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RANGES OF TRACES ON K_o OF REDUCED CROSSED PRODUCTS BY FREE GROUPS

Mihai V. Pimsner

The aim of this paper is to give a formula for computing the range of a trace on K_o of a reduced crossed product by a free group on m generators. Even the case when the group has one generator, i.e. the case of crossed products by $\mathbb{Z}$, is of interest since it includes the case of the irrational rotation C*-algebras A_θ, which were the initial example for the general problem we consider. In [17] M.A. Rieffel had found examples of projections in A_θ which showed that the range of the trace on $K_o(A_\theta)$ containes $\mathbb{Z}+\theta\mathbb{Z}$ and led him to conjecture that this range actually coincided with $\mathbb{Z}+\theta\mathbb{Z}$. This was proved later to be the case in [13] by an embedding argument. After the computations of the K-groups of crossed products by $\mathbb{Z}$ in [14], it seemed that the computation of the range of the trace on K_o would be much easier. Indeed new proofs for computing the range of the trace on $K_o(A_\theta)$ appeared in [7] and [14], both of which however used some particular feature of the irrational rotation algebra. The first natural approach to our problem is due to A. Connes, who combined his formula for crossed products by $\mathbb{R}$ with the "dual trace" [2], to get results for crossed products by $\mathbb{Z}$. Moreover his "differential geometry" approach to the problem [3] and the discovery that the traces are the elements of order zero in a cohomology theory for algebras [4], [5], [6] are crucial for this problem.

In the present paper we combine the results of [14], [15] with those of [5], [6] to get results for the case of reduced crossed products by free groups.

Section 0 recalls very briefly the ~~results~~ concerning the Toeplitz extension of [15].

For the convenience of the reader we have treated the 0-dimensional case separately in Section 1. We show that, the range of a trace τ on $K_o(A \times_{\alpha_r} F_m)$ is very roughly speaking the subgroup generated by the values of τ on $K_o(A)$ and of a certain 1-trace on $K_1(A)$. This already shows that higher dimensional traces naturally occur in this problem. To make this section as selfcontained as possible, we have avoided any reference to [4], [5], [6], and have used instead the notion of determinant associated to τ, introduced by P. de la Harpe and G. Skandalis in [10]. As a corollary of the above results we get a (slight) generalization of a theorem of N. Riedel [16] and the computation of the range of the trace on $K_o(C(\mathbb{T}) \times_{\alpha_T} \mathbb{T})$ where $T:\mathbb{T} \to \mathbb{T}$ is any orientation preserving homeomorphism of the unit circle, in terms of the rotation number of T. This section serves as well as an illustration of the basic ideas that are behind the proofs of Section 2.

Section 2 treats the case of higher dimensional traces and relies heavily on A. Connes' papers [5] and [6]. To illustrate our results we show that the theorem of G.A. Elliott concerning the range of the trace on K_o of a noncommutative torus [9] can be obtained by a simple induction from our Theorem 15, in the same way their K-groups are obtained by an iteration of the exact sequence of [14].

§0

By F_m we shall denote the free group on m generators $g_1,\ldots,g_m$. Consider a C*-algebra A with an action

$$\alpha:F_m \to \mathrm{Aut}(A).$$

The reduced crossed product of A by α, $A \times_{\alpha_r} F_m$ will be identified with the C*-algebra of operators on $\ell^2(F_m,H) \simeq \ell^2(F_m) \otimes H$, generated by the operators

$$1 \otimes \rho(a) \qquad a\varepsilon A$$

$$(u_g k)h = v_g k(g^{-1}h) \qquad g,h\varepsilon F_m \text{ and } k\varepsilon \ell^2(F_m,H)$$

where (ρ,v_g) is any covariant faithful representation of A on the Hilbert space H [12].

By $\Gamma_k \subset F_m$ we shall denote the subset of F_m consisting of elements $g_{i_1}^{m_1}\cdots g_{i_s}^{m_s}$ ($s\geq 0$, $i_1\neq i_2,\ldots,i_{s-1}\neq i_s$, $m_1\neq 0,\ldots,m_s\neq 0$) such that $m_s>0$ if $i_s=k$. Remark that the neutral element e of F_m is in Γ_k and $g_j\Gamma_k=\Gamma_k$ if $j\neq k$, while $g_k\Gamma_k=\Gamma_k\setminus\{e\}$.

On $\bigoplus_{k=1}^{m} \ell^2(\Gamma_k, H) \subset \bigoplus_{k=1}^{m} \ell^2(F_m, H)$, denote by $d(a)$ and s_i the restrictions of $\bigoplus_{k=1}^{m} (1\otimes p(a))$ respectively $\bigoplus_{k=1}^{m} u_{g_i}$.

The Toeplitz algebra T is by definition the C*-algebra generated by $d(a)$ and $s_1, \ldots, s_m$.

It is shown in [15] that the closed two-sided ideal generated in T by the projections $1-s_i s_i^*$ is isomorphic to $\bigoplus_{k=1}^{m} A\otimes K(\ell^2(\Gamma_k))$ and that the quotient of T by this ideal is isomorphic to $A \times_{\alpha_r} F_m$. Thus there is an exact sequence

$$0 \longrightarrow (A\otimes K)^m \xrightarrow{i} T \xrightarrow{\pi} A \times_{\alpha_r} F_m \longrightarrow 0$$

called the Toeplitz extension of $A \times_{\alpha_r} F_m$.

Moreover the map $\pi\circ d: A \to A \times_{\alpha_r} F_m$ coincides with the usual embedding of A into the crossed product. The main result of [15] shows that d_* induces isomorphisms between $K_i(A)$ and $K_i(T)$ and that if $j: A^m \to (A\otimes K)^m = \bigoplus_{k=1}^{m} A\otimes K(\ell^2(\Gamma_k))$ denotes the map

$$j((a_k)_{k=1}^{m}) = \bigoplus_{k=1}^{m} a_k \otimes e(e,e)$$

($e(g,g')$ is the natural matrix unit of $K(\ell^2(\Gamma_k))$) that induces an isomorphism of $K_*(A)^m$ with $K_*((A\otimes K)^m)$, then the sequence

$$\begin{array}{ccccc} K_0(A)^m & \xrightarrow{\beta} & K_0(A) & \xrightarrow{(\pi\circ d)_*} & K_0(A \times_{\alpha_r} F_m) \\ \uparrow \partial & & & & \downarrow \partial \\ K_0(A \times_{\alpha_r} F_m) & \xleftarrow{(\pi\circ d)_*} & K_1(A) & \xleftarrow{\beta} & K_1(A)^m \end{array}$$

where $\beta((x_i)_{i=1}^{m}) = \sum_{i=1}^{m} (x_i - \alpha_{g_i^{-1}*}(x_i))$ and $\partial = j_*^{-1}\circ\delta$ (δ being the boundary map of the Toeplitz extension) is exact [15, Theorem 3.5].

§1

Let A be a C*-algebra and $M_n(A)$ the C*-algebra of nxn matrices over A. If A has a unit we shall denote by $U_n(A)$ the group of unitary elements in $M_n(A)$. If A has no unit, let $\tilde{A}$ be the C*-algebra obtained by adjoining a unit to A. Then $U_n(A)$ denotes the subgroup of $U_n(\tilde{A})$ consisting of elements of the form $1-x$ with $x \in M_n(A)$. $U_n(A)$ is a topological group with the topology induced by the norm topology of $M_n(A)$. Its connected component of the identity will be denoted by $U_n^o(A)$, while the (discrete) group of connected components will be denoted by $\pi_o(U_n(A))$. Considering $U_n(A)$ as a subgroup of $U_{n+1}(A)$ via the map $u \to \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$ which sends $U_n^o(A)$ into $U_{n+1}^o(A)$, we get maps

$$\varphi_{n,n+1}:\pi_o(U_n(A)) \to \pi_o(U_{n+1}(A))$$

and

$$\varphi_n:\pi_o(U_n(A)) \to K_1(A)$$

where $K_1(A)$ is by definition the inductive limit of the system $(\pi_o(U_n(A)),\varphi_{n,n+1})$. Moreover we shall denote by $K_o(A)$ the algebraic K group of A. If $\alpha:A \to B$ is a *-homomorphism we shall denote by α_* the maps induced on either K_o or K_1, keeping the notation α_{*n} for the map

$$\alpha_{*n}:\pi_o(U_n(A)) \to \pi_o(U_n(B))$$

induced by α. Moreover we shall denote also by α the map

$$\alpha \otimes 1:A \otimes M_n \to B \otimes M_n .$$

If τ is a trace state on A, we shall denote also by τ its extension $\tau \circ Tr$ to $M_n(A)$, where $Tr:M_n(A) \to A$ is the usual map $Tr((a_{ij}))=\sum a_{ii}$ and by $\underline{\tau}$ the induced group homomorphism from $K_o(A)$ to $\mathbb{R}$. Corresponding to τ P. de la Harpe and G. Skandalis defined a determinant with values in $\mathbb{R}/K_o(A)$. Let us briefly recall their construction [10].

If $[\alpha_1,\alpha_2]$ is a compact interval of the real line and $\xi:[\alpha_1,\alpha_2] \to U_n(A)$ a piecewise continuous differentiable path, they defined

$$\tilde{\Delta}_\tau(\xi)=(1/2\pi i)\int_{\alpha_1}^{\alpha_2} \tau(\dot{\xi}(\alpha)\xi(\alpha)^{-1})d\alpha .$$

Since $\tilde{\Delta}_\tau(\xi)$ rests unchanged when ξ is composed with the inclusion $U_n(A) \to U_{n+1}(A)$, $\tilde{\Delta}_\tau(\xi)$ depends only on the homotopy class (with fixed end points) of ξ, and $\tilde{\Delta}_\tau$ restricted to loops around the identity coincides via the Bott isomorphism with

$$\underline{\tau}: K_o(A) \simeq \lim_{\to} \pi_1(U_n(A)) \to \mathbb{R}$$

one gets a well defined map

$$\Delta_\tau : \bigcup_n U_n^o(A) \to \mathbb{R}/\underline{\tau}(K_o(A))$$

by $\Delta_\tau(u) = q(\tilde{\Delta}_\tau(\xi))$, where $\xi: [\alpha_1, \alpha_2] \to U_n(A)$ is any piecewise continuous differentiable path such that $\xi(\alpha_1) = 1$, $\xi(\alpha_2) = u$ and where

$$q: \mathbb{R} \to \mathbb{R}/\underline{\tau}(K_o(A))$$

is the natural projection.

The determinant is a group homomorphism and if u is of the form $\exp(2\pi i\, a)$ with $a = a^*$ then $\Delta_\tau(u) = q(\tau(a))$.

Let us prove two easy properties of the determinant.

