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ON THE DEFINITION OF M-FLOWCHARTS

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This paper is a natural continuation of the following Căzănescu and Ungureanu result: the M-flowcharts over an algebraic theory with iterate T form a T-module with iterate freely generated by M.

The definition of M-flowcharts uses an artificial linear order on the set of internal vertices. We remove this deficency by a simple factorization. In this way we obtain a new free algebraic structure which differs from the previous one by a new axiom.

1. Introduction

Let us begin with some notation. Let T be an S-sorted algebraic theory. Let M be a monoid and let $r : M \longrightarrow S^*$ be a monoid morphism. The letter m will denote an element of M. If $f \in N(n,p)$ is a morphism in the initial (one-sorted) algebraic theory and $m_1, m_2, \dots, m_p$ are p elements of M then we denote by

$$r(f;m_1,m_2,\dots,m_p) \in T(r(m_{f(1)}m_{f(2)}\dots m_{f(n)}),r(m_1m_2\dots,{}_p))$$

the unique morphism of T such that for each $i \in [n]$

$$(0_{r(m_{f(1)}\dots m_{f(i-1)})}+1_{r(m_{f(i)})}+0_{r(m_{f(i+1)}\dots m_{f(n)})})r(f;m_1,\dots,m_p)=$$

$$=0_{r(m_1m_2\dots m_{f(i)-1})}+1_{r(m_{f(i)})}+0_{r(m_{f(i)+1}\dots m_p)}\ .$$

Notice that

$$r(f;m_1,m_2,\dots,m_p) = 1_{S^*}(f;r(m_1),r(m_2),\dots,r(m_p))$$

where 1_{S^*} is the identity morphism of S^*.

The following equalities are easy to prove:

(1.1) $r(fg;m_1,m_2,\ldots,m_q)=r(f;m_{g(1)},m_{g(2)},\ldots,m_{g(p)})r(g;m_1,\ldots,m_q)$
for every $f\in N(n,p)$ and $g\in N(p,q)$,

(1.2) $r(\langle f,g\rangle;m_1,m_2,\ldots,m_p)=\langle r(f;m_1,m_2,\ldots m_p),r(g;m_1,m_2,\ldots,m_p)\rangle$
for every $f\in N(n,p)$ and $g\in N(q,p)$,

(1.3) $r(f+g;m_1,m_2,\ldots m_p,m'_1,m'_2,\ldots,m'_q)=r(f,m_1,\ldots,m_p)+r(g;m'_1,\ldots,m'_q)$
for every $f\in N(n,p)$ and $g\in N(n',q)$,

(1.4) $r(1_n;m_1,m_2,\ldots,m_n)=1_{r(m_1m_2\ldots m_n)}$

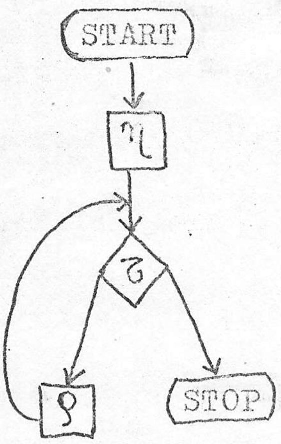
If $f\in N(n,n)$ is an isomorphism then $r(f;m_1,m_2,\ldots,m_n)$ is an isomorphism too and

(1.5) $r(f;m_1,m_2,\ldots,m_n)^{-1}=r(f^{-1};m_{f(1)},m_{f(2)},\ldots,m_{f(n)})$

If $f\in N(2,2)$, $f(1)=2$ and $f(2)=1$ then

$$r(f;m,m') = S^{r(m)}_{r(m')}$$

Let as regard again the following exemple



The set of statements $\Sigma = \{\eta, \zeta, \rho\}$ was naturally endowed with the ranking junction $r' : \Sigma \longrightarrow \omega$ defined by $r'(\eta) = r'(\rho) = 1$ and $r'(\zeta) = 2$.

