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Sorin POPA

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Sorin POPA*)

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*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd.Pacii 220, 79622 Bucharest, Romania.

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INTRODUCTION

In [3] B.Johnson and S.Parrott proved the beautiful result that if an operator $T \in B(H)$ commutes modulo the set of compact operators $K(H)$ with all the operators in a von Neumann algebra $M \subset B(H)$ which does not contain certain type II_1 factors, then T is a compact perturbation of an operator in the commutant of M. In what follows we say that a von Neumann algebra M has the property (JP) if the preceding theorem holds true for M. The argument used in their paper is quite elementary. The idea is to prove first that the abelian algebras have property (JP). Next one uses that a unitary operator with absolutely continuous spectrum cannot commute with a nonzero compact operator. So, if $A \subset M$ is a diffuse abelian von Neumann subalgebra and $\delta : M \to K(H)$ is a derivation such that $\delta(a) = [K,a]$ for some $K \in K(H)$ and all $a \in A$, then the derivation $\delta' = \delta - \mathrm{ad}\,K$ takes values in $K(H)$ as well and in addition vanishes on A. Thus if $u \in M$ is a unitary element that normalizes A, $uAu^* = A$, and if $\sigma(a) = uau^*$, then $ux = \sigma(x)u$ so that for all $a \in A$, $\delta'(u)a = \delta'(ua) = \delta'(\sigma(a)u) = \sigma(a)\delta'(u)$. This shows that $|\delta'(u)|$ is in A' and since it is compact it follows that $|\delta'(u)| = 0$, hence $\delta'(u) = 0$ and $\delta(u) = [K,u]$. This simple argument and the important fact that δ is automatically weakly continuous show that if a von Neumann algebra M has a diffuse abelian subalgebra A such that its normalizer in M, $N(A)$, generates M then M has the property (JP). It then follows immediately that the von Neumann algbras with diffuse center and the properly infinite von Neumann algebras do have the property (JP). The problem is so reduced to the study of the type II_1 factor case. Furthermore by the above remarks if M is a type II_1 factor that has a Cartan subalgebra (i.e. a maximal abelian subalgebra with normalizer generating M) or splits such a factor (i.e. $M = M_1 \otimes M_2$ where M_1 is of type II_1 with a Cartan subalgebra) then M has the property (JP).

Our main purpose in this paper is to introduce and discuss an intrinsic property for II_1 factors that implies the property (JP). This property, that we call (C), is weaker than property Γ of Murray and von Neumann, is stable under tensor products (i.e. if M has (C) then $M \otimes N$ has (C) for any N) and holds true for factors with Cartan subalgebras.

Like the property Γ, the property (C) is an "asymptotic commutativity" type property. Unfortunately it seems to be difficult to verify. Actually, besides the factors known (aprioricaIly) to have Cartan subalgebras or the propety Γ, we don't know other examples of factors with propety (C). For instance, are the factors coming from the free groups property (C) factors?

The main technical result of the paper allowing us to prove that (C)$\Longrightarrow$(JP) is that any derivation of M into the compacts is continuous from the unit ball of M with the strong operator topology into $K(H)$ with the norm topology. For the proof of this result we use the Johnson and Parrott result that the abelian algebras have the property (JP). Note that if a derivation is implemented by a compact operator then, of course, it is so-normic continuous.

1. A PROPERTY OF II_1 FACTORS

In what follows M will always be a type II_1 factor acting on a separable Hilbert space H. The unique normalized trace on M is denoted by τ and the corresponding norm by $\|\cdot\|_2$. If $B \subset M$ is a von Neumann subalgebra (always assumed to have the same unity as M) then E_B denotes the unique trace preserving conditional expectation on B. If $\varepsilon > 0$ then an ε-partition of the unity in M is a finite dimensional abelian von Neumann subalgebra $A_o \subset M$ with $\tau(e) < \varepsilon$ for any minimal projection $e \in A_o$.

