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ON THE CONVERGENCE OF THE GAUSS-SEIDEL ITERATION FOR SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS WITH PARTITIONED MATRIX

by

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Abstract

The coefficient matrix of the system of linear equations is supposed to be partitioned such that the submatrices on the diagonal have the number of columns equal to the number of lines.

In appling the Gauss-Seidel iteration, the operation with submatrices insted of elements leads to solving some systems of linear equations, the matrices of which are submatrices on the diagonal of the initial matrix. The approximative solutions of these sets of equations is suggested, performing for each of them a certain number of Gauss-Seidel iterations which operate with elements.

The total error is estimated for this double iterative process, and the number of iterations is determined for every subsystem of equations, provided that the error obtained and the computing time be in an optimal relation.

This optimal number of iterations depends neither on the total imposed error, nor on computing time , but on the matrix partitioning and on the ratio of the computing time relation .

1. Introduction

In literature, there are a lot of iterative methodes for linear equations.We can mention ,for instance , methods of the SOR type , of the gradient type a.s.o.(see [4], [5], [6]).It was noticed that a better convergence is generally obtained when the coefficient matrix is partitioned and when the operation is performed with submatrices instead of elements .

In this paper we shall consider the set of linear equations.

$$Ax = b , \qquad (1.1)$$

where the matrix $A = (a_{ij})_{1 \leq i , j \leq N}$ is assumed to be symmetric and positive definite.Moreover , we shall assume a partitioning of the matrix of the form .

$$A =(A_{ij})_{1 \leq i,j \leq M} , \qquad (1.2.)$$

M being the number of submatrices on the diagonal .

In [1] it was studied the convergence of the Jacobi and Gawss - Seidel methods , the coefficient matrix being thus partitioned .

One reached to the conclusion that the Jacobi method converges only if $M = 2$, a particular case being when the matrix has property A (see [9]).

As to the Gauss - Seidel method , this converges the faster, the smaller M is.But in this case, for every itteration, we have to solve M sets of equations ,

$$A_{ii}x_i^{(m)} = Bi , 1 \leq i \leq M , \qquad (1,3,)$$

where $x^{(m)}=(x_i^{(m)})_{1 \leq i \leq M}$ is the solution to the iteration m .

For the SOR method (SBOR methods) or for implicit procedures (ADIP, SIP) , it is suggested in [3] ,[6] and [7] to solve the sets of equations (1.3.) by direct methodes.

In this paper we shall study the convergence when the sets of equations (1.3.) are solved by the standard Gauss-Seidel method , i.e. the submatrices are of order 1 x 1. For a fixed (1.2.) partition, the comple of iterations (n,m) to obtain the $x_i^{(n,m)}$ approximation of $x_i^{(m)}$, $1 \leq i \leq M$, will be found in the condition that the computing time and the total error be in an optimal relation .

The iterations for solving simulataneons equations (1.3.) will be called inner iterations because they are performed within the iterations which use submatrices.The latter will be called outer iterations .

2.Error, optimal number of iterations

The Gauss - Seidel iteration attached to the simultaneons equations(1.1.) with the partitioned matrix under the form (1.2.) will be written for m = 1,2 ,...

$$A_{ii} x_i^{(m)} + \sum_{j=1}^{i-1} A_{ij} x_j^{(m)} + \sum_{j=i+1}^{M} A_{ij} x_j^{(m-1)} = b_i \,, \; 1 \leq i \leq m, \qquad (2.1.)$$

where $x^{(0)} \in \mathbb{R}^N$ is any initial vector, and the right - hand vector and the solution were segmented according to the coefficient matrix partitioning

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_M \end{pmatrix}, \quad x^{(m)} = \begin{pmatrix} x_1^{(m)} \\ \vdots \\ x_M^{(m)} \end{pmatrix}.$$

Obviously , equations (2.1.) can be written as :

$$\sum_{i=1}^{M} a(x_i^{(m)}, y_i) + \sum_{\substack{i,j=1 \\ i<j}}^{M} a(x_i^{(m)}, y_j) + \sum_{\substack{i,j=1 \\ i>j}}^{M} a(x_i^{(m-1)}, y_j) = (b, y), \quad (2.2.)$$

satisfied for any

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_M \end{pmatrix} \in \mathbb{R}^N,$$

where

$$a(x,y) = y^T A x,$$
$$a(x_i, y_j) = y_j^T A_{ji} x_i, \quad 1 \le i, j \le M,$$
$$(b, y) = y^T b.$$

