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ISSN 0250 3638

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PREPRINT SERIES IN MATHEMATICS

No.55/1984

BUCUREȘTI

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September 1984

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by

Dan POLIŠEVSKI*)

The Boussinesq system which governs in a certain approximation the thermal flows of viscous fluids [5] has been considered in [1] and [2] where the macroscopic equations of the phenomenon in porous media have been obtained by the multiple scale method of the homogenization theory. The aim of the present paper is to prove the convergence of the homogenization process in this case, moreover avoiding any symmetry hypothesis for the tensor of thermal diffusion. A convergence theorem of a homogenization process (Stokes equations - Darcy law) can be found in [10], and as we know it have not been extended yet. Here, with the constructions of [8] we treat a more realistic periodic model of porous media that is, tridimensional, with connex phases and biphasic boundary.

In the first section we discuss the passage to the variational formulation of the natural convection problem. Then we prove the existence of the weak solutions and we give some estimates, inter alia a maximum principle. Thus we can attain the uniqueness result which permit us to start the homogenization process. In the last section we prove the convergence theorem and some properties of the homogenized coefficients.

1. THE BOUSSINESQ SYSTEM

Let $Y=[0,\delta[^3$ for some $\delta>0$, and let Γ be a surface of class C^2 included in $\overline{Y}$, which cross the boundary of the cube following some regular curves which are reproduced identically on opposite faces. Also Γ separate Y into two domains, Y_s - the solid part and Y_f - the fluid part, with the property that repeating Y by periodicity, the reunion of all the fluid parts , respectively the solid parts, are connex in $\mathbb{R}^3$ and of class C^2. We assume only that the intersection of $\bar{Y}_S$ with the edges of $\bar{Y}$ is empty.

Let Ω be an open connected bounded set in $\mathbb{R}^3$, locally located on one side of the boundary $\partial\Omega$ - a manifold of class C^2, composed of a finite number of connex components, and let $\varphi : \mathbb{R} \to [0,1[$ be the function which associates to any real number its fractional part; defining the function $\phi : \mathbb{R}^3 \to Y$ by $\phi(x) = \delta\varphi(\frac{1}{\delta}x)$ then we say that a function $f : \mathbb{R}^3 \to \mathbb{R}$ is Y-periodic if $f = f\circ\phi$. Also, for any $\varepsilon \in (0,1)$ we denote

$$\Omega_f^\varepsilon = \left\{x\in\Omega \,\middle|\, \phi(\tfrac{1}{\varepsilon}x)\in Y_f\right\}$$
$$\Omega_S^\varepsilon = \left\{x\in\Omega \,\middle|\, \phi(\tfrac{1}{\varepsilon}x)\in Y_S\right\}$$
$$\Gamma^\varepsilon = (\bar{\Omega}_S^\varepsilon \cap \bar{\Omega}_f^\varepsilon)$$
$$(\partial\Omega)_f^\varepsilon = \bar{\Omega}_f^\varepsilon \cap \partial\Omega$$

If $\underset{\sim}{u}^\varepsilon$, p^ε and T^ε stand, respectively, for velocity, pressure and temperature, then they have to satisfy in some way the following system:

(1.1) $\operatorname{div} \underset{\sim}{u}^\varepsilon = 0$ in Ω_f^ε

(1.2) $-\nu\Delta\underset{\sim}{u}^\varepsilon + (\underset{\sim}{u}^\varepsilon\nabla)\underset{\sim}{u}^\varepsilon + \nabla p^\varepsilon = \left[1-\alpha(T-T_o)\right]\underset{\sim}{g}$ in Ω_f^ε

(1.3) $\operatorname{div}(\underset{\sim}{a}^\varepsilon\nabla T^\varepsilon) = \underset{\sim}{u}^\varepsilon\nabla T^\varepsilon$ in Ω_f^ε

(1.4) $\operatorname{div}(\underset{\sim}{a}^\varepsilon\nabla T^\varepsilon) = 0$ in Ω_S^ε $\left(\underset{\sim}{a}^\varepsilon\nabla = a_{ij}^\varepsilon\frac{\partial}{\partial x_j}\right)$

with the boundary conditions

(1.5) $\underset{\sim}{u}^\varepsilon = \underset{\sim}{0}$ on $\partial(\Omega_f^\varepsilon) = \Gamma^\varepsilon \cup (\partial\Omega)_f^\varepsilon$

(1.6) $T^\varepsilon = \tau$ on $\partial\Omega$

and transmission conditions

(1.7) $\left[T^\varepsilon\right]_\varepsilon = 0$ ($[\ ]_\varepsilon$ - the jump across Γ^ε)

(1.8) $\left[\underset{\sim}{a}^{\varepsilon}\nabla T^{\varepsilon}\right]_{\varepsilon}\underset{\sim}{n}^{\varepsilon}=0$ ($\underset{\sim}{n}^{\varepsilon}$ - the unit normal on Γ^{ε}, exterior to Ω^{ε}_{f})

where $\alpha>0$, $\nu>0$ are, respectively, the volumetric coefficient of thermal expansion and the kinematic viscosity of the fluid, $\underset{\sim}{g}\in \underset{\sim}{L}^2(\Omega)$ is the gravitational acceleration, $\tau\in H^{3/2}(\partial\Omega)$ is the non-uniform temperature of the boundary (the case τ uniform is not interesting) and $T_o>0$ is a uniform reference temperature, by convenience

$$T_o=\frac{1}{2}\left(\sup_{x\in\partial\Omega}\tau+\inf_{x\in\partial\Omega}\tau\right).$$

From the subsequent maximum principle (2.5) and "a priori" estimates (2.6)-(2.8), we shall learn that, independently of $\underset{\sim}{a}^{\varepsilon}$, the velocity $\underset{\sim}{u}^{\varepsilon}$ is of order ε^{-2}, while the temperature T^{ε} is of order 1. As we look for a model of natural convection, both hand sides of (1.3) have to be of the same order; consequently the tensor of thermal diffusion $\underset{\sim}{a}^{\varepsilon}$ have to be of order ε^{-2}. That is why we assume (the physical meaning of this assumption can be given, as in [2], by a dimensional analysis) that

$$\underset{\sim}{a}^{\varepsilon}(x)=\varepsilon^2\underset{\sim}{a}\left(\frac{1}{\varepsilon}x\right)$$

where $\underset{\sim}{a}=(a_{ij})_{i,j}\in\underset{\sim}{L}'^{\infty}(\mathbb{R}^3)$ is Y-periodic and

$(\exists)$ $a>0$ such that $a_{ij}\xi_i\xi_j\geq a\xi_i^2$ $(\forall)$ $\xi_i\in\mathbb{R}$, $i=1,2,3$.

As usual, the scalar products and norms in $L^2(\Omega)$, $H^m(\Omega)$ and $H^1_o(\Omega)$ are, respectively, denoted by

$$(u,v)=\int_{\Omega}u.v\;dx \qquad |u|=(u,u)^{1/2}$$

$$((u,v))_m=\sum_{|j|\leq m}\left(\frac{\partial u}{\partial x_j},\frac{\partial v}{\partial x_j}\right) \qquad \|u\|_m=((u,u))_m^{1/2}$$

$$((u,v))=(\nabla u,\nabla v) \qquad \|u\|=(u,u))^{1/2}$$

and the norm in $L^p(\Omega)$ $(p\neq 2)$ by $|\cdot|_p$. We agree to use the same notations for the scalar products and norms in $\underset{\sim}{L}^2(\Omega)=[L^2(\Omega)]^3$, $\underset{\sim}{H}^m(\Omega)$, $\underset{\sim}{H}^1_o(\Omega)$ and $\underset{\sim}{L}^p(\Omega)$. To the corresponding notations associated to Ω_f^ε we attach the index ε.