DEFINITION. M has the *property* (C) if for any finite set of elements $x_1, \ldots, x_n \in M$ and $\alpha > 0$ there exists a finite set $y_1, \ldots, y_m \in M$ such that

(a) any x_i is α-contained in Span $\{y_j\}_j$;

(b) for any $\varepsilon > 0$ there exist mutually commuting ε-partitions of the unity $A_1, \ldots, A_m$ such that

$$\|E_{A_j' \cap M}(y_j) - y_j\|_2 < \varepsilon, \qquad j = 1, \ldots, m.$$

(an element $x \in M$ is α-contained in a set $S \subset M$ if there exists $s \in S$ such that $\|x - s\|_2 < \alpha$).

It is clear that the conditions in the preceding definition are very difficult to verify. This is mainly because the asymptotic commutativity condition (b) concerns the intermediate elements y_j. These, in turn, are related to the initial x_i's by the "bad" condition (a).

PROPOSITION. *If* M *has any of the following properties then it has property* (C).

(i) $M \simeq M_o \otimes N$ *with* M_o *a type* II_1 *factor with property* (C).

(ii) *There exists an increasing sequence of subfactors* $M_n \subset M$ *such that* $M = \overline{\bigcup_n M_n}^w$ *and each* M_n *has property* (C).

(iii) M *has property* Γ *(see [6])*.

(iv) M *has a Cartan subalgebra, i.e. a maximal abelian subalgebra* A *with the normalizer* $N(A)$ *generating* M.

PROOF. (i) and (ii) are immediate consequence of the definition.

Suppose M has property Γ. Let ω be a free ultrafilter on $\mathbb{N}$ and denote by M^ω the corresponding ultrapower algebra (see [5]). Then $M' \cap M^\omega$ is a diffuse algebra (cf.[2]). So, given $x_1, \ldots, x_n \in M$ and $\varepsilon > 0$ there exists an ε-partition of the unity A_o in M^ω such that $E_{A_o' \cap M^\omega}(x_i) = x_i$. If $e_1, e_2, \ldots, e_r \in M^\omega$ are the minimal projections of A_o and $e_i = (e_i^k)_k$ for some projections $e_i^k \in M$ with $\sum_i e_i^k = 1$ and $\tau(e_i^k) = \tau_\omega(e_i)$ for all k and i, it follows that for k large enough the algebra A_1 generated by $e_1^k, e_2^k, \ldots, e_r^k$ is an ε--partition of the unity in M and satisfies $\| E_{A_1' \cap M}(x_i) - x_i \|_2 < \varepsilon$. This shows that M has property (C).

Suppose now that $A \subset M$ is a Cartan subalgebra and $x_1, \ldots, x_n \in M$, $\alpha > 0$. Then there exist $a_1, \ldots, a_m \in A$, $u_1, \ldots, u_m \in N(A)$ such that x_i are α- contained in span $\{a_j u_j\}_j$. Let $\varepsilon > 0$. Using Rohlin's theorem one can easily construct for each u_j and ε--partition A_j in A such that $\| E_{A_j' \cap M}(u_j) - u_j \|_2 < \varepsilon (\sum_i \| a_i \|_2)^{-1}$. Then $y_j = a_j u_j$ and A_j are easily seen to satisfy conditions (a) and (b) in the preceding definition. **Q.E.D.**

2. A CONTINUITY RESULT

As before $M \subset B(H)$ is a type II_1 factor acting on a separable Hilbert space H. A derivation $\delta : M \to B(H)$ is a linear application satisfying $\delta(xy) = x\delta(y) + \delta(x)y$. Such a derivation is automatically norm continuous ([3]) and weakly continuous ([4]). Since the set of compact operators $K(H) \subset B(H)$ is a two sided ideal in $B(H)$ it make sense to consider derivations δ of M into $K(H)$. Such a derivation is called inner if there exists $K \in K(H)$ such that $\delta(x) = [K,x] = Kx - xK$ for all $x \in M$.