We shall denote the Euclidian norm of the vector $y \in \mathbb{R}^N$ by

$$|y| = (y, y)^{1/2} = \left[\sum_{i=1}^{N} y_i^2 \right]^{1/2}.$$

Since the matrix A is symmetric and positive definite, we can consider some other two norms,

$$|y|_a = a(y, y)^{1/2},$$
$$\|y\|_a = \left[\sum_{i=1}^{M} a(y_i, y_i) \right]^{1/2}.$$

Let us denote by λ and μ the smallest and, respectively, the largest eigenvalue of the problem $Az = \lambda D z$, where

$$D = \begin{pmatrix} A_{11} & & 0 \\ & \ddots & \\ 0 & & A_{MM} \end{pmatrix}.$$

We shall also denote by α' and β' (α'' and β'') the smallest and, respectively, the largest eigenvalue of the matrix A (D).

It results that

$$\lambda \|y\|_a^2 \le |y|_a^2 \le \mu \|y\|_a^2 , \qquad (2.3.)$$

$$\alpha' |y|^2 \le |y|_a^2 \le \beta' |y|^2 ,$$
$$\alpha'' |y|^2 \le \|y\|_a^2 \le \beta'' |y|^2 . \qquad (2.4.)$$

In the following , for a vector $y = \begin{pmatrix} y_1 \\ \vdots \\ y_M \end{pmatrix} \in \mathbb{R}^N$, instead of

$$y = \begin{pmatrix} y_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ y_M \end{pmatrix} ,$$

we shall write :

$$y = y_1 + y_2 + \cdots + y_M .$$

Remark.

The eigenvalues defined above satisfy the relations :

$$\alpha' \le \alpha'' \le \beta'' \le \beta' ,$$
$$\alpha'/\beta'' \le \lambda \le 1 \le \mu \le \beta'/\alpha'' , \qquad (2.5.)$$
$$1/(M-1) \le (\mu-1)/(1-\lambda) \le M-1 .$$

Indeed, the first relation is obtained by using the fact that the characteristic polynomials of the matrices, derived one from the other by eliminating one line and the corresponding column, form a Sturm sequence .It can be easily seen that

$$\frac{\alpha'}{\beta''} \|y\|_a^2 \le |y|_a^2 \le \frac{\beta}{\alpha''} \|y\|_a^2$$

for any $y \in \mathbb{R}^N$. Therefore, $\alpha'/\beta'' \le \lambda \le \mu \le \beta'/\alpha''$

We can deduce that

$$M\|y\|_a^2 - |y|_a^2 = \sum_{\substack{i,j=1\\ i<j}}^{M} a(y_i - y_j, y_i - y_j) \geq \lambda \sum_{\substack{i,j=1\\ i<j}}^{M} \left[a(y_i, y_i) + a(y_j, y_j)\right] =$$
$$= \lambda(M-1)\|y\|_a^2 .$$

Therefore ,

$$[\lambda + (1-\lambda)M]\|y\|_a^2 \geq |y|_a^2 \geq \lambda\|y\|_a^2 ,$$

and, consequently,

$$\lambda \leq 1 \text{ and } \mu \leq \lambda + (1-\lambda)M.$$

Analogously , it results that $\mu + (1-\mu)M \leq \lambda$ and $\mu \geq 1$, and therefore inequalities (2.5) are true .

Suppose that matrices Aii, $1 \leq i \leq M$, are of the order $N_i \times N_i$, $\sum_{i=1}^{M} N_i = N$, and are partitioned such that a submatrix is formed only by one element. Obviously , for every $i = 1, 2, \ldots, M$ we can consider $\mathbb{R}^{N_i}$, with matrix A_{ii} instead of $\mathbb{R}^N$ and A , respectively . If $\lambda_i, \mu_i, \alpha'_i, \beta'_i, \alpha''_i, \beta''_i, 1 \leq i \leq M$, correspond to $\lambda, \mu, \alpha', \beta', \alpha'', \beta''$, we shall have :

$$\alpha'' = \min_{1\leq i\leq M} \alpha'_i , \quad \beta'' = \max_{1\leq i\leq M} \beta'_i , \text{ and if}$$

$$\alpha''' = \min_{1\leq i\leq M} \alpha''_i , \quad \beta''' = \max_{1\leq i\leq M} \beta''_i ,$$

thery are the smallest and, respectively , the largest elements on the diagonal of matrix A .