In order to pass to homogeneous boundary conditions, like in [6] we introduce for any $h>0$ an element $w_h\in H^2(\Omega)$ with the properties

$$(1.9)\quad w_h=\zeta-T_o \quad \text{on } \partial\Omega$$

$$(1.10)\quad |S\nabla w_h|\leq h\cdot\|S\| \qquad (\forall)\, S\in H^1_o(\Omega)$$

Puting $S^\varepsilon=T^\varepsilon-(w_h+T_o)$ and keeping the right to choose later, in a proper way, the parameter $h>0$, the system (1.1)-(1.8) becomes:

$$(1.11)\quad \operatorname{div} \underset{\sim}{u}^\varepsilon=0 \quad \text{in } \Omega_f^\varepsilon$$

$$(1.12)\quad -\nu\Delta\underset{\sim}{u}^\varepsilon+(\underset{\sim}{u}^\varepsilon\nabla)\underset{\sim}{u}^\varepsilon+\nabla p^\varepsilon=\left[1-\alpha(S^\varepsilon+w_h)\right]\underset{\sim}{g} \text{ in } \Omega_f^\varepsilon$$

$$(1.13)\quad \operatorname{div}\left[\underset{\sim}{a}^\varepsilon\nabla(S^\varepsilon+w_h)\right]=\underset{\sim}{u}^\varepsilon\nabla(S^\varepsilon+w_h) \quad \text{in } \Omega_f^\varepsilon$$

$$(1.14)\quad \operatorname{div}\left[\underset{\sim}{a}^\varepsilon\nabla(S^\varepsilon+w_h)\right]=0 \qquad \text{in } \Omega_s^\varepsilon$$

$$(1.15)\quad \underset{\sim}{u}^\varepsilon=\underset{\sim}{0} \qquad \text{on } \partial(\Omega_f^\varepsilon)$$

$$(1.16)\quad S^\varepsilon=0 \qquad \text{on } \partial\Omega$$

$$(1.17)\quad [S^\varepsilon]_\varepsilon=0 \text{ on } \Gamma^\varepsilon$$

$$(1.18)\quad \left[\underset{\sim}{a}^\varepsilon\nabla(S^\varepsilon+w_h)\right]_\varepsilon\cdot\underset{\sim}{n}^\varepsilon=0 \qquad \text{on } \Gamma^\varepsilon$$

Let $\underset{\sim}{V}_f^\varepsilon$ be the closure in $\underset{\sim}{H}^1_o(\Omega_f^\varepsilon)$ of

$$\underset{\sim}{\mathcal{V}}(\Omega_f^\varepsilon)=\left\{\underset{\sim}{v}\in\underset{\sim}{\mathcal{D}}(\Omega_f^\varepsilon)\;\middle|\;\operatorname{div}\underset{\sim}{v}=0\right\}$$

Thus $X_\varepsilon=\underset{\sim}{V}_f^\varepsilon\times H^1_o(\Omega)$ is a Hilbert space with the scalar product

$$((\underset{\sim}{u},S),(\underset{\sim}{v},T))_{X_\varepsilon}=((\underset{\sim}{u},\underset{\sim}{v}))_\varepsilon+((S,T))$$

Defining the mapping $G_\varepsilon : X_\varepsilon \rightarrow X'_\varepsilon$ by

$$(1.19) \quad \langle G_\varepsilon(\underset{\sim}{u},S),\ (\underset{\sim}{v},T)\rangle = \nu((\underset{\sim}{u},\underset{\sim}{v}))_\varepsilon + ((\underset{\sim}{u}\nabla)\underset{\sim}{u}+\alpha S\underset{\sim}{g},\underset{\sim}{v})_\varepsilon + $$
$$+\lambda(\underset{\sim}{a}^\varepsilon\nabla S,\nabla T)+\lambda(\underset{\sim}{u},T\nabla(S+w_h))_\varepsilon\ ,\quad (\forall)\ (\underset{\sim}{v},T)\in X_\varepsilon\ ,$$

where the coupling-parameter $\lambda>0$ will be chosen conveniently, we have arrived to the variational formulation of the problem (1.11)-(1.18):

<u>To find</u> $(\underset{\sim}{u}^\varepsilon,S^\varepsilon)\in X_\varepsilon$ <u>such that</u>

$$(1.20) \quad \langle G_\varepsilon(\underset{\sim}{u}^\varepsilon,S^\varepsilon),(v,T)\rangle = (1-\alpha w_h,\underset{\sim}{g}\cdot\underset{\sim}{v})-\lambda(\underset{\sim}{a}^\varepsilon\nabla w_h,\nabla T)$$
$$(\forall)\ (\underset{\sim}{v},T)\in X_\varepsilon\ .$$

It is clear that if $(\underset{\sim}{u}^\varepsilon,S^\varepsilon,p^\varepsilon)$ is a smooth solution of (1.11)-(1.18) then $(\underset{\sim}{u}^\varepsilon,S^\varepsilon)$ is a solution of (1.20).

Conversely, if $(\underset{\sim}{u}^\varepsilon,S^\varepsilon)\in X_\varepsilon$ satisfy (1.20) then choosing the test functions in a proper manner we get

$$(1.21) \quad \langle -\nu\Delta\underset{\sim}{u}^\varepsilon+(\underset{\sim}{u}^\varepsilon\nabla)\underset{\sim}{u}^\varepsilon-[1-\alpha(S^\varepsilon+w_h)]\underset{\sim}{g},\underset{\sim}{v}\rangle=0,\quad (\forall)\ \underset{\sim}{v}\in \underset{\sim}{V}_f^\varepsilon$$

$$(1.22) \quad (\underset{\sim}{a}^\varepsilon\nabla(S+w_h),\nabla T)+(\underset{\sim}{u}^\varepsilon,T\nabla(S^\varepsilon+w_h))_\varepsilon=0,\quad (\forall)\ T\in H_0^1(\Omega)$$

Supposing, possibly for a subsequence of $\varepsilon\rightarrow 0$, that $\partial(\Omega_f^\varepsilon)$ is Lipschitz, then we can give the following characterizations of $\underset{\sim}{V}_f^\varepsilon$ [11]:

$$(1.23) \quad \underset{\sim}{V}_f^\varepsilon=\left\{\underset{\sim}{v}\in \underset{\sim}{H}_0^1(\Omega_f^\varepsilon)\ \middle|\ \operatorname{div}\underset{\sim}{v}=0\right\}$$

(1.24) <u>If</u> $\underset{\sim}{f}\in \underset{\sim}{H}^{-1}(\Omega_f^\varepsilon)$ <u>satisfy</u> $\langle\underset{\sim}{f},\underset{\sim}{v}\rangle=0$ $(\forall)\ \underset{\sim}{v}\in\underset{\sim}{V}_f^\varepsilon$ <u>then</u> $(\exists)$ $p\in L^2(\Omega_f^\varepsilon)$ <u>such that</u> $\underset{\sim}{f}=\nabla p$.