We recall from [3] that M is said to have the property (JP) if given any normal representation of it on a separable Hilbert space H, any derivation of M into $K(H)$ is inner.

THEOREM. *Let* $\delta : M \to K(H)$ *be a derivation. Then* δ *is continuous from the unit ball of* M *with the strong operator topology to* $K(H)$ *with the uniform norm topology.*

PROOF. We have to show that if $(x_n)_n$ is a bounded sequence in M with $\|x_n\|_2 \to 0$ then $\|\delta(x_n)\| \to 0$. It is clear that we only need to prove this implication in the case x_n are selfadjoint elements. Moreover, since $\| |x_n| \|_2 = \|x_n\|_2$, it follows that if $\|x_n\|_2 \to 0$ then $\|(x_n)_+\|_2 \to 0$ and $\|(x_n)_-\|_2 \to 0$, so that it is sufficient to prove that if $x_n \in M_+$, $\|x_n\| \leq 1$ and $\|x_n\|_2 \to 0$ then $\|\delta(x_n)\| \to 0$. For each such x_n let e_n be the spectral projection corresponding to the interval $[\|x_n\|_2^{1/2}, \infty)$. Then we have

$$\|x_n(1-e_n)\| \leq \|x_n\|_2$$

$$\tau(e_n) \leq \|x_n\|_2 .$$

Thus, if $x_n e_n = \sum_k 2^{-k} e_n^k$ is the diadic decomposition of $x_n e_n$, with e_n^k projections, $e_n^k \leq e_n$, then we get

$$\|\delta(x_n)\| \leq \|\delta\| \, \|x_n(1-e_n)\| + \sum_k 2^{-k} \|\delta(e_n^k)\| \leq \|\delta\| \, \|x_n\|_2 + \sum_k 2^{-k} \|\delta(e_n^k)\| .$$

This shows that it is actually sufficient to prove that if $(f_n)_n$ are projections in M such that $\tau(f_n) \to 0$ then $\delta(f_n) \to 0$. Indeed, because then, if x_n and e_n^k are as above we get $\sup_k \|\delta(e_n^k)\| \to 0$. Thus

$$\|\delta(x_n)\| \leq \|\delta\| \, \|x_n\|_2 + \sup_k \|\delta(e_n^k)\| \to 0 .$$

Let us suppose on the contrary that there exists a sequence of projections $(f_n)_n$ in M such that $\tau(f_n) \to 0$ but $\|\delta(f_n)\| \geq c > 0$ for all n. We may further assume that $\sum \tau(f_n) < \infty$. Let g_n be the supremum of the projections $(f_k)_{k \geq n}$, $g_n = \bigvee_{k \geq n} f_k$. Then $\tau(g_n) \leq \sum_{k \geq n} \tau(f_k)$ tends to zero with n. Let s_{nm} be the support of the element $f_m g_n f_m$. Then $s_{nm} \leq f_m$ and s_{nm} is majorized by g_n so that $\tau(s_{nm}) \leq \tau(g_n) \to 0$, for each fixed m. Moreover since g_n is decreasing, $f_m g_n f_m$ is decreasing in n so that s_{nm} is also decreasing in n. Thus $(f_m - s_{nm}) \uparrow f_m$ and so, by the inferior semicontinuity of the norm, for n big enough we have

$$\|\delta(f_m - s_{nm})\| \geq c/2.$$

It follows that we can find inductively an increasing sequence of integers $n_1, \ldots, n_k, \ldots$, such that the projections $h_k = f_{n_k} - s_{n_{k+1}, n_k}$ satisfy $\|\delta(h_k)\| \geq c/2$. These projections also satisfy $\tau(h_k) \leq \tau(f_{n_k}) \to 0$.