Let $u_i, v_i \in U(A)$ $i = 1, \ldots, m$, *be unitaries such that* $\prod_1^m u_i v_i \in U^o(A)$. *Let* s_i, $i = 1, \ldots, m$ *be isometries and denote by* p_i *the projections* $1 - s_i s_i^*$. *Then*

$$\left(\prod_1^m u_i (s_i v_i s_i^* + p_i)\right) \oplus 1_3 \in U_4^o(A)$$

and

$$\Delta_\tau\left(\prod_1^m u_i (s_i v_i s_i^* + p_i)\right) = \Delta_\tau\left(\prod_1^m v_i v_i\right).$$

PROOF. Let $s_i' \in U_2(A)$ be the unitaries

$$s_i' = \begin{bmatrix} s_i & p_i \\ 0 & s_i^* \end{bmatrix}.$$

Since

$$\begin{bmatrix} s_i v_i s_i^* + p_i & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s_i & p_i \\ 0 & s_i^* \end{bmatrix} \begin{bmatrix} v_i & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s_i^* & 0 \\ p_i & s_i \end{bmatrix}$$

we may write

$$\left(\prod_1^m u_i (s_i v_i s_i^* + p_i)\right) \oplus 1 = \prod (u_i \oplus 1) s_i' (v_i \oplus 1) s_i'^* .$$

Passing now to $U_4(A)$ we may replace s_i' with $s_i'' = \begin{bmatrix} s_i' & 0 \\ 0 & s_i'^* \end{bmatrix} \in U_4^o(A)$ to get

$$(\prod_1^m v_i(s_i v_i s_i^* + p_i)) \oplus 1_3 = \prod_1^m (u_i \oplus 1_3) s_i''(v_i \oplus 1_3) s_i''^* .$$

Choosing a path joining s_i'' to the identity we get a path ξ connecting $\prod_1^m u_i(s_i v_i s_i^* + p_i) \oplus 1_3$ with $(\prod_1^m u_i v_i) \oplus 1_3$ such that $\tilde{\Delta}_\tau(\xi)=0$. Q.E.D.

If A is commutative and $\det : M_n(A) \to A$ *is the usual determinant, then* $\Delta_\tau \circ \det = \Delta_\tau$.

PROOF. If $u \in U_k^o(A)$, then u is a finite product of exponentials [8], so it is enough to suppose $u = \exp(2\pi i\ a)$. Then

$$\Delta_\tau(\det u) = \Delta_\tau(\exp(Tr(2\pi i\ a))) = q(\tau \circ Tr(a)) = \Delta_\tau(\exp(2\pi i\ a)) = \Delta_\tau(u).$$

Q.E.D.

LEMMA 1. *Let*

$$0 \longrightarrow J \xrightarrow{i} B \xrightarrow{\pi} A \longrightarrow 0$$

be an exact sequence of C-algebras and* τ *a trace state on A. If* $u \in U_n(J)$ *is a unitary such that* $[i(u)]_1 = 0$ *in* $K_1(B)$, *then* $\Delta_{\tau \circ \pi}(i(u))$ *depends only on the class of u in* $K_1(J)$.

PROOF. It is enough to show that if $u \in U_n^o(J)$ then $\Delta_{\tau\circ\pi}(i(u))=0$. Let $\xi : [0,1] \to U_n(J)$ be a path of class C^1 such that $\xi(0)=1$, $\xi(1)=u$. Since $\xi(\alpha) - 1 \in M_n(J)$ for $\alpha \in [0,1]$ it follows that $\dot{\xi}(\alpha) \in M_n(J)$, so that $\tau\circ\pi(i(\dot{\xi}(\alpha)) i(\xi(\alpha)^{-1})) = 0$ for every $\alpha \in (0,1)$. Thus $\tilde{\Delta}_{\tau\circ\pi}(i \circ \xi) = 0$. Q.E.D.

The preceding lemma shows that the map

$$\underline{\Delta}_\tau : \ker i_* \to \mathbb{R}/\underline{\tau\circ\pi(K_o(B))}$$

defined for $[u]_1 \in \ker i_* \subset K_1(J)$ by

$$\underline{\Delta}_\tau([u]_1) = \Delta_{\tau\circ\pi}(i(u))$$

is a well defined group homomorphism. (The condition $[u]_1 \in \ker i_*$ is needed for $\Delta_{\tau\circ\pi}(i(u))$ to make sense.)

PROPOSITION 2. *Let*

$$0 \longrightarrow J \xrightarrow{i} B \xrightarrow{\pi} A \longrightarrow 0$$

be an exact sequence of C-algebras and τ a trace state on A.*
Then

$$0 \longrightarrow \underline{\tau\circ\pi}(K_0(B)) \longrightarrow \underline{\tau}(K_0(A)) \xrightarrow{q} \underline{\Delta}_\tau(\ker i_*) \longrightarrow 0$$

where the first map is the inclusion of the two subgroups of $\mathbb{R}$ and q is the restriction of the map

$$q:\mathbb{R} \to \mathbb{R}/\underline{\tau\circ\pi}(K_0(B))$$

to $\underline{\tau}(K_0(A))$, is an exact sequence.
Moreover if $p\in M_n(A)$ is a projection then

$$q(\tau(p))\in\underline{\Delta}_\tau(\ker i_{*n})$$

where

$$i_{*n}:\pi_0(U_n(J)) \to \pi_0(U_n(B)) .$$

PROOF. If $p\in M_n(A)$ is a projection and $a\in M_n(B)$ is a selfadjoint element such that $\pi(a)=p$, then $\exp(2\pi i\ a)\in U_n^0(J)$ and since $\pi(\exp(2\pi i\ a))=$ $=\exp(2\pi i\ p)=1$ there is a unitary $u\in U_n(J)$ such that $i(u)=\exp(2\pi i\ a)$. By definition $\underline{\Delta}_\tau(u)=\Delta_{\tau\circ\pi}(\exp(2\pi i\ a))=q(\tau\circ\pi(a))=q(\tau(p))$. This proves that $q(K_0(A))\subset\Delta_\tau(\ker i_*)$ and the last part of the proposition. Finally recall that the boundary map $\delta:K_0(A)\to K_1(J)$ is defined exactly by $\delta[p]_0=[u]_1$ where u is related to p in the above manner, so that the equality $q(K_0(A))=\underline{\Delta}_\tau(\ker i_*)$ follows from the fact that $\delta(K_0(A))=\ker i_*$ which is part of the exactness of the six term sequence in K-theory.

Q.E.D.

We shall now apply the preceding proposition to crossed products by free groups.

THEOREM 3. *Let $\alpha:F_m\to \mathrm{Aut}(A)$ be an action of the free group on m generators $g_1,\dots,g_m$ on the C*-algebra A and $A\times_{\alpha_r}F_m$ the correspond ind reduced crossed-product. Suppose τ is a trace state on $A\times_{\alpha_r}F_m$, and denote also by τ its restriction to A.*

a) *If $\beta:K_1(A)^m\to K_1(A)$ denotes the map*

$$\beta((x_i)_{i=1}^m)=\sum_{i=1}^m(x_i-\alpha_{g_i^{-1}*}(x_i)),$$

then the map

$$\underline{\Delta}^{\alpha}_{\tau} : \ker \beta \to \mathbb{R}/\underline{\tau}(K_0(A))$$

defined by

$$\underline{\Delta}^{\alpha}_{\tau}(([u_1],\ldots,[u_m])) = \Delta_\tau\Big(\prod_1^m u_i \alpha^{-1}_{g_i}(u_i^{-1})\Big)$$

where u_i *are unitaries in some* $U_n(A)$, *such that* $\prod_1^m u_i \alpha^{-1}_{g_i}(u_i^{-1}) \in U_n^0(A)$ *is a well defined group homomorphism.*

b) *The sequence*

$$0 \to \underline{\tau}(K_0(A)) \to \underline{\tau}(K_0(A \times_{\alpha_r} F_m)) \xrightarrow{q} \underline{\Delta}^{\alpha}_{\tau}(\ker \beta) \to 0$$

(where the first map is the inclusion of the two subgroups of $\mathbb{R}$*) is exact.*

PROOF. It is an easy exercise to get a) from the properties of the determinant using the fact that τ is a F_m invariant trace on A. However both a) and b) follow from the preceding Proposition applied to the Toeplitz extension

$$0 \to (A \otimes \mathcal{R})^m \xrightarrow{i} \mathcal{T} \xrightarrow{\pi} A \times_{\alpha_r} F_m \to 0$$

and from the results of [15].

Since $d : A \to \mathcal{T}$ induces an isomorphism of $K_0(A)$ with $K_0(\mathcal{T})$ and $\pi \circ d$ is the usual embedding of A into $A \times_{\alpha_r} F_m$, all we have to prove is that

$$\underline{\Delta}^{\alpha}_{\tau} = \underline{\Delta}_\tau \circ j_* .$$

Let $u_i \in U_n(A)$, $1 \le i \le m$, be unitaries such that

$$\prod_1^m u_i \alpha^{-1}_{g_i}(u_i^{-1}) \in U_n^0(A) .$$

Then

$$i \circ j((u_i)_{i=1}^m) = \prod_{i=1}^m d(u_i)(s_i d(\alpha^{-1}_{g_i}(u_i^{-1}))s_i^* + 1 - s_i s_i^*)$$

so that the desired equality follows from the previously proved property of the determinant. Q.E.D.

REMARK. Since the ideal appearing in the Toeplitz extension is

$(A\otimes K)^m$ we lost the information of Proposition 2 concerning the possibility of choosing the unitary corresponding to a projection in $M_k(A \times_{\alpha_r} F_m)$ in $U_k(A)^m$. However if the C*-algebra A is commutative one can get the following result.

COROLLARY 4. *In the conditions of the preceding theorem suppose A is commutative. Then*

$$q(\tilde{\underline{\tau}}(K_o(A \times_{\alpha_r} F_m))) = \underline{\Delta}^*_\tau(\ker \beta_1)$$

where $\beta_o : \pi_o(U_1(A))^m \to \pi_o(U_1(A))$ *is defined by*

$$\beta_o((x_i)_{i=1}^m) = \sum_1^n (x_i - (\alpha_{g_i^{-1}})_{*1}(x_i)) .$$

Thus in order to compute the range of the trace on K_o of the crossed product we need only to know the action induced on $\pi_o(U(A)) \simeq \check{H}^1(X,\mathbb{Z})$ where $\check{H}^1(X,\mathbb{Z})$ is the first Cech cohomology group of the maximal ideal space of A and to compute determinants.

The corollary is a direct consequence of the theorem once we show that $\Delta^*_\tau(\ker \beta_1) \supset \Delta^*_\tau(\ker \beta)$, the other inclusion being obvious.

Let $u_1,\dots,u_n$ be unitaries in some $U_k(A)$ such that $\prod_1^m u_i \alpha_{g_i^{-1}}(u_i)^{-1} \in U_k^o(A)$. Then the usual determinant $\det : M_n(A) \to A$ (acting pointwise when $A \simeq C(X)$) maps u_i in $U(A)$ and $\prod_1^m u_i \alpha_{g_i^{-1}}(u_i)^{-1}$ in $U^o(A)$ so that $(\det(u_i))_{i=1}^m \in \ker \beta_1$, and since $\Delta_\tau \circ \det = \Delta_\tau$ the conclusion follows.

Q.E.D.

The preceding results are especially easy to apply either if $K_1(A)$ (respectively $\check{H}^1(X,\mathbb{Z})$) is trivial or if the action of F_m has discrete spectrum.

For example one gets the following extension of a result of N. Riedel [16].

Let Γ be a discrete abelian group, G its dual (compact) group and μ the normalized Haar measure on G. Denote by $\alpha_\rho \in \mathrm{Aut}(C(G))$ the automorphism determined by translation with a fixed element $\rho \in G$ and by $A_{\Gamma,\rho}$ the crossed product $C(G) \times_{\alpha_g} \mathbb{Z}$. The trace state τ determined by μ extends to a trace state $\tilde{\tau}$ of $A_{\Gamma,\rho}$. For each finite group $F \subset \Gamma$ (including $F=\{0\}$), let $|F|$ denote its cardinality and p_F the period of F, that is the smallest positive integer p such that $(\rho|F)^p$ is the trivial character of F.