From (1.23) we get (1.15) in $H^{1/2}(\partial(\Omega_f^\varepsilon))$ and (1.11) in $\underset{\sim}{L}^2(\Omega_f^\varepsilon)$; as $\Delta\underset{\sim}{u}^\varepsilon\in \underset{\sim}{H}^{-1}(\Omega_f^\varepsilon)$,

$$(\underset{\sim}{u}^{\varepsilon}\nabla)\underset{\sim}{u}^{\varepsilon}\in \underset{\sim}{L}^{3/2}(\Omega_f^{\varepsilon}) \text{ and } \left[1-\alpha(S^{\varepsilon}+w_h)\right]\underset{\sim}{g}\in \underset{\sim}{L}^{3/2}(\Omega_f^{\varepsilon}),$$

according to (1.24) it follows that $(\exists)$ $p^{\varepsilon}\in L^2(\Omega_f^{\varepsilon})$ (determined up to an additive constant) such that (1.12) is satisfied in $\underset{\sim}{H}^{-1}(\Omega_f^{\varepsilon})$. Now, if we continue $\underset{\sim}{u}^{\varepsilon}$ (initially defined on Ω_f^{ε}) to Ω with zero out of Ω_f^{ε}, then (1.22) becomes:

(1.25) $\left\langle -\operatorname{div}\left[\underset{\sim}{a}^{\varepsilon}\nabla(S^{\varepsilon}+w_h)\right] + \underset{\sim}{u}^{\varepsilon}\nabla(S^{\varepsilon}+w_h), T\right\rangle = 0$, $(\forall) T\in H_o^1(\Omega)$

Because $\operatorname{div}\left[\underset{\sim}{a}^{\varepsilon}\nabla(S^{\varepsilon}+w_h)\right]\in H^{-1}(\Omega)$ and $\underset{\sim}{u}^{\varepsilon}\nabla(S^{\varepsilon}+w_h)\in L^{3/2}(\Omega)$, then (1.13)-(1.14) are satisfied in $H^{-1}(\Omega)$. Also $S^{\varepsilon}\in H_o^1(\Omega)$ implies (1.16)-(1.17) in the usual trace senses. Unfortunately, in order to find the sense of (1.18) we have to make a digression, slight generalisation of Theorem 1.2 [11].

<u>Theorem 1.1.</u> Let Ω be an open bounded set of class C^2 in $\mathbb{R}^N$ and let $E(\Omega)=\left\{\underset{\sim}{u}\in \underset{\sim}{L}^2(\Omega)\,\middle|\, \operatorname{div}\underset{\sim}{u}\in L^p(\Omega)\right\}$, where $p=2N/(N+2)$ if $N\geq 3$ and $p>1$ if $N=2$. Then $E(\Omega)$ is a Banach space equipped with the norm $|\underset{\sim}{u}|_E=|\underset{\sim}{u}|+|\operatorname{div}\underset{\sim}{u}|_p$. Also, there exists a linear continuous operator $\Gamma_{\underset{\sim}{n}}\in\mathcal{L}(E(\Omega), H^{-1/2}(\partial\Omega))$ such that

(1.26) $\Gamma_{\underset{\sim}{n}}u$=the restriction of $\underset{\sim}{u}.\underset{\sim}{n}$ to $\partial\Omega$, $(\forall)$ $\underset{\sim}{u}\in\underset{\sim}{\mathcal{D}}(\bar{\Omega})$ ($\underset{\sim}{n}$-the unit outward normal on $\partial\Omega$) and for $(\forall)\underset{\sim}{u}\in E(\Omega)$, $(\forall) q\in H^1(\Omega)$ the generalized Stokes formula holds

(1.27) $(\underset{\sim}{u},\nabla q)+(\operatorname{div}\underset{\sim}{u},q)=\langle\Gamma_{\underset{\sim}{n}}\underset{\sim}{u}, \Gamma_o q\rangle$.

where $\Gamma_o\in\mathcal{L}(H^1(\Omega), H^{1/2}(\partial\Omega))$ is the usual trace operator.

The proof of this theorem [7] proceeed in the classical way, with the "Lions-Magenes" method and the adequate Sobolev imbedding theorems.

Coming back to our problem, as $\underset{\sim}{a}^{\varepsilon}\nabla(S^{\varepsilon}+w_h)\in\underset{\sim}{L}^2(\Omega)$ and $\operatorname{div}\left[\underset{\sim}{a}^{\varepsilon}\nabla(S^{\varepsilon}+w_h)\right] = \underset{\sim}{u}^{\varepsilon}\nabla(S^{\varepsilon}+w_h)\in L^{3/2}(\Omega)$, it is clear from Theorem 1.1 that (1.18) is satisfied in the $E(\Omega)$ trace sense.

2. THE WEAK SOLUTIONS

In proving the existence theorem for the variational problem (1.20) we shall make use of the following result [3]:

Theorem 2.1. Let X a reflexive Banach space and $G:X \to X'$ a continuous mapping between the corresponding weak topologies. If

$$\lim_{|u|_X \to \infty} \frac{\langle Gu,u\rangle}{|u|_X} = \infty$$

then G is a surjection.

Now we can prove

Theorem 2.2. The problem (1.20) has at least one solution.

Proof. Taking in account that $(1-\alpha w_h)\underset{\sim}{g} \in \underset{\sim}{L}^2(\Omega_f^\varepsilon)$ and $\operatorname{div}(\underset{\sim}{a}^\varepsilon \nabla w_h) \in H^{-1}(\Omega)$ we have only to prove that $G_\varepsilon : X_\varepsilon \to X_\varepsilon'$ defined by (1.19) satisfy the hypothesis of Theorem 2.1.

First we check the weak continuity of G_ε.

Let $(\underset{\sim}{u}_k, S_k) \rightharpoonup (\underset{\sim}{u}, S)$ weakly in X_ε. Then for any $(\underset{\sim}{v}, T) \in X_\varepsilon$ we have:

$$\begin{aligned} \left|\langle G_\varepsilon(\underset{\sim}{u}_k, S_k) - G_\varepsilon(\underset{\sim}{u}, S), (\underset{\sim}{v}, T)\rangle\right| \leq \nu \left|((\underset{\sim}{u}_k - \underset{\sim}{u}, v))_\varepsilon\right| + \\ + \alpha|\underset{\sim}{g}|\cdot|S_k - S|_4 |\underset{\sim}{v}|_{4,\varepsilon} + |\underset{\sim}{u}_k - \underset{\sim}{u}|_{4,\varepsilon} \cdot \|\underset{\sim}{u}_k\|_\varepsilon \cdot |\underset{\sim}{v}|_{4,\varepsilon} + |\underset{\sim}{u}|_{4,\varepsilon} \|\underset{\sim}{v}\|_\varepsilon \cdot |\underset{\sim}{u}_k - \underset{\sim}{u}|_{4,\varepsilon} + \\ + \lambda\left|(\nabla(S_k - S), {}^t\underset{\sim}{a}^\varepsilon \nabla T)\right| + \lambda|\underset{\sim}{u}_k|_{4,\varepsilon} \|T\| \cdot |S_k - S|_4 + \lambda|\underset{\sim}{u}_k - \underset{\sim}{u}|_{4,\varepsilon} \|T\| \cdot |S + w_h|_4 \end{aligned}$$

and the property follows because $\left\{|\underset{\sim}{u}_k|_{4,\varepsilon}\right\}_k$ and $\left\{\|\underset{\sim}{u}_k\|_\varepsilon\right\}_k$ are bounded and the imbeddings of $\underset{\sim}{L}^4(\Omega_f^\varepsilon)$, respectively $L^4(\Omega)$, in $\underset{\sim}{V}_f^\varepsilon$, $H_0^1(\Omega)$ are compact.

Now we shall prove that G_ε is coercitive. This is the moment when λ and h will become more precise.

