INSTITUTUL
DE
MATEMATICĂ

INSTITUTUL NAȚIONAL
PENTRU CREAȚIE
ȘTIINȚIFICĂ ȘI TEHNICĂ

ISSN 0250 3638

POSITIVE-DEFINITE KERNELS ON Z (AND NONSTATIONARY PROCESSES)

by

Tiberiu CONSTANTINESCU

PREPRINT SERIES IN MATHEMATICS
No.54/1984

BUCUREȘTI

POSITIVE-DEFINITE KERNELS ON Z
(AND NONSTATIONARY PROCESSES)

by

Tiberiu CONSTANTINESCU *)

August 1984

*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd.Pacii 220, 79622 Bucharest,Romania

POSITIVE-DEFINITE KERNELS ON $\mathbb{Z}$
(AND NONSTATIONARY PROCESSES)

by T. Constantinescu

I INTRODUCTION

In a classical work ([9]), I.Schur obtains an algorithm for describing the structure of the Fourier coefficients of an analytic contractive function on the unit disc by the use of a sequence of free parameters (Schur parameters). There are many other "equivalent" problems where one finds again the structure of the Schur algorithm. The contractive intertwining dilation theory (see ([1]) for details) , provides a very general setting for all these problems. Moreover, one obtains the general notion of Schur parameters called, in this general context, a choice sequence.

Roughly speaking , the construction of a contractive intertwining dilation (as it was done in ([1])) consists of an analysis of " the first step of the dilation" and of the general algorithm. This second step is equivalent to describe the structure of a positive Toeplitz form. In this language , the whole matter is reduced to a diligent usage of the well-known structure of a 2x2 positive matrix. At the same time, the study of a general positive form along this line is imposed.

The last problem leads to a new generalization of the Schur parameters (see ([4])).

This note is only an appendix to ([4]). We obtain the Kolmogorov decomposition for a positive-definite kernel on the set of integers, using the generalized choice sequence.

To conclude, we use (as usual) these facts for nonstationary processes. Moreover, as we showed in ([2]) , the two Szegö limit theorems (at least the existence of the limits), became a matter of some simple computations. We note here , variants for the nonstationary case.

II PRELIMINARIES

In this section we recall the notation and some results from ([4]). The notation is rather involved, so that it can become tedious and not only for the reader. But , after a moments' thought, its significance becomes clear enough.

To begin with, let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and $\mathcal{L}(\mathcal{H},\mathcal{K})$ be the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$.

For a contraction $T\in\mathcal{L}(\mathcal{H},\mathcal{K})$ (i.e. $\|T\|\leq 1$), $D_T=(I-T^*T)^{\frac{1}{2}}$ and $\mathcal{D}_T=\overline{D_T\mathcal{H}}$ will be called the defect operator and the defect space of T. The following unitary operator can be thought as the fundamental cell for the dilation theory:

$$J(T):\mathcal{H}\oplus\mathcal{D}_{T^*}\longrightarrow\mathcal{K}\oplus\mathcal{D}_T$$

$$J(T)=\begin{bmatrix} T & , & D_{T^*} \\ D_T & , & -T^* \end{bmatrix}.$$

We call generalized choice sequence ((gc)-sequence), a familly $\{G_{ij}/\ i\geqslant 1,\ j\geqslant i+1\}$ of contractions, so that for $j=i+1$, $i\geqslant 1$, G_{ij} acts on $\mathcal{H}$ and otherwise, G_{ij} acts from $\mathcal{D}_{G_{i+1,j}}$ into $\mathcal{D}_{G^*_{i,j-1}}$.

Note that if G_{ij} is a (gc)-sequence so that $G_{ij}=G_{i+k,j+k}$ for $k\geqslant 1$, then $\{G_n=G_{1,n+1}\}_{n=1}^{\infty}$ is a choice sequence.