Using a linear order on the set of internal vertices reflected by the order of statements in the word $m = \eta\zeta\rho \in \Sigma^*$, the above flowchart was represented by the triple (i,t,m) where the function $i : [1] \longrightarrow [3+1]$ is defined by $i(1)=1$ and the function $t : [1+2+1] \longrightarrow [3+1]$ is defined by the table

j	1	2	3	4
$t(j)$	2	3	4	2

Now let us change the linear order on the set of internal vertices. Let us put for example $m' = \rho\eta\zeta$. The corresponding input function $i' : [1] \longrightarrow [3+1]$ is defined by $i'(1) = 2$. The corresponding function $t' : [1+1+2] \longrightarrow [3+1]$ is defined by the next table

j	1	2	3	4
$t'(j)$	3	3	1	4

Hence we have in $Fl_{\Sigma, N}(1,1)$ six elements representing the same flowchart. The triples (i,t,m) and (i',t',m') are two of them. We have to do something to obtain a bijection between flowcharts and their abstract representations.

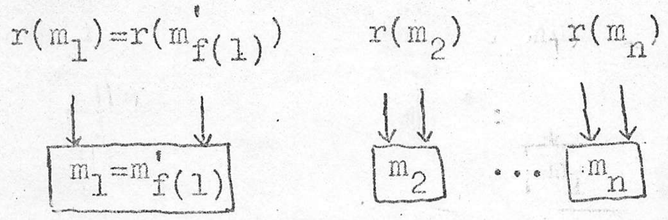
Coming back to our example, it is natural to introduce the unique isomorphism between the two above linear orders. This is - morphism $f : [3] \longrightarrow [3]$ is defined by

j	1	2	3
$f(j)$	2	3	1

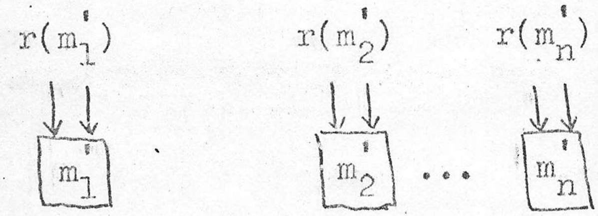
Let us remark that the i-the letter of m is equal to the $f(i)$-th letter of m'. It is obvious that $i(f+1_1) = i'$. Another equality for t and t' may be written but we ~~shall do~~ it only in the general case where the understanding is easier.

Let M be a monoid and let $r : M \longrightarrow S^*$ and $r' : M \longrightarrow S^*$ be two monoid morphisms.

Let (i,t,m) and (i',t',m') be in $Fl_{M,T}(a,b)$. Suppose $m = m_1 m_2 \ldots m_n$, $m' = m'_1 m'_2 \ldots m'_n$ and $f \in N(n,n)$ is an isomorphism such that $m_i = m'_{f(i)}$ for every $i \in [n]$. Now think of (i,t,m) and (i',t',m') as two flowcharts with n internal vertices which differ only by the order of their internal vertices and try to write the conditions to be imposed on i and i' and on t and t'. Let us think of f as an isomorphism between the orders of the internal vertices. If we regard the imputs of the statements in the first flowchart



and in the second flowchart respectively



we see that $r(f;m'_1,m'_2,\ldots,m'_n)$ is the isomorphism between the inputs of the statements induced by f, therefore i and i' must satisfy

$$i(r(f;m'_1,m'_2,\ldots,m'_n)+1_b)=i'$$

If we regard the outputs of the statements we see that $r'(f;m'_1,m'_2,\ldots,m'_n)$ in the isomorphism between the outputs of the statements induced by f, therefore t and t' must satisfy

$$t(r(f;m'_1,m'_2,\ldots,m'_n)+1_b)=r'(f;m'_1,m'_2,\ldots,m'_n)t' .$$

This is our motivation for definition 2.4 below.

2. Quotient T-modules

Let T be and S-sorted algebraic theory with iterate.