Let $\Gamma_{\rho,F}=\{t\in\mathbb{R} \mid \exp(2\pi i\frac{|F|}{p_F}t)<\Gamma,\rho>\}$ and note that $\Gamma_{\rho,F_1}\subset\Gamma_{\rho,F_2}$ whenever $F_1\subset F_2$, since $\frac{|F_1|}{p_{F_1}}$ divides $\frac{|F_2|}{p_{F_2}}$. Define $\Gamma_\rho=\bigcup_{F\subset\Gamma}\Gamma_{\rho,F}$ which is a group that depends functorial on (Γ,ρ). Moreover if ρ is faithful then each finite subgroup $F\subset\Gamma$ is cyclic, $p_F=|F|$, and $\Gamma_\rho=\{t\in\mathbb{R} \mid \exp(2\pi it)\in <\Gamma|\rho>\}$ so that in this case the next result is Theorem 3.6 of [16].

THEOREM 5. *The range of the trace* $\tilde{\tau}$ *on* $K_o(A_{\Gamma,\rho})$ *is* Γ_ρ.

PROOF. Since both $A_{\Gamma,\rho}$ and Γ_ρ commute with direct limits, it is sufficient to consider the case when Γ is finitely generated, so that we may suppose $\Gamma=\mathbb{Z}^m\times F$, for some $m\in\mathbb{N}\cup\{0\}$ and some finite group F. (Note that in this case $\Gamma_\rho=\Gamma_{\rho,F}$.) It follows that G, the dual of Γ, is isomorphic to $\mathbb{T}^m\times\hat{F}$, where $\mathbb{T}^m$ is the m-dimensional torus and $\hat{F}$ is isomorphic to F. This shows on one hand that $\tau(K_o(C(G)))=\frac{1}{|F|}\mathbb{Z}$ and on the other hand that $\pi_o(U_1(C(G)))$ is isomorphic to $(\mathbb{Z}^m)^{\hat{F}}$ by regarding an $|\hat{F}|$-tuple $(m_f)_{f\in\hat{F}}$ as the continuous function

$$(m_f)_{f\in\hat{F}}(t,h)=<m_h|t>\in\mathbb{T}. \qquad (\mathbb{Z}^m=\hat{\mathbb{T}}^m).$$

Note that if $\rho=(\rho_1,\rho_2)\in\mathbb{T}^m\times\hat{F}$ is the decomposition of ρ then

$$(\alpha_\rho)^{-1}((m_f)_{f\in\hat{F}})(t,h)=(m_f)_{f\in\hat{F}}(\rho_1 t,\rho_2 h)=$$

$$=<m_{\rho_2 h}|\rho_1 t>=<m_{\rho_2 h}|\rho_1><m_{\rho_2 h}|t>=<m_{\rho_2 h}|\rho_1>(m_{\rho_2 f})_{f\in\hat{F}}(t,h).$$

We are now in position to apply Corollary 4 for $F_1=\mathbb{Z}$. The above computation of α_ρ shows first that an $|\hat{F}|$-tuple $(m_f)_{f\in\hat{F}}$ is in $\ker\beta_o$ iff $m_{\rho_2 f}=m_f$ for every $f\in\hat{F}$, and in this case

$$(m_f)_{f\in\hat{F}}\cdot\alpha_\rho^{-1}((m_f)_{f\in\hat{F}}^{-1})(t,h)=<m_h|t>\cdot\overline{<m_{\rho_2 h}|\rho_1>}\,\overline{<m_{\rho_2 h}|t>}=$$

$$=\overline{<m_h|(\rho_1)>}=<-m_h|\rho_1>.$$

Since m_f only depends on $\tilde{f}\in\hat{F}/(\rho_2)$, the orbit of f, ($(\rho_2)$ is the subgroup generated by $\rho_2=\rho|F$) it follows that

$$(m_f)_{f\in\hat{F}}\cdot\alpha_\rho^{-1}((m_f)_{f\in\hat{F}}^{-1})=\sum_{\tilde{f}\in\hat{F}/(\rho_2)}<-m_{\tilde{f}}|\rho_1>\chi(\mathbb{T}^m\times\tilde{f}).$$

Thus

$$\underline{\Delta}_\tau^*([(m_{\tilde f})_{\tilde f\in \hat F}])=q(\sum_{\tilde f\in \hat F/(\rho_2)}\frac{p_F}{|\hat F|}t_{\tilde f})=q(\sum_{f\in F/(\rho_2)}\frac{p_F}{|F|}t_{\tilde f})$$

where $t_{\tilde f}\in \mathbb{R}$ satisfies $\exp(2\pi\, it_{\tilde f})=\langle -m_{\tilde f}|\rho_1\rangle$.

Collorary 4 now implies that $\underline{\tilde\tau}(K_o(A_{\Gamma,\rho}))$ is the subgroup of $\mathbb{R}$ generated by $\frac{1}{|F|}$ and $\frac{p_F}{|F|}t$ where $\exp(2\pi\, it)\in\langle \mathbb{T}^m|\rho_1\rangle=\langle \mathbb{T}^m\times\{0\}|\rho\rangle$.

Since the period of $\rho_2=\rho|F$ is p_F, it follows that there exists $a\in F$ such that $\langle a|\rho\rangle=\langle a|\rho_2\rangle=\exp(2\pi\, i\frac{1}{p_F})=\exp(2\pi\, i\frac{|F|}{p_F}\cdot\frac{1}{|F|})$. This shows that the above group is contained in $\Gamma_\rho=\Gamma_{\rho,F}$. Conversely if $t\in\mathbb{R}$ satisfies $\exp(2\pi i\frac{|F|}{p_F}t)=\langle\gamma|\rho\rangle$ for some $\gamma\in\Gamma$, write $\gamma=(\gamma_1,\gamma_2)$ so that $\langle\gamma|\rho\rangle=$
$=\langle\gamma_1|\rho_1\rangle\langle\gamma_2|\rho_2\rangle=\langle\gamma_1|\rho_1\rangle\exp(2\pi i\frac{a}{p_F})=\langle\gamma_1|\rho_1\rangle\exp(2\pi i\frac{|F|}{p_F}\cdot\frac{a}{|F|})$ for some $a\in\mathbb{Z}$.

It follows that $t=\frac{a}{|F|}+\frac{p_F}{|F|}t_1$ where $\exp(2\pi it_1)=\langle\gamma_1|\rho_1\rangle$ so that $\Gamma_\rho=\Gamma_{\rho,F}\subset\underline{\tilde\tau}(K_o(A_{\Gamma,\rho}))$ which concludes the proof. Q.E.D.

Another application of Theorem 3 is the following.

Let $\mathbb{T}$ be the one dimensional torus and $T:\mathbb{T}\to\mathbb{T}$ an orientation preserving homeomorphism. The rotation number of T is defined in the following way (see for instance [19]): choose a continuous increasing function $f:\mathbb{R}\to\mathbb{R}$ such that $e^{2\pi if(x)}=T(e^{2\pi ix})$ for every $x\in\mathbb{R}$ (this function is unique up to an integer constant). One shows that the limit $\lim_{n\to\infty}\frac{f^{(n)}(x)}{n}=\theta$ (where $f^{(n)}$ is the n-times composition of f with itself) exists and is independent of the point x_o. Since $f(x+m)=f(x)+m$ for every $m\in\mathbb{Z}$, θ is uniquely defined by T up to an integer. The particular $\theta\in[0,1)$ is called the rotation number of T.

PROPOSITION 6. *Let $T:\mathbb{T}\to\mathbb{T}$ be an orientation preserving homeomorphism of the unit circle with rotation number θ, and μ any T invariant probability measure on $\mathbb{T}$.*

Let α_T be the automorphism $\alpha_T(h)=h\circ T^{-1}$ of $C(\mathbb{T})$ and $\tau=\tau_\mu$ be the induced trace on $C(\mathbb{T})\times_{\alpha_T}\mathbb{Z}$.

Then the range of the trace τ on $K_o(C(\mathbb{T})\times_{\alpha_T}\mathbb{Z})$ is $\mathbb{Z}+\theta\mathbb{Z}$.

PROOF. We shall apply Theorem 3. Let $z:\mathbb{T} \to \mathbb{T}$ be the identity function which is the generator of $K_1(C(\mathbb{T}))=K^1(\mathbb{T})$. Since T is orientation preserving $[z]_1=[\alpha_{T^{-1}}(z)]_1$, so that $\ker \beta = K_1(C(\mathbb{T}))=\mathbb{Z}$. Moreover $\underline{\tau}(K_0(C(\mathbb{T})))=\mathbb{Z}$ and since $\underline{\Delta}^*_\tau$ is a group homomorphism it is sufficient to compute $\underline{\Delta}^*_\tau([z]_1)$. Let $f:\mathbb{R} \to \mathbb{R}$ be the continuous increasing function such that $e^{2\pi i f(x)}=T(e^{2\pi i x})$ and $\lim_{n\to\infty} \frac{f^{(n)}(x)}{x}=\theta$. Defining $g:\mathbb{T} \to \mathbb{R}$ by $g(e^{2\pi i x})=x-f(x)$ it follows that $e^{2\pi i g}=z\cdot\alpha_{T^{-1}}(z)^{-1}$ and so

$$\underline{\Delta}^*_\tau([z]_1)=\Delta_\tau(z\cdot\alpha_{T^{-1}}(z)^{-1})=q(\tau(g))=q(\int_{\mathbb{T}} g\,d\mu)$$

where $q:\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the natural projection. Note that $g\circ T^k(e^{2\pi i x})= =g(e^{2\pi i f^{(k)}(x)})=f^{(k)}(x)-f^{(k+1)}(x)$ and since μ is T invariant it follows that

$$\int_{\mathbb{T}} g\,d\mu=\frac{1}{k}\int_{\mathbb{T}}(g+g\circ T+\ldots+g\circ T^{k-1})\,d\mu=-\theta$$

since $\frac{1}{k}\sum_{0}^{k-1} g\circ T^i(e^{2\pi i x})=\frac{x-f^{(k)}(x)}{k}$ are uniformly bounded and converge to $-\theta$ for every x.

§2

Recall from [5] that for any algebra A, $\Omega(A)$ denotes the universal graded differential algebra associated to A. Thus if τ is an n+1 linear functional on A, it extends to a linear functional $\hat{\tau}$ on $\Omega^n(A)$, with

$$\hat{\tau}(a^0da^1da^2\ldots da^n)=\tau(a^0,a^1,\ldots,a^n) \qquad a^i\in A \quad [5].$$

τ is called a cyclic cocycle iff $\hat{\tau}$ is a closed graded trace, i.e. if it satisfies

$$\hat{\tau}(d\omega)=0, \quad \forall\, \omega\in\Omega^{n-1} \qquad \text{and} \qquad \hat{\tau}(\omega\omega')=(-1)^{\deg\omega\,\deg\omega'}\,\hat{\tau}(\omega'\omega).$$

The set of cyclic cocycles of A is denoted by $Z^n_\lambda(A)$. Let us quote from [6] the definition of an n-trace on a Banach algebra.

DEFINITION. Let B be a Banach algebra. By an *n-trace on* B we mean an n+1 linear functional τ on a dense subalgebra A of B such that

a) τ is a cyclic cocycle on A.

b) For any $a^i \in A$, $i=1,\dots,n$, there exists $C=C_{a^1,\dots,a^n}<\infty$ such that

$$|\hat{\tau}((x^1da^1)(x^2da^2)\dots(x^nda^n))|\le C\|x^1\|\dots\|x^n\| \qquad \forall x^i\in\tilde{A}.$$

Note that if τ is the character of an n-dimensional cycle $(\Omega,d,\int)$ [5] that satisfies b) (in Ω), then τ is an n-trace. The main property of n-traces that we will be concerned with, is that they determine maps from $K_*(B)$ to $\mathbb{C}$ (see [6], Theorem 7).

