, the tangent map $T_y\Delta \longrightarrow T_{\varphi(y)}M$ is injective hence $\Delta \longrightarrow M$ is locally injective around y, in particular $\varphi(\Delta)$ is uncountable, contradiction. The theorem is proved.

(4.7) Remark. In the proof of (4.6) we used in an essential way the fact that M from definition (4.2) is an algebraic space and not only an analytic space. Consequently the results of Popp [29] are essential in our approach.

(4.8) We close this section by showing that smooth projective Fuchs models defined over C may be described in a quite simple manner. Let $K \subset L$ be a differential extension (K being algebraically closed) with smooth projective Fuchs model V defined over C. Fix an isomorphism $\beta : V \xrightarrow{\sim} Z \times \mathrm{Spec}(K)$, Z a smooth projective C-variety. Fix also a basis $\theta_1, \ldots, \theta_g$ of the C-vector space $H^0(Z, T_{Z/C})$ and consider the "structure constants" $c_{ijk} \in C$ defined by

$$(*) \quad [\theta_i, \theta_j] = \sum_{k=1}^{g} c_{ijk}\theta_k$$

Regard R(Z) and K embedded in L=R(V) via the projections $p_1 : V \xrightarrow{\beta} Z \times \mathrm{Spec}(K) \longrightarrow Z$ and $p_2 : V \xrightarrow{\beta} Z \times \mathrm{Spec}(K) \longrightarrow \mathrm{Spec}(K)$. Now we have

$$H^0(V, T_{V/C}) = \mathrm{Hom}_V(\Omega_{V/C}, \mathcal{O}_V) = \mathrm{Hom}_V(p_1^*\Omega_{Z/C} \oplus p_2^*\Omega_{K/C}, \mathcal{O}_V) =$$

$$= \mathrm{Hom}_Z(\Omega_{Z/C}, \mathcal{O}_Z \otimes_C K) \oplus \mathrm{Hom}_K(\Omega_{K/C}, K) =$$

$$= (H^0(Z, T_{Z/C}) \otimes_C K) \oplus T_{K/C}$$

(we used of course the fact that $\Omega_{Z/C}$ is locally free and H^0 commutes with flat base change). Consequently there exists a unique matrix (a_{jk}), $a_{jk} \in K$, $1 \leq j \leq r$, $1 \leq k \leq g$ such that:

$$(**) \qquad \delta_j x = \sum_{k=1}^{g} a_{jk}\theta_k x \quad \text{for all} \quad x \in R(Z) \quad \text{and all } j.$$

As an immediate consequence of the commutation of δ_j's we get:

$$(***) \qquad \delta_j a_{pk} - \delta_p a_{jk} + \sum_{q,m=1}^{g} a_{jq} a_{pm} c_{qmk} = 0, \text{for all } j,p,k.$$

(see [17] p.421 for an analog computation). Clearly (a_{jk}) depends on the choice of β and $\theta_1,\ldots,\theta_g$. Conversely, if one is given an algebraically closed differential field K, a smooth projective C-variety Z a basis $\theta_1,\ldots,\theta_g$ of $H^0(Z,T_{Z/C})$, constants c_{ijk} satisfying (*) and a matrix (a_{jk}), $a_{jk} \in K$ satisfying (***) then one can construct a smooth projective Fuchs model $V = Z \times \mathrm{Spec}(K)$ using formulae (**).

(4.9) The conclusion of this section should be that at least in the "abstract case" classification of smóoth projective Fuchs models is done once we know they are defined over C, hence by Theorem (4.6) once we know they have C-moduli. So the following (non-differential) question becomes important: does any smooth projective K-variety (K algebraically closed and containing as usual C) have C-moduli ? Due to the discussion at (4.8) it would be very interesting to know this at least for varieties V with $H^0(T_{V/K})=0$; note that there exist structure theorems for varieties V with $H^0(T_{V/K}) \neq 0$ [21] which could be brought into the setting of Fuchs models, provided of course C-moduli exist.

5. Clairaut models and Poincaré models.

(5.1) Let $K \subset L$ be a differential extension with K algebraically closed and with smooth projective Fuchs model V.

1) Say that V is a Clairaut model if there exists a K-isomorphism $\beta: V \xrightarrow{\sim} Z \times \mathrm{Spec}(K)$ with Z a smooth projective C-variety such that (regarding $R(Z)$ embedded in L via β as in (4.8)) we have $\underline{R(Z) \subset L_{\Omega}}$.

2) Say that V is a Poincaré model if there exists a K-isomorphism $\beta: V \xrightarrow{\sim} Z \times \mathrm{Spec}(K)$ with Z an abelian C-variety

Terminolcgy above is inspired from [23] see (5.3) below. Clairaut and Poincaré models will play an important role in classification; by the very definition they are defined over C, hence classified in (4.8). We shall say that a differential extension $K \subset L$ is a Clairaut (Poincaré) extension if it has a Clairaut (Poincaré) model; we also say L is Clairaut (Poincaré) over K.

(5.2) Notations being as in (4.8) we have:

1) V is a Clairaut model if and only if either $H^0(V,T_{V/K})=0$ or for a suitable β, $a_{jk}=0$ for all j,k. Note on the other hand that $H^0(V,T_{V/K})=0$ provided V is of one of the following types: a variety with ample canonical bundle, a minimal surface of general type, a variety with q=0 and $\varkappa \neq -\infty$ a hypersurface in projective space, of degree ≥ 3 and dimension ≥ 2; this follows from [29] pp 47 and 51 and from [16] Lemma 14.2.

2) If V is a Poincaré model then $\delta_p a_{jk} = \delta_j a_{pk}$ for all p,j,k.

(5.3) In the case n=r=1 ($n=\text{tr.deg.}_K L$) concepts of Clairaut and Poincaré extensions were introduced in [23].

Our Clairaut and Poincaré extensions do not reduce for $n=r=1$ to Matsuda's, although they are very close to them. Note that if $K \subset L$ is a differential extension then:

1) If $K \subset L$ is Clairaut in our sense then it is also Clairaut in Matsuda's sense.

2) If $K \subset L$ is Poincaré but not Clairaut in our sense then it is Poincaré in Matsuda's sense.

(5.4) Remarks. 1) If V is a Clairaut model then V_C is infinite and consists only of K-rational points. Indeed if, $p: V \simeq Z \times \mathrm{Spec}(K) \to Z$ is the projection then for any C-point $y \in Z$, $p^{-1}(y)$ consists of a K-point belonging to V_C. Furthermore for $p \in V_C$, $R(p)=K((R(p))_o)$ so $R(p)=K$.

2) If V is an abelian K-variety and is defined over C then it is a Poincaré model. Indeed if $V \simeq Z \times \mathrm{Spec}(K)$ then to see that Z is an abelian C-variety one can use either [25] p.124 or the fact that any projective complex manifold with trivial canonical bundle and irregularity equal to the dimension must be an abelian variety (the latter fact is a corollary of the classification in [3]).

3) Notations being as in (5.1),1) note that it does not followthat for all isomorphism $\eta : V \xrightarrow{\sim} Z \times \mathrm{Spec}(K)$ the embedding of $R(Z)$ in L via η is contained in L_o (see (5.6)).

Here is the main result of this section:

(5.5) <u>Theorem</u>. Let $K \subset L$ be a differential extension with Poincaré model V.

1) V is a Clairaut model if and only if V_D contains a K-rational point.

2) If $L_o = C$ then V_D consists only of the generic point of V.

3) There exists a differential extension K^* of K such that $V \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K^*)$ is a Clairaut model.

Before giving the proof we would like to note that statement 2) above will be used in an essential way inChapter III; this statement combined with Proposition (3.7) on discriminants will give,very roughly speaking, the "smoothness of the Albanese map" of a Fuchs model.

Note also that statement 3) above was proved in [23] for $n=r=1$; our method is more conceptual.

(5.6) Let's prove (5.5),1) By the general remark (5.4),1) we only have to prove that V is a Clairaut model provided there exists a K-rational point in V_D. Let p be such a point. Let $\beta: V \xrightarrow{\sim} Z \times \mathrm{Spec}(K)$ be an isomorphism as in (5.1),2) with Z an abelian variety, fix a basis $\theta_1,\ldots,\theta_g$ of $H^0(Z, T_{Z/C})$ and let (a_{jk}) be the matrix associated to $\beta, \theta_1,\ldots,\theta_g$ as in (4.8). Using (2.7) we may find a smooth affine C-variety S, an algebraic extension $R(S) \subset K$ and a section $t: S \longrightarrow Z \times S$ of the projection $p_2: Z \times S \longrightarrow S$ such that the following hold:

1) If $t_K: \mathrm{Spec}(K) \longrightarrow Z \times \mathrm{Spec}(K)$ is the morphism deduced from t then the image of t_K equals p.

2) The elements a_{jk} belong to $H^0(\mathcal{O}_S)$.

3) $R(S)$ is a differential subfield of K and S is a Fuchs model.

By the above conditions $R(Z \times S)$ is a differential subfield of L and $Z \times S$ is a Fuchs model. Now consider the S-morphism

$$\varphi : Z \times S \xrightarrow{f} Z \times Z \times S \xrightarrow{g} Z \times S$$

where $f = 1_Z \times t$ and $g = a \times 1_S$ with $a : Z \times Z \longrightarrow Z$ the addition map for the group law on Z. Note that for any $y \in S(C)$ the induced morphism $\varphi_y : Z \times \{y\} \longrightarrow Z \times \{y\}$ is precisely the automorphism $\varphi_y(x) = x + p_1(t(y))$ where $p_1 : Z \times S \longrightarrow Z$ is the natural projection. In particular φ is an isomorphism and let ψ its inverse. Denote by η the composed morphism

$$\eta : V \xrightarrow{\beta} Z \times \mathrm{Spec}(K) \longrightarrow Z \times S \xrightarrow{\psi} Z \times S \xrightarrow{p_1} Z.$$

Claim. $\eta^*(R(Z))$ is contained in the field of constants of $L=R(V)$.