If we put $(\underset{\sim}{v},T)=(\underset{\sim}{u},S)$ in (1.19) then

$$(2.1)\qquad \langle G_\varepsilon(\underset{\sim}{u},S),(\underset{\sim}{u},S)\rangle=\nu\|\underset{\sim}{u}\|_\varepsilon^2+\alpha(S,\underset{\sim}{g}\underset{\sim}{u})_\varepsilon+\varepsilon^2\lambda(\underset{\sim}{a}\nabla S,\nabla S)+$$
$$+\lambda(\underset{\sim}{u},S\nabla w_h)_\varepsilon\geq\nu\|\underset{\sim}{u}\|_\varepsilon^2+\lambda a\varepsilon^2\|S\|^2-C_1(\alpha|\underset{\sim}{g}|+\lambda h)\ |\underset{\sim}{u}|_{4,\varepsilon}\|S\|$$

where we have used (1.10) and some Sobolev inequalities. It seems from (2.1) that λ and h will not be uniform with respect to ε; but in [10] the following inequality have been proved

$$|\underset{\sim}{u}|_\varepsilon\leq C\,\varepsilon\,\|\underset{\sim}{u}\|_\varepsilon\quad,\quad(\forall)\,\underset{\sim}{u}\in\underset{\sim}{H}_0^1(\Omega_f^\varepsilon)$$

Without any difficulty, as a matter-of-fact with the same proof, we can prove for any $r\leq 6$

$$(2.2)\qquad |\underset{\sim}{u}|_{r,\varepsilon}\leq\frac{\varepsilon\delta}{\gamma_r}\|\underset{\sim}{u}\|_\varepsilon\qquad,\qquad(\forall)\,\underset{\sim}{u}\in\underset{\sim}{H}_0^1(\Omega_f^\varepsilon)$$

where $\gamma_r>0$ are independent of ε, δ and $\underset{\sim}{u}$. Using (2.2), from (2.1) it results

$$(2.3)\qquad \langle G_\varepsilon(\underset{\sim}{u},S),(\underset{\sim}{u},S)\rangle\geq\nu\|\underset{\sim}{u}\|_\varepsilon^2+\lambda a\varepsilon^2\|S\|^2-$$
$$-C_2(\alpha|\underset{\sim}{g}|+\lambda h)\varepsilon\|\underset{\sim}{u}\|_\varepsilon\cdot\|S\|$$

and in order to get the desired property of G_ε we have to impose

$$(2.4)\qquad C_2^2(\alpha|\underset{\sim}{g}|+\lambda h)^2<4\nu a\lambda$$

Obviously, (2.4) can be satisfied with a sufficiently small h and a proper λ. $\square$

The following weak maximum principle is formulated in terms of inequalities in the sense of $H^1(\Omega)$. That is why we start by recalling this notion and some properties, following [4].

Let $u \in H^1(\Omega)$ and $E \subseteq \bar{\Omega}$; we say that u is non-negative on E in the sense of $H^1(\Omega)$, or briefly, $u \geq 0$ on E in $H^1(\Omega)$, if there exists a sequence $u_k \in W^{(1)}_{\infty}(\Omega)$ such that $u_k(x) \geq 0$ for $x \in E$ and $u_k \to u$ in $H^1(\Omega)$. Let $v \in H^1(\Omega)$; naturally, we say that $u \leq v$ on E in $H^1(\Omega)$ if $v-u \geq 0$ on E in $H^1(\Omega)$. As v may be a constant, we define

$$\sup_E u = \inf\left\{ m \in \mathbb{R} \,\middle|\, u \leq m \text{ on } E \text{ in } H^1(\Omega) \right\}$$

<u>Proposition 2.1.</u> If $u \geq 0$ on E in $H^1(\Omega)$ then $u \geq 0$ a.e. on E.

<u>Proposition 2.2.</u> If $\sup\limits_{\partial\Omega} u < \infty$ then for any $M \geq \sup\limits_{\partial\Omega} u$ we have:

$$\max\left\{u-M, 0\right\} \in H^1_0(\Omega) \text{ and } \max\left\{u-M, 0\right\} \geq 0 \text{ on } \Omega \text{ in } H^1(\Omega).$$

<u>Proposition 2.3.</u> Let $u \in W^{(1)}_p(\Omega)$ $(p \geq 1)$; then $v = \max\left\{u, 0\right\} \in W^{(1)}_p(\Omega)$ and we have in the sense of distributions:

$$\nabla v = \begin{cases} \nabla u & \text{in } \left\{ x \in \Omega \,\middle|\, u > 0 \text{ on } \{x\} \text{ in } H^1(\Omega) \right\} \\ 0 & \text{in } \left\{ x \in \Omega \,\middle|\, u \leq 0 \text{ on } \{x\} \text{ in } H^1(\Omega) \right\} \end{cases}$$

We can pass now to our maximum principle:

<u>Theorem 2.3.</u> If $(\underset{\sim}{u}^{\varepsilon}, S^{\varepsilon}) \in X_{\varepsilon}$ is a solution of the problem (1.20) then $S^{\varepsilon} \in L^{\infty}(\Omega)$ and

$$(2.5) \qquad \left| S^{\varepsilon} + w_h \right|_{\infty} \leq \beta = \frac{1}{2}\left(\sup_{\partial\Omega} \tau - \inf_{\partial\Omega} \tau \right)$$

<u>Proof.</u> As $-\beta \leq w_h(x) \leq \beta$ for a.a. $x \in \partial\Omega$ from Proposition 2.2 and Proposition 2.3 it follows that

$$R = \max\{S^{\varepsilon}+w_h-\beta, 0\} \in H^1_o(\Omega) \text{ and that}$$

$$\nabla R = \begin{cases} (S^{\varepsilon}+w_h) & \text{when } R\neq 0 \\ 0 & \text{when } R=0 \end{cases}$$

Puting T=R in (1.22) we obtain

$$a\varepsilon^2\|R\|^2 \leqslant (\underset{\sim}{a}^{\varepsilon}\nabla R, \nabla R) = (\underset{\sim}{a}^{\varepsilon}\nabla(S^{\varepsilon}+w_h), \nabla R) =$$

$$= -(\underset{\sim}{u}^{\varepsilon}, R\nabla(S^{\varepsilon}+w_h))_{\varepsilon} = -(\underset{\sim}{u}^{\varepsilon}, R\nabla R)_{\varepsilon} = 0$$

and hence R=0 in $H^1_o(\Omega)$, that is $S^{\varepsilon}+w_h \leqslant \beta$ on Ω in $H^1(\Omega)$. According to Proposition 2.1, it results $S^{\varepsilon}+w_h \leqslant \beta$ a.e. on Ω. Analogously, with

$$R = \min\{S^{\varepsilon}+w_h+\beta, 0\}$$

we get $S^{\varepsilon}+w_h \geqslant -\beta$ a.e. on Ω. Thus (2.5) is proved and concomitantly the whole theorem because $w_h \in H^2(\Omega) \subseteq L^{\infty}(\Omega)$. □

<u>Proposition 2.4.</u> If $(\underset{\sim}{u}^{\varepsilon}, S^{\varepsilon}) \in X_{\varepsilon}$ is a solution of the problem (1.20) then

(2.6) $\|\underset{\sim}{u}^{\varepsilon}\|_{\varepsilon} \leqslant \varepsilon\delta C_o$ where $C_o = (1+\alpha\beta)|\underset{\sim}{g}|/\nu\gamma_2$

(2.7) $|u^{\varepsilon}|_{r,\varepsilon} \leqslant \varepsilon^2 C_o \delta^2/\gamma_r$ for $r \leqslant 6$

(2.8) $\|S^{\varepsilon}\| \leqslant C_o\delta^2\beta/a\gamma_2 + |\underset{\sim}{a}|_{\infty} \cdot |\nabla w_h|/a$

<u>Proof.</u> Puting $\underset{\sim}{v}=\underset{\sim}{u}^{\varepsilon}$, T=0 in (1.20) we obtain

$$\nu\|\underset{\sim}{u}^{\varepsilon}\|^2_{\varepsilon} = (\underset{\sim}{u}^{\varepsilon}\cdot\underset{\sim}{g}, 1-\alpha(S^{\varepsilon}+w_h))_{\varepsilon} \leqslant (1+\alpha\beta)|\underset{\sim}{g}|\cdot|\underset{\sim}{u}^{\varepsilon}|$$

and using succesively (2.2) it follows (2.6) and (2.7).