Recall also from [5] the defintion of the cup product $\varphi \# \psi \in Z_\lambda^{n+m}(A\otimes B)$ of the two cyclic cocycles $\varphi\in Z_\lambda^n(A)$ and $\psi\in Z_\lambda^m(B)$, and that the map $S:H_\lambda^n(A)\to H_\lambda^{n+2}(A)$ is induced by the cup product with the generator $\sigma\in H_\lambda^2(\mathbb{C})$ of $H_\lambda^*(\mathbb{C})$ characterized by

$$\sigma(1,1,1)=2i\pi.$$

Let us note for further use the explicit formula for $S^k:H_\lambda^n(A)\to H_\lambda^{n+2k}(A)$.

LEMMA 7. *Let* $\varphi\in Z_\lambda^n(A)$ *be a cyclic cocycle and* k *a natural number. We shall denote by* $\mathcal{D}_{n,k}$ *the collection of subsets* $D\subset\{1,\dots,n+2k\}$ *consisting of* 2k *elements with the property that the maximal intervals appearing in the decomposition of* D *contain an even number of elements. If* $\omega=x^0dx^1\dots dx^{n+2k}$ *and* $D\in\mathcal{D}_{n,k}$ *denote by* ω_D *the n-form obtained by replacing for each* $i\in D$ dx^i *with* x^i *in the expression of* ω. *Then*

$$S^k\varphi(\omega)=(2i\pi)^k k!\sum_D\hat{\varphi}(\omega_D)$$

the sum running over all elements of $\mathcal{D}_{n,k}$.

PROOF. The properties of the cup product imply that S^k is given by the cup product with $\sigma^k\in H^{2k}(\mathbb{C})$. By [5], Corollary 10, σ^k is the 2k-cocycle characterized by

$$\sigma^k(1,1,\dots,1)=(2i\pi)^k k!\ .$$

The above formula is now an easy consequence of the definition of the cup product.

Q.E.D.

The n-traces that we will consider on crossed products will arise as characters of some particular cycles, so let us state some of their properties.

LEMMA 8. *Let* $(\Omega, d, \hat{\varphi}_n)$ *be a cycle of dimension* n. *Suppose that* Ω^o *is unital and that* $d1=0$. *Let* A *be a subalgebra of* Ω^o *containing the unit of* Ω^o *and* $s\varepsilon\Omega^o$ *an invertible element that normalizes* A. *Suppose that* $sdas^{-1}=d(sas^{-1})$ $\forall a\varepsilon A$. *Then*

1) 1 *is the unit of* Ω; $ds^{-1}=-s^{-1}dss^{-1}$.

2) *If* ω *is a form in the graded algebra generated by* A *then* $s^{-1}ds\omega=(-1)^{\deg\omega}\omega s^{-1}ds$.

3) *For each* $k\varepsilon\mathbb{N}$, $k<n$ *the equality*

$$\varphi_{n-k}(a^o,a^1,\ldots,a^{n-k})=\hat{\varphi}_n(a^o da^1\ldots da^{n-k}\underbrace{s^{-1}ds\ldots s^{-1}ds}_{k\text{-times}})$$

$a^i\varepsilon A$, *defines a cyclic cocycle* $\varphi_{n-k}\varepsilon Z_\lambda^{n-k}(A)$. *However* $\varphi_{n-k}=0$ *when* k *is even.*

PROOF. 1) Follows from $1dx=dx-d1x=dx$ and similarly $dx1=dx$, and from $0=d1=d(ss^{-1})=dss^{-1}+sds^{-1}$.

2) Since $sdas^{-1}=d(sas^{-1})$ one gets $dsas^{-1}+sads^{-1}=0$. This shows that $s^{-1}dsa=as^{-1}ds$ $\forall a\varepsilon A$. Differentiating the same quality one gets $0=dsdas^{-1}-sdads^{-1}=dsdas^{-1}+sdas^{-1}dss^{-1}$ so that we also have $s^{-1}dsda=-das^{-1}ds$, $\forall a\varepsilon A$.

3) The equality

$$\hat{\varphi}_n(a^o da^1\ldots da^{n-k}s^{-1}\underbrace{ds\ldots s^{-1}ds}_{k\text{-times}})=$$

$$=(-1)^{n-k}\hat{\varphi}_n(s^{-1}ds\, a^o da^1\ldots da^{n-k}\underbrace{s^{-1}ds\ldots s^{-1}ds}_{k\text{-}1\text{-times}})=$$

$$=(-1)^{n-k}(-1)^{n-1}\hat{\varphi}_n(a^o da^1\ldots da^{n-k}\underbrace{s^{-1}ds\ldots s^{-1}ds}_{k\text{-times}})$$

where the first follows from 2) and the second from the fact that $\hat{\varphi}_n$ is a graded trace, shows that $\varphi_{n-k}=0$ when k is even. Note next that $d((s^{-1}ds)^m)=-m(s^{-1}ds)^{m+1}$ so that $\hat{\varphi}_n(d\omega(s^{-1}ds)^k)=(-1)^k k\hat{\varphi}_n(\omega(s^{-1}ds)^{k+1})$ for any form ω in the graded algebra generated by A. Thus if k is odd then $\hat{\varphi}_{n-k}$ is closed. To show that $\hat{\varphi}_{n-k}$ is graded let ω and ω' be forms in the graded algebra generated by A. Then

$$\hat{\varphi}_n(\omega\omega'(s^{-1}ds)^k)=(-1)^{\deg\omega'\cdot k}\hat{\varphi}_n(\omega(s^{-1}ds)^k\omega')=$$

$$=(-1)^{\deg \omega' \cdot k}(-1)^{\deg \omega'(\deg \omega + k)}\hat{\varphi}_n(\omega'\omega(s^{-1}ds)^k) =$$

$$=(-1)^{\deg \omega' \deg \omega}\hat{\varphi}_n(\omega'\omega(s^{-1}ds)^k,$$

which gives the desired result. Q.E.D.

Let now $\mathbb{T}$ denote the one dimensional torus and ε the 1-trace on on $C(\mathbb{T})$ (the C*-algebra of continuous functions on $\mathbb{T}$) given by

$$\varepsilon(f^o, f^1) = \int f^o df^1$$

defined on the dense subalgebra of smooth functions. Given any Banach algebra A and any n-trace φ on $\mathcal{A}$, the cup product $\varphi \# \varepsilon$ defines an n+1-trace on $\mathcal{A} \otimes C(\mathbb{T})$. The explicit formula of this cup product is as follows. If $a^i(t)$ are smooth functions on $\mathbb{T}$ with values in the domain of φ, then

$$(\varphi \# \varepsilon)(a^o, a^1, \ldots, a^{n+1}) =$$

$$= \sum_{i=1}^{n+1} (-1)^{n+1-i} \int \hat{\varphi}(a^o(t)da^1(t)\ldots da^{i-1}(t)\dot{a}^i(t)da^{i+1}\ldots da^{n+1})dt .$$

The next lemma is the main computational result of this section.

LEMMA 9. *Let* φ_n *be an* n-*trace on the Banach algebra* B *with domain* $\mathcal{B}$ *and* $A \subseteq B$ *a Banach subalgebra such that* $\mathcal{A} = A \cap \mathcal{B}$ *is dense in* A. *Suppose that* B *is unital, that the unit lies in* $\mathcal{A}$ *and that there exists an invertible* $s \in \mathcal{B}$ *that normalizes* $\mathcal{A}$. *Suppose moreover that* φ_n *arises as the character of an* n-*cycle* (Ω, d, φ_n) *with the properties*

a) $\Omega^o = \mathcal{B}$

b) $d1 = 0$; $d(sas^{-1}) = sdas^{-1}$ *for every* $a \in \mathcal{A}$.

Let $v_t \in M_2(\mathcal{B})$ *be the invertible element*

$$v_t = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} s^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

and for any $x \in \mathcal{A}$ *let* $x_t = v_t \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} v_t^{-1} \in M_2(\mathcal{B})$. *Then for every* $x^o, x^1, \ldots, x^{n+1}$ *belonging to* $\mathcal{A}$ *one gets*

$$\sum_{i=1}^{n+1} (-1)^{n-i+1} \int_o^{\pi/2} \varphi_n \# \mathrm{Tr}(x_t^o dx_t^1 \ldots dx_t^{i-1} \dot{x}_t^i dx_t^{i+1} \ldots dx_t^{n+1}) dt =$$

$$= \sum_{k=1}^{[\frac{n+1}{2}]} c_k S^k \varphi_{n-2k+1}(x^o, x^1, \ldots, x^{n+1})$$

where $c_k = (-1)^k \frac{1}{(2i\pi)^k} \frac{1}{2^{k-1}} \frac{1}{(2k-1)(2k-3)\ldots 3\cdot 1}$ *and* φ_{n-2k+1} *is defined in the preceding lemma. In particular the left hand side of the above equality defines an* n+1-*trace.*

PROOF. Let us denote by p_t the projection $\begin{bmatrix} s^2 & sc \\ sc & c^2 \end{bmatrix}$ by $\dot{p}_t$ its derivative which is the unitary $\begin{bmatrix} 2sc & c^2-s^2 \\ c^2-s^2 & 2sc \end{bmatrix}$ and by $e = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, where s=sint t and c=cos t. Note that

$$p_t^2 = p_t \ , \ p_t\dot{p}_t = \dot{p}_t(1-p_t) \ , \ (1-p_t)\dot{p}_t = \dot{p}_t p_t$$

$$ep_t = (1-p_t)e \ , \ e\dot{p}_t = -\dot{p}_t e$$

and that $Tr(p_t\dot{p}_t e) = 1$ for every t. A direct computation shows that

$$x_t = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} p_t \otimes x \begin{bmatrix} s^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\dot{x}_t = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} \dot{p}_t \otimes x \begin{bmatrix} s^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

$$dx_t = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} (p_t \otimes dx + sce \otimes (s^{-1} ds\ x)) \begin{bmatrix} s^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

so that we have to compute

$$\sum_{i=1}^{n+1} (-1)^{n+1-i} \varphi_n \# Tr(p_t \otimes x^o (p_t \otimes dx^1 + sce \otimes (s^{-1} dsx^1)) \ldots$$

$$\ldots (\dot{p}_t \otimes x^i) \ldots (p_t \otimes dx^{n+1} + sce \otimes (s^{-1} dsx^{n+1})))$$

In order to expand the above sum recall from Lemma 1 the definition of the sets belonging to $\mathcal{D}_{n,k}$ and note that the formulae connecting p_t, $\dot{p}_t$ and e ensure that only terms for which $\dot{p}_t$ and all the e's belong to some $\mathcal{D}_{n-2k+1,k}$ appear. To be more precise, denote by $\mathcal{D}_{n-2k+1,k,i}$ the collection of pairs (D,i) with $D\varepsilon\mathcal{D}_{n-2k+1,k}$, $i\varepsilon\{1,\dots,n+1\}$, such that $i\varepsilon D$. For each such pair denoty by $r_{t,D,i}$ the matrix obtained from $r^0r^1\dots r^{n+1}$ by replacing r^i with $\dot{p}_t$, r^j for $j\varepsilon D\setminus\{i\}$ with e and r^j for $j\notin D$ with p_t. Similarly denote by $\omega_{D,i}$ the n-form obtained from $x^0dx^1\dots dx^{n+1}$ by replacing dx^j with

$$\begin{cases} x^i & \text{for } j=i \\ s^{-1}dsx^j & \text{for } j\varepsilon D\setminus\{i\} \\ dx^j & \text{for } j\notin D . \end{cases}$$

The definition of the cup product with Tr and the above discussion shows that the desired sum equals

$$\sum_{(D,i)} (-1)^{n+1-i}(sc)^{2k-1}\mathrm{Tr}(r_{t,D,i})\varphi_n(\omega_{D,i}) ,$$

the sum running over $\underset{k,i}{\mathcal{D}}_{n-2k+1,k,i}$.