Indeed, for any $u \in R(Z)$ and any $x \in Z$ the function $p_1^* u$ takes the same value at any point of $\{x\} \times S$ hence hence $\psi^* p_1^* u$ takes the same value at any point of $t(S) + x$ (here Z acts on $Z \times S$ by translations via the first factor). Since $p \in V_D$ it follows that the generic point of $t(S)$ in $Z \times S$ belongs to $(Z \times S)_D$ hence by (2.9),1), $t(S)$ is an integral subvariety for $\delta_1, \dots, \delta_r$. Now by the special form of derivations (**) in (4.8) and since $\theta_1, \dots, \theta_g$ are Z-invariant, we see that the vector fields on $Z \times S$ corresponding to $\delta_1, \dots, \delta_r$ are invariant under the action of Z

on $Z \times S$ and consequently $\underbrace{t(S)+x}_{\text{for all } x \in Z}$ will be integral

subvarieties for $\delta_1,\ldots,\delta_r$. In particular by (2.9),2)

$\delta_j(\psi^* p_1^* u)=0$ for all j hence $\delta_j(\eta^* u)=0$ for all j

so the claim is proved.

Now consider the isomorphism

$$\gamma : V \xrightarrow{\ \beta\ } Z \times \mathrm{Spec}(K) \xrightarrow{\ \psi_K\ } Z \times \mathrm{Spec}(K)$$

where ψ_K is naturally induced by ψ. Then if we embed

$R(Z)$ in $R(V)=L$ via $V \xrightarrow{\ \gamma\ } Z \times \mathrm{Spec}(K) \longrightarrow Z$

we get by our claim that $\delta_j x=0$ for $x \in R(Z)$ and all j

and we are done.

(5.7) Let's prove (5.5),2). We shall proceed by induction

on the codimension of points in V_D. We prove first that if

$L_0=C$ then V_D cannot contain any point p of codimension one.

Indeed if V_D would contain such a point p then let Y be the corresponding prime divisor on V and let B be the component of the unity of the algebraic group $\ker(\lambda_Y : V \longrightarrow \hat{V})$ where $\lambda_Y(a) = [\mathcal{O}_V(Y_a - Y)]$, $Y_a = Y + a$. By [26] we have a morphism $f : V \longrightarrow V/B = W$ and an ample divisor H on W such that $f^* H = Y$. Since f has connected fibres, by (3.7),1) $R(W)$ is a differential subfield of L and W is a Fuchs model. Furthermore $(R(W))_o = C$ and by (2.4),2) the generic point of H belongs to W_D. So we may reduce ourselves to the case when Y itself is ample. Let S be an affine smooth C-variety such that $R(S)$ is a differential subfield of K over which K is algebraic and such that S is a Fuchs model and $a_{jk} \in H^o(\mathcal{O}_S)$ where (a_{jk}) is the matrix associated as in (4.8) to some isomorphism $V \xrightarrow{\sim} Z \times \mathrm{Spec}(K)$ and to a basis of $H^o(Z, T_{Z/C})$. We may choose S such that in addition there exists a (flat) relatively ample (over S) divisor X on $Z \times S$ such that the projection $X \longrightarrow S$ gives a C-presentation of Y. If $\dim(Z) = g$ there exist $x_1, \ldots, x_g \in Z$ such that after replacing

S by some open Zariski subset of it, every irreducible component T_i of $T=(X+x_1)\cap(X+x_2)\cap\ldots\ldots\cap(X+x_g)$ dominates and is generically finite over S; note that by ampleness, $T \neq \phi$ .Now exactly as in (5.6) X is an integral subvariety of $Z \times S$ for $\delta_1,\ldots,\delta_r$ and $X+x_j$ will still be integral subvarieties. By (2.3),3) the generic points of T_i belong to $(Z \times S)_D$. This immediately implies by (2.4),3) that V_D contains closed points, contradicting (5.5),1).

Now let's make the induction step. Suppose we know V_D does not contain points of codimension $k-1$ and suppose V_D contains a point p of codimension $k \geq 2$. Let S be again a smooth C-variety, $R(S) \subset K$ algebraic, S a Fuchs model, $a_{jk} \in H^0(\mathcal{O}_S)$. Suppose in addition that we chose S such that there exists an irreducible reduced closed subscheme X in $Z \times S$ dominating S and such that $(X \longrightarrow S, R(S) \subset K)$ gives a C-presentation of Y. There exists an irreducible curve B in Z such that if $X^* = \bigcup_{b \in B} (X+b)$ then $\dim(X^*)=\dim(X)+1$ (here again Z acts on $Z \times S$ via the first factor. Now since $F = \bigcup_{b \in B} (X_{sing} + b)$ is a proper closed subset of X^* we get that for any $x \in X^* \setminus F$ there exists $b \in B$ such that $x \in X_{reg} + b = (X + b)_{reg}$. We conclude by (2.9),1) that the generic point of X^* belongs to $(Z \times S)_D$. By (2.4),3) again we will find a point of codimension k-1 in V_D, contradiction.

(5.8) Let's prove (5.5),3). Take simply K^* to be the algebraic closure of L. Then $V \times_{Spec(K)} Spec(K^*) = V^*$ will still be a Poincaré model; but V^*_D will contain a K^*-rational point and we conclude by (5.5),1).

(5.9) One can give easier alternative proofs for (5.5),2) in cases dim(V)=1 and dim(V)=2 (see (7.3) and (8.6)). The idea is to use Proposition (3.2) on cyclic coverings.

We close this section by summarizing what we know up to now about classification of Fuchs smooth projective models in the "abstract case". Theorems below may be deduced by simply putting together (3.6),(4.4),(4.6), (5.2),(5.4),(5.5).

(5.10) Theorem. Let $K \subset L$ be a differential extension with K algebraically closed and with smooth projective Fuchs model V. Let $K_o = C$.

1) Suppose V is of one of the following types: a variety with ample canonical bundle, a minimal surface of general type, a variety with q=0 and $\varkappa \neq -\infty$, a hypersurface of dimension $\geqslant 2$ and degree $\geqslant 3$ in projective space. Then V is a Clairaut model; in particular V_C consists only of K-rational points.

2) Suppose V is an abelian K-variety. Then V is a Poincaré model; in particular if $L_o = C$ then $V_D = V_C = \{\xi_V\}$ where ξ_V is the generic point of V.

(5.11) Theorem. Suppose $K \subset L$ is a differential extension with model a simple abelian variety. Suppose K is algebraically closed, $K_o = C$ and the group of differential K-automorphisms of L is infinite. Then $K \subset L$

is a Poincaré extension.

Note that (except information about V_D) case $n=r=1$ ($n=\text{tr.deg.}_K L$) in (5.10) was proved in [23] pp.37 and 91 by Matsuda and Nishioka, and has its roots in Poincaré's paper [28]. For $n=1$ Theorem (5.11) was proved by Kolchin [18] (see also [23] p.70).

6. The analytic case. Kolchin extensions.

In this section we make the connection between results obtained in the abstract case (see (5.10)) and the analytic discussion from the Introduction, (see also (1.6)).

(6.1) Theorem. Let K be a differential subfield of $\mathrm{Mer}(\mathcal{A})$, $K \subset L$ a differential extension with smooth projective Fuchs model V and $f: R(p) \longrightarrow \mathrm{Mer}(\mathcal{B})$ an element of V_A. Let $\overline{V}=V\times_{\mathrm{Spec}(K)}\mathrm{Spec}(\overline{K})$.

1) Suppose $\overline{V}$ is of one of the following types: a variety with ample canonical bundle, a minimal surface of general type, a variety with $q=0$ and $\varkappa\neq\infty$, a hypersurface of dimension ≥ 2 and degree ≥ 3 in projective space. Then the extension $K\subset f(R(p))$ is finite.

2) Suppose $\overline{V}$ is an abelian $\overline{K}$-variety. Then $K\subset f(R(p))$ has a finite embedding into a Kolchin extension $K^*\subset L^*$ such that $L^*=K^*(L)$.

(6.2) Note that Theorem (6.1) immediately implies Theorems 1,4,5 from the Introduction. To prove Theorem 5, use the fact that if V is a Fuchs model birationally isomorphic to an abelian K-variety A then the rational map $\overline{V} \dashrightarrow \overline{A}=A\times_{\mathrm{Spec}(K)}\mathrm{Spec}(\overline{K})$ must be everywhere defined [20] p.20; hence by (3.7),1) $\overline{A}$ is a Fuchs model and by (2.4),1) A will be a Fuchs model, so Theorem (6.1) will apply to A.

(6.3) If one looks at V in Theorem (6.1) as obtained by glueing (1.7) algebraic differential systems S_i: $F_{i1}=\ldots=F_{in_i}=0$ then (6.1),1) says that any meromorphic solution of any system S_i has its components algebraic over the differential field $C\langle a_1,\ldots,a_N\rangle$ where the a_i's are all coefficients appearing in the F_{ij}'s. This concept of algebraicity of solutions is clearly much stronger than the one in [12] p.215. On the other hand results of Painlevé-Malmquist type [12] pp.232-240 do not suffice to prove our algebraicity results. One of the main points which make the method in [12] fail in our problem is the following. Let (f,j), $f:X \longrightarrow S$ be a differential C-presentation of a Fuchs model V; we may suppose that for any $x \in X$ the dimension of $C\delta_1(x)+\ldots+C\delta_r(x) \subset T_xX$ is constant equal to $r_0 \leq r$ so we get in this way a foliation $\mathcal{E} \subset T_{X/C}$. We may suppose $\mathcal{E}$ is transverse to f [12] p.206. But it will generally happen that $\dim(S) > \dim(\mathcal{E})$ so the theory in [12] will not apply (the simplest example is provided by equations $F(y,y')=0$ with $F \in C(\varphi,\psi)[t_0,t_1]$ where $\varphi,\psi \in Mer(C)$ are algebraically independent; take for instance $\varphi(w)=w$ and $\psi(w)=\exp(w)$).