Taking in (1.22) $T=S^{\varepsilon}$ we get

$$(\underset{\sim}{a}^{\varepsilon}\nabla S^{\varepsilon},\nabla S^{\varepsilon})=(\underset{\sim}{u}^{\varepsilon},(S^{\varepsilon}+w_h)\nabla S^{\varepsilon})_{\varepsilon}-(\underset{\sim}{a}^{\varepsilon}\nabla w_h,\nabla S^{\varepsilon})$$

and therefore

$$\varepsilon^2 a\,\|S^{\varepsilon}\|\leqslant\beta|\underset{\sim}{u}^{\varepsilon}|_{\varepsilon}+\varepsilon^2|\underset{\sim}{a}|_{\infty}|\nabla w_h|$$

Finally, with (2.7) it results (2.8). $\square$

Theorem 2.4. If $\delta>0$ is sufficiently small then the solution of the problem (1.20) is unique for any $\varepsilon\in(0,1)$.

Proof. Let $(\underset{\sim}{u}_1,S_1)$ and $(\underset{\sim}{u}_2,S_2)$ be solutions of (1.20); denoting with $\underset{\sim}{u}=\underset{\sim}{u}_1-\underset{\sim}{u}_2$, $S=S_1-S_2$ and subtracting the corresponding relations we have

$$(2.9)\quad \nu((\underset{\sim}{u},\underset{\sim}{v}))_{\varepsilon}+(\alpha S\underset{\sim}{g}+(\underset{\sim}{u}\nabla)\underset{\sim}{u}_2,\underset{\sim}{v})+((\underset{\sim}{u}_1\nabla)\underset{\sim}{u},\underset{\sim}{v})=0,\quad (\forall)\underset{\sim}{v}\in\underset{\sim}{V}_t^{\varepsilon}$$

$$(2.10)\quad (\underset{\sim}{a}^{\varepsilon}\nabla S,\nabla T)+(\underset{\sim}{u},T\nabla(S_2+w_h))_{\varepsilon}+(\underset{\sim}{u}_1,T\nabla S)=0,\quad (\forall)T\in H_o^1(\Omega).$$

Taking in (2.9) $\underset{\sim}{v}=\underset{\sim}{u}$ and in (2.10) $T=S$ one can easily obtain with (2.2) and (2.6)-(2.7) the following estimations:

$$(2.11)\quad \nu\|\underset{\sim}{u}\|_{\varepsilon}\leqslant C_1(\varepsilon\delta\|S\|+\varepsilon^3\delta^3\|\underset{\sim}{u}\|_{\varepsilon})$$

$$(2.12)\quad a\varepsilon\|S\|\leqslant\beta\delta\|\underset{\sim}{u}\|_{\varepsilon}/\gamma_2^{\nu}$$

Introducing (2.12) in (2.11), as $\varepsilon\in(0,1)$ it results

$$(2.13)\quad \nu\|\underset{\sim}{u}\|_{\varepsilon}\leqslant C_2(\delta^2+\delta^3)\cdot\|\underset{\sim}{u}\|_{\varepsilon}$$

Obviously, the Theorem follows as soon as

$$(2.14) \qquad \delta^2+\delta^3 < \nu / C_2 ,$$

this relation being ensured by the hypothesis. $\square$

As the parameter $\delta > 0$ (initial size of the period) has no significance for our main purpose, the homogenization process, we shall consider from now on that it was chosen "a priori" - satisfying (2.14). That is the way in which we surpass the well known "uniqueness problem" of the viscous thermal flows. Here, we have to remark that the homogenization process in a case of non-unicity seems to be far more difficult.

3. THE CONVERGENCE OF THE HOMOGENIZATION PROCESS

Denoting the mean operator of any Y-periodic function by

$$m(f) = \frac{1}{\mu(Y)} \int_Y f(y)\,dy$$

where $\mu(Y)$ is the measure of Y, we introduce the following Hilbert spaces:

$$\underset{\sim}{V}^f_{per} = \left\{ \underset{\sim}{v} \in \underset{\sim}{H}^1(Y_f) \,\middle|\, \underset{\sim}{v} \text{ is Y-periodic, div } \underset{\sim}{v}=0,\ \underset{\sim}{v}\Big|_{\Gamma} = \underset{\sim}{0} \right\}$$

$$\tilde{H}^1_{per}(Y) = \left\{ T \in H^1(Y) \,\middle|\, T \text{ is Y-periodic, } m(T)=0 \right\}$$

with the corresponding scalar products

$$(\underset{\sim}{u},\underset{\sim}{v})_V = \int_{Y_f} \nabla \underset{\sim}{u} \cdot \nabla \underset{\sim}{v}\; dy$$

$$(S,T)_H = \int_Y \nabla S \nabla T\; dy$$

For any $k \in \{1,2,3\}$ we consider the following local-periodic problems:

<u>To find</u> $\underset{\sim}{u}^k \in \underset{\sim}{V}^f_{per}$ <u>and</u> $s^k \in \tilde{H}^1_{per}(Y)$ <u>such that</u>

(3.1) $\nu(\underset{\sim}{u}^k, \underset{\sim}{v})_V = \int_{Y_f} v_k \, dy \quad , \quad (\forall)\, \underset{\sim}{v} \in \underset{\sim}{V}^f_{per}$

(3.2) $\int_Y a_{ij} \frac{\partial s^k}{\partial y_i} \frac{\partial T}{\partial y_j} dy = - \int_Y a_{kj} \frac{T}{y_j} dy \quad , \quad (\forall)\, T \in \tilde{H}^1_{per}(Y)$

where v_k is the k-component of $\underset{\sim}{v}$.

Obviously, by the Lax-Milgram theorem, there exists unique $\underset{\sim}{u}^k \in \underset{\sim}{V}^f_{per}$ and $s^k \in \tilde{H}^1_{per}(Y)$ satisfying respectively, the problems (3.1) and (3.2). Also, with the regularity theorem of the Stokes problem [11] we obtain $\underset{\sim}{u}^k \in \underset{\sim}{H}^2(Y_f)$; moreover, using the flux property as in [10] Ch.7 we can prove that there exists a unique $p^k \in H^1_{per}(Y_f)/\mathbb{R}$ such that

(3.3) $\quad -\nu \Delta \underset{\sim}{u}^k + p^k = \underset{\sim}{e}^k \quad$ in Y_f

(3.4) $\quad \operatorname{div} \underset{\sim}{u}^k = 0 \quad$ in Y_f

where $\underset{\sim}{e}^k$ is the unit vector of the k-axis.

With methods similar to those at the end of the first section we obtain

(3.5) $\quad \operatorname{div} \left[{}^t\underset{\sim}{a} \nabla (s^k + y_k) \right] = 0 \quad$ in $H^{-1}(Y)$

and in the sense of $E(Y)$

(3.6) $\quad \left[{}^t\underset{\sim}{a} \nabla (s^k + y_k) \right]_1 \cdot \underset{\sim}{n} = 0 \quad$ <u>on</u> Γ

where ${}^t a_{ij} = a_{ji}$, $[\ ]_1$ is the jump across Γ and $\underset{\sim}{n}$ is the unit normal on Γ, exterior to Y_f.

Before the main result of this section we have to present a result [8] which will be fundamental for the continuation of the pressure p^{ε} to Ω.