2.1. Definition. Let Q be a T-module with iterate. We say that $\sim$ is a congruence on Q if for every $a,b \in S^*$ an equivalence relation on Q(a,b) is given such that:

a) $\alpha \sim \gamma$ in Q(a,b) and $\beta \sim \delta$ in Q(b,c) imply $\alpha\beta \sim \gamma\delta$ in Q(a,c),

b) $\alpha \sim \gamma$ in Q(a,c) and $\beta \sim \delta$ in Q(b,c) imply $\langle\alpha,\beta\rangle \sim \langle\gamma,\delta\rangle$ in Q(ab,c),

c) $\alpha \sim \beta$ in Q(a,ab) implies $\alpha^\dagger \sim \beta^\dagger$ in Q(a,b).

Let us denote by $C(\alpha)$ the equivalence class of $\alpha \in Q(a,b)$.

2.2. Definition. Let $H : T \longrightarrow Q$ be a T-module with iterate and let $\sim$ be a congruence on Q. The quotient T-module with iterate $H/\sim : T \longrightarrow Q/\sim$ of Q by $\sim$ is defined by

$Q/\sim (a,b) = Q(a,b)/\sim$		for all $a,b \in S^*$,
composition	$C(\alpha)C(\beta) = C(\alpha\beta)$	for all $\alpha \in Q(a,b)$ and $\beta \in Q(b,c)$,
tupling	$\langle C(\alpha), C(\beta)\rangle = C(\langle\alpha,\beta\rangle)$	for all $\alpha \in Q(a,c)$ and $\beta \in Q(b,c)$,
iterate	$C(\alpha)^\dagger = C(\alpha^\dagger)$	for all $\alpha \in Q(a,ab)$,
$H/\sim (f) = C(H(f))$		for all $f \in T(a,b)$.

The correctness of the composition, tupling and iterate operations in the above definition follows from the conditions a), b) and c) of definition 2.1.

Obviously $C : Q \longrightarrow Q/\sim$ is a T-module with iterate morphism. It is easy to prove the following property:

2.3. Proposition. If $\sim$ is a congruence on the T-module with iterate Q and if $F : Q \longrightarrow Q'$ is a morphism of T-modules with iterate such that

$$\alpha \sim \beta \text{ implies } F(\alpha) = F(\beta)$$

then there exists a unique T-module with iterate morphism $G : Q/\sim \longrightarrow Q'$ such that $CG = F$.

2.4. Definition. Let (i,t,m) and (i',t',m') be two elements of $Fl_{M,T}(a,b)$. We say that (i,t,m) is equivalent to (i',t',m') and we write $(i,t,m)\sim(i',t',m')$ if there exists a positive integer n, an isomorphism $f\in N(n,n)$ and $m_1,m_2,\dots,m_n,m'_1,m'_2,\dots,m'_n\in M$ such that

$$m = m_1m_2\dots m_n$$
$$m' = m'_1m'_2\dots m'_n$$
$$m_j = m'_{f(j)} \quad \text{for every} \quad j\in[n]$$
$$i(r(f;m'_1,m'_2,\dots,m'_n)+1_b) = i' \quad \text{and}$$
$$t(r(f;m'_1,m'_2,\dots,m'_n)+1_b) = r'(f;m'_1;m'_2,\dots,m'_n)t' .$$

In this definition we may replace every element m_k or m'_k by a new product. For example if we replace m_k by $m^k_1m^k_2\dots m^k_j = m_k$ then we replace $m'_{f(k)}$ by $m^k_1m^k_2\dots m^k_j$ and f by the isomorphism $g\in N(n-1+j,n-1+j)$ defined by

$$g(u) = \begin{cases} f(u) & \text{if} \quad u<k \text{ and } f(u)<f(k) \\ f(u)+j-1 & \text{if} \quad u<k \text{ and } f(u)>f(k) \\ f(k)+u-k & \text{if} \quad k\le u<k+j \\ f(u-j+1) & \text{if} \quad k+j\le u \text{ and } f(u-j+1)<f(k) \\ f(u-j+1)+j-1 & \text{if} \quad k+j\le u \text{ and } f(u-j+1)>f(k) . \end{cases}$$

It follows that

$$r(f;m'_1,m'_2,\dots,m'_n)=r(g;m'_1,\dots,m'_{f(k)-1},m^k_1,m^k_2,\dots,m^k_j,m'_{f(k)+1},\dots,m'_n)$$

therefore all the conditions of definition 2.4 are again fulfilled.