In particular if $r=1$

theory in [12] applies only if $tr.deg._C K=1$. But inthis case C-moduli always exist (4.3) so by (4.6) smooth projective Fuchs models over K ($K_o=C$) are always defined over C hence completely classified (4.8).

Finally (6.1),2) says that any meromorphic solution of the corresponding systems may be written as u/v, where both u and v have the form $\varphi_1 \mathcal{P}(\beta)+\ldots+\varphi_N \mathcal{P}_N(\beta)$ $\beta=(\beta_1,\ldots,\beta_g)$, $\mathcal{P}_j$ being abelian functions and φ_i, $\partial\beta_j/\partial w_k$ being algebraic over K.

(6.4) Let's prove (6.1),1). Clearly $L\otimes_K \bar{K}$ will be a differential extension of $\bar{K}$ with Fuchs model $\bar{V}$. By (2.1), $(\bar{K})_o=C$. By (5.10),1) $(\bar{V})_C$ consists of closed points hence by (2.4),4) V_C consists of closed points and we are done.

(6.5) Let's prove (6.1),2). By (5.10),2) $\bar{V}$ is a Poincaré model. Suppose $\bar{V}$ is $\bar{K}$-isomorphic to $Z\times \mathrm{Spec}(\bar{K})$ where Z is some abelian C-variety, $Z=C^g/\Lambda$, Λ a lattice in C^g. Let $z_1,\ldots,z_g$ be holomorphic global coordinates on C^g and $\pi : C^g \longrightarrow Z$ the canonical projection such that if $\theta_1, \ldots, \theta_g$ is a basis of $H^0(Z, T_{Z/C})$ as in (4.8) then $(\theta_j u) \circ \pi = (\partial/\partial z_j)(u \circ \pi)$ for all j and all $u \in R(Z)$. Now let (a_{jk}) be the matrix defined in (4.8). There exists a finite extension K^* of K, $K^* \subset \overline{K}$ such that $V^* = V \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K^*)$ is K^*-isomorphic to $Z \times \mathrm{Spec}(K^*)$ and such that all a_{jk} belong to K^*.

Let $p^* \in V^*$ be any point lying over $p \in V$ and take $U = \mathrm{Spec}(A) \subset Z$ be an open subset such that $p^* \in U \times \mathrm{Spec}(K^*)$.

By (2.4),5) there exists a differential extension $h : R(p^*) \longrightarrow \mathrm{Mer}(\mathcal{B}^*)$ in V_A^* extending $f : R(p) \longrightarrow \mathrm{Mer}(\mathcal{B})$.

Write $A = C[T_1, \ldots, T_N]/J = C[u_1, \ldots, u_N]$, J being a prime ideal in the polynomial ring $C[T_1, \ldots, T_N]$. Let $\alpha_{jk}, \psi_q \in \mathrm{Mer}(\mathcal{B}^*)$ be the images of a_{jk} and $u_q \otimes 1$ via h.

Since $\theta_k(A) \subset A$ there exist polynomials $P_{qk} \in C[T_1, \ldots, T_N]$ such that $\theta_k u_q = P_{qk}(u_1, \ldots, u_N)$ for $1 \leq k \leq r$, $1 \leq q \leq N$.

Applying h to (**) and (***) in (4.8) we get

$$\partial \alpha_{jk}/\partial w_p = \partial \alpha_{pk}/\partial w_j$$

$$\partial \psi_q/\partial w_j = \sum_{k=1}^{g} \alpha_{jk} P_{qk}(\psi_1, \ldots, \psi_N)$$

for all j,p,k,q. Furthermore we have

$$F(\psi_1,\ldots,\psi_N)=0 \quad \text{for all} \quad F\in J$$

By the holomorphic Poincaré lemma [9] p.448 there exists a subdomain $\mathcal{B}^{**}$ of $\mathcal{B}^*$ and functions $\beta_1,\ldots,\beta_g\in$ $\in \mathrm{Mer}(\mathcal{B}^{**})$ such that $\partial\beta_k/\partial w_j=\alpha_{jk}$ for all j,k. Choose $w^o=(w^o_1,\ldots,w^o_r)\in\mathcal{B}^{**}$ such that all β_k and ψ_q are holomorphic around w^o. Since $y^o=(\psi_1(w^o),\ldots,\psi_N(w^o))$ is a C-point of $U\subset \mathrm{Spec}(C[T_1,..,T_N])$ we may write $\psi_q(w^o)=u_q(y^o)$ for all q. Take $x^o\in C^g$ such that $\pi(x^o)=y^o$ and define $\eta_1,\ldots,\eta_N\in \mathrm{Mer}(\mathcal{B}^{**})$ by the formula:

$$\eta_q(w)=\mathcal{P}_q(\beta_1(w)-\beta_1(w^o)+x^o_1,\ldots,\beta_g(w)-\beta_g(w^o)+x^o_g)$$

where $\mathcal{P}_q=u_q\circ\pi$. By construction, $\eta_q(w^o)=\psi_q(w^o)$ for all q and it is easy to see that

$$\partial\eta_q/\partial w_j=\sum_{k=1}^{g}\alpha_{jk}P_{qk}(\eta_1,\ldots,\eta_N)$$

for all j,q. One obtains inductively that the Taylor expansions of η_q and ψ_q around w^o coincide , hence

$\eta_q = 1/q$. Now the differential extension $h(K^*) \subset h(R(p^*))$ is generated by $\eta_1, \ldots, \eta_N$ and $h(R(p^*)) = h(K^*)(h(R(p)))$ so we are done.

The following corollary will be useful to deduce Theorems 2 and 3 in the Introduction from the results in Chapter III.

(6.6) Corollary. Suppose $K \subset L$ is a differential extension of differential subfields of $\mathrm{Mer}(\mathcal{A})$, admitting a model V. Suppose $L \otimes_K \overline{K}$ has a finite extension M which is Poincaré over $\overline{K}$. Then $K \subset L$ has a finite embedding into a Kolchin extension $K^* \subset L^*$.

Proof. Let $W \simeq Z \times \mathrm{Spec}(\overline{K})$ be a Poincaré model of $\overline{K} \subset M$. The extension $L \otimes_K \overline{K} \subset M$ yelds a rational map $Z \times \mathrm{Spec}(\overline{K}) \dashrightarrow \overline{V} = V \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$ over $\overline{K}$. This map will be induced by a rational map $W_1 = Z \times \mathrm{Spec}(K_1) \dashrightarrow V_1 = V \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K_1)$ over K_1 where K_1 is some finite extension of K contained in $\overline{K}$. Let $D \subset W_1$ be the open

Zariski subset where the latter map is defined. Applying (2.4),5) to the morphism $D \to V_1 \to V$ we get that the inclusion $L \longrightarrow Mer(\mathcal{A})$ extends to a differential extension $R(W_1) \longrightarrow Mer(\mathcal{A}_1)$, $\mathcal{A}_1 \subset \mathcal{A}$. Now apply (6.1),2) to the model W_1.

Chapter III

7. Classification: curves.

As we already remarked, Fuchs extensions of transcendence degree one were classified in [23] in the ordinary case (i.e. r=1). The case r=1 of the following theorem is essentially contained in [23]; the general case $r \geqslant 1$ follows immediately from Theorem (5.10).

(7.1) Theorem. Let $K \subset L$ be a differential extension with Fuchs model V a smooth projective curve of genus g. Suppose K is algebraically closed and $K_0=C$. Then V is defined over C. More precisely, for g=1 V is a Poincaré model and for $g \geqslant 2$ it is a Clairaut model.

(7.2) We would like to note that in [23] no hypothesis is made on the ground field C. We are able to generalize our arguments to obtain results also for arbitrary ground fields, in the case of curves and abelian varieties; one has to make use of moduli for such varieties over any ground field [25].

(7.3) If V is as in Theorem (7.1) and $g=1$ it follows from (5.5),1) that V is a Clairaut model if and only if V_D (or equivalently V_C) contains a closed point. We would like to prove here directly a weaker statement (it is (5.5),2) for dimension one) namely that V_D cannot contain closed points provided $L_o=C$. We shall use (7.1) which in its turn is independent of (5.5). Suppose indeed there exists a closed point $p \in V_D$. Take an étale double covering $f: V^* \longrightarrow V$ of V. By (3.1), V^* is a Fuchs model and of course is an elliptic curve. If $f^*(p)=p_1+p_2$ then by (2.4),3) $p_1,p_2 \in V^*_D$. Since $Pic^o(V^*)$ is divisible there exists $\mathcal{L} \in Pic(V^*)$ such that $p_1+p_2 \in |\mathcal{L}^2|$. Let $W=Spec(\mathcal{O} \oplus \mathcal{L}^{-1}) \longrightarrow V^*$ be the double covering of V^* ramified along p_1+p_2. By (3.2), W is a smooth projective Fuchs model. But $g(W)=2$ hence by (7.1) W is Clairaut, hence

W_C consists of closed points. By (2.4),4) V_C cannot contain the generic point of V, contradicting $L_o=C$ and we are done.

(7.4) *Proposition.* Let $K \subset L$ be a differential extension with Fuchs model V a projective curve. Suppose K is algebraically closed and $L_o=C$. Then $g=g(V^{nor}) \leq 1$ and

1) If $g=0$, $(V^{nor})_D$ has at most 2 K-rational points. In particular if $m:V^{nor} \longrightarrow V$ is the normalization morphism then $m^{-1}(V_{sing})$ has at most 2 K-rational points.