Note next that property 2) of Lemma 2 implies that

$$\varphi_n(\omega_{D,i})=(-1)^{n-i+\varepsilon}\varphi_{n-2k+1}(\omega_D)$$

where

$$\varepsilon=\begin{cases} 0 & \text{if there is an odd number of elements in D greater than i} \\ 1 & \text{otherwise} \end{cases}$$

(ω_D is defined in Lemma 1).

Note further that since $e^2=-1$ and $\mathrm{Tr}(p_t\dot{p}_te)=1=-\mathrm{Tr}(p_te\dot{p}_t)$ one gets

$$\mathrm{Tr}(r_{t,D,i})=(-1)^{k-1+\varepsilon}$$

so that out sum becomes

$$\sum_{(D,i)}(-1)^k(sc)^{2k-1}\varphi_{n-2k+1}(\omega_D)=$$

$$=\sum_k(-1)^k(sc)^{2k-1}2k(\sum_{D\varepsilon\mathcal{D}_{n-2k+1,k}}\varphi_{n-2k+1}(\omega_D))=$$

$$=\sum_k(-1)^k(sc)^{2k-1}(2i\pi)^{-k}2k(k!)^{-1}S^k\varphi_{n-2k+1}(x^0,x^1,\dots,x^{n+1})$$

by Lemma 1.

The proof is complete once we notice that

$$\int_0^{\pi/2} (\sin t \cos t)^{2k-1} dt = \frac{1}{2^k} \cdot \frac{(k-1)!}{(2k-1)\cdot(2k-3)\ldots 3\cdot 1} \cdot \qquad \text{Q.E.D.}$$

The next lemma expresses the fact that the cup product with ε is compatible with Bott periodicity.

LEMMA 10. *Let* φ *be an n-trace on the Banach algebra* B, *with domain* B.

a) $n=2m$. *Let* $p\in \mathrm{Proj}\ M_k(B)$ *be a projection and* $u_p \in Gl_k(B\otimes C^\infty(\mathbb{T}))$ *the invertible element* $u_p(t)=\exp(2i\pi t)p+(1-p)$. *Then*

$$<[p],[\varphi]> = <[u_p],[\varphi \# \varepsilon]> .$$

b) $n=2m-1$. *Let* $u\in Gl_k(B)$ *be an invertible element and* $v_t\in Gl_{2k}(B\otimes C^\infty([0,1]))$ *any differentiable path such that*

$$v_0=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad v_1=\begin{bmatrix}u^{-1} & 0\\ 0 & u\end{bmatrix} .$$

If e *denotes the constant projection* $e=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$ *and* p_u *denotes the projection* $p_u(t)=v_t e v_t^{-1} \in \mathrm{Proj}\ M_{2k}(B\otimes C^\infty(\mathbb{T}))$, *then*

$$<[u],[\varphi]> = <[e]-[p_u],[\varphi]>$$

($<,>$ *denotes the pairing of* $H^*(A)$ *with* $K_*(A)$ *defined in* [5] *and* [6].)

PROOF. a) Since $\mathrm{Tr} \# (\varphi \# \varepsilon) = (\mathrm{Tr} \# \varphi) \# \varepsilon$ it is sufficient to consider $k=1$. Note that $u_p-1=(\exp(2i\pi t)-1)p$, $u_p^{-1}-1=(\exp(-2i\pi t)-1)p$ so that

$$(\varphi \# \varepsilon)(u_p^{-1}-1, u_p-1, \ldots, u_p^{-1}-1, u_p-1) =$$

$$= \sum_{i=1}^{n+1} (-1)^{n+i-1} \hat{\varphi}(p\, dp..dp\ p\ dp..dp) \int_0^1 (\exp(2i\pi t)-1)^m (\exp(-2i\pi t)-1)^m \cdot$$

$$\cdot (-1)^{i-1} 2i\pi \exp((-1)^{i-1} 2i\pi t)(\exp((-1)^i 2i\pi t)-1)\,dt .$$

Since

$$\hat{\varphi}(p\, dp..dp\, p\, dp..dp) = \begin{cases} 0 & \text{when } i \text{ is even} \\ \varphi(p,\dots,p) & \text{when } i \text{ is odd} \end{cases}$$

[5] the above sum becomes

$$\sum_{\substack{i=1 \\ i=\text{odd}}}^{n+1} \varphi(p,..,p) \int_0^1 (\exp(-2i\pi t)-1)^{2m+1} (-1)^m \exp(2i\pi t(m+1))\, 2i\pi dt =$$

$$= 2i\pi(m+1)\frac{(2m+1)!}{(m+1)!\,m!}\varphi(p,\dots,p) = 2i\pi\frac{(2m+1)!}{m!\;m!}\varphi(p,\dots,p) \;.$$

Thus

$$<[u_p],[\varphi \# \varepsilon]> = \frac{1}{(2i\pi)^{m+1}} \cdot \frac{1}{2^{2m+1}(m+\frac{1}{2})(m-\frac{1}{2})\dots\frac{1}{2}} \;.$$

$$\cdot 2i\pi\frac{(2m+1)!}{m!\;m!}\varphi(p,..,p) = \frac{1}{(2i\pi)^m m!}\varphi(p,..,p) = <[p],[\varphi]> \;.$$

b) Again it is sufficient to consider the case $k=1$. Since the result does not depend on the path of invertible elements, we will choose the following particular one

$$v_t = \begin{cases} \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} u^{-1} & 0 \\ 0 & 1 \end{bmatrix} & \text{for } t\varepsilon[0,\frac{\pi}{2}] \\ \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix} \begin{bmatrix} 0 & u \\ -u^{-1} & 0 \end{bmatrix} & \text{for } t\varepsilon[\frac{\pi}{2},\pi]. \end{cases}$$

Note next that we may use the same trick as in [5] to suppose that $1\varepsilon\mathcal{B}$ and that $d1=0$. That is, we replace (even if $\mathcal{B}$ is already unital) $\mathcal{B}$ by $\tilde{\mathcal{B}}$ and extend φ to $\tilde{\mathcal{B}}$ by

$$\tilde{\varphi}(x^o+\lambda^o 1, x^1+\lambda^o 1, .., x^n+\lambda^n 1) = \varphi(x^o,\dots,x^n) \;.$$

In this case $<[e],[\varphi]>=0$.

To compute $<[p_u],[\varphi]>$ we shall apply Lemma 9 with Λ equal to the scalar multiples of the identity. The integral up to $\frac{\pi}{2}$ appearing in the formula of $\varphi \# \varepsilon$ thus equals

$$\sum_{k=1}^{m} c_k S^k \varphi_{n-2k+1}(1,1,\dots,1)$$

and again since $d1=0$, only

$$c_m S^m \varphi_o(1,\dots,1)$$

is different from 0. The integral from $\frac{\pi}{2}$ to π is again of the type considered in Lemma 9 but this time with s equal to 1. Using once again $d1=0$ it follows that $\varphi_{n-k}=0$ for every k. Thus

$$(\varphi \# \varepsilon)(p_u,\dots,p_u)=c_m S^m \varphi_o(1,\dots,1) \; .$$

Since the operator S does not affect the value of $<[p],[\varphi]>$ ([5], Proposition 14)

$$<[p_u],[\varphi \# \varepsilon]>=c_m<[1],[\varphi_o]>=c_m\hat{\varphi}(u^{-1}du\dots u^{-1}du) \; .$$

Using the fact that $du^{-1}=-u^{-1}du\, u^{-1}$ the last term equals

$$(-1)^{m-1} c_m \varphi(u^{-1},u,\dots,u^{-1},u)=\pm<[u],[\varphi]> ,$$

which concludes the proof. Q.E.D.

Let us restate the preceding lemma as

PROPOSITION 11. *If* φ *is an* n-*trace on the Banach algebra* B, *and* $b:K_n(B) \to K_{n+1}(B\otimes C(\mathbb{T}))$ *is the Bott map, then*

$$<b(x),[\varphi \# \varepsilon]>=<x,[\varphi]>$$

for every $x\in K_n(B)$.

We have now the technical tools in order to extend the results of the preceding paragraph to higher dimensional traces.

Let us first consider the case of short exact sequences

$$0 \to I \xrightarrow{i} B \xrightarrow{\pi} A \to 0$$

of Banach algebras.

Denote by B_I the algebra of continuous functions $f:[0,1] \to B$ such that $f(0)=0$, $f(1)\in I$, and denoty by $\rho:B_I \to I$ the map given by evaluation at 1. The kernel of ρ is $C_o(\mathbb{T},B)$, the algebra of continuous functions on $\mathbb{T}$ with values in B that vanish at 1. Note that the exact sequence

$$0 \longrightarrow C_o(\mathbb{T},B) \longrightarrow B_I \xrightarrow{\rho} I \longrightarrow 0$$

gives rise to the exact sequence

$$(*) \qquad K_m(C_o(\mathbb{T},B)) \longrightarrow K_m(B_I) \xrightarrow{\rho_*} \ker i_* \longrightarrow 0$$

for every m.

Suppose that φ is an n-trace on A. Since there is an obvious map

$$\tilde{\pi}: B_I \to A \otimes C(\mathbb{T})$$

we get an n+1-trace on B_I by $\tilde{\pi}^*(\varphi \# \varepsilon)$. Let us denote this n+1-trace by φ_π.

PROPOSITION 12. a) *The range of* φ_π *restricted to* $K_{n+1}(C_o(\mathbb{T},B))$ *coincides with the range of* $\pi^*\varphi$ *on* $K_n(B)$.

b) *The* n+1-*trace* φ_π *determines a well defined group homomorphism*

$$K_{n+1}(I) \supset \ker i_* \xrightarrow{\underline{\varphi}_\pi} \mathbb{C}/\langle K_n(B),[\pi^*\varphi]\rangle$$

by the formula

$$\underline{\varphi}_\pi(\rho_*(x)) = q(\langle x,[\varphi_\pi]\rangle)$$

where $q:\mathbb{C} \to \mathbb{C}/\langle K_n(B),[\pi^*\varphi]\rangle$ *is the natural projection.*

c) *If* $\delta: K_n(A) \to K_{n+1}(I)$ *denotes the boundary map determined by the considered exact sequence, then*

$$q(\langle x,[\varphi]\rangle) = \underline{\varphi}_\pi(\delta(x)).$$

In particular, the sequence

$$0 \longrightarrow \langle K_n(B),[\pi^*\varphi]\rangle \longrightarrow \langle K_n(A),[\varphi]\rangle \xrightarrow{q} \underline{\varphi}_\pi(\ker i_*) \to 0$$

is exact.

PROOF. a) Follows from the preceding proposition once we notice that the restriction of φ_π to $K_{n+1}(C_o(\mathbb{T},B))$ coincides with $(\pi^*\varphi) \# \varepsilon$.

b) Is a direct consequence of a) and of the exact sequence (*).

c) Recall the definition of δ to see that with the above notations $\delta(x)=\rho_*(z)$, where $z\in K_{n+1}(B_I)$ is any element such that $\tilde{\pi}_*(z)=b(x)$. Thus

$$\underline{\varphi}_\pi(\delta(x))=q(<z,[\varphi_\pi]>)=q(<\tilde{\pi}_*(z),[\varphi\# \varepsilon]>)=q(<b(x),[\varphi\#\varepsilon]>)$$

so that by the preceding proposition

$$\underline{\varphi}_\pi(\delta(x))=q(<x,[\varphi]>) .$$

The exact sequence now follows from the six term exact sequence associated to the considered short sequence. Q.E.D.

As in Section 1 we shall apply the above proposition to the Toeplitz extension associated to reduced crossed products by free groups.