From (2.5) the following inequalites result

$$\begin{aligned} &\alpha'' \leq \alpha''' \leq \beta''' \leq \beta'' , \\ &\alpha''/\beta'' \leq \nu \leq 1 \leq \eta \leq \beta''/\alpha'' , \qquad (2.6) \\ &1/(L-1) \leq (\eta - 1)/(1-\nu) \leq L-1 , \end{aligned}$$

where

$$\nu = \min_{1\le i\le M} \lambda_i \,,\; \eta = \max_{1\le i\le M} \mu_i \,,\; L = \max_{1\le i\le M} N_i \,.$$

Now we shall modify the Gauss- Seidel iteration (2.1), taking into account that for every outer iteration $m=1,2,\ldots$, simultaneons equation (1.3.) will be solved approximately. Suppose that for every set of equations (1.3) we shall performe n inner iterations .

We obtain the approximation $x^{(n,m)}$ to the solution $x^{(m)}$ of the equation.

$$\sum_{i=1}^{M} a(x_i^{(m)}, y_i) + \sum_{\substack{i,j=1\\ i<j}}^{M} a(x_i^{(n,m)}, y_j) + \sum_{\substack{i,j=1\\ i>j}}^{M} a(x_i^{(n,m-1)}, y_j) = (b, y), \qquad (2.7)$$

satisfed for any $y \in \mathbb{R}^N$, where $x^{(n,m-1)}$ is the approximation to $x^{(m-1)}$ after n inner iterations for simultaneons equations (1.3.).

Obviously , when sets of equations (1.3.) are exactly solved, equation (2.7) becomes equation (2.2).

Remark

For any two vectors, $x=(x_i)_{1\le i\le M}$, $y=(y_i)_{1\le i\le M}$, segmented according to the partitioning of matrix A , the following inequalities are satisfied

$$\left|a(x_i, \sum_{j=1}^{i-1} y_j)\right| \le (1-\lambda)^{1/2}(\mu-1)^{1/2} a(x_i,x_i)^{1/2} \left[\sum_{j=1}^{i-1} a(y_j,y_j)\right]^{1/2}, \quad 2\le i\le M,$$

and

$$\left|a(\sum_{i=j+1}^{M} x_i, y_j)\right| \le (1-\lambda)^{1/2}(\mu-1)^{1/2} a(y_j,y_j)^{1/2} \left[\sum_{i=j+1}^{M} a(x_i,x_i)\right]^{1/2}, \quad 1\le j\le M-1. \qquad (2.8)$$

Indeed , from relations (2.3), for any $\varepsilon \in \mathbb{R}$ and $2 \le i \le M$ we have :

$$\lambda\Big[\varepsilon^2 a(x_i,x_i)+\sum_{j=1}^{i-1} a(y_j,y_j)\Big] \le a\Big(\varepsilon x_i+\sum_{j=1}^{i-1} y_j,\ \varepsilon x_i+\sum_{j=1}^{i-1} y_j\Big) \le$$
$$\le \mu\Big[\varepsilon^2 a(x_i,x_i)+\sum_{j=1}^{i-1} a(y_j,y_j)\Big],$$

whence it results that ,

$$\Big|a\Big(x_i,\sum_{j=1}^{i-1} y_j\Big)\Big| \le (\mu-1)^{1/2} a(x_i,x_i)^{1/2}\Big[\mu\sum_{j=1}^{i-1} a(y_j,y_j) - a\Big(\sum_{j=1}^{i-1} y_j,\sum_{j=1}^{i-1} y_j\Big)\Big]^{1/2}$$

and

$$\Big|a\Big(x_i,\sum_{j=1}^{i-1} y_j\Big)\Big| \le (1-\lambda)^{1/2} a(x_i,x_i)^{1/2}\Big[a\Big(\sum_{j=1}^{i-1} y_j,\sum_{j=1}^{i-1} y_j\Big) - \lambda\sum_{j=1}^{i-1} a(y_j,y_j)\Big]^{1/2}.$$

From the last two inequalities , the first of relations (2.8) is obtained. The second of relations (2.8) is analogous .