2) If $g=1$ then V is smooth.

Proof. By (2.3),4) V^{nor} is a Fuchs model. So by (7.1) $g \leq 1$. Suppose $V^{nor}=\mathbb{P}^1$. If $\mathbb{P}^1$ contains 3 K-rational points p_1,p_2,p_3 then consider the cyclic 3-fold covering of $\mathbb{P}^1$, $W \longrightarrow \mathbb{P}^1$ ramified at $p_1+p_2+p_3$. By (3.2) W is a Fuchs model of genus one. By (2.4),3) W_D will contain K-rational points hence by (7.3) (or by (5.5)) W is a Clairaut model. But this again imples that $L_o \neq C$, contradiction. The statement about V_{sing} follows from (2.8) and (2.4),3). The case when V^{nor} is an elliptic curve may be treated in the same way.

(7.5) We would like to remark that the case $K=C$ is again trivial in (7.4). Indeed if $K=C$, (7.4),1) reduces to the obvious fact that a global vector field on $\mathbb{P}^1$ which vanishes at three points must be identically zero.

8. Surfaces.

(8.1) Let $K \subset L$ be a differential extension with K algebraically closed and $n=tr.deg._K L=2$ and suppose there is a projective Fuchs model V. Then by (3.4) there will exist also a smooth projective Fuchs model V^* and hence by (2.3),1) any minimal model V_o dominated by V^* will be a Fuchs model.

(8.2) Theorem. Let $K \subset L$ be a differential extension with Fuchs model V a smooth projective minimal surface. Suppose K is algebraically closed and $K_0=C$. Then V is defined over C. More precisely V is K-isomorphic to $Z \times \mathrm{Spec}(K)$ where either

1) Z is a minimal model of rational surface or

2) Z is a $\mathbb{P}^1$-bundle over an elliptic curve or

3) Z is an abelian surface or

4) Z is a bielliptic surface.

We shall refer to the types above as types 1),2),3),4).

(8.3) Note that by (4.8) our classification is complete; it depends only on the "structure constants" of the Lie algebra $H^0(Z,T_{Z/C})$ in each of the four cases. Of course in case 3) V is a Poincaré model and in case 4), V has a finite étale covering $V^* \longrightarrow V$ with V^* a Poincaré model.

(8.4) Corollary. Suppose $K \subset L$ is a Fuchs extension of transcendence degree $n=2$ with K algebraically closed $L_0=C$ and $\varkappa \neq -\infty$. Then L has a finite extension L^* which is Poincaré over K.

Note that (8.4) and (6.6) immediately imply Theorem 2 from the Introduction.

(8.5) Let's prove Theorem (8.2). By classification of surfaces [1] (the complex case suffices because K is algebraically closed and of finite transcendence degree over $\mathbb{C}$ hence K is abstractly isomorphic to $\mathbb{C}$) and applying Theorem (5.10),1) we get that V must be of one of the following types:

a) rational

b) a $\mathbb{P}^1$-bundle over a curve B of genus $g(B)=q(V) \geqslant 1$.

c) abelian

d) elliptic with $q(V) \geqslant 1$ and $\varkappa \neq -\infty$, but not abelian.

In case a) we are done because minimal models of rational surfaces [1] p.79 are defined over rationals. Suppose we are in case b). By (3.7),1) $R(B)$ is a differential subfield of L and B is a Fuchs model. Since $(R(B))_0 \subset L_0=\mathbb{C}$ we get by (7.1) that B is a Poincaré model, so $B=E \times \mathrm{Spec}(K)$, E an elliptic curve over $\mathbb{C}$.

Now $V=\mathbb{P}(\mathcal{E}^K)$ where $\mathcal{E}^K$ is a rank 2 normalized vector bundle on B [10] p.373. By Atyiah's classification [10] p.377 two situations may occur. The first is when $\mathcal{E}^K$ is indecomposable and there are exactly two ruled surfaces of the form $\mathbb{P}(\mathcal{E}^K)$; by the explicit construction of these surfaces [10] p.377 one sees that both of them are defined over the same field as the base curve B; in our case they are defined over C. The second case is when $\mathcal{E}^K=\mathcal{O}_B\oplus\mathcal{L}^K$ where $\mathcal{L}^K\in\mathrm{Pic}(B)$, $\deg(\mathcal{L}^K)=d\leq 0$. The giving of $\mathcal{L}^K$ is equivalent to the giving of a morphism $\varphi:\mathrm{Spec}(K)\longrightarrow P=\mathrm{Pic}^d_{E/C}$ [25] p.23. Let $\mathcal{L}^P$ be the universal sheaf on $P\times E$, let S be a smooth open subscheme of the closure of $\varphi(\mathrm{Spec}(K))$ in P and let $\mathcal{L}^S\in$ $\in\mathrm{Pic}(S\times E)$ the restriction of $\mathcal{L}^P$ to $S\times E$. Consider the natural projection $f:\mathbb{P}(\mathcal{O}_{S\times E}\oplus\mathcal{L}^S)\longrightarrow S$ and the embedding $j:R(S)\longrightarrow K$. Then (f,j) is a C-presentation of V and we claim that the identity map $S\dashrightarrow S$ is a moduli map for f (4.2). Indeed if we have an abstract isomorphism $\mathbb{P}(\mathcal{O}_E\oplus\mathcal{L}_1)\simeq\mathbb{P}(\mathcal{O}_E\oplus\mathcal{L}_2)$ with $\deg(\mathcal{L}_1)=$ $=\deg(\mathcal{L}_2)\leq 0$ then it is an easy exercise to see that either

$\mathcal{L}_1 \simeq \mathcal{L}_2$ or $\mathcal{L}_1 \simeq \mathcal{L}_2^{-1}$. Consequently V has C-moduli and by (4.6) V is defined over C so case b) is done.

In case c) we conclude by (5.10),2).

Suppose we are in case d).

Claim. There exists a morphism $h:V \longrightarrow G$ onto an elliptic curve G such that $h_* \mathcal{O}_V = \mathcal{O}_G$.

Indeed take an elliptic fibration $f:V \longrightarrow B$. As in case b) one sees that $g(B) \leq 1$. By [1] p.150 $g(B) \leq q(V) \leq g(B)+1$. Now if $g(B)=1$ we take $h=f$, $G=B$, and the claim follows. If $g(B)=0$ we have $q(V)=1$ and we may take $h:V \longrightarrow G$ to be the Stein factorization of the Albanese map $V \longrightarrow Alb(V)$ (once again the fact that $g(G)=1$ will follow as in case b) via (7.1)).

Now we claim that h has smooth fibres. Indeed by (3.7),1) R(G) is a differential subfield of L and G is a Fuchs model hence a Poincaré model by (7.1). Furthermore $(R(G))_o \subset L_o = C$. Now if $h^{-1}(y)$ is singular for some closed point $y \in G$, it follows by (3.7),2) that $y \in G_D$ and this contradicts (5.5),2) (see also (7.3)). Now if h is smooth it follows from [1] p.99 that V must have an étale covering which is a product of curves. By (3.1), (3.7), 1) and (7.1) these curves are elliptic so V is bielliptic. So by [1] p.113 there exist elliptic curves B_1, B_2 over K and a finite group $F \subset B_1(K)$ acting freely on $B_1 \times B_2$ such that $V \simeq (B_1 \times B_2)/F$. By (3.1) $B_1 \times B_2$ is a Fuchs model so by (3.7), 1) and (7.1) $B_i = E_i \times \mathrm{Spec}(K)$ with E_i elliptic curves over C.

Claim. The action of F on $B_1 \times B_2$ is induced by an action of F on $E_1 \times E_2$.

Indeed by the Bagnera-DeFranchis list [1] p.113 F has generators whose action is either

$$(b_1, b_2) \longrightarrow (b_1 + \beta_1, b_2 + \beta_2) \quad \text{or}$$

$$(b_1, b_2) \longrightarrow (b_1 + \beta_1, \omega(b_2))$$

with $\beta_1 \in B_1(K)$, $\beta_2 \in B_2(K)$, $\omega \in \mathrm{Aut}_o(B_2)$. Now if we think $E_i(C)$ as embedded in $B_i(K)$ it is well known that $E_i(C)$ contains all points of finite order of $B_i(K)$, in particular $\beta_i \in E_i(C)$. Furthermore since the cardinals of $\mathrm{Aut}_o(E_i)$ and $\mathrm{Aut}_o(B_i)$ are equal (they depend only on the j-invariant [10] p.321) it follows that any automorphism in $\mathrm{Aut}_o(B_i)$ is induced by some automorphism in $\mathrm{Aut}_o(E_i)$. The claim follows.

By the claim above $V=((E_1 \times E_2)/F) \times Spec(K)$ and case d) is also done. The theorem is proved.

(8.6) Let's show how one can use the method from (7.3) and Theorem (8.2) to prove (5.5),2) for $\dim(V)=2$. Let V be a smooth projective Fuchs model which is an abelian surface and suppose $(R(V))_o=C$. We want to show that there is no irreducible curve on V whose generic point belongs to V_D. Indeed if Y was such a curve, consider first a finite étale covering $f:V^* \longrightarrow V$ such that $f^*Y \in |\mathcal{L}^2|$ for some $\mathcal{L} \in Pic(V^*)$ (such an f may be constructed as follows: take $V^*=V$, f=multiplication by 2; then by [20] pp. 86 and 92, f^*Y is algebraically equivalent to $4Y$ so we may conclude by the fact that $Pic^o(V^*)$ is divisible). By (3.1) V^* is a Fuchs model and by (2.4),3) all irreducible components of f^*Y have their generic points in V^*_D. Let V' be the double covering of V^* constructed with the help of $\mathcal{L}$ and f^*Y. Since f^*Y has no multiple components, V' has only isolated singularities , hence it is normal and connected. By (3.2), V' is a Fuchs model. By (3.4) there is a birational morphism $W \longrightarrow V'$ with W a smooth projective Fuchs model. Since W dominates V^*, $q(W) \geq q(V^*)=2$ hence by our classification in (8.2) the minimal model W_{min} of W, which will still dominate V^*, is an abelian variety. Hence the morphism $W_{min} \longrightarrow V^*$ must be étale, which clearly contradicts the fact that $V' \longrightarrow V^*$ is a ramified double covering, so we are done.