Associated with a fixed (gc)-sequence $\{G_{ij}\}$ we consider the operators: for $i\geqslant 1$ and $j\geqslant i+1$,

$$J^1_{ij}:(\mathcal{H}\oplus\mathcal{D}_{G^*_{i,i+1}})\oplus\mathcal{D}_{G_{i+1,i+3}}\oplus\mathcal{D}_{G_{i+1,i+4}}\oplus\ldots\oplus\mathcal{D}_{G_{i+1,j}}\longrightarrow$$

$$\longrightarrow(\mathcal{H}\oplus\mathcal{D}_{G_{i,i+1}})\oplus\mathcal{D}_{G_{i+1,i+3}}\oplus\ldots\oplus\mathcal{D}_{G_{i+1,j}}$$

$$J^1_{ij}=J(G_{i,i+1})\oplus I\ ;$$

for $j-1\geqslant k>1$,

$$J^k_{ij}:\mathcal{H}\oplus\mathcal{D}_{G_{i+1,i+2}}\oplus\ldots\oplus(\mathcal{D}_{G_{i+1,i+k}}\oplus\mathcal{D}_{G^*_{i,i+k}})\oplus\ldots\oplus\mathcal{D}_{G_{i+1,j}}\longrightarrow$$

$$\longrightarrow\mathcal{H}\oplus\mathcal{D}_{G_{i+1,i+2}}\oplus\ldots\oplus(\mathcal{D}_{G^*_{i,i+k-1}}\oplus\mathcal{D}_{G_{i,i+k}})\oplus\ldots\oplus\mathcal{D}_{G_{i+1,j}}$$

$$J^k_{ij}=I\oplus J(G_{i,i+k})\oplus I.$$

For a choice sequence, these operators appear in the contractiv intertwining dilation theory. We also define the operators (Euler transformations):

$$V_{ij}:\mathcal{H}\oplus\mathcal{D}_{G_{i+1,i+2}}\oplus\mathcal{D}_{G_{i+1,i+3}}\oplus\ldots\oplus\mathcal{D}_{G_{i+1,j}}\oplus\mathcal{D}_{G^*_{ij}}\longrightarrow$$

$$\longrightarrow\mathcal{H}\oplus\mathcal{D}_{G_{i,i+1}}\oplus\ldots\oplus\mathcal{D}_{G_{i,j-1}}\oplus\mathcal{D}_{G_{ij}}.$$

$$V_{ij}=J^1_{ij}J^2_{ij}\ldots J^{j-i}_{ij}\ .$$

We define the contractions:

$$X_{ij}:\mathcal{H}\oplus\mathcal{D}_{G_{i+1,i+2}}\oplus\ldots\oplus\mathcal{D}_{G_{i+1,j}}\longrightarrow\mathcal{H}$$

$$X_{ij}=(G_{i,i+1},D_{G^*_{i,i+1}}G_{i,i+2},\ldots,D_{G^*_{i,i+1}}\ldots D_{G^*_{i,j-1}}G_{ij})$$

$$\widetilde{X}_{ij}:\mathcal{H}\longrightarrow\mathcal{H}\oplus\mathcal{D}_{G^*_{j-2,j-1}}\oplus\ldots\oplus\mathcal{D}_{G^*_{i,j-1}}$$

$$\widetilde{X}_{ij}=(G_{j-1,j},G_{j-2,j}D_{G_{j-1,j}},\ldots,G_{ij}D_{G_{i+1,j}}\ldots D_{G_{j-1,j}})^t$$

("t" standing for matrix transpose).

We consider the operators: $U_{ii}=I_{\mathcal{H}}$, $i\geqslant 1$ and

$$U_{ij}:\mathcal{H}\oplus\mathcal{D}_{G^*_{i,i+1}}\oplus\ldots\oplus\mathcal{D}_{G^*_{i,j-1}}\oplus\mathcal{D}_{G^*_{ij}}\longrightarrow$$

$$\longrightarrow\mathcal{H}\oplus\mathcal{D}_{G_{i,i+1}}\oplus\ldots\oplus\mathcal{D}_{G_{ij}}$$

$$U_{ij}=V_{ij}(U_{i+1,j}\oplus I_{\mathcal{D}_{G^*_{ij}}})$$

$F_{ii}=I_{\mathcal{H}}$, $i\geq 1$ and

$$F_{ij}=\begin{bmatrix} F_{i,j-1} & , U_{i,j-1}\widetilde{X}_{ij} \\ \\ 0 & , D_{G_{ij}}\ldots D_{G_{j-1,j}} \end{bmatrix}.$$

Now, let us consider a familly $\{S_{ij}/\ i\geq 1, j>i\}$ of operators on $\mathcal{H}$ so that

$$B_{1n}=\begin{bmatrix} I & ,S_{12},\ldots\ldots,S_{1n} \\ S_{12}^{*} & ,\ I,\ldots\ldots,S_{2n} \\ \\ S_{1n}^{*} & ,\ldots\ldots\ldots\ ,\ I \end{bmatrix}$$

are positive operators for $n\geq 2$.