Theorem 3.1. For any $\varepsilon > 0$ sufficiently small there exists a restriction operator $R_{\varepsilon} \in \mathcal{L}(\underset{\sim}{H}^1_0(\Omega), \underset{\sim}{H}^1_0(\Omega^{\varepsilon}_f))$ such that

1°. If $\underset{\sim}{u} \in \underset{\sim}{H}^1_0(\Omega^{\varepsilon}_f)$ is continued with zero in $\Omega \setminus \Omega^{\varepsilon}_f$ then $R_{\varepsilon}\underset{\sim}{u} = \underset{\sim}{u}$

2°. If $\underset{\sim}{u} \in \underset{\sim}{H}^1_0(\Omega)$ and $\operatorname{div} \underset{\sim}{u} = 0$, then $\operatorname{div}(R_{\varepsilon}\underset{\sim}{u}) = 0$

3°. For any $\underset{\sim}{u} \in \underset{\sim}{H}^1_0(\Omega)$ the following estimations hold:

$$|R_{\varepsilon}\underset{\sim}{u}|_{\varepsilon} \leqslant C(\Omega, Y_f)\ (|\underset{\sim}{u}| + \varepsilon\|\underset{\sim}{u}\|) \tag{3.7}$$

$$\|R_{\varepsilon}\underset{\sim}{u}\|_{\varepsilon} \leqslant C(\Omega, Y_f)\ (|\underset{\sim}{u}|/\varepsilon + \|\underset{\sim}{u}\|) \tag{3.8}$$

From now on we suppose $\varepsilon \in (0,1)$ sufficiently small, as the Theorems 3.1 can be used. Also, introducing the following subspace of $\underset{\sim}{L}^2(\Omega)$

$$\underset{\sim}{H} = \left\{ \underset{\sim}{u} \in \underset{\sim}{L}^2(\Omega) \,\middle|\, \operatorname{div} \underset{\sim}{u} = 0,\ \underset{\sim}{u}.\underset{\sim}{n} = 0 \text{ on } \partial\Omega \right\}$$

we can state our convergence theorem.

Theorem 3.2. If $(\underset{\sim}{u}^{\varepsilon}, S^{\varepsilon}, p^{\varepsilon})$ is the weak solution of the problem (1.11)-(1.18), and if we consider $\underset{\sim}{u}^{\varepsilon}$ continued to Ω with value zero out of Ω^{ε}_f, then there exists a continuation of p^{ε} to Ω (denoted with $\tilde{p}^{\varepsilon}$) such that

$$\frac{1}{\varepsilon^2}\underset{\sim}{u}^{\varepsilon} \longrightarrow \underset{\sim}{u} \qquad \text{weakly in } \underset{\sim}{L}^6(\Omega) \tag{3.9}$$

$$S^{\varepsilon} \longrightarrow S \qquad \text{weakly in } H^1_0(\Omega) \tag{3.10}$$

$$\tilde{p}^{\varepsilon} \longrightarrow p \qquad \text{strongly in } L^2(\Omega) \tag{3.11}$$

where $(\underset{\sim}{u},S,p)\in \underset{\sim}{H}\times H_0^1(\Omega)\times L^2(\Omega)/\mathbb{R}$ satisfy "weakly" in Ω

$$(3.12)\qquad u_i = K_{ij}\left\{-\frac{\partial p}{\partial X_j}+\left[1-\alpha(S+w_h)\right]g_j\right\}$$

$$(3.13)\qquad -\mathrm{div}\left[\underset{\sim}{A}\nabla(S+w_h)\right]+\underset{\sim}{u}\nabla(S+w_h)=0$$

the homogenized coefficient being defined by:

$$A_{ij}=m(a_{ij}+\frac{\partial s^i}{\partial Y_k}a_{kj}),$$

$$K_{ij}=\frac{1}{\mu(Y)}\int_{Y_f}u_j^i(y)\,dy$$

(u_j^i is the j-component of the vector $\underset{\sim}{u}^i$).

Remark 3.1. The problem (3.12)-(3.13) was studied in the series [9]. Unless the case when the corresponding non-dimensional Rayleigh number is sufficiently small, the solution of this problem is probably not unique; that is why, in case of non-uniqueness, the convergences (3.9)-(3.11) hold only on some subsequences.

Proof. From (2.7)-(2.8) it results that $\left\{\frac{1}{\varepsilon^2}\underset{\sim}{u}^\varepsilon\right\}_\varepsilon$ and $\left\{S^\varepsilon\right\}_\varepsilon$ are bounded in $\underset{\sim}{L}^6(\Omega)$ and respectively, in $H_0^1(\Omega)$; hence there exists $\underset{\sim}{u}\in\underset{\sim}{L}^6(\Omega)$ and $S\in H_0^1(\Omega)$ for which, passing, just in case, to a subsequence, the convergences (4.9) and (4.10) hold. Also, for any $q\in H^1(\Omega)$, with (1.27) we get

$$(3.14)\qquad 0=(\underset{\sim}{u}^\varepsilon,\nabla q)\longrightarrow(\underset{\sim}{u},\nabla q)$$

As the space $\underset{\sim}{L}^2(\Omega)$ admits the orthogonal decomposition $\underset{\sim}{L}^2(\Omega)=$ $=\underset{\sim}{H}\oplus\underset{\sim}{H}^\perp$, where

$$\underset{\sim}{H}^\perp=\left\{\underset{\sim}{u}\in\underset{\sim}{L}^2(\Omega)\;\middle|\;(\exists)q\in H^1(\Omega)\text{ such that }\underset{\sim}{u}=\nabla q\right\}\quad[11]$$

from (3.14) it follows that $\underset{\sim}{u}\in\underset{\sim}{H}$.

We construct now the continuation of $p^{\varepsilon}\in L^2(\Omega_f^{\varepsilon})$ defined after (1.24). As (1.12) is satisfied in $\underset{\sim}{H}^{-1}(\Omega_f^{\varepsilon})$, with the restriction operator of Theorem 3.1, for any $\underset{\sim}{v}\in\underset{\sim}{H}_o^1(\Omega)$ we have

$$\langle \nabla p^{\varepsilon}, R_{\varepsilon}\underset{\sim}{v}\rangle = -\nu((\underset{\sim}{u}^{\varepsilon}, R_{\varepsilon}\underset{\sim}{v}))_{\varepsilon} - ((\underset{\sim}{u}^{\varepsilon}\nabla)\underset{\sim}{u}^{\varepsilon}, R_{\varepsilon}\underset{\sim}{v})_{\varepsilon} + \tag{3.15}$$
$$+ (1-\alpha(S^{\varepsilon}+w_h), \underset{\sim}{g}.R_{\varepsilon}\underset{\sim}{v})_{\varepsilon}$$

Using the estimations (3.7)-(3.8) it follows that the functional $\underset{\sim}{F}^{\varepsilon}=\langle\nabla p^{\varepsilon}, R_{\varepsilon}(.)\rangle$ is bounded on $\underset{\sim}{H}_o^1(\Omega)$, that is $\underset{\sim}{F}^{\varepsilon}\in\underset{\sim}{H}^{-1}(\Omega)$. If we continue $\underset{\sim}{v}\in\underset{\sim}{H}_o^1(\Omega_f^{\varepsilon})$ with value zero in Ω_S^{ε}, from Theorem 3.1 (1^o) it results $\underset{\sim}{F}^{\varepsilon}\big|_{\Omega_f^{\varepsilon}}=\nabla p^{\varepsilon}$. Moreover, if div $\underset{\sim}{v}=0$ with Theorem 3.1 (2^o) we get $\langle\underset{\sim}{F}^{\varepsilon},\underset{\sim}{v}\rangle=0$. Thus (1.24) implies the existence of a continuation of p^{ε} to Ω, $\tilde{p}^{\varepsilon}\in L^2(\Omega)/\mathbb{R}$ such that $\underset{\sim}{F}^{\varepsilon}=\nabla\tilde{p}^{\varepsilon}$. Also, from (4.12) we get

$$\left|\langle\nabla\tilde{p}^{\varepsilon},\underset{\sim}{v}\rangle\right|\leq\nu\|\underset{\sim}{u}^{\varepsilon}\|_{\varepsilon}\cdot\|R_{\varepsilon}\underset{\sim}{v}\|_{\varepsilon}+C\|\underset{\sim}{u}^{\varepsilon}\|_{\varepsilon}^2\|R_{\varepsilon}\underset{\sim}{v}\|+(1+\alpha\beta)\left|\underset{\sim}{g}\right|\cdot\left|R_{\varepsilon}\underset{\sim}{v}\right|.$$