(8.7) <u>Proposition</u>. Suppose V is as in (8.2)

1) If V is of type 2),3),4) then V_D does not contain closed points.

2) If V is of type 3),4), V_D does not contain codimension one points.

3) If V is of type 2) and Y is an irreducible curve on V whose generic point lies in V_D then Y is a smooth elliptic curve.

Proof. 1) If V is of type 2) let $f:V \longrightarrow B$ be the rulling with B an elliptic curve (B will be then a Poincaré model). Now if V_D contains a closed point, so will do B_D (2.4),2). But $(R(B))_o=C$ and this contradicts (5.5),1). If p is a codimension one point in V_D and Y is the corresponding divisor, then the same argument as above shows that Y cannot be a fibre of f, hence Y dominates B. Now by (2.3),2), Y is a Fuchs model of $R(Y)$. It is easy to see that $R(B)$ is a differential subfield of $R(Y)$ hence by (2.1) $(R(Y))_o=C$. Now by (7.4) Y must be a smooth elliptic curve.

Suppose now V is of type 3),4) and let $V^* \longrightarrow V$ be a finite étale covering with V^* a Poincaré model (8.3). Any point of V_D, different from the generic point of V, yelds by (2.4),3) a point of V^*_D which is different from the generic point of V^*. Since by (2.1) $(R(V^*))_o=C$ we conclude by (5.5),2) that the proposition holds also for types 3),4).

(8.8) <u>Proposition.</u> Let $K \subset L$ be a differential extension of transcendence degree 2 and V a projective Fuchs model. Suppose K is algebraically closed, $L_o=C$ and V is not rational.

1) If V is smooth then V is minimal.

2) If V is non-ruled then V is smooth (hence by 1) it is minimal hence by (8.2) it is of type 3) or 4)).

Proof. 1) Let V_{min} be a minimal model dominated by V. If $V \longrightarrow V_{min}$ is not an isomorphism, then by (3.3),2), $(V_{min})_D$ will contain closed points, contradicting (8.7).

2) By (3.4) there exists a morphism $f:V^* \longrightarrow V$ with V^* a smooth projective Fuchs model. By (8.8),1) V^* will be minimal hence of type 3) or 4) (in notations from (8.2)). If p is a generic point of V_{sing} then by (2.8) $p \in V_D$ hence by (2.4),3) V^*_D will contain points of codimension 1 or 2 which contradicts (8.7). So V is smooth.

(8.9) Information on the structure of the set V_D for a smooth projective Fuchs model V with $\dim(V)=2$ is essential to investigate Fuchs models of dimension 3. Proposition (8.7) gives a complete answer for non-rational surfaces. The problem seems more difficult in the rational case.

We shall look only at the case $V=\mathbb{P}^2$. Again, if we are in the "trivial case" $K=C$, describing V_D means precisely to describe the integral subvarieties of global vector fields on $\mathbb{P}^2$; but this is a classically solved problem [12] Exposé 1. For non-constant K we do not dispose of a complete classification of points in V_D and V_C. Here is what we can prove (and this will be useful in Section 9):

(8.10) Proposition. Let V be as in (8.2) and suppose $V=\mathbb{P}^2$. Then:

1) If the set S of K-rational points of V belonging to V_D has at least 4 elements then S is contained in some line.

2) Let E be an irreducible curve in V whose generic point belongs to V_D. Then E is a rational curve (i.e. $E^{nor}=\mathbb{P}^1$).

3) If Y is a curve in V such that the generic points of the irreducible components of Y belong to V_D and if Y has at most ordinary double points as singularities then Y has degree ≤ 3 (hence Y is either a line or a conic or a nodal cubic or a line plus a conic or a sum of 3 lines).

Proof. 1) Suppose S contains 3 points $\underbrace{}_{p_1,p_2,p_2}$ not belonging to a line and that $\#S \geqslant 4$. Choose $p_4 \in S \setminus \{p_1,p_2,p_3\}$. There exists an index $i \in \{1,2,3\}$ such that the lines $\overline{p_i p_j}$, $\overline{p_i p_k}$, $\overline{p_i p_m}$ are distinct, where $\{i,j,k,m\}=\{1,2,3,4\}$. Let $f:W \longrightarrow V$ be the blowing up of V at p_i; by (3.3) W is a Fuchs model. Put $q_j=f^{-1}(p_j)$, $q_k=f^{-1}(p_k)$, $q_m=f^{-1}(p_m)$. By (2.4),3) $q_j,q_k,q_m \in W_D$. Now W is a $\mathbb{P}^1$-bundle over $B=f^{-1}(p_i)\simeq \mathbb{P}^1$. Hence by (3.7) $R(B)$ is a differential subfield of L and B is a Fuchs model. If $h:W \longrightarrow B$ is the canonical projection then by (2.4),2) $h(q_j)$, $h(q_k)$, $h(q_m) \in$ $\in B_D$ and clearly are distinct. But $(R(B))_o \subset L_o=C$ and this contradicts (7.4).

2) By (2.3),2) E is a Fuchs model hence so is E^{nor} (2.3),4). Suppose that $g(E^{nor}) \geqslant 2$. Then by (7.1) E^{nor} is a Clairaut model; in particular $(E^{nor})_D$ (and hence also E_D) contains infinitely many K-rational points. Since $E_D \subset V_D$ it follows by (8.10),1) that there exists a line $Y \subset V$ such that $E \cap Y$ is infinite; consequently $E=Y$ contradicting $g(E^{nor}) \geqslant 2$. Suppose now $g(E^{nor})=1$. If E is not

smooth it follows by (2.8), (2.4),3) ,(7.1) and (5.5),1) that E^{nor} is a Clairaut model and exactly as above we get a contradiction. So E is a smooth cubic of equation $f_3(x_o,x_1,x_2)=0$ in $\mathbb{P}^2$. Consider the cubic surface W given by $x_3^3-f_3(x_o,x_1,x_2)=0$ in $\mathbb{P}^3$. Since W is the cyclic 3-fold covering of $\mathbb{P}^2$ ramified along E we know (3.2) that W is a Fuchs model. By (2.1) $(R(W))_o=C$ and by (5.10),1) W is a Clairaut model, contradiction.

3) Let $f_d(x_o,x_1,x_2)=0$ be the equation of Y, f_d being a homogenous polynomial of degree $d \geqslant 4$. As above consider the surface W in $\mathbb{P}^3$ defined by the equation $x_3^d-f_d=0$; W will be a Fuchs model with $(R(W))_o=C$. Choose (3.4) a desingularization $W^* \longrightarrow W$ with W^* a Fuchs model. Now W has only rational singularities (of type A_{d+1}) hence a standard computation shows that $q(W^*)=0$ and the geometric genus $p_g(W^*) \geqslant 1$. By (5.10),1) W_C^* consists of closed points, contradiction. The proposition is proved.

9. Three-folds.

Theorem (5.10) already yelds some information about the structure of smooth projective Fuchs models V of dimension 3. For instance if $(R(V))_0=C$ then V cannot have an ample canonical bundle, cannot have $q=0, \varkappa \neq -\infty$ and cannot be a hypersurface of degree $\geqslant 3$ in $\mathbb{P}^4$. In this section we treat three-folds more systematically. We would like to note that Iitaka's birational theory [34] cannot be applied directly to our problem; the main obstruction is that if V is a Fuchs model and V^* is birationally isomorphic to V then V^* may not be a Fuchs model (this was already clear in the case of surfaces: just blow up a Poincaré model !)

(9.1) It is convenient to recall the following fact [7]: If $f: X \longrightarrow S$ is a smooth projective morphism of smooth K-varieties such that f has connected fibres then $Pic^o_{X/S}$ exists as a projective abelian scheme over S, its dual

$A=Alb^{o}_{X/S}$ is a projective abelian scheme and there exists a principal homogenous space $P=Alb^{1}_{X/S} \longrightarrow S$ over $A \longrightarrow S$ and an S-morphism $X \longrightarrow P$ such that for any $y \in S$ the induced morphism on the fibres $X_y \longrightarrow P_y$ is the Albanese map of X_y; P will be smooth and proper over S [24] p.120.

(9.2) Lemma. Let S be a smooth C-variety, A/S a projective abelian scheme and P/S a principal homogenous space over A/S. Suppose either dim(S)=1 or the fibres of P/S are curves. Then there exists a Zariski open set $S_o \subset S$ and a finite étale morphism $S^* \longrightarrow S_o$ such that:

1) $S \setminus S_o$ has codimension ≥ 2 in S.

2) $P \times_S S^*$ is a projective abelian scheme over S^*.