Theorem 2.4 in ([4]) asserts that there exists a one-to-one correspondence between the set of the sequences of positive operators B_{1n} and the set of (gc)-sequences, given by the formulas:

$$(*)\begin{cases} S_{i,i+1}=G_{i,i+1} \\ S_{ij}=X_{i,j-1}U_{i+1,j-1}\widetilde{X}_{i+1,j}+D_{G^{*}_{i,i+1}}\ldots D_{G^{*}_{i,j-1}}G_{ij}D_{G_{i+1,j}}\ldots D_{G_{j-1,j}} \end{cases}$$

When the (gc)-sequence is a choice sequence, the algorithm reduces to the one written in ([2]) for positive Toeplitz forms.

The main point in proving $(*)$ is the factorization:

$$B_{1n}=F_{1n}^{*}\,F_{1n}$$

which has an other consequence formulas for computing $\det B_{1n}$.

2.1 PROPOSITION If S_{ij} are rxr matrices, then

$$\det B_{1n}=(\det D_{G_{12}})^{2}(\det D_{G_{13}}\det D_{G_{23}})^{2}\ldots(\det D_{G_{1n}}\ldots\det D_{G_{n-1,n}})$$

for $n\geq 2$. ∎

III POSITIVE-DEFINITE KERNELS ON $\mathbb{Z}$

In this section we shall use the (gc)-sequences in order to obtain a construction for the minimal Kolmogorov decomposition of a positive-definite kernel on $\mathbb{Z}$. We call a positive-definite kernel on

$\mathbb{Z}$, a map $K:\mathbb{Z}\times\mathbb{Z} \longrightarrow \mathcal{L}(\mathcal{H})$ with the property that for each $n\in\mathbb{N}$ and each choice of vectors $h_1,\dots,h_n$ in $\mathcal{H}$ and $p_1,\dots p_n$ in $\mathbb{Z}$, the following inequality holds:

$$\sum_{i,j=1}^{n}(K(p_i,p_j)h_j,h_i) \geqslant 0 .$$

Without loss of generality, we can suppose $K(p,p)=I$, $p\in\mathbb{Z}$, and it is easy to see that K is positive-definite if and only if the matrices

$$B_{ij}=\begin{bmatrix} I & ,S_{i,i+1}, & S_{i,i+2}, & \dots, & S_{ij} \\ \cdot & I & ,S_{i+1,i+2}, & \dots, & S_{i+1,j} \\ \vdots & & & \vdots & \\ & & & & I \end{bmatrix}$$

are positive, $i\in\mathbb{Z}$, $j>i$, and $S_{ij}=K(i,j)$.

A Kolmogorov decomposition of K will be a map

$$V:\mathbb{Z} \longrightarrow \mathcal{L}(\mathcal{H},\mathcal{H}_V)$$

where $\mathcal{H}_V$ is a Hilbert space and $K(i,j)=V(i)^*V(j)$, $i,j\in\mathbb{Z}$.

If $\mathcal{H}_V=\bigvee_{n\in\mathbb{Z}}V(n)\mathcal{H}$, then the decomposition is said to be minimal and two minimal decompositions are equivalent in an appropiate sense (see for instance (6)). We obtain:

<u>3.1 THEOREM</u> There exists a one-to-one correspondence between the set of positive-definite kernels on $\mathbb{Z}$ (with $K(p,p)=I$, $p\in\mathbb{Z}$) and the set of double generalized choice sequences ((dgc)-sequences) $\{G_{ij}/\ i\in\mathbb{Z},\ j\geqslant i+1\}$ given by the same formulas as in $(*)$. ∎

Now, we use the (dgc)-sequence associated by Theorem 3.1 in order to describe the Kolmogorov decomposition of K. Actually, the construction will be an adapatation of the construction of the Naimark dilation given in ([3]).

We note that for $i\in\mathbb{Z}$, there exists the contraction

$$X_{i,\infty}:\mathcal{H}\oplus\bigoplus_{j=i+2}^{\infty}\mathcal{D}_{G_{i+1,j}} \longrightarrow \mathcal{H}$$

$$X_{i,\infty}=\text{s-}\lim_{j\to\infty}X_{ij}$$

In ([3]) one obtained proper identifications for the defect spaces of $X_{i,\infty}$.