Let $\alpha:F_m \to Aut(A)$ be an action of the free group on m generators $g_1,\dots,g_m$ on the unital C*-algebra A and $A \times_{\alpha_r} F_m$ the corresponding reduced crossed product.

Denote by $A_\alpha \subset M_m(C([0,1],A))$ the C*-algebra of those $M_m(A)$ valued functions f such that f(0) and f(1) are diagonal matrices whose entries satisfy the conditions

$$f(1)_{ii}=\alpha_{g_i}(f(0)_{ii}) .$$

Evaluation at 1 determines a map $\eta:A_\alpha \to A^m$, whose kernel is $M_m(C_o(T,A))$, the C*-algebra of continuous functions on T with values in $M_m(A)$ that vanish at 1. Note that the sequence

$$0 \to M_m(C_o(T,A)) \to A_\alpha \xrightarrow{\eta} A^m \to 0$$

gives rise to the exact sequence

$$(**) \quad K_n(C_o(T,A)) \to K_n(A_\alpha) \xrightarrow{\eta_*} \ker\beta \to 0$$

where $\beta:K_n(A)^m \to K_n(A)$ is the map

$$\beta((x)_{i=1}^m)=\sum_{i=1}^m (x_i-\alpha_{g_{i*}}^{-1}(x_i)).$$

Suppose that φ is an n-trace on $A \times_{\alpha_r} F_m$, arising as the character of an n-cycle $(\Omega,d,\hat{\varphi})$ with the following properties:

i) $\Omega^o \subset A \times_{\alpha_r} F_m$ contains the immage of the free group (in particular it contains the unit of $A \times_{\alpha_r} F_m$).

ii) the algebra $\mathcal{A}=A \cap \Omega^o$ is dense in A and is normalised by u_g, $\forall g \in F_m$.

iii) $d1=0$; $d(u_g a u_g^{-1})=u_g da u_g^{-1}$ $\forall a \in \mathcal{A}$ and $\forall g \in F_m$.

This implies in particular that the restriction of φ to $\mathcal{A}$ (still denoted by φ) is an invariant n-trace, i.e.

$$\varphi(u_g a^o u_g^{-1},\dots,u_g a^n u_g^{-1})=\varphi(a^o,a^1,\dots,a^n)$$

for every $a^i \in \mathcal{A}$ and $g \in F_m$.

An n-trace with the above properties on a crossed product will be called natural.

LEMMA 13. *If* φ *is a natural* n-*trace on* $A \times_{\alpha_r} F_m$, *then the formula*

$$\varphi_\alpha(f^o,f^1,\dots,f^{n+1})=$$

$$=\sum_{i=1}^{n+1}(-1)^{n-i+1}\int\hat{\varphi} \# \mathrm{Tr}(f^o(t)df^1(t)\dots df^{i-1}(t)\dot{f}^i(t)df^{i+1}(t)\dots df^{n+1}(t))dt$$

$\forall f^i \in M_m(C^\infty[0,1],\mathcal{A})$, *defines an* n+1-*trace on* A_α, *whose restriction to* $M_m(C_o(T,\mathcal{A}))$ *coincides with* $(\varphi \# \mathrm{Tr}) \# \varepsilon$.

PROOF. It is an easy computation to show that φ_α is a Hochschild cocycle. (In fact it is the restriction to A_α of $(\varphi \# \mathrm{Tr}) \# \tilde{\varepsilon}$, where $\tilde{\varepsilon}$ is the Hochschild cocycle

$$\tilde{\varepsilon}(f^o,f^1)=\int f^o df^1 \quad \text{on} \quad C^\infty[0,1].)$$

Note next that since 1 is the unit of Ω, and $d1=0$

$$\varphi_\alpha(1,f^1,\dots,f^{n+1})=\sum_1^{n+1}(-1)^{n-i+1}\int\varphi \# \mathrm{Tr}(df^1\dots df^{i-1}\dot{f}^i df^{i+1}\dots df^{n+1})=$$

$$=\sum_1^{n+1}(-1)^n\int\varphi \# \mathrm{Tr}(f^1,\dots,f^{i-1},\dot{f}^i,f^{i+1},\dots,f^{n+1})=$$

$$=(-1)^n(\varphi \# \mathrm{Tr}(f^1(1),\dots,f^{n+1}(1))-\varphi \# \mathrm{Tr}(f^1(0),\dots,f^{n+1}(0)))=0$$

(since φ is invariant). This now implies that φ_α is cyclic:

$$\varphi_\alpha(f^o,f^1,\ldots,f^{n+1})=\varphi_\alpha(1,f^of^1,\ldots,f^{n+1})-\varphi_\alpha(1,f^o,f^1f^2,\ldots,f^{n+1})+\ldots$$

$$\ldots+(-1)^{n+1}\varphi_\alpha(f^{n+1},f^o,\ldots,f^n) .$$

The rest is now easy. Q.E.D.

Recall that $\mathcal{T}_{(A\otimes K)^m}$ is the algebra of continuous functions $f:[0,1] \to \mathcal{T}$ such that $f(0)=0$ and $f(1)\in(A\otimes K)^m$. Let us denote it from now on with $\mathcal{T}_*$. Consider also $\mathcal{T}_{**}$ the algebra of continuous functions $f:[0,1] \to \mathcal{T}$ such that $f(0)-f(1)\in(A\otimes K)^m$. One thus gets an exact sequence

$$0 \to \mathcal{T}_* \to \mathcal{T}_{**} \to \mathcal{T} \to 0 .$$

The obvious map $\tilde{\tilde{\pi}}$ from $\mathcal{T}_{**}$ to $(A \times_{\alpha_r} F_m)\otimes C(\mathbb{T})$ induces the n+1-trace $\tilde{\tilde{\pi}}^*(\varphi \# \varepsilon)$ which will be denoted by φ_π (note that its restriction to $\mathcal{T}_*$ coincides with the previous φ_π).

In order to get from Proposition 12 the result for the reduced crossed product by the free group, we shall construct an embedding of A_α into $M_{2m+1}\otimes\mathcal{T}_{**}$. Let (e_{ij}), $i,j=0,1,\ldots,2m$, be the matrix units in M_{2m+1} and $f\in A_\alpha$. The embedding will be done in four steps.

1) $t\in[-2,-\frac{\pi}{2}]$. Consider the diagonal matrix $f(1)\in M_m(A)$ and define

$$\sigma_o(f)=\sum_{i=1}^{m} d(f(1)_{ii})\otimes e_{ii}$$

where $d:A \to \mathcal{T}$ is the natural inclusion. It is then easily seen that there exists a smooth path of unitaries $w_t\in M_{2m+1}\otimes\mathcal{T}$, $t\in[-2,-\frac{\pi}{2}]$ such that $w_{-2}=1$, $(1\otimes\pi)(w_t)=1$ for $t\in[-2,-\frac{\pi}{2}]$ and

$$w_{-\frac{\pi}{2}}\sigma_o(f(1))w^*_{-\frac{\pi}{2}}=i\circ j(f(1))\otimes e_{oo}+\sum_{i=1}^{m} d(f(1)_{ii})s_is_i^*\otimes e_{ii}$$

where $i:(A\otimes K)^m \to \mathcal{T}$ is the inclusion map, $j:A^m \to (A\otimes K)^m$ the natural embedding (so that $i\circ j((a_i)_i)=\sum_{i=1}^{m}(1-s_is_i^*)d(a_i))$, and s_i the isometries that satisfiy $\pi(s_i)=u_i$, $s_id(a)=d(\alpha_{g_i}(a))s_i$.

Let

$$\sigma_t'(f)=w_t\sigma_o(f(1))w_t^* .$$

2) $t\in[-\frac{\pi}{2},0]$. Let w_t be the isometries

$$w_t=1\otimes e_{oo}+\sum_{i=1}^{m}(s_i\otimes e_{ii}+1\otimes e_{2i2i})(\cos(-t)\otimes e_{ii}+$$

$$+\sin(-t)\otimes e_{i,2i}-\sin(-t)\otimes e_{2i,i}+\cos(-t)\otimes e_{2i,2i})(s_i^*\otimes e_{ii}+1\otimes e_{2i,2i}).$$

Note that the isometries w_t satisfy the following relations

$$w_o(i\circ j(f(1))\otimes e_{oo}+\sum_{i=1}^{m}d(f(0)_{ii})\otimes e_{2i,2i})w_o^*=$$

$$=i\circ j(f(1))\otimes e_{oo}+\sum_{i=1}^{m}d(f(0)_{ii})\otimes e_{2i,2i}$$

and

$$w_{-\frac{\pi}{2}}(i\circ j(f(1))\otimes e_{oo}+\sum_{i=1}^{m}d(f(0)_{ii})\otimes e_{2i,2i})w_{-\frac{\pi}{2}}^*=$$

$$=i\circ j(f(1))\otimes e_{oo}+\sum_{i=1}^{m}d(\alpha_{g_i}(f(0)_{ii}))s_is_i^*\otimes e_{ii}=$$

$$=i\circ j(f(1))\otimes e_{oo}+\sum_{i}^{m}d(f(1)_{ii})s_is_i^*\otimes e_{ii}.$$

Define

$$\sigma_t'(f)=w_t(i\circ j(f(1))\otimes e_{oo}+\sum_{1}^{m}d(f(0))_{ii})\otimes e_{2i,2i})w_t^*.$$

3) $t\in[0,1]$. Define

$$\sigma_t'(f)=i\circ j(f(1))+\sum_{i,j=1}^{m}d(f(t)_{ij})\otimes e_{2i,2j}.$$

4) $t\in[1,2]$. Let w_t be the unitaries

$$w_t=1\otimes e_{oo}+\sum_{i=1}^{m}(\sin\tfrac{\pi}{2}t\otimes e_{ii}+\cos\tfrac{\pi}{2}t\otimes e_{i,2i}-\cos\tfrac{\pi}{2}t\otimes e_{2i,i}+\sin\tfrac{\pi}{2}t\otimes e_{2i,2i})$$

and define

$$\sigma_t'(f)=w_t(i\circ j(f(1)+\sum_{i=1}^{m}d(f(1)_{ii})\otimes e_{2i,2i})w_t^*.$$

The desired embedding is given by the formula

$$\sigma_t(f)=\sigma_{3t-2}'(f).$$

Note that

$$\sigma_1(f)-\sigma_0(f)=i\circ j(f(1))\otimes e_{00}.$$

LEMMA 14. *Let* φ *be a natural* n-*trace on* $A\times_{\alpha_r}F_m$ *and* φ_π, φ_α, $\eta:A_\alpha\to A^m$ *and* $\sigma:A_\alpha\to M_{2m+1}\otimes T_{**}$ *as above. For each* $k\leq n$ *and* $i=1,\ldots,m$ *consider the* n-k-*traces on* A

$$\varphi^i_{n-k}(a^0,a^1,\ldots,a^{n-k})=\varphi(a^0da^1\ldots da^{n-k}(u_{g_i}du^{-1}_{g_i})^k)$$

(defined in Lemma 8*), and* φ_{n-k} *the* n-k *trace on* A^m

$$\varphi_{n-k}=\sum_{i=1}^m\varphi^i_{n-k}.$$

Then

$$\sigma^*(Tr\#\varphi_\pi)=\varphi_\alpha+\sum_{k=1}^{[\frac{n+1}{2}]}c_k\eta^*(S^k\varphi_{n-2k+1})$$

where $c_k=(-1)^k\frac{1}{(2i\pi)^k}\frac{1}{2^{k-1}}\frac{1}{(2k-1)(2k-3)\ldots3\cdot1}$.