Proof. There exists an irreducible reduced closed subscheme T in P dominating S such that dim(T)=dim(S). If S is a curve, $T \longrightarrow S$ is finite and flat; let n be its degree and put in this case $S_o=S$. If the fibres of P/S are curves then T is a divisor in P hence it is Cohen-Macaulay. Let $S_1 \subset S$ be the closed set of points in S where the fibre of $p:T \longrightarrow S$ is not finite. Clearly S_1 has codimension $\geqslant 2$ in S and put $S_o = S \setminus S_1$; then $p^{-1}(S_o) \longrightarrow S_o$ will be finite and flat of degree n [10] p.276. So replacing S by S_o we may suppose that $T \longrightarrow S$ is finite and flat. By [11] p.23 there exists a section σ of the projection $(T/S)^{(n)} \longrightarrow S$ where $(T/S)^{(n)}$ denotes the relative symmetric product of $T \to S$. By a theorem of Deligne [15] p.183 the relative symmetric product $(P/S)^{(n)}$ exists as a separated algebraic space. Consider the morphism

$$\delta : P \longrightarrow S \xrightarrow{\sigma} (T/S)^{(n)} \longrightarrow (P/S)^{(n)}$$

and denote by $f:P \longrightarrow S$ the canonical projection. Denote by $\mathcal{E}: A \times_S P \longrightarrow P$ the action of A on P; on the fibres write $\mathcal{E}(a,x) = a+x$. There is a morphism $\mathcal{E}^n : (A/S)^n \times_S P \to (P/S)^n$ deduced from $\mathcal{E}$ which on the fibres looks like

$$(a_1, \ldots, a_n, x) \longrightarrow (a_1 + x, \ldots, a_n + x)$$

Then $\mathcal{E}^n$ induces a morphism $\alpha : (A/S)^{(n)} \times_S P \longrightarrow (P/S)^{(n)}$ of algebraic spaces. Denote by $\beta : (A/S)^{(n)} \longrightarrow A$ the "sum" morphism and by $\delta : A \longrightarrow A$ the multiplication by n.

Then consider the algebraic scheme

$$Z = P \times_S A \times_S (A/S)^{(n)} \times_S P$$

Denote by p_1, p_{23}, a.s.o the projections of Z onto P, $A \times_S (A/S)^{(n)}$, a.s.o., and let Z', Z'', Z''' be the inverse images of the diagonals via each of the following morphisms:

$$p_1 \times (\varepsilon \circ p_{24}) : Z \longrightarrow P \times P$$

$$(\beta \circ p_3) \times (\delta \circ p_2) : Z \longrightarrow A \times A$$

$$(\delta \circ p_4) \times (\alpha \circ p_{34}) : Z \longrightarrow (P/S)^{(n)} \times (P/S)^{(n)}$$

By separatedness, Z', Z'', Z''' are closed subschemes of Z. Put $S^* = p_1(Z' \cap Z'' \cap Z''')$. In down-to earth terms S^* has the following description: for each fibre F of $P \longrightarrow S$ we fix an arbitrary point $x \in F$ and consider the Albanese map $\varphi_x : F \longrightarrow \mathrm{Alb}(F)$ (which of course is an isomorphism) for which $\varphi_x(x) = 0$. Consider the 0-cycle $\sum_{i=1}^{n} x_i = F \cap T$. Let $a_1, \ldots, a_{n^{2g}} \in \mathrm{Alb}(F)$ be the solutions of the equation $na = \sum_{i=1}^{n} \varphi_x(x_i)$, where $g = \dim(F)$ and for any $j = 1, \ldots, n^{2g}$

put $y_j = \varphi_x^{-1}(a_j)$. The correspondence $\sum_{i=1}^{n} x_i \to \sum_{j=1}^{n^{2g}} y_j$ will not depend on the choice of $x \in F$. Furthermore S^* is spread out by the cycles $\sum_{j=1}^{n^{2g}} y_j$ as F runs through all fibres of $P \longrightarrow S$. We conclude that S^* is finite and étale over S. Now $P \times_S S^* \longrightarrow S^*$ is a principal homogenous space over the projective abelian scheme $A \times_S S^* \longrightarrow S^*$ and has a section hence it is isomorphic to $A \times_S S^* \longrightarrow S^*$ [24] p.120 and we are done.

The main result of this section in the following:

(9.3) Theorem. Let $K \subset L$ be a differential extension with smooth projective Fuchs model V of dimension 3 and Kodaira dimension $\varkappa(V) \neq -\infty$. Suppose K is algebraically closed, $L_o = C$ and suppose there exists a morphism $f: V \longrightarrow W$ onto a projective K-variety W with $1 \leq \dim(W) \leq 2$ and $\varkappa(W) \neq -\infty$. Then L has a finite extension L^* which is Poincaré over K.

Proof. Let's examine first the case $\dim(W)=2$. Taking Stein factorization we may suppose that $f_* \mathcal{O}_V = \mathcal{O}_W$ and

W is normal. Let g be the genus of the generic fibre $f^{-1}(y)$, $y \in W$. We cannot have $g=0$ because of Iitaka's inequality $\varkappa(V) \leqslant \dim(W)+\varkappa(f^{-1}(y))$ [29] p.10. We also cannot have $g \geqslant 2$; indeed $V \times_W \mathrm{Spec}(\overline{R(W)})$ will be a smooth projective curve of genus $\geqslant 2$; it will be a Fuchs model of its function field $\tilde{L}$ over $\overline{R(W)}$ hence by (7.1) it will be a Clairaut model. But this yelds a contradiction because $\tilde{L}$ is algebraic over L hence by (2.1) $\tilde{L}_o = C$. So we conclude that $g=1$. Now by (3.7) $R(W)$ is a differential subfield of L and W is a Fuchs model. By (8.8) and (8.2) W is smooth minimal of type 3) or 4) in the terminology of Section 8. So there exists a finite étale morphism $e: A \longrightarrow W$ where A is an abelian surface (and even a Poincaré model). By (3.7),2) and (5.5),2) (or alternatively by (8.6)) we deduce that the discriminant of f consists of a finite set of closed points $S \subset W$; this is the key point of the present proof. Put $F_1 = e^{-1}(S) \subset A$. Then $f_1 : V_1 = V \times_W (A \setminus F_1) \longrightarrow A_1 = A \setminus F_1$ is a smooth morphism with fibres elliptic curves so the morphism $V_1 \longrightarrow \mathrm{Alb}^1_{V_1/A_1}$ is bijective hence an isomorphism.

Now K is abstractly isomorphic to the complex field and we shall choose a structure of complex field on K; the analytic arguments we are going to use in the sequel refer to this complex structure on K, which of course has no connection with the structure of complex field existing on the subfield $K_o=C$ of K. By (9.2) there exists a finite set F_2, with $F_1 \subset F_2 \subset A$ and an étale finite morphism $A_3 \longrightarrow A_2=A \setminus F_2$ such that $V_3=V_1 \times_{A_1} A_3$ is a projective abelian scheme over A_3. Choose an integer $n \geqslant 3$; there exists (use the argument in [1] p.100) a finite étale covering $A_4 \longrightarrow A_3$ such that the family $V_4=V_3 \times_{A_3} A_4 \longrightarrow A_4$ has a level n structure([29] p.131). Consequently $V_4 \longrightarrow A_4$ is the pull back of the universal family $U_{1,n} \longrightarrow M_{1,n}$ of principally polarized abelian schemes of dimension 1 with level n structure([29] p.134 or [25])via some morphism $A_4 \longrightarrow M_{1,n}$. We claim there exists an abelian K-variety B and a finite set $F_4 \subset B$ such that $A_4=B \setminus F_4$. Indeed, $A_4 \longrightarrow A_2=$ $=A \setminus F_2$ is a finite étale morphism and since F_2 has codimension 2 in A, the category $FEt(A)$ of finite étale coverings of A is equivalent to the corresponding category $FEt(A \setminus F_2)$

[24] pp. 42-43 but on the other hand any étale covering of an abelian variety is an abelian variety. Now look at the map $h: B \setminus F_4 \longrightarrow M_{1,n} \longrightarrow M_{1,1}$, where $M_{1,1}$ is the coarse moduli of abelian varieties of dimension 1. Since $M_{1,1}$ identifies with the complex line (via the j-invariant) by Hartogs' theorem h extends to all of B hence it is constant. Consequently the map $A_4 \longrightarrow M_{1,n}$ will also be constant, hence $V_4 \simeq (B \setminus F_4) \times_{\mathrm{Spec}(K)} E$ for some elliptic curve E. But V_4 is a Fuchs model by (3.1) and if $X = B \times_{\mathrm{Spec}(K)} E$ then $X \setminus V_4$ has codimension 2 in X so by (2.3),1) X is a Fuchs model. By (5.10) X is a Poincaré model and the theorem in proved in the case dim(W)=2.

Let's consider now the case dim(W)=1. Again ba Stein factorization we may suppose $f_* \mathcal{O}_V = \mathcal{O}_W$ and W is a smooth curve. As above W will be a Fuchs model with $(R(W))_0 = \mathbb{C}$ so by (7.1) W is a Poincaré model. Now $V \times_W \mathrm{Spec}(\overline{R(W)})$ is a smooth projective Fuchs model which again by Iitaka's inequality must be non-ruled and ~~hence by~~ (2.1) and (8.2) the generic fibre of $V \longrightarrow W$ has irregularity 1 or 2 and Kodaira dimension zero. Hence all fibres of $V \longrightarrow W$

will have $1 \leq q \leq 2$ and $\varkappa = 0$. Let $h: V \longrightarrow P = \mathrm{Alb}^1_{V/W}$ be the W-morphism from (9.1). If $q=1$, P is a smooth complete surface, hence a projective surface and clearly $\varkappa(P) \neq -\infty$; so we may conclude by the first part of the proof. Let's suppose $q=2$. In this case $V \longrightarrow P$ is birational, so by (3.7),1) P is a Fuchs model. Now exactly as in the first part of the proof we may use (9.2) to find an étale finite map $W^* \longrightarrow W$ such that $P^* = P \times_W W^*$ is a projective abelian scheme over W^* carrying a level n structure, $n \geq 3$. Choose any polarization of the family $P^* \longrightarrow W^*$ and let d^2 be its degree [29] p.86. Then $P^* \longrightarrow W^*$ is the pullback of the universal family $U_{2,d,n} \longrightarrow M_{2,d,n}$ of abelian schemes of dimension 2 with polarization of degree d^2 and level n structure ([29] p.134) via some morphism $\mathcal{E}: W^* \longrightarrow M_{2,d,n}$. We claim $\mathcal{E}$ is constant; it will be sufficient to see that all fibres of $P^* \longrightarrow W^*$ are isomorphic. But just look at the period map ([29] p.113): it will be a holomorphic map from the universal covering of W^* (which is the complex affine line) to the Siegel half space of genus 2(which is a bounded domain). Hence the period map is constant and we conclude by"Torelli for abelian varieties" [29]p.115. The constance of $\mathcal{E}$ implies that $P^* \simeq W^* \times_{\mathrm{Spec}(K)} B$ for some abelian surface B over K, hence P^* is an abelian threefold which by (3.1) is a Fuchs model. Now we are done by (5.10),2).