So we consider

$$\alpha_{i,+}:\mathcal{D}_{X_{i,\infty}} \longrightarrow \bigoplus_{k=1}^{\infty}\mathcal{D}_{G_{i,i+k}}$$

$$\alpha_{i,+}D_{X_{i,\infty}} = D_{i,\infty}$$

where

$$D_{i,\infty} = \text{s-}\lim_{j\to\infty} D_{ij}$$

and

$$D_{ij} = \begin{bmatrix} D_{G_{i,i+1}}, & -G^*_{i,i+1}G_{i,i+2}, & \ldots, & -G^*_{i,i+1}D_{G^*_{i,i+2}} \cdots G_{ij} \\ 0, & D_{G_{i,i+2}} & \ldots & \\ \vdots & & & \\ & \ldots & & D_{G_{ij}} \end{bmatrix}$$

In order to identify $\mathcal{D}_{X^*_{i,\infty}}$ we consider

$$\tilde{\alpha}_{i,+}:\mathcal{D}_{X^*_{i,\infty}} \longrightarrow \mathcal{D}_{i,*} = \overline{\mathrm{Ran}H_{(i)}}$$

where $H_{(i)} = \text{s-}\lim_{j\to\infty} D_{G^*_{i,i+1}} \ldots D^2_{G^*_{ij}} \ldots D_{G^*_{i,i+1}}$

Now, we define

$$W_{i,red}:\mathcal{D}_{i,*}\oplus\mathcal{K}\oplus\bigoplus_{k=2}^{\infty}\mathcal{D}_{G_{i+1,i+k}} \longrightarrow \mathcal{K}\oplus\bigoplus_{k=1}^{\infty}\mathcal{D}_{G_{i,i+k}}$$

$$W_{i,red} = \begin{bmatrix} I, & 0 \\ 0, & \alpha_{i,+} \end{bmatrix} J(X_{i,\infty}) \begin{bmatrix} 0, & I \\ \tilde{\alpha}_{i,+} & 0 \end{bmatrix}$$

and

$$W_i:\ldots\oplus\mathcal{D}_{i-2,*}\oplus\mathcal{D}_{i-1,*}\oplus\mathcal{D}_{i,*}\oplus\mathcal{K}\oplus\bigoplus_{k=2}^{\infty}\mathcal{D}_{G_{i+1,i+k}} \longrightarrow$$

$$\longrightarrow \ldots\oplus\mathcal{D}_{i-2,*}\oplus\mathcal{D}_{i-1,*}\oplus\mathcal{K}\oplus\bigoplus_{k=1}^{\infty}\mathcal{D}_{G_{i,i+k}}$$

(the first space will be denoted by $\mathcal{K}_{i+1}$ and the second one by $\mathcal{K}_i$

$$W_i = I \oplus W_{i,red}$$

with respect to the decompositions:

$$\left(\bigoplus_{j=-\infty}^{i-1}\mathcal{D}_{j,*}\right)\oplus\left(\mathcal{D}_{i,*}\oplus\mathcal{K}\oplus\bigoplus_{\nu=2}^{\infty}\mathcal{D}_{G_{i+1,i+k}}\right) \quad \text{and}$$

$$\left(\bigoplus_{j=-\infty}^{i-1}\mathcal{D}_{j,*}\right)\oplus\left(\mathcal{H}\oplus\bigoplus_{k=1}^{\infty}\mathcal{D}_{G_{i,i+k}}\right).$$

3.2 THEOREM Let K be a positive-definite kernel on $\mathbb{Z}$ (with $K(p,p)=I$, $p\in\mathbb{Z}$) and $\{G_{ij}\}$ the associated (dgc)-sequence. Then

$$V:\mathbb{Z}\longrightarrow\mathcal{L}(\mathcal{H},\mathcal{K}_0),$$

$$V(n)=\begin{cases} W_{-1}^{*}W_{-2}^{*}\dots W_{n}^{*}/\mathcal{H} & ,n<0\\ P_{\mathcal{H}}^{\mathcal{K}_0} & n=0\\ W_0W_1\dots W_{n-1}/\mathcal{H} & n>0\end{cases}$$

is the Kolmogorov decomposition of K.