We shall write $y^{(m)} = x - x^{(m)}$, $y^{(n,m)} = x - x^{(n,m)}$, where x is the solution to set of equations (1.1), $x^{(m)}$ and $x^{(n,m)}$ are the previously described approximations. We have the following

<u>Lemma 2.1.</u> - For the iterative process (2.7) the following relations are satisfied :

$$\|y^{(m)}-y^{(n,m-1)}\|_a^2 - \|y^{(n,m)}-y^{(m)}\|_a^2 + |y^{(n,m)}|_a^2 - |y^{(n,m-1)}|_a^2 = 0, \quad (2.9)$$

$$|y^{(n,m)}|_a \le [(1-\lambda)(\mu-1)(M-1)/\lambda]^{1/2}\big(\|y^{(n,m)}-y^{(m)}\|_a +$$
$$+\|y^{(n,m-1)}-y^{(m)}\|_a\big) + (1/\lambda)^{1/2}\|y^{(n,m)}-y^{(m)}\|_a, \quad (2.10)$$

and when $x^{(m)} = x^{(n,m)}$, the case of process (2.2), we have :

$$|y^{(m)}|_a^2 + \|y^{(m)}-y^{(m-1)}\|_a^2 - |y^{(m-1)}|_a^2 = 0, \quad (2.11)$$

$$|y^{(m)}|_a \le \left[(1-\lambda)(\mu-1)(M-1)/\lambda\right]^{1/2} \|y^{(m)} - y^{(m-1)}\|_a . \qquad (2.12)$$

<u>Proof</u> : Subtracting equation (2.7) from equation

$$a(x,y) = (b,y), \text{ for any } y \in \mathbb{R}^N,$$

we obtain the equation

$$\sum_{i=1}^{M} a(y_i^{(m)}, y_i) + \sum_{\substack{i,j=1 \\ i<j}}^{M} a(y_i^{(n,m)}, y_j) + \sum_{\substack{i,j=1 \\ i>j}}^{M} a(y_i^{(n,m-1)}, y_j) = 0, \qquad (2.13)$$

for any $y \in \mathbb{R}^N$.

Substituting $y = y^{(n,m)} - y^{(n,m-1)}$ into this equation, we obtain another form of relation (2.9). Substituing $y = y^{(n,m)}$ into the same equation (2.13) , we obtain

$$a(y^{(n,m)}, y^{(n,m)}) = \sum_{i=1}^{M} a(y_i^{(n,m)} - y_i^{(m)}, y_i^{(n,m)}) + \sum_{\substack{i,j=1 \\ i>j}}^{M} a(y_i^{(n,m)} - y_i^{(n,m-1)}, y_j^{(n,m)}),$$

and, making use of (2.8), it results ,

$$|y^{(n,m)}|_a^2 \le \left[\sum_{j=1}^{M-1} (1-\lambda)(\mu-1) \sum_{i=j+1}^{M} a(y_i^{(n,m)} - y_i^{(n,m-1)}, y_i^{(n,m)} - y_i^{(n,m-1)})\right]^{1/2} \|y^{(n,m)}\|_a + \|y^{(n,m)} - y^{(m)}\|_a \|y^{(n,m)}\|_a .$$

In order to obtain (2.1o), in the last inequality , we increase the right member tacking into account (2.3). Relations (2.11) and (2.12) are particular cases of (2.9) and (2.1o), respectively .

Note that, in the previons lemma, $y^{(n,m)}$ can be any approximation of $y^{(m)}$. In every case, interpreting (2.9), the iterative process diverges if $\|y^{(m)} - y^{(n,m-1)}\|_a \le \|y^{(n,m)} - y^{(m)}\|_a$
occurs systematically.

The error estimate for the iterations of simultaneons equations (1.3.) is given by

Proposition 2.1.

With the notations from the previous lemma to any outer iteration m of the iterative process (2.7) , after performing n inner iterations for every simultaneons equations of the type (1.3.), we have

$$\|y^{(m)} - y^{(n,m)}\|_a \leq \Delta^n \|y^{(m)} - y^{(n,m-1)}\|_a , \qquad (2.14)$$

where $\Delta = \dfrac{\gamma}{(\gamma^2+\lambda)^{1/2}}$, $\gamma = [(L-1)(1-\nu)(\eta-1)]^{1/2}$.