(9.4) Corollary. Let $K \subset L$ be a differential extension with smooth projective Fuchs model V of dimension 3. Suppose K is algebraically closed, $L_o = C$ and one of the following conditions holds:

1) $\varkappa(V)=0$

2) $\varkappa(V) \geqslant 1$ and $\alpha(V) \neq 3$.

Then L has a finite extension which is Poincaré over K.

Proof. If $q(V)=0$ we are done by (5.10),1). If $q(V) \geqslant 1$ consider the Albanese map $f: V \longrightarrow A = \mathrm{Alb}(V)$ and put $W=f(V)$. If $1 \leqslant \dim(W) \leqslant 2$ we are done by Theorem (9.3) because an abelian variety cannot contain rational (singular) curves so $\varkappa(W) \neq -\infty$. If $\varkappa(V)=0$ then a theorem of Ueno [29]p.6

says that f is surjective and a birational map if $q=3$. so we are done again by (3.7),1) and (5.10),2).

(9.5) Remark that (9.4) and (6.6) immediately imply Theorem 3 in the Introduction.

(9.6) The problem of determining all smooth projective Fuchs models of dimension 3 with $\varkappa = -\infty$ should be easier than the classification of all three-folds with $\varkappa = -\infty$. To illustrate this, suppose V is a Fuchs model and that V has a structure of a conic bundle over $\mathbb{P}^2$ [2]. By (3.7) and (8.10),3) the discriminant of the projection $V \longrightarrow \mathbb{P}^2$ must have degree ≤ 3 (and cannot be a smooth cubic). So by [2] pp. 317 and 334 the intermediate Jacobian of V is trivial.

Appendix A. Infinite transcendence degree over C.

Throughout Chapters I, II, III we supposed (see (1.1), (1.3)) that:

a) C=the complex field.

b) All differential fields under consideration (except those of the form Mer($\mathcal{A}$)) have finite transcendence degree over C.

Theese assumptions are quite reasonable in analytic applications; for instance if r=1 and $\mathcal{A} \subset C$ is a domain, then a differential subfield of Mer($\mathcal{A}$) containing C and finitely generated over C (as a differential field) will satisfy condition b) above if and only if it contains no differential transcendental function in Mer($\mathcal{A}$) [17] p.112. Of course there are plenty of differentially transcendental functions in Mer($\mathcal{A}$), but from the point of view of integration of algebraic differential equations theese should be viewed as quite pathological.

Now definitions from Section 1 make sense without the hypothesis b) and one expects that all our results hold without this hypothesis. Moreover there are many partial results whose proofs do not use hypothesis b). But unfortunately the proof of the "key point" (4.6) does use it; indeed we used in an essential way Lemma(2.6) which is obviously false if K is not supposed of finite transcendence degree over C.

So what can be recaptured if we do not make the hypothesis b)?

Surprisingly enough, we are able to recapture the "key point" (4.6) in the "most differentially transcendental" case, namely when $K=C\langle y_1,\dots,y_N\rangle$, where $y_1,\dots,y_N$ are differential indeterminates [17] p.69 (although we are not able to recapture it in "less differentially transcendental" cases). Our method will be to reduce the case $K=C\langle y_1,\dots,y_N\rangle$ to the case $\operatorname{tr.deg.}_C K < \infty$; the way we do it somewhat reminds of the reduction from characteristic zero to characteristic $p > 0$ in algebraic geometry. Here is our result:

(A.1) <u>Theorem</u>. Let $K=C\langle y_1,\ldots,y_N\rangle$, $y_1,\ldots,y_N$ being differential indeterminates and let $K\subset L$ be a differential extension with smooth projective Fuchs model V. Suppose V has C-moduli. Then $\bar{V}=V\times_{\mathrm{Spec}(K)}\mathrm{Spec}(\bar{K})$ is defined over C.

(A.2) <u>Lemma</u>. Let K be as in the Theorem. Then for any finite set $u_1,\ldots,u_s\in K$ there exists a prime differential ideal P in $C\{u_1,\ldots,u_s\}$ such that:

1) $P\cap C[u_1,\ldots,u_s]=0$.

2) $M=R(C\{u_1,\ldots,u_s\}/P)$ has finite transcendence degree over C and $M_o=C$.

(A.3) Let's prove the theorem assuming the Lemma holds; we shall prove in fact more, namely that the Theorem holds for any field K for which the Lemma holds. Therefore it would be very convenient to dispose of the Lemma above for any differential field K containing C.

Let (f,j) be a C-presentation of V, $f:X\longrightarrow S$, $j:R(S)\longrightarrow K$ such that f has a moduli map. We may suppose $S=\mathrm{Spec}(A)$

$A=C[x_1,\ldots,x_m]$. We claim that there exist $a_1,\ldots,a_n \in K$ such that if $B=C\{x_1,\ldots,x_m,a_1,\ldots,a_n\}$ and if $X_B=X\times_S \mathrm{Spec}(B)$ then for any $q\in X_B$ we have $\delta_j(\mathcal{O}_{X_B,q})\subset \mathcal{O}_{X_B,q}$ for all j; note that $\mathcal{O}_{X_B,q}\subset L$ by flatness of f. To prove the claim, cover X with affine open sets $\mathrm{Spec}(R_i)$ and use the fact that, since V is a Fuchs model, we have $\delta_j(R_i\otimes_A K)\subset R_i\otimes_A K$ (see also (2.6)). Now by the Lemma (A.2) applied to $x_1,\ldots,x_m,a_1,\ldots,a_n$ there exists a prime differential ideal P in B such that $P\cap A=0$, $R(B/P)$ has finite transcendence degree over C and $(R(B/P))_0=C$. By (2.7) $R(B/P)$ is finitely generated as a field extension over C. Let T be any smooth Fuchs model of $C\subset R(B/P)$ such that T is an open subscheme in $\mathrm{Spec}(B/P)$ (such a T exists for very simple reasons and may be choosen of the form $\mathrm{Spec}((B/P)_b)$ $b\in B$; we leave this to the reader). Put $X_T=X\times_S T$. Since $P\cap A=0$, $T\longrightarrow S$ is dominant; furthermore $X_T\longrightarrow T$ is smooth projective and has a moduli map. Finally note that since $X_T=X_B\times_{\mathrm{Spec}(B)} T$, the derivations in the local rings of X_B induce derivations in the local rings of X_T, making $R(X_T)$

a differential extension of $R(T)=R(B/P)$ such that X_T is a Fuchs model. Now $W=X_T \times_T \mathrm{Spec}(\overline{R(T)})$ is a smooth projective Fuchs model of its function field, $X_T \longrightarrow T$ being a presentation of W. So W has C-moduli. Furthermore by (2.1) $(\overline{R(T)})_o=C$ so we may apply (4.6) to W and conclude that $W \simeq Z \times \mathrm{Spec}(\overline{R(T)})$ over $\overline{R(T)}$ for some smooth projective C-variety Z. By representability of the functor $S' \longrightarrow \mathrm{Isom}_{S'}(X \times_S S', Z \times S')$ we easily get that there is a finite extension M of $R(S)$ such that $X \times_S \mathrm{Spec}(M) \simeq Z \times \mathrm{Spec}(M)$ over M; hence because M embeds in $\overline{K}$, we get $\overline{V} \simeq X \times_S \mathrm{Spec}(\overline{K}) \simeq Z \times \mathrm{Spec}(\overline{K})$ as desired.

(A.4) Let's prove the Lemma. We proceed in two steps.

Step 1. Put $B=C\{y_1,\dots,y_N\}$ and for any integer k put $B_k=C\left[\delta_1^{a_1}\dots\delta_r^{a_r}y_j;\ 0\le a_i\le k-1,\ 1\le j\le N\right]$. We shall construct differential prime ideals P_k in B such that $P_k \cap B_k=0$, $R(B/P_k)$ has finite transcendence degree over C and $(R(B/P_k))_o=C$. To construct such P_k's choose first an integer $m > kN-1$, choose a prime integer $p > 2m^{r+1}$, let $\varepsilon \in C$ be

$\delta_1^{a_1}\cdots\delta_r^{a_r}y_j$ with $0\leq a_i\leq p-1$, $1\leq j\leq N$.

Step 2. Notations being as in (A.2) and as in Step 1, there exists a nonzero element $b\in B$ such that $C\{u_1,\dots,u_s\}\subset B[b^{-1}]$. One can find an integer $k\geq 1$ such that $b\in B_k$ and $C[u_1,\dots,u_s]\subset B_k[b^{-1}]$. Now since $P_k\cap B_k=0$ we have $b\notin P_k$ so $P_kB[b^{-1}]$ is a prime differential ideal in $B[b^{-1}]$. Put $P=C\{u_1,\dots,u_s\}\cap P_kB[b^{-1}]$. We have $M\subset R(B/P_k)$ hence M has finite transcendence degree over C and $M_o=C$. Finally, $P\cap C[u_1,\dots,u_s]\subset$ $\subset P_kB[b^{-1}]\cap B_k[b^{-1}]=0$.