PROOF We have only to use Theorem 3.3 in ([4]). ■

IV NONSTATIONARY PROCESSES

For the general theory of the (nonstationary) processes, we can refer to the book ([5]). In this section we call nonstationary process a sequence $\{\mathcal{V}_n\}_{n\in\mathbb{Z}}$ of operators in $\mathcal{L}(\mathcal{H},\mathcal{K})$ and we define the covariance kernel,

$$K:\mathbb{Z}\times\mathbb{Z}\longrightarrow\mathcal{L}(\mathcal{H})$$

$$K(i,j)=\mathcal{V}_i^{*}\mathcal{V}_j.$$

From the very begining, we suppose $K(n,n)=I$, $n\in\mathbb{Z}$ and $\mathcal{K}=\bigvee_{n\in\mathbb{Z}}\mathcal{V}_n\mathcal{H}$. As K is a positive-definite kernel on $\mathbb{Z}$, we consider its Kolmogorov decomposition (in particular, in the form given by Theorem 3.2),

$$K(i,j)=V(i)^{*}V(j).$$

As usual, we define:

$$X\left(\sum_i\mathcal{V}_ih_i\right)=\sum_i V(i)h_i \qquad , \quad h_i\in\mathcal{H}$$

for finite sums, then

$$\left\|\sum_i\mathcal{V}_ih_i\right\|^2=\sum_{i,j}(\mathcal{V}_ih_i,\mathcal{V}_jh_j)=$$
$$=\sum_{i,j}(h_i,\mathcal{V}_i^{*}\mathcal{V}_jh_j)=\sum_{i,j}(h_i,K(i,j)h_j)=$$
$$=\sum_{i,j}(h_i,V(i)^{*}V(j)h_j)=\left\|\sum_i V(i)h_i\right\|^2$$

so, X extends to a unitary operator from $\mathcal{K}$ onto $\mathcal{K}_0 (= \bigvee_{i\in\mathbb{Z}} V(i)\mathcal{H}$ and $X\mathcal{V}_i = V(i)$, so

$$\mathcal{V}_i = X^* V(i).$$

We say that we associated to the process $\{\mathcal{V}_n\}_{n\in\mathbb{Z}}$ an (essentially unique) "operatorial model" $\{V(n)\}_{n\in\mathbb{Z}}$.

In recent years it was recognized (see for details ([8])) that some notions (reflection coefficients,...) and specific methods (Schur and Levinson algorithms,...) to the stationary processes have a natural counterpart for the nonstationary case. We should continue along the classical line (in ([10]), for instance), but we prefer an aspect connected with the " embeding of a nonstationary process into a stationary one". We give now a very crude result in this direction, mainly a reformulation of some of the precedent results, sending a more detailed analysis to a forthcoming paper.

Let $\{\mathcal{V}_n\}_{n=-\infty}^{\infty}$ be a nonstationary process, K its covariance kernel and $\{G_{ij}\}$ the associated (dgc)-sequence. Let $\tilde{\mathcal{H}} = l^2(\mathbb{Z}, \mathcal{H})$,

$$\tilde{G}_1 \in \mathcal{L}(\tilde{\mathcal{H}}),$$

$$\tilde{G}_1\Big(\sum_{k=-\infty}^{\infty} h_k\Big) = \sum_{k=-\infty}^{\infty} G_{k,k+1} h_{k+1} \quad .$$

Having the definition of a (dgc)-sequence, the following operators determine a choice sequence: for $n \geq 2$,

$$\tilde{G}_n : \mathcal{D}_{G_{n-1}} \Big(= \bigoplus_{k=-\infty}^{\infty} \mathcal{D}_{G_{k,k+n-1}}\Big) \longrightarrow \mathcal{D}_{G^*_{n-1}} \Big(= \bigoplus_{k=-\infty}^{\infty} \mathcal{D}_{G^*_{k,k+n-1}}\Big)$$

$$\tilde{G}_n\Big(\sum_{k=-\infty}^{\infty} d_k\Big) = \sum_{k=-\infty}^{\infty} G_{k,k+n} d_{k+n} \quad .$$

We define $\tilde{W} = W(\{\tilde{G}_n\}_{n=1}^{\infty})$, the unitary operator given by (2.2) in ([3]), and let $\tilde{\mathcal{K}}$ be the spacewhere this operator acts.