Proof:

Relations (2.11) and (2.12) give

$$|y^{(m)}|_a \leq \Gamma^m |y^{(0)}|_a , \qquad (2.15)$$

where $\Gamma = \dfrac{\delta}{(\delta^2+\lambda)^{1/2}}$, $\delta = [(1-\lambda)(\mu-1)(M-1)]^{1/2}$.

For every simultaneons equations (1.3.) solved by the standard Gauss- Seidel iteration, we can establish an inequality of the type (2.15) .Summing up these inequalities and taking account of (2.6) we obtain (2.14).

A total error estimate is given by the following

Theorem 2 .1.

An error estimate of the iterative process (2.7) making use of the previous noations can be given by

$$|y^{(n,m)}|_a \leq E^m |y^{(0)}|_a , \qquad (2.16)$$

where $$E = \frac{\delta(1+\Delta^n) + \Delta^n}{\{[\delta(1+\Delta^n)+\Delta^n]^2 + \lambda(1-\Delta^{2n})\}^{1/2}}$$

Proof

From relations (2.9) and (2.14) it results that

$$|y^{(n,m)}|_a^2 - |y^{(n,m-1)}|_a^2 + (1-\Delta^{2n})\|y^{(m)} - y^{(n,m-1)}\|_a^2 \leq 0,$$

and from (2.10) and (2.14),

$$|y^{(n,m)}|_a^2 \leq [\delta + \Delta^n(\delta+1)]^2 \|y^{(m)} - y^{(n,m-1)}\|_a^2 / \alpha .$$

Estimation (2.16) is obtained making use of the last two inequalities .

In the following we shall determine the number n of inner iterations and the number m of outer iterations, provided that the computing time is given by the formula

$$t = m(\tau + \theta n), \qquad (2.17)$$

where $\tau, \theta > 0$, and the error estimate given by (2.16) be in an optimal relation .More precisely , se shall state the following optimal problems.

Problem 2.1.

$$\min_{n,m} E^{m}(n),$$

with constraints $m(\tau + \theta n), n > 0, m > 0,$
where $T > 0$.

Problem 2.2.

$$\min_{n,m} m(\tau + \theta n),$$

with constraints $E^{m}(n) \leq \varepsilon, n > 0, m > 0,$
where $0 < \varepsilon < 1$.

For the sake of simplicity we assume that m and n can take any positive values.

Notice that for the two problems the minimum is reached on the boukdary , and the value of n is the same, Consider the function

$$f(\pi_6) = E(\pi_6)^{1/(\tau+\theta\pi_6)} \tag{2.18}$$

Notice that the point $\bar{\pi}_6 \in \mathbb{R}$, $\bar{\pi}_6 > 0$, if it exists , which gives the minimum of this function is the optimal number of inner iteration of the problems under consideration .The functions $\frac{df(\pi_6)}{d\pi_6}$ and

$$h(\pi_6) = -(\tau+\theta\pi_6)/\theta + E(\pi_6)\ln E(\pi_6)/\left(\frac{dE(\pi_6)}{d\pi_6}\right) \tag{2.19}$$

are of the same sign.The derivative of function (2.19) is

$$\frac{dh(\pi_6)}{d\pi_6} = \left[\left(\frac{dE(\pi_6)}{d\pi_6}\right)^2 - \frac{d^2E(\pi_6)}{d\pi_6^2}E(\pi_6)\right]\ln E(\pi_6)/\left(\frac{dE(\pi_6)}{d\pi_6}\right)^2,$$

and it can be shown that this has at most one zero, and for larger values of π_6 it is positive.Since

$$\lim_{\pi_6\to 0} h(\pi_6) = -\tau/\theta, \quad \lim_{\pi_6\to\infty} h(\pi_6) = \infty,$$

it results that $h(\pi_6)$ has a uniqre zero, which is also the minimal point of function (2.18). Therefore, problems 2.1. and 2.2. have a unique solution .

Finally , considering the inverse of the time - error product as an objective function we can show that,

<u>Problem 2.3.</u> $$\max_{\tau_6>0}\ \min_{\pi_6>0}\ E(\pi_6)^{-\pi_6}\left[\pi_6(\tau+\theta\pi_6)\right]^{-1},$$

has a unique solution and the ~~value of~~ π_6 in the optimal point is $\bar{\pi}_6$, which is the optimal number of inner iterations for problems 2.1. an 2.2.

Summing up, we obtain the following ...

Proposition 2.2.