We suppose in the sequel that the monoid M is equidivisible [[2]]. The following property is equivalent to the definition of equidivisibility; the reader may take it as a definition even if the original definition is more elegant. ~~If~~

$$m'_1m'_2\dots m'_n = m''_1m''_2\dots m''_p$$

then there exists $m_1, m_2, \ldots, m_q$, $0 = i_0 < i_1 < \ldots < i_{n-1} < i_n = q$ and $0 = j_0 < j_1 < \ldots < j_{p-1} < j_p = q$ such that

$$m'_k = m_{i_{k-1}+1} m_{i_{k-1}+2} \ldots m_{i_k} \quad \text{for every } k \in [n] \text{ and}$$

$$m''_k = m_{j_{k-1}+1} m_{j_{k-1}+2} \ldots m_{j_k} \quad \text{for every } k \in [p].$$

2.5 <u>Proposition.</u> If the monoid M is equidivisible then the relation $\sim$ from definition 2.4 is a congruence on $Fl_{M,T}$.

<u>Proof.</u> To show that $\sim$ is an equivalence relation on $Fl_{M,T}(a,b)$ we prove only transitivity.

Let us suppose $(i,t,m) \sim (i',t',m')$ with the same notations as in definition 2.4 and $(i',t',m') \sim (i'',t'',m'')$. From the above remark and equidivisibility we may accept without loss of generality that we have only one decomposition of m' as a product, i.e. there exists an isomorphism $g \in N(n,n)$ and the equality $m'' = m''_1 m''_2 \ldots m''_n$ such that

$$m'_i = m''_{g(i)} \quad \text{for every } i \in [n],$$

$$i'(r(g;m''_1,m''_2,\ldots,m''_n)+1_b) = i'' \text{ and}$$

$$t'(r(g;m''_1,m''_2,\ldots,m''_n)+1_b) = r'(g;m''_1,m''_2,\ldots,m''_n)t''$$

Then, using (1.1) we deduce

$$i(r(fg;m''_1,m''_2,\ldots,m''_n)+1_b) =$$

$$= i(r(f;m'_1,m'_2,\ldots,m'_n)+1_b)(r(g;m''_1,m''_2,\ldots,m''_n)+1_b) =$$

$$= i'(r(g;m''_1,m''_2,\ldots,m''_n)+1_b) = i'' \text{ and}$$

$$t(r(fg;m''_1,m''_2,\ldots,m''_n)+1_b) =$$

$$= t(r(f;m'_1,m'_2,\ldots,m'_n)+1_b)(r(g;m''_1,m''_2,\ldots,m''_n)+1_b) =$$

$$= r'(f;m'_1,m'_2,\ldots,m'_n)t'(r(g;m''_1,m''_2,\ldots,m''_n)+1_b) =$$

$$= r'(f;m'_1,m'_2,\ldots,m'_n)r'(g;m''_1,m''_2,\ldots,m''_n)t'' =$$

$$= r'(fg;m'_1,m'_2,\ldots,m'_n)t''.$$

We still have to prove that conditions a), b) and c) of definition 2.1 are fulfilled.

Let us suppose $(i,t,m) \sim (i',t',m')$ with the same notations as in definition 2.4 and $(j,u,q) \sim (j',u',q')$ in $Fl_{M,T}(b,c)$, i.e. there exist $p \geq 1$, an isomorphism $g \in N(p,p)$ and $q_1, q_2, \ldots q_p, q'_1, q'_2, \ldots, q'_p \in M$ such that

$$q = q_1 q_2 \ldots q_p$$

$$q' = q'_1 q'_2 \ldots q'_p