Moreover since $h_k \leq f_{n_k}$ and s_{n_{k+1}, n_k} is the support of $f_{n_k} g_{n_{k+1}} f_{n_k}$, by the definition of h_k we get

$$h_k g_{n_{k+1}} h_k = h_k f_{n_k} g_{n_{k+1}} f_{n_k} h_k \leq h_k s_{n_{k+1}} n_k h_k = 0.$$

Thus $h_k g_{n_{k+1}} = 0$, in particular $h_k f_{n_{k+1}} = 0$ and so $h_k h_{k+1} = 0$ which means that h_k are all mutually orthogonal projections. Since we also have $\|\delta(h_k)\| \geq c/2$ we obtain a contradiction, cf. [3]. Q.E.D.

COROLLARY. *If M and δ are as in the preceding theorem and $S = \overline{co}^w\{\delta(u)u^* \mid u$ unitary element in $M\}$ then for any $\beta>0$ there exists $\alpha>0$ such that if $x \in M$, $\|x\| \leq 1$, $\|x\|_2 < \alpha$ then $\|Tx\| < \beta$ and $\|xT\| < \beta$ for any $T \in S$.*

PROOF. By the theorem there exists $\alpha>0$ such that $\|y\| \leq 1$, $\|y\|_2 < \alpha$ implies $\|\delta(y)\| < \beta/3$. Since $\delta(u)u^*y = \delta(y) - u\delta(u^*y)$ and $\|u^*y\|_2 = \|y\|_2$ it follows that

$$\|\delta(u)u^*y\| \leq \|\delta(y)\| + \|\delta(u^*y)\| < 2\beta/3$$

for any unitary element u in M. Taking convex combinations and weak closure we get $\|Ty\| < \beta$ for all $T \in S$. Similarly $\|yT\| < \beta$ Q.E.D.

3. (C)$\Longrightarrow$(JP)

THEOREM. *Let M be a type II_1 factor acting on a separable Hilbert space H. If M has the property (C) then any derivation of M into the set of compact operators $K(H)$ is inner.*

PROOF. Let $\delta : M \to K(H)$ be a derivation and denote $S = \overline{co}^w\{\delta(u)u^* \mid u$ unitary element in $M\}$. Fix $\{x_n\}_n$ a sequence of elements in the unit ball of M, dense in the norm $\|\cdot\|_2$. Let $n \geq 1$ and $\alpha>0$. By the definition of property (C) there exist $y_1, \ldots, y_m \in M$ such that $x_1, \ldots, x_n$ are α-contained in span $\{y_j\}_{m \geq j \geq 1}$ with y_j satisfying property (b) of that definition. So, for arbitrary $\varepsilon>0$ there are ε-partitions $A_1, \ldots, A_m$ mutually commuting and such that $\|E_{A_j' \cap M}(y_j) - y_j\|_2 < \varepsilon$. Let A be the abelian subalgebra of M generated by $A_1, \ldots, A_m$ and let $K \in K(H)$ be such that $\delta(a) = [K,a]$, for all $a \in A$, and such that $K \in S$ (cf. [3]). By the continuity result and its corollary we have $\|\delta(y_j - E_{A_j' \cap M}(y_j))\| < \varepsilon'$ and $\|[K, y_j - E_{A_j' \cap M}(y_j)]\| < \varepsilon'$ where $\varepsilon' \to 0$ when $\varepsilon \to 0$. Denote by $e_1, \ldots, e_k$ the minimal projections of A. Since $(\delta - \text{ad}\, K)|A = 0$ we get $(\delta - \text{ad}\, K)\cdot(E_{A' \cap M}(y_j)) = \sum_i e_i(\delta - \text{ad}\, K)(y_j)e_i$.