Further, using (2.6) and (3.7)-(3.8) we obtain

$$\left|\langle\nabla\tilde{p}^{\varepsilon},\underset{\sim}{v}\rangle\right|\leq C(\varepsilon\|R_{\varepsilon}\underset{\sim}{v}\|_{\varepsilon}+\varepsilon^2\|R_{\varepsilon}\underset{\sim}{v}\|_{\varepsilon}+|R_{\varepsilon}\underset{\sim}{v}|)\leq C(|\underset{\sim}{v}|+\varepsilon\|\underset{\sim}{v}\|)$$

that is $\left|\nabla\tilde{p}^{\varepsilon}\right|_{\underset{\sim}{H}^{-1}(\Omega)}\leq C$. Consequently, with the inequality

$$\left|\tilde{p}^{\varepsilon}\right|_{L^2(\Omega)/\mathbb{R}}\leq C(\Omega)\cdot\left|\nabla\tilde{p}^{\varepsilon}\right|_{\underset{\sim}{H}^{-1}(\Omega)} \quad [11] \tag{3.16}$$

we find that $\{\tilde{p}^{\varepsilon}\}_{\varepsilon}$ is bounded in $L^2(\Omega)/\mathbb{R}$ and therefore $(\exists)p\in L^2(\Omega)/\mathbb{R}$ such that on some subsequence $\tilde{p}^{\varepsilon}\rightharpoonup p$ weakly in $L^2(\Omega)/\mathbb{R}$ and $\nabla\tilde{p}^{\varepsilon}\rightharpoonup\nabla p$ weakly in $\underset{\sim}{H}^{-1}(\Omega)$.

Let us notice that for any $\underset{\sim}{w}^{\varepsilon}\rightharpoonup\underset{\sim}{w}$ weakly in $\underset{\sim}{H}_o^1(\Omega)$ we have

$$\left|\langle\nabla\tilde{p}^{\varepsilon},\underset{\sim}{w}^{\varepsilon}\rangle-\langle\nabla p,\underset{\sim}{w}\rangle\right| \leqslant \left|\langle\nabla\tilde{p}^{\varepsilon},\underset{\sim}{w}^{\varepsilon}-\underset{\sim}{w}\rangle\right|+\left|\langle\nabla p^{\varepsilon}-\nabla p,\underset{\sim}{w}\rangle\right| \leqslant$$

$$\leqslant C(|\underset{\sim}{w}^{\varepsilon}-\underset{\sim}{w}|+\varepsilon\|\underset{\sim}{w}^{\varepsilon}-\underset{\sim}{w}\|)+ \text{ (terms which} \to 0),$$

using the Rellich's Theorem it follows that

$$\langle\nabla\tilde{p}^{\varepsilon},\underset{\sim}{w}^{\varepsilon}\rangle \longrightarrow \langle\nabla p,\underset{\sim}{w}\rangle$$

that is $\nabla\tilde{p}^{\varepsilon}\to\nabla p$ strongly in $\underset{\sim}{H}^{-1}(\Omega)$. Finally, (3.11) can be obtained by recalling (3.16).

Now, it only remains to prove that $(\underset{\sim}{u},S,p)$ satisfy (3.12) and (3.13), for which we apply a standard method.

We write the local equations (3.3)-(3.5) in terms of $x=\varepsilon y$, puting $\underset{\sim}{v}^{\varepsilon}(x)=\underset{\sim}{u}^{k}(\frac{1}{\varepsilon}x)$,

$T^{\varepsilon}(x)=x_k+\varepsilon S^{k}(\frac{1}{\varepsilon}x)$ and $q^{\varepsilon}(x)=p^{k}(\frac{1}{\varepsilon}x)$:

(3.17) $\operatorname{div}\underset{\sim}{v}^{\varepsilon}=0$

(3.18) $-\varepsilon^2\nu\Delta\underset{\sim}{v}^{\varepsilon}+\varepsilon\nabla q^{\varepsilon}=\underset{\sim}{e}^{k}$

(3.19) $\operatorname{div}({}^{t}\underset{\sim}{a}\nabla T^{\varepsilon})=0$

Because $\underset{\sim}{u}^{k}$, S^{k} and p^{k} are independent of ε, by straight integrations we find that

(3.20) $\|\underset{\sim}{v}^{\varepsilon}\|_{\varepsilon}\leqslant C/\varepsilon$, $\|T^{\varepsilon}\|\leqslant C$, $|q^{\varepsilon}|_{\varepsilon}\leqslant C$

Let $\varphi\in\mathcal{D}(\Omega)$; making the duality product of (1.12) and (3.18) with $\varphi\underset{\sim}{v}^{\varepsilon}$ and respectively, with $\frac{1}{\varepsilon^2}\varphi\underset{\sim}{u}^{\varepsilon}$, by subtraction we get

(3.21) $\nu\left(\underset{\sim}{v}^{\varepsilon}\frac{\partial\underset{\sim}{u}^{\varepsilon}}{\partial x_i}-\underset{\sim}{u}^{\varepsilon}\frac{\partial\underset{\sim}{v}^{\varepsilon}}{\partial x_i},\frac{\partial\varphi}{\partial x_i}\right)+((\underset{\sim}{u}^{\varepsilon}\nabla)\underset{\sim}{u}^{\varepsilon},\varphi\underset{\sim}{v}^{\varepsilon})+$

$+\frac{1}{\varepsilon}(q^{\varepsilon},\operatorname{div}(\varphi\underset{\sim}{u}^{\varepsilon}))=-(\frac{1}{\varepsilon^2}\underset{\sim}{u}^{\varepsilon},\varphi\underset{\sim}{e}^{k})+(\tilde{p}^{\varepsilon},\underset{\sim}{v}^{\varepsilon}\nabla\varphi)+$

$+(1-\alpha(S^{\varepsilon}+w_h),\varphi\underset{\sim}{g}.\underset{\sim}{v}^{\varepsilon})$

With the estimations (2.5)-(2.8) and (3.20) we have

$$\left|\left(\underset{\sim}{u}^{\varepsilon}\frac{\partial \chi^{\varepsilon}}{\partial x_i} - \underset{\sim}{v}^{\varepsilon}\frac{\partial u^{\varepsilon}}{\partial x_i}, \frac{\partial \varphi}{\partial x_i}\right)\right| \leqslant C\left(|\underset{\sim}{v}^{\varepsilon}|_{\varepsilon}\,\|\underset{\sim}{u}^{\varepsilon}\|_{\varepsilon} + |\underset{\sim}{u}^{\varepsilon}|_{\varepsilon}\,\|\underset{\sim}{v}^{\varepsilon}\|_{\varepsilon}\right) \leqslant C\,\varepsilon \to 0$$

$$\left|((\underset{\sim}{u}^{\varepsilon}\nabla)\underset{\sim}{u}^{\varepsilon}, \varphi\underset{\sim}{v}^{\varepsilon})\right| \leqslant C\|\underset{\sim}{u}^{\varepsilon}\|_{\varepsilon}^{2}\,\|\varphi\underset{\sim}{v}^{\varepsilon}\|_{\varepsilon} \leqslant C\,\varepsilon \to 0$$

$$\left|\frac{1}{\varepsilon}(q^{\varepsilon}, \operatorname{div}(\varphi\underset{\sim}{u}^{\varepsilon}))\right| = \left|\frac{1}{\varepsilon}\int_{\Omega_f^{\varepsilon}} q^{\varepsilon}\underset{\sim}{u}^{\varepsilon}\nabla\varphi\, dx\right| \leqslant \frac{C}{\varepsilon}|q^{\varepsilon}|_{\varepsilon}|\underset{\sim}{u}^{\varepsilon}|_{\varepsilon} \leqslant C.\varepsilon \to 0$$