PROOF. Let $f^0,\ldots,f^{n+1}\in A_\alpha$ be smooth functions with values in A. Since $\sigma(f^i)$ are piecewise smooth functions we have to compute

$$\sum_{i=1}^{n+1}(-1)^{n-i+1}\int_0^1(Tr\#\hat\varphi)(\pi(\sigma_t(f^0))d(\pi(\sigma_t(f^1)))\ldots\overbrace{\pi(\sigma_t(f^i))}^{\cdot}\ldots d(\pi(\sigma_t(f^{n+1}))))dt=$$

$$=\sum_{i=1}^{n+1}(-1)^{n-i+1}\int_{-2}^{2}(Tr\#\hat\varphi)(\pi(\sigma'_t(f^0))d(\pi(\sigma'_t(f^1)))\ldots\overbrace{\pi(\sigma'_t(f^i))}^{\cdot}\ldots d(\pi(\sigma'_t(f^{n+1}))))dt.$$

Note that the integral from -2 to $-\frac{\pi}{2}$ is 0 since in this case $\pi(\sigma'_t(f))$ is a constant function.

The integral from $-\frac{\pi}{2}$ to 0 is of the type considered in Lemma 9. More precisely if $u\in M_m\otimes(A\times_{\alpha_r}F_m)$ denotes the unitary

$$u=\sum_{i=1}^m u_i\otimes e_{ii}$$

and we identify M_{2m} with $M_2\otimes M_m$, then we have to compute

$$\sum_{i=1}^{n+1}(-1)^{n-i+1}\int_{\pi/2}^{0}Tr\#(Tr\#\hat\varphi)(v_{-t}x^0v^{-1}_{-t}d(v_{-t}x^1v^{-1}_{-t})\ldots)dt=$$

$$=\sum_{i=1}^{n+1}(-1)^{n-1}\int_0^{\pi/2}Tr\#(Tr\#\hat\varphi)(v_tx^0v_t^{-1}d(v_tx^1v_t^{-1})\ldots)dt.$$

where

$$v_t = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} u^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

and $x^i \varepsilon M_m \otimes (A \times_{\alpha_r} F_m)$ is

$$x^i = \sum_{j=1}^{m} f^i(0)_{jj} \otimes e_{jj} .$$

Lemma 9 now applies to yield

$$\sum_{k=1}^{[\frac{n+1}{2}]} -c_k S^k (\mathrm{Tr} \# \varphi)'_{n-2k+1}(x^o, x^1, \ldots, x^{n+1}) =$$

$$\sum_{k=1}^{[\frac{n+1}{2}]} -c_k S^k \left(\sum_{i=1}^{m} \varphi'^{\,i}_{n-2k+1} \right)) (f(0)_{ii}, \ldots, f^{n+1}(0)_{ii}) ,$$

where

$$\varphi'^{\,i}_{n-2k+1}(a^o, \ldots, a^{n-k}) = \varphi(a^o da^1 \ldots da^{n-k} (u_{g_i}^{-1} du_{g_i})^k) .$$

But since

$$\varphi'^{\,i}_{n-2k+1}(u_{g_i}^{-1} a^o u_{g_i}, u_{g_i}^{-1} a^1 u_{g_i}, \ldots, u_{g_i}^{-1} a^{n-2k+1} u_{g_i}) =$$

$$= \varphi(a^o da^1 \ldots da^{n-2k+1} u_{g_i} (u_{g_i}^{-1} du_{g_i})^{2k+1} u_{g_i}^{-1}) =$$

$$= -\varphi^i_{n-2k+1}(a^o, a^1, \ldots, a^{n-2k+1})$$

we get that the integral from $-\frac{\pi}{2}$ to 0 is equal to

$$\sum_{k=1}^{[\frac{n+1}{2}]} c_k S^k \left(\sum_{i=1}^{m} \varphi^i_{n-2k+1} \right)) (f^o(1)_{ii}, \ldots, f^{n+1}(1)_{ii}) .$$

The integral from 0 to 1 obviously coincides with

$$\varphi_\alpha(f^o, \ldots, f^{n+1})$$

while the integral from 1 to 2 is again of the type considered in Lemma 9, but this time with the invertible element equal to 1 and so equals 0. Q.E.D.

THEOREM 15. *Let* φ *be a natural* n-*trace on* $A \times_{\alpha_r} \mathbb{F}_m$. *Let* $A_\alpha = \{f \in M_m(C([0,1],A));\ f(1)_{ii} = \alpha_{g_i}(f(1)_{ii}),\ f(0)_{ij} = f(1)_{ij} = 0$ *for every* $i,j = 1,\dots,m,\ i \neq j\}$ *and* φ_α the n+1 *trace on* A_α *defined in* Lemma 13. *Then:*

a) *The* n+1 *trace* φ_α *induces a well defined group homomorphism*

$$\underline{\varphi}_\alpha : \ker \beta \to \mathbb{C}/\langle K_n(A),[\varphi]\rangle$$

by the formula

$$\underline{\varphi}_\alpha(\eta_*(x)) = q(\langle x,[\varphi_\alpha]\rangle)$$

where $\beta : K_{n+1}(A)^m \to K_{n+1}(A)$ *is the map* $\beta((x_i)_i) = \sum_{i=1}^{m} (x_i - \alpha_{g_{i*}^{-1}}(x_i))$, $\eta : A_\alpha \to A^m$ *is the map given by evaluation at* 1 *and* $q : \mathbb{C} \to \mathbb{C}/\langle K_n(A),[\varphi]\rangle$ *is the natural projection.*

b) *For each generator* $g_i \in \mathbb{F}_m$ *consider the* n-k *traces on* A, $\varphi^i_{n-k}(a^0,\dots,a^{n-k}) = \varphi(a^0 da^1 \dots da^{n-k}(u_g du_g^{-1})^k)$ *(see* Lemma 8) *and let* φ_{n-k} *be the* n-k *trace on* A^m, $\varphi_{n-k} = \sum_{i=1}^{m} \varphi^i_{n-k}$. *Let also* $\partial : K_n(A \times_{\alpha_r} \mathbb{F}_m) \to K_{n+1}(A)^m$ *be the map which composed with* $j_* : K_{n+1}(A)^m \to K_{n+1}((A\otimes K)^m)$ *gives the boundary map* δ *associated to the Toeplitz extension. Then*

$$q(\langle x,[\varphi]\rangle) = \underline{\psi}_\alpha(\partial(x))$$

where

$$\underline{\psi}_\alpha = \underline{\varphi}_\alpha + q\Big(\sum_{k=1}^{[\frac{n+1}{2}]} c_k \varphi_{n-2k+1}\Big)$$

and

$$c_k = (-1)^k \frac{1}{(2i\pi)^k} \frac{1}{2^{k-1}} \frac{1}{(2k-1)(2k-3)\dots 3\cdot 1}.$$

In particular the sequence

$$0 \to \langle K_n(A),[\varphi]\rangle \to \langle K_n(A \times_{\alpha_r} \mathbb{F}_m),[\varphi]\rangle \to \underline{\psi}_\alpha(\ker \beta) \to 0$$

is exact.

PROOF. a) Follows imediately from the exact sequence (**), from Lemma 13 and from Proposition 12.

b) Since the inclusion of A in T induces an isomorphism of $K_n(A)$

with $K_n(T)$, [15, Lemma 3.4], we know already by Proposition 12 that

$$q(\langle x,[\varphi]\rangle)=\underline{\varphi}_\pi(\delta(x)) .$$

So all we have to do is to compute $\underline{\varphi}_\pi(j_*\partial(x))$.

Consider the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & K_*(T_*) & \longrightarrow & K_*(T_{**}) & \longrightarrow & K_*(T) & \longrightarrow 0 \\
 & \rho_* \downarrow & & \uparrow \sigma_* & & & \\
 & K_*((A\otimes K)^m) & & K_*(A_\alpha) & & &
\end{array}$$

and note that the horizontal sequence is exact and split. This leads to a map $r:K_*(T_{**}) \to K_*(T_*)$. Recall the definition of σ and η to see that

$$\sigma_1(f)=i\circ j\circ\eta(f)\otimes e_{oo}\oplus\sigma_o(f) .$$

This easily implies that $\rho_*\circ r\circ\sigma_*=j_*\circ\eta_*$ so that $\underline{\varphi}_\pi(j_*\circ\eta_*(x))=$ $=q(\langle r\circ\sigma_*(x),[\varphi_\pi]\rangle)$. Note next that since $y-r(y)$ may be represented by a constant function, for every $y\in K_*(T_{**})$,

$$\langle y,[\varphi_\pi]\rangle=\langle r(y),[\varphi_\pi]\rangle$$

so that

$$\underline{\varphi}_\pi(j_*\ \eta_*(x))=q(\langle\sigma_*(x),[\varphi_\pi]\rangle)=q(\langle x,[\sigma^*\varphi_\pi]\rangle) ,$$

so that the conclusion follows by the preceding lemma.

Again the exact sequence is now a direct consequence of the six term exact sequence in K-theory. Q.E.D.

REMARKS.- Since the proof depends only on the results concerning the Toeplitz extension [15] and on Proposition 12 it works without the naturality condition on the n-trace φ. However, $\sigma^*\varphi_\pi$ is no longer a sum of traces obtained in an easy way from φ.

- The only term in the expression of $\underline{\psi}_\alpha$ that does not stem from a trace on A is $\underline{\varphi}_\alpha$. However if there is a cross-section to the map $\eta:A_\alpha \to A^m$ (for example if there is for each generator $g_i\in F_m$ a continuous path of automorphisms connecting α_{g_i} to the identity) this term also appears as an n+1 trace on A (the pull back of φ_α).

- The analogue of Corollary 4 is a consequence of the interpretation given in [5], [6] to the cohomology $H^*_\lambda(A)$, in the case A is the C*-algebra of continuous functions on a smooth manifold, in terms of the homology (with complex coefficients) of the manifold.

As an application of the preceding theorem we shall give a new proof of the result of G.A. Elliott [9] concerning the range of the trace on noncommutative tori. Given a discrete abelian group G and a character ρ of the second exterior power $G\wedge G$ one defines the C*-algebra A_ρ as follows (see [9] and the references given there). Choose a 2-cocycle α on G with values in $\mathbb{T}$ such that

$$\alpha(g,h)\alpha(h,g)^{-1}=\rho(g\wedge h)$$

and consider the envelopping C*-algebra of unitaries $(u_g)_{g\in G}$ with the relations

$$u_g u_h=\alpha(g,h)u_{g+h}$$

(see for example [12]). The resulting C*-algebra is independent of the cocycle α and is denoted A_ρ. It is the universal C*-algebra generated by a projective representation of G such that

$$u_g u_h u_g^{-1} u_h^{-1}=\rho(g\wedge h) .$$

When G is isomorphic to $\mathbb{Z}^n$ it is easy to see that A_ρ is obtained by an iteration of n ordinary crossed products by actions of $\mathbb{Z}$, the first action on $\mathbb{C}$. Thus we may apply Theorem 15 recursively.

Let $\{e_i\}$ be the canonical basis in $\mathbb{Z}^n$, and $(\theta_{ij})_{i,j}\in M_n(\mathbb{R})$ the antisymetric matrix given by

$$\rho(e_i\wedge e_j)=e^{2i\pi\theta_{ij}} .$$

In this case A_ρ is generated by n unitaries $u_1,\ldots,u_n$ satisfying the relations

$$u_i u_j=e^{2i\pi\theta_{ij}}u_j u_i .$$

We shall denote by $A_{\rho,k}$, $0\le k\le n$ the C*-algebra generated by $\{u_{k+1},\ldots,u_n\}$ ($A_{\rho,n}=\mathbb{C}$). This is a decomposition series of A_ρ in the sense that

$$A_\rho=A_{\rho,o}\supset A_{\rho,1}\supset\ldots\supset A_{\rho,n}=\mathbb{C}$$

and

$$A_{\rho,k-1}=A_{\rho,k}\times_{\alpha_k}\mathbb{Z}$$

where $\alpha_k\in Aut(A_{\rho,k})$ is the automorphism induced by the unitary u_k, which normalizes $A_{\rho,k}$.