Moreover $\|e_i(\delta - \text{ad } K)(y_j)e_i\| = \|(\delta - \text{ad } K)(e_i y_j e_i)\| \leq \|\delta(e_i y_j e_i)\| + \|Ke_i y_j e_i\| + \|e_i y_j e_i K\|$ and applying again the preceding theorem and its corollary, since $\tau(e_i) < \varepsilon$, we may also suppose $\|e_i(\delta - \text{ad } K)(y_j)e_i\| < \varepsilon'$ with the same ε' as before. But e_i are mutually orthogonal so that we get $\|(\delta - \text{ad } K)(E_{A'\cap M}(y_j)\| = \|\sum e_i(\delta - \text{ad } K)(y_j)e_i\| < \varepsilon'$ and thus $\|(\delta - \text{ad } K)(y_j)\| \leq \|\delta(y_j - E_{A'\cap M}(y_j))\| + \|[K, y_j - E_{A'\cap M}(y_j)]\| + \|(\delta - \text{ad } K)(E_{A'\cap M}(y_j))\| < 3\varepsilon'$.

Note that ε (and thus ε') is independent of a and of the choice of $y_1, \ldots, y_m$. Therefore if ε is small enough (to insure that ε' is small enough) and for appropriate α, the compact operator $K \in S$ will satisfy $\|(\delta - \text{ad } K)(x_i)\| < 2^{-n}$ for all $i=1,\ldots,n$. We denote such a K by K_n.

Let now $T \in S$ be a weak limit point of $(K_n)_n$. It follows that $\delta(x_i) = [T, x_i]$ for all $i \geq 1$, so that by the weak continuity of δ we have $\delta = \text{ad } T$.

Suppose T is not a compact operator and let $B \subset M$ be a maximal abelian $*$ - subalgebra. By [3] there exists a compact operator $C \in S$ such that $[T,b] = \delta(b) = [C,b]$ for all $b \in B$. Thus, if we denote $T' = T - C$, then $[T',b] = 0$.

Since T' is not compact but commutes with the projections in B it follows that there exists a decreasing sequence of projections $\{e_n\}_n$ in B such that $\tau(e_n) = 2^{-n}$ and $\|e_n T' e_n\| = \|T'\|$. But this contradicts the corollary.

This shows that T must be a compact operator. Q.E.D.

REMARKS 1°. As pointed out in [3], if one could show that the set S defined in the corollary of Section 2 and in the proof of the preceding theorem is contained in $K(H)$ then the Ryll - Nardjewski fixed point theorem would apply to obtain that δ is inner. The proof of the above theorem shows that whenever $T \in S$ is such that it commutes modulo $K(H)$ with M (actually with a completely nonatomic subalgebra of M), T is indeed a compact operator.

2°. Let $R \subset M$ be a hyperfinite subfactor of M with trivial relative commutant $R' \cap M = \mathbb{C}$ (cf. [7]). If $\delta : M \to K(H)$ is a selfadjoint derivation then let $K \in S \cap K(H)$ be such that $\delta|R = \text{ad } K$ (cf. [3]). Then the kernel of the derivation $\delta' = \delta - \text{ad } K$ contains R and, since δ' is selfadjoint and weakly continuous, it is a von Neumann subalgebra of M, to be denoted by N. Since $N' \cap M \subset R' \cap M = \mathbb{C}$, N is a factor. Let $B \subset N$ be a completely nonatomic von Neumann subalgebra. By the argument in [3] and explained here in the introduction if a unitary element u in M normalizes B then $u \in N$. Thus $N(B) \subset N$. In particular all maximal abelian $*$-subalgebras in N are maximal abelian in M. By the preceding theorem N is either non Γ or a maximal Γ subfactor in M. Moreover, if $N_0 \subset M$ is any von Neumann subalgebra with property (JP) (for instance a hyperfinite subfactor or more generally a Γ subfactor) and if $N_0 \cap N$ is an algebra without atoms, then $N_0 \subset N$. So, if one would know that such properties of the pair $N \subset M$ necessarily imply $N = M$ then the general theorem would be true. Unfortunately this is not the case. Indeed if M is the algebra associated with the left regular representation of the free group on three generators and $N \subset M$ is the subfactor corresponding to the subgroup on two generators then the pair $N \subset M$ satisfies all these patologies (cf.[8], [9]).

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Sorin Popa

Department of Mathematics
INCREST
Bdul Păcii 220, 79622 Bucharest
Romania.