Problems 2.1, 2.2 and 2.3 have unique solutions $(\overline{\tau}, \overline{m}_1)$, $(\overline{\tau}, \overline{m}_2)$, and $(\overline{\tau}, \overline{m}_3)$, respectively , where $\overline{\tau}$ is the solution of equation

$$h(\tau) = 0, \qquad (2.20)$$

and $\overline{m}_1 = T/(\tau + \theta\overline{\tau})$, $\overline{m}_2 = \ln\varepsilon / \ln E(\overline{\tau})$, $\overline{m}_3 = 1/\ln E(\overline{\tau})$.

3. Practical checking of the error, conclusions

The following propositon allows us to check the error by means of the last two approximations of the solution .

Proposition 3.1.

For the iterative process (2.7) the error estimate (2.16) holds true when we note $y^{(n,m)} = x^{(n,m+1)} - x^{(n,m)}$, $y^{(0)} = x^{(n,1)} - x^{(0)}$. The relation corresponding to the error estimate for the inner iterations is $\|x^{(n+1,m)} - x^{(n,m)}\|_a \leq \Delta \|x^{(n,m)} - x^{(n-1,m)}\|_a$ for any $m \geq 1$.

Proof

If relation (2.13) were satisfied when $y^{(m)} = x^{(m+1)} - x^{(m)}$, $m \geq 1$, and $y^{(n,m)}$ with the new significance , then relations (2.9) - (2.12) having the new significations can be deduced analogously to the proof to Lemma 2.1. The new relation (2.13) results substracting equation (2,7) of iteration m from that of iteration m+1 .In order to prove the relation corresponding to inequality (2.14), let the k-th set of equations (1.3) of the m-th outer iterations

$$\sum_{i=1}^{N_k} a(x_{ki}^{(n,m)}, y_{ki}) + \sum_{\substack{i,j=1 \\ i<j}}^{N_k} a(x_{ki}^{(n,m)}, y_{kj}) + \sum_{\substack{i,j=1 \\ i>j}}^{N_k} a(x_{ki}^{(n-1,m)}, y_{kj}) = (b_k^{(m)}, y_k).$$

Substracting this equation from

$$a(x_k^{(m)}, y_k) = (b_k^{(m)}, y_k),$$

we obtain

$$\sum_{i=1}^{N_k} a(x_{ki}^{(m)} - x_{ki}^{(n,m)}, y_{ki}) + \sum_{\substack{i,j=1\\ i<j}}^{N_k} a(x_{ki}^{(m)} - x_{ki}^{(n,m)}, y_{kj}) + \sum_{\substack{i,j=1\\ i>j}}^{N_k} a(y_{ki}^{(m)} - y_{ki}^{(n-1,m)}, y_{kj}) = 0$$

Substracting this equation from the one corresponding to iteration m+1 we have

$$\sum_{i=1}^{N_k} a(y_{ki}^{(m)} - y_{ki}^{(n,m)}, y_{ki}) + \sum_{\substack{i,j=1\\ i<j}}^{N_k} a(y_{ki}^{(m)} - y_{ki}^{(n,m)}, y_{kj}) + \sum_{\substack{i,j=1\\ i>j}}^{N_k} a(y_{ki}^{(m)} - y_{ki}^{(n-1,m)}, y_{kj}) = 0$$

Starting from this equation and considering $y^{(0,m)} = y^{(n,m-1)}$, as in the previons proofs, we can verify

$$\|y^{(m)} - y^{(n,m)}\|_a \leq \Delta^n \|y^{(m)} - y^{(n,m-1)}\|_a .$$

Respeating the reasoning from the proof of Theorem 2.1. we can obtain the relation analogous to estimation (2.16). If we consider $x^{(n,m)} = x^{(m)}$, $\Delta = 0$ in the last estimation, one obtains the relation corresponding to inequality (2.15). As in the proof to Proposition 2.1. we obtain the second relation we had to prove.

Remarks

1. From what we stated above it results that (2.15) is the error estimate when sets of equations (1 .3) are exactly solved and (2.16) gives the error estimate when (1.3.) are iteratively , solved. As to these estimations se shall notice that :

a) If M = 1, then $\lambda = \mu = 1$ and (2,16) is analogous to (2.15). This is in confirmation that for M=1, the iterative process (2,7) becomes the very standard Gauss- Seidel iterative process.

b) If $M=1$, from (2.15) we obtain $x = x^{(m)}$. Hence, obviously , when we have only one submatrix on the diagonal and the corresponding system of equations is exactly solved , we obtain the exact solution from the first step .

c) Estimation (2.16) becomes of the (2.14) type for $n \longrightarrow \infty$. This reflects the fact that in this case systems of equations (1.3.) have an exact solution .

d) If in estimation (2.15) we replace $\lambda = \alpha'/\beta''$ and $M = \beta'/\alpha''$ we obtain a weaker error estimate which depends only on the number of submatrices on the diagonal, M.