Then, passing (3.21) to the limit ($\varepsilon\to 0$) we obtain

$$(\underset{\sim}{u}\underset{\sim}{e}^{k}, \varphi) = (p, m(\underset{\sim}{u}^{k})\nabla\varphi) + (1-\alpha(S+w_h), \varphi\underset{\sim}{g}.m(\underset{\sim}{u}^{k}))$$

because $\underset{\sim}{v}^{\varepsilon} \rightharpoonup m(\underset{\sim}{u}^{k})$ weakly in $\underset{\sim}{L}^2(\Omega)$, and hence $(\underset{\sim}{u}, S, p)$ satisfy (3.12) in the sense of distributions.

On the other hand, remarking that the vector

$$\sigma_i^{\varepsilon} = a_{ij}\left(\frac{1}{\varepsilon}(.)\right)\frac{\partial}{\partial x_j}(S^{\varepsilon}+w_h)$$

is bounded in $\underset{\sim}{L}^2(\Omega)$, it follows that there exist $\sigma_i \in L^2(\Omega)$ such that $\sigma_i^{\varepsilon} \rightharpoonup \sigma_i$ weakly in $L^2(\Omega)$ ($i=1,2,3$), on some subsequence in case of need. As (1.22) can be rewritten in the form

$$(3.22) \qquad (\underset{\sim}{\sigma}^{\varepsilon}, \nabla T) + \left(\frac{1}{\varepsilon^2}\underset{\sim}{u}^{\varepsilon}, T\nabla(S^{\varepsilon}+w_h)\right) = 0 \qquad (\forall)\, T \in H_0^1(\Omega)$$

and using the already known convergences we get

$$(3.23) \qquad (\underset{\sim}{\sigma}, \nabla T) + (\underset{\sim}{u}, T\,\nabla(S+w_h)) = 0 \qquad (\forall)\, T \in H_0^1(\Omega)$$

Making the duality product of (3.19) with $\varphi(S^{\varepsilon}+w_h)$ we have

$$(3.24) \qquad (\underset{\sim}{a}\nabla(S^{\varepsilon}+w_h), \varphi\nabla T^{\varepsilon}) + (\underset{\sim}{a}(S^{\varepsilon}+w_h)\nabla\varphi, \nabla T^{\varepsilon}) = 0$$

Also, taking in (3.22) $T=\varphi T^\varepsilon$ and subtracting this from (3.23) it results

$$(3.25) \quad (\underset{\sim}{\Gamma}^\varepsilon, T^\varepsilon \nabla\varphi) - ({}^t\underset{\sim}{a}\nabla T^\varepsilon, (S^\varepsilon + w_h)\nabla\varphi) = -(\frac{1}{\varepsilon^2}\underset{\sim}{u}^\varepsilon, \varphi T^\varepsilon \nabla(S^\varepsilon + w_h)) =$$

$$= \int_\Omega \frac{1}{\varepsilon^2}\underset{\sim}{u}^\varepsilon (S^\varepsilon + w_h)\nabla(\varphi x_k)\,dx + \int_\Omega \frac{1}{\varepsilon}\underset{\sim}{u}^\varepsilon S^k(\frac{1}{\varepsilon}x)\nabla(S^\varepsilon + w_h)\,dx$$

As $({}^t\underset{\sim}{a}\nabla T^\varepsilon) \longrightarrow m({}^t\underset{\sim}{a}\nabla(S^k + y_k))$ weakly in $\underset{\sim}{L}^2(\Omega)$ and $T^\varepsilon \to x_k$ strongly in $L^2(\Omega)$, it follows

$$(\underset{\sim}{\Gamma}^\varepsilon, T^\varepsilon\nabla\varphi) - ({}^t\underset{\sim}{a}\nabla T^\varepsilon, (S^\varepsilon + w_h)\nabla\varphi) \longrightarrow (\underset{\sim}{\Gamma}, x_k\nabla\varphi) + (m({}^t\underset{\sim}{a}\nabla(S^k + y_k))\cdot\nabla(S + w_h), \varphi)$$

From (2.7) and (2.8) we find $\frac{1}{\varepsilon}\underset{\sim}{u}^\varepsilon \to \underset{\sim}{0}$ strongly in $\underset{\sim}{L}^4(\Omega)$ and $\left\{|\nabla(S^\varepsilon + w_h)|\right\}_\varepsilon$ bounded; also, $S^k(\frac{1}{\varepsilon}(.)) \longrightarrow m(S^k)$ weakly in $L^4(\Omega)$ and therefore

$$(\frac{1}{\varepsilon^2}\underset{\sim}{u}^\varepsilon, \varepsilon S^k(\frac{1}{\varepsilon}(.))\nabla(S^\varepsilon + w_h)) \longrightarrow 0$$

and thus (3.25) at the limit becomes

$$(3.26) \quad (\underset{\sim}{\Gamma}, x_k\nabla\varphi) + (m({}^t\underset{\sim}{a}\nabla(S^k + y_k))\nabla(S + w_h), \varphi) = (\underset{\sim}{u}, (S + w_h)\nabla(\varphi x_k))$$

Finally, subtracting from (3.26) the relation (3.23) in which $T=\varphi x_k$ we obtain

$$(\Gamma_k, \varphi) = (A_{ki}\frac{\partial}{\partial x_i}(S + w_h), \varphi) \qquad (\forall)\varphi \in \mathcal{D}(\Omega)$$

Hence $\Gamma_k = A_{ki}\frac{\partial}{\partial x_i}(S + w_h)$, that is, recalling (3.23), $(\underset{\sim}{u}, S, p)$ satisfy (3.13) also. □

We have to remark that, besides (3.1), the second local-periodical problem which is obtained with the "heuristic" two-scale method is not exactly (3.2), but the following:

<u>To find</u> $\chi^\ell \in \tilde{H}^1_{per}(Y)$ $(\ell=1,2,3)$ <u>such that</u>

$$(3.27) \quad \int_Y a_{ij}\frac{\partial \chi^\ell}{\partial y_j}\frac{\partial T}{\partial y_i}dy = -\int_Y \frac{\partial T}{\partial y_j}a_{j\ell}\, dy, \quad (\forall)\, T\in \tilde{H}^1_{per}(Y)$$

Consequently, the definition in [2] of the tensor $\underset{\sim}{A}$ is

$$A_{ij} = m(a_{ij} + a_{jk}\frac{\partial \chi^i}{\partial y_k})$$

But, we see that there is no difference between this definition and ours, by puting in (3.2) $T=\chi^\ell$ and in (3.27) $T=S^k$. This means that the homogenized coefficients $\underset{\sim}{K}$ and $\underset{\sim}{A}$ have the following properties, already proved in [1] and [2]: they are positively defined tensors, $\underset{\sim}{K}$ being also symmetric. Moreover, if $\underset{\sim}{a}^\varepsilon$ is symmetric then $\underset{\sim}{A}$ is also symmetric [7], because

$$A_{ij} = \frac{1}{\mu(Y)}\int_Y a_{k\ell}\frac{\partial(S^i+y_i)}{\partial y_k}\frac{\partial(S^j+y_j)}{\partial y_\ell}\,dy .$$

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