Let us define $\mathcal{H}_{(n)} = \ldots \oplus 0 \oplus \underset{\substack{\uparrow \\ n}}{\mathcal{H}} \oplus 0 \oplus \ldots$

and $$k_n : \mathcal{H} \longrightarrow \tilde{\mathcal{K}}$$

$$k_n h = \ldots 0 \oplus \underset{\substack{\uparrow \\ n}}{h} \oplus 0 + \ldots$$

4.1 THEOREM Let $\{\mathcal{V}_n\}_{n=-\infty}^{\infty}$ be a nonstationary process on $\mathcal{K}_o$, then there exist a Hilbert space $\tilde{\mathcal{K}} \supset \mathcal{K}_o$, a unitary operator $\tilde{W} \in \mathcal{L}(\tilde{\mathcal{K}})$, a family $\{\mathcal{H}_{(n)}\}_{n\in \mathbb{Z}}$ of subspaces of $\tilde{\mathcal{K}}$ and a family $\{k_n : \mathcal{H} \to \mathcal{H}_{(n)}\}_{n\in\mathbb{Z}}$ of isomtric operators, so that the given process has the following operatorial model $\{P^{\tilde{\mathcal{K}}}_{\mathcal{K}_o} \tilde{W}^n k_n\}_{n=-\infty}^{\infty}$. ∎

V NONSTATIONARY SZEGÖ LIMIT THEOREMS

The two Szegö theorems concern in the asymptotic of the positive Toeplitz determinants. More precisely, when

$$B_{1n} = \begin{bmatrix} I, & S_1, \ldots, & S_n \\ S_1^*, & I, \ldots, & S_{n-1} \\ & & \\ S_n^*, & \ldots & , I \end{bmatrix}$$

the first Szegö limit theorem asserts the existence of the limit:

$$\lim_{n\to\infty} \frac{\det B_{1,n+1}}{\det B_{1,n}} = G(B),$$

where G(B) is the geometrical mean of the positive measure μ having the coefficients S_n as Fourier coefficients.

The second Szegö limit theorem (The strong Szegö limit theorem states that there exists the limit:

$$\lim_{n\to\infty} \frac{\det B_{1,n}}{G(B)^{n+1}}$$

and gives an expression for this limit.

Using the choice sequence, in ([2]) , the following variants of the Szegö limit theorems were obtained:

$$\lim_{n\to\infty} \frac{\det B_{1,n+1}}{\det B_{1,n}} = \prod_{n=1}^{\infty} \det D^2_{G_n}$$

$$\lim_{n\to\infty} \frac{G(B)^{n+1}}{\det B_{1,n}} = \prod_{n=1}^{\infty} \det D^{2n}_{G_n}$$

where $\{G_n\}$ is the choice sequence associated to the given Toeplitz form.

For the nonstationary case, using Proposition 2.1, we have the following statements:

5.1 THEOREM For $i \in \mathbb{Z}$,

$$\lim_{j\to\infty} \frac{\det B_{ij}}{\det B_{i+1,j}} = \prod_{n=i+1}^{\infty} \det D^2_{G_{i,n}} \quad (=G_i(B)).$$

5.2 THEOREM There exists:

$$\lim_{n\to\infty} \frac{G_1(B)G_0(B)\ldots G_{-n}(B)}{\det B_{-n,1}}.$$

In a forthcoming paper , we shall indicate the geometrical significance of these theorems.

REFERENCES

1. GR. ARSENE, Z. CEAUSESCU AND C. FOIAS, On intertwining dilation.VIII, J.Operator Theory,4(1980),55-91.

2: T. CONSTANTINESCU, On the structure of positive Toeplitz forms, in Dilation theory,Toeplitz operators and other topics, Birkhäuser Verlag,1983,127-149.

3. T. CONSTANTINESCU, On the structure of the Naimark dilation, J.Operator Theory,12(1984),159-175.

4. T. CONSTANTINESCU, The structure of nxn positive operator matrices, INCREST preprint, No.14/1984.

5. J.L. DOOB, STOCHASTIC PROCESSES, Wiley,New York,1953.

6. D.E. EVANS AND J.T. LEWIS, Dilations of irreversible evolutions in algebraic quantum theory, Dublin, 1977.

7. U. GRENANDER AND G.SZEGÖ, TOEPLITZ FORMS AND THEIR APPLICATIONS, Univesity of California Press,1958.

8. H. LEV-ARI AND T. KAILATH, Lattice filter parametrization and modeling of nonstationary processes, IEEE Trans.Inform.Theory, vol.IT-30,2-16.Jan.1984.

9. I. SCHUR, Uber Potenzreihen, die im innern des Einheitkreses beschränkt sind,J.Reine Angew.Math.,148(1918),122-145.

10. N. WIENER AND P. MASANI, Theprediction theory of multivariate stochastic processes.I;II, Acta Math.,98(1957),111-150; 99(1958),93-139.