2. From Proposition 2.2. it results that the optimal number of inner iterations , $\bar{\tau}_0$, depends neither on the time T, nor on the error ε . This number depends only on the matrix partitioning and on the ratio τ/θ .

3. We can show that for T large enough or ε small enough the optimal number of inner iterations is $[\bar{\tau}_0]$ or $[\bar{\tau}_0] + 1$, where we denoted by ([ ]) the integral part.

By means of the following tests one tried to find out to what extent the real error has the optimal properties, pointed out for the error estimate found by us.

For the considered examples, as will be seen, there is an optimal number of inner iterations, but it is smaller than the number determined by equation (2.2o).

The first set of tests was carried out by means of the simmultaneons equations with coefficient matrix of the type

$$\begin{pmatrix} 1 & a & a & \dots & a & a \\ a & 1 & a & \dots & a & a \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & a & a & & a & 1 \end{pmatrix}.$$

These matrices are positive definite for $-\frac{1}{N-1} < a < 1$ (see [2], [6]).

We made partitions of the type $L=2, M=2$; $L=10, M=2$; $L=5, M=5$.

For a fixed a and a given partition, starting from the same initial vector, we found the optimal number of inner iterations such that the computing time be minimum, in order that the absolute error be smaller than 10^{-5}. Then, under the same conditions, we measured the computing time for the error to be smaller than 10^{-10}. In both cases the minimum time was found for the same number of inner iterations. For all tests of this type the optimal number of inner iterations was equal to 1 for small absolute values of a, and equal to 2 for large absolute values of a. We illustrate this type of tests taking as an example $a = .65$ $L = 5$, $M = 5$, the absolute error smaller than 10^{-5}.

no of inner iterations	no of outer iterations	computing time(sec.)	error
1	634	5.17	.98570E-05
2	251	3.03	.9771E-05
3	221	3.57	.97151E-05
4	213	4.40	.93277E-05
5	209	5.52	.95399E-05
6	207	5.98	.96362E-05
7	206	6.90	.95435E-05
8	205	7.76	.97556E-05
9	205	8.28	.94744E-05
10	205	9.40	.93535E-05

We notice that the optimal number of inner iterations is equal to 2.

The second set of tests was performed by the equations of the finite element system that correspond to the following Dirichlet problem

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - u = -x_1 x_2 (x_1 - 1)(x_2 - 1) + 2x_1(x_1 - 1) + 2x_2(x_2 - 1),$$

$$(x_1, x_2) \in D = (0,1) \times (0,1),$$

with the boundary conditon $u|_{\partial D} = 0$.

The solution of this problem is

$u = x_1 x_2 (x_1 - 1)(x_2 - 1)$ (see [8]).

Considering the side of the square D divided into 4o equal segments, the resulting finite element system has 1521 equations and a half- bandwidth of 4o. The coefficient matrix was partitioned such that on the diagonal resulted 3 submatrices of order 5o7 x 5o7 each .We looked for the optimal number of inner iterations such that we may obtain a minimum error in an interval of 12oo sec. The error wa determined by means of the vector norm

$$|x| = \max_{1 \le i \le N} |x_i| \quad , \; x \in \mathbb{R}^N .$$

The following table was obtained :

no of inner iterations	no of outer iterations	error
1	251	.13139E-1
5	1o7	.26o31E-2
7	83	.22851E-2
9	68	.22342E-2
1o	63	.21911E-2
11	58	.227o1E-2
15	44	.2745oE-2
2o	34	.3539oE-2

1	2	3
36	2o	.72344E-2
84	9	.198o2E-1

We notice that the optimal number of inner iterations is equal to 1o .If we make the computing time of 18oo sec. we obtain :

no of inner iterations	no of outer iterations	error
9	1o2	.455o7E-3
1o	94	.45456E-3
11	87	.46625E-3

We notice that the optimal number of inner iterations does not depend on the computing time .

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