(A.5) Theorem (A.1) plus Theorem (4.4) and the discussion at(4.8) imply that smooth projective Fuchs models V over $K=C\langle y_1,\dots,y_N\rangle$ are classified provided V is of one of the types listed in (4.4).

Appendix B. The divisor $V \setminus V_F$.

Here we work again under the assumption made at (1.3) about the finite transcendence degree over C.

We shall prove that $V \setminus V_F$ enjoys a remarkable non-negaticity property:

(B.1) Theorem. Let $K \subset L$ be a differential extension with smooth projective model V of dimension $\geqslant 2$ and K algebraically closed. Let E be the reduced divisor whose support is $V \setminus V_F$ (see (2.3)) and suppose E is irreducible smooth and non-zero. Then the conormal bundle N_E^{-1} cannot be ample on E unless the following happens: there exist a smooth projective Fuchs model W of $K \subset L$ and a closed point $p \in W \setminus W_D$ such that V is the blowing up of W at p and E is the exceptional locus.

Proof. By (2.5) and (2.6) we may choose a differential C-presentation (f,j) of V, $f: X \longrightarrow S$ with S a smooth

Fuchs model of $C \subset R(S)$, j algebraic and f smooth. -

Put $Y=X \setminus X_F$; by (2.3),1) Y has pure codimension 1 in X. Consider the exact sequences:

$$(1) \qquad 0 \longrightarrow f_*(T_{X/S} \otimes N_{mY}) \longrightarrow f_*(T_{X/C} \otimes N_{mY}) \longrightarrow T_{S/C} \otimes f_*(N_{mY})$$

$$(2) \qquad 0 \longrightarrow f_*(T_{X/C}) \longrightarrow f_*(T_{X/C}(mY)) \longrightarrow f_*(T_{X/C} \otimes N_{mY})$$

where we identify the Cartier divisor mY with the subscheme of X with ideal sheaf $\mathcal{O}_X(-mY)$. Let η be the generic point of S and put $X_\eta = X \times_S \mathrm{Spec}(R(S))$, $Y_\eta = Y \times_S \mathrm{Spec}(R(S))$ and $g: V \longrightarrow X_\eta$ the canonical projection. Using (2.4),1) it is easy to see that $g^* Y_\eta = E$ as Cartier divisors. For any $\mathcal{O}_S$-module M write $M_K = g^*(M_\eta)$. By flatness of $\mathrm{Spec}(K) \longrightarrow S$ we get

$$(f_*(T_{X/S} \otimes N_{mY}))_K = H^0(T_{V/K} \otimes N_{mE}) \quad \text{and}$$

$$(f_*(N_{mY}))_K = H^0(N_{mE})$$

Now the exact sequences ($k \geqslant 1$)

$$(3) \qquad 0 \longrightarrow N_{kE} \longrightarrow N_{(k+1)E} \longrightarrow N_E^{k+1} \longrightarrow 0$$

and ampleness of N_E^{-1} show that $H^0(N_{mE})=0$. Three cases may occur:

Case 1. (E,N_E^{-1}) is different from both $(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$ and $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$.

Then by a theorem of Wahl [36] (which is the main ingredient of the present proof) we have $H^0(T_{E/K}\otimes N_E^k)=0$ for any $k \geqslant 1$; but this implies, using the standard sequence

$$(4) \qquad 0 \longrightarrow T_{E/K} \longrightarrow T_{V/K}\otimes \mathcal{O}_E \longrightarrow N_E \longrightarrow 0$$

that $H^0(T_{V/K}\otimes N_E^k)=0$ for any $k \geqslant 1$. Using (3) we get by induction that $H^0(T_{V/K}\otimes N_{mE})=0$. Now by the sequences (1) and (2) we get that for any $m \geqslant 1$ we have

$$(5) \qquad (f_*(T_{X/C}))_\eta = (f_*(T_{X/C}(mY)))_\eta$$

Since $\delta_1,\ldots,\delta_r \in H^0(X\setminus Y, T_{X/C}) = \bigcup_m H^0(X, T_{X/C}(mY))$ we get by (5) that $\delta_1,\ldots,\delta_r \in H^0(T_{X_\eta/C})$ which clealy implies $V=V_F$, contradiction.

Case 2. $(E,N_E^{-1})=(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$.

In this case, by Artin's work on contractions V may be obtained as the blowing up of a projective normal surface at some ordinary double point p. By (2.3),1) W will be a Fuchs model and by (2.8) $p \in W_D$. So by (3.3) V must be a Fuchs model, contradiction.

Case 3. $(E, N_E^{-1}) = (\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$.

In this case one knows, after Kodaira, that V is the blowing up of a smooth projective variety W at some point. We conclude again by (2.3),1) and (3.3).

(B.2) The proof above involved two different arguments: Wahl's theorem on one hand and Artin contraction plus (2.8) and (3.3), on the other. It would be interesting to extend these arguments to cover also the case when E is not necessarily smooth or irreducible.

(B.3) If $K \subset L$ is a differential extension and V is a normal model one can define in a natural way multiplicities associated to the irreducible components of $V \setminus V_F$ and get in this way a (not necessarily reduced)

Weil divisor V_E on V whose support is $V \setminus V_F$. One can do this as follows, (see also [23] p.95 for the case n=r=1). If p is a codimension 1 point in $V \setminus V_F$ we define

$$\tilde{n}_p = \sup\{-v_p(\delta_j x);\ 1 \leq j \leq r,\ x \in \mathcal{O}_{V,p}\}$$

$$n_p = \sup\{0, \tilde{n}_p\}$$

where v_p is the valuation on L corresponding to p and then we put

$$V_E = \sum n_p \overline{\{p\}}$$

The whole theory developed in the present paper may be adapted to quasiprojective (instead of projective) Fuchs models V. Results will change accordingly, involving the divisor $\tilde{V}_E$ where $\tilde{V}$ is a projective compactification of V. A key role will be played then by the K-vector space

$$\ker(H^1(\tilde{V}, T_{\tilde{V}/K}) \longrightarrow H^1(\tilde{V}, T_{\tilde{V}/K}(\tilde{V}_E)))$$

which will be a "deformation theoretic measure" of $\tilde{V}_E$; The results one can obtain will be close to the results in the case when V is projective, provided the dimension of the above vector space is small; this will happen if $\tilde{V}_E$ is not too "positive". On the other hand the moral of Theorem (B.1) is that $\tilde{V}_E$ cannot be too "negative".

To illustrate what we said, we give a corollary of analytic nature, in the simplest case, the case of curves:

(B.4) Theorem. Let $K \subset L$ be a differential extension of differential subfields of $\mathrm{Mer}(\mathcal{A})$ with smooth projective model V of dimension 1 and genus g. Suppose that $e = \deg(V_E) < 2g-2$. Then there is a finite embedding of $K \subset L$ into $K^* \subset L^* \subset \mathrm{Mer}(\mathcal{A}^*)$ such that L^* is generated over K^* by functions of the form $\mathcal{F}(\beta_1,..,\beta_e)$ with $\mathcal{F}$ holomorphic functions of e variables and $\beta_1,...,\beta_e \in$ $\in \mathrm{Mer}(\mathcal{A}^{**})$, $\mathcal{A}^{**} \subset \mathcal{A}^*$, $\delta_i \beta_j \in K^*$ for all i,j.

Appendix C. Models with infinitesimal Torelli property.

In Section 4 we used algebraicity results of Popp [29] to prove that some classes of Fuchs models are defined over C. In this Appendix we show how one can use an argument of Viehweg (which has behind it a deep algebraicity result of Sommese) [35] to prove that some new classes of Fuchs models are defined over C. We still work under the assumption(1.3

(C.1) Let V be a smooth projective K-variety of dimension n (no differential structure is assumed). We say V has the infinitesimal Torelli property if the cup-product map

$$H^1(V, T_{V/K}) \longrightarrow \mathrm{Hom}_K(H^0(V, \Omega^n_{V/K}), H^1(V, \Omega^{n-1}_{V/K}))$$

is injective.

It is well known and quite clear that varieties with trivial canonical bundle satisfy the above property. Many other classes of varieties were proved to satisfy this property [29]p.115 but the area of varieties satisfying it seems much smaller than that of varieties with C-moduli. Here is our result:

(C.2) Theorem. Let $K \subset L$ be a differential extension with smooth projective Fuchs model V. Suppose K is algebraically closed and $K_o = C$. Suppose furthermore that V has the infinitesimal Torelli property. Then V is defined over C.

Proof. Choose a differential C-presentation (f,j) of V as in (4.6), $f: X \longrightarrow S$. Semicontinuity theorem and Grauert's theorem [10]p.288 easily imply that there is a Zariski open set $S_o \subset S$ such that for any $y \in S_o$, the fibre $X_y = f^{-1}(y)$ has the infinitesimal Torelli property. By [35]p.574 there exists an étale morphism $\pi: U \longrightarrow S$ a dominant morphism $\sigma: U \longrightarrow Z$ and $\phi: F \longrightarrow Z$ such that $U \times_S X$ is U-isomorphic to $U \times_Z F$ and for $u \in U$, in $T_u U$ we have $T_u(\sigma^{-1}(\sigma(u))) = \pi^*(\ker(\rho_{\pi(u)})$ where $\rho_{\pi(u)}: T_{\pi(u)}S \longrightarrow H^1(X_{\pi(u)}, T_{X_{\pi(u)}/C})$ is the Kodaira-

Spencer map (here Z and F are algebraic schemes). But now one can immitate the reasoning from the proof of (4.6) (with Z instead of M) and the proof of (C.2) may be concluded without any problem.

(C.3) Theorem (C.2) would not have been sufficient to prove the classification results in Chapters II or III; indeed counterexamples are known (for instance) to the "infinitesimal Torelli" for surfaces of general type !

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