Moreover the universal property of A_ρ shows that there exists an action

$$\alpha:\mathbb{R}^n \to \mathrm{Aut}\,(A_\rho)$$

defined on each generator by

$$\alpha_{t_1,\ldots,t_n}(u_l)=(\exp(2\pi i\sum_j \theta_{jl}t_j))u_l \ .$$

Note that α_k is the restriction of α_{f_k} on $A_{\rho,k}$, where $f_1,\ldots,f_n$ is the canonical basis in $\mathbb{R}^n$. The algebra of smooth elements under this action, i.e. of those elements a such that

$$\mathbb{R}^n \ni t \to \alpha_t(a)\varepsilon A_\rho$$

is smooth, will be denoted by A_ρ^∞.

As in [2] and [6] we shall consider the action of the Lie algebra of $\mathbb{R}^n$ on A_ρ^∞, i.e. the derivations

$$\delta_j:A_\rho^\infty \to A_\rho^\infty \quad , \quad j=1,\ldots,n \ ,$$

corresponding for fixed j to the one parameter group of automorphisms

$$\mathbb{R} \ni t \to \alpha_{f_j t}\varepsilon\mathrm{Aut}\,(A_\rho)$$

and the corresponding graded diferential algebra

$$\Omega=A_\rho^\infty\otimes\Lambda\mathbb{R}^n$$

where the differential $d:A_\rho^\infty\otimes\Lambda^p\mathbb{R}^n \to A_\rho^\infty\otimes\Lambda^{p+1}\mathbb{R}^n$ is defined in the following way:

$$d(a\otimes\omega)=\sum_{i=1}^n \delta_i(a)\otimes(f_i\wedge\omega) \ .$$

Note that the generators $u_1,\ldots,u_n \varepsilon A_\rho^\infty$ and that

$$\delta_j(u_i)=2i\pi\theta_{ji}u_i \ .$$

Suppose now that τ is any trace state on A_ρ.

Again as in [6] we shall consider for each $\omega^*\varepsilon\Lambda^p(\mathbb{R}^n)^*$ the closed graded p-trace τ_{ω^*} on Ω defined by

$$\tau_{\omega^*}(a\otimes\omega)=\tau(a)\cdot\omega^*(\omega) .$$

Let us denote the dual basis in $(\mathbb{R}^n)^*$ by $f_1^*,\ldots,f_n^*$.

The p-traces corresponding to the elements

$$f_{i_1}^*\wedge f_{i_2}^*\wedge\ldots\wedge f_{i_p}^* \qquad p>0$$

will be denoted by $\tau_{i_1,\ldots,i_p}$. Note that the explicit formula of $\tau_{i_1,\ldots,i_p}$ is:

$$\tau_{i_1,\ldots,i_p}(a^o,a^1,\ldots,a^p)=\sum_{\sigma\in\mathfrak{S}_p}\varepsilon(\sigma)\tau(a^o\delta_{i_{\sigma(1)}}(a^1)\ldots\delta_{i_{\sigma(p)}}(a^p))$$

$a^o,a^1,\ldots,a^p\in A_\rho^\infty$.

For $p=0$ we get $\tau_{1^*}=\tau$.

The inclusion $A_{\rho,k-1}^\infty \to A_\rho^\infty$ thus defines p-dimensional cycles $(\Omega,d,\tau_{i_1,\ldots,i_p})$ satisfying

$$d\,1=0$$

$$d(u_k a u_k^{-1})=u_k da u_k^{-1} \qquad a\in A_{\rho,k}^\infty ,$$

so that the restriction of $\tau_{i_1,\ldots,i_p}$ to each $A_{\rho,k-1}$ is a natural p--trace with respect to the decomposition

$$A_{\rho,k-1}=A_{\rho,k}\times_{\alpha_k}\mathbb{Z} .$$

Moreover since $u_k du_k^{-1}=2i\pi\sum_{i=1}^{n}-\theta_{jk}1\otimes f_i$, from all the p-m traces

$$\hat{\tau}_{i_1,\ldots,i_p}(a^o da^1 \ldots da^{p-m}(u_k du_k^{-1})^m)$$

only $\hat{\tau}_{i_1,\ldots,i_p}(a^o da^1..da^{p-1}(u_k du_k^{-1}))$ is different from 0 and equals

$$2i\pi\sum_{j=1}^{p}(-1)^{p-j+1}\theta_{i_j,k}\tau_{i_1,\ldots,\hat{i}_j,\ldots,i_p}(a^o da^1\ldots.da^{p-1}).$$

Let us also compute the map φ_{α_k} of Theorem 15. Note that there exists a map

$$\gamma:A_{\rho,k} \to (A_{\rho,k})_{\alpha_k}$$

(where $(A_{\rho,k})_{\alpha_k}$ is by definition the C*-algebra of continuous functions

$f:[0,1] \to A_{\rho,k}$ such that $f(1)=\alpha_k(f(0))$ given by the formula

$$\gamma(a)(t)=\alpha_{k,t}(a)$$

where with our previous notations $\alpha_{k,t}=\alpha_{f_k t}$.

Recall the definition of $\underline{\varphi}_{\alpha_k}$ to see that we have to compute

$$\tau_{i_1,\ldots,i_p;\alpha_k}(x^0,x^1,\ldots,x^{p+1})=$$

$$=\sum_{i=1}^{p+1}(-1)^{p-i+1}\int_0^1 \tau_{i_1,\ldots,i_p}(\alpha_{k,t}(x^0)d(\alpha_{k,t}(x^1))\ldots d(\alpha_{k,t}(x^{i-1}))\cdot$$

$$\cdot\widehat{\alpha_{k,t}(x^i)}d(\alpha_{k,t}(x^{i+1}))\ldots d(\alpha_{k,t}(x^{p+1}))dt=\sum_{i=1}^{p+1}(-1)^{p-i+1}\int_0^1\sum_{\sigma\in\mathfrak{S}_p}\varepsilon(\sigma)\cdot$$

$$\cdot\tau(\alpha_{k,t}(x^0)\delta_{i_{\sigma(1)}}(\alpha_{k,t}(x^1))\ldots\delta_{i_{\sigma(i-1)}}(\alpha_{k,t}(x^{i-1}))\delta_k(\alpha_{k,t}(x^i))\cdot$$

$$\cdot\delta_{i_{\sigma(i)}}(\alpha_{k,t}(x^{i+1}))\ldots\delta_{i_{\sigma(p)}}(x^{p+1}))dt=\int_0^1\sum_{\sigma\in\mathfrak{S}_{p+1}}\varepsilon(\sigma)\cdot$$

$$\cdot\tau(\alpha_{k,t}^{(x^0)}\delta_{i_{\sigma(1)}}(\alpha_{k,t}(x^1))\ldots\delta_{i_{\sigma(p+1)}}(\alpha_{k,t}(x^{p+1}))dt=$$

$$=\tau_{i_1,\ldots,i_p,k}(x^0,x^1,\ldots,x^{p+1}),$$

since the derivations commute with the automorphisms and τ is α invariant.

Thus $\underline{\tau}_{i_1,\ldots,i_p;\alpha_k}=q\circ\tau_{i_1,\ldots,i_p,k}$.

Theorem 15 thus says that the range of $\tau_{i_1,\ldots,i_p}$ on $K_p(A_{\rho,k-1})$ is equal to the subgroup generated by the range of $\tau_{i_1,\ldots,i_p}$ on $K_p(A_{\rho,k})$ and the range of

$$\tau_{i_1,\ldots,i_p,k}+\sum_{j=1}^{p}(-1)^{p-j}\theta_{i_j,k}\tau_{i_1,\ldots,\hat{i}_j,\ldots,i_p}$$

on $\ker\beta=K_{p+1}(A_{\rho,k})$.

THEOREM 16. (G.A. Elliott). *Let* G *be a torsion free abelian group and* $\theta:G\wedge G\to\mathbb{R}$ *be a homomorphism. Let* ρ *be the character* $\rho=\exp\circ\theta$ *of* $G\wedge G$, *and* A_ρ *the corresponding* C*-*algebra.*

Given any trace τ *on* A_ρ, *the range of* τ *on* $K_0(A_\rho)$ *is the range* *of the map*

$$\exp_\Lambda \theta = 1 + \theta + \frac{1}{2}(\theta\wedge\theta) + \ldots : \Lambda G \to \mathbb{R}.$$

PROOF. It is sufficient to consider the case when $G=\mathbb{Z}^n$ for some n. Consider then the decomposition series of A_ρ

$$A_\rho = A_{\rho,o} \supset A_{\rho,1} \supset \cdots \supset A_{\rho,n} = \mathbb{C},$$

$$A_{\rho,k-1} = A_{\rho,k} \times_{\alpha_k} \mathbb{Z}.$$

An easy induction argument, using the above form of Theorem 15, shows that the range of τ on A_ρ is the subgroup generated by the following 2^n traces on K_o of $A_{\rho,n}=\mathbb{C}$. For each $i_1<i_2<\ldots<i_p$

$$\varphi_{i_1,\ldots,i_p} = \tau_{i_1,\ldots,i_p} + \frac{1}{(p-2)!2!}\sum_{\sigma\in\mathfrak{S}_p} \varepsilon(\sigma)\tau_{i_{\sigma(1)}\cdots i_{\sigma(p-2)}} \cdot$$

$$\cdot\theta_{i_{\sigma(p-1)}i_{\sigma(p)}} + \frac{1}{2!}\,\frac{1}{(p-4)!2!2!}\sum_{\sigma\in\mathfrak{S}_p}\varepsilon(\sigma)\tau_{i_{\sigma(1)}\cdots i_{\sigma(p-4)}}\theta_{i_{\sigma(p-1)}i_{\sigma(p-2)}} \cdot$$

$$\cdot\theta_{i_{\sigma(p-1)}i_{\sigma(p)}} + \ldots \frac{1}{m!}\,\frac{1}{(p-2m)\cdot(2!)^m}\sum_{\sigma\in\mathfrak{S}_p}\varepsilon(\sigma)\tau_{i_{\sigma(1)}\cdots i_{\sigma(p-2m)}} \cdot$$

$$\cdot\theta_{i_{\sigma(p-2m+1)}i_{\sigma(p-2m+2)}}\cdots\theta_{i_{\sigma(p-1)}i_{\sigma(p)}} + \ldots$$

where $\mathfrak{S}_p$ is the symmetric group and $\varepsilon(\sigma)$ the signature of the permutation σ. Since $K_o(\mathbb{C})=\mathbb{Z}$ with generator 1 all traces $\tau_{i_1\ldots i_p}$ for $p>0$ vanish on $K_o(\mathbb{C})$ so that only the traces $\varphi_{i_1,\ldots,i_p}$ with p even, $p=2k$, are different from 0 on $K_o(\mathbb{C})$ and their value equals

$$\frac{1}{k!(2!)^k}\sum_{\sigma\in\mathfrak{S}_p}\varepsilon(\sigma)\theta_{i_{\sigma(1)}i_{\sigma(2)}}\cdots\theta_{i_{\sigma(p-1)}i_{\sigma(p)}} =$$

$$= \frac{1}{k!}\underbrace{\theta\wedge\theta\wedge\ldots\wedge\theta}_{k\text{-times}}(e_{i_1}\wedge\ldots\wedge e_{i_p}).$$

Q.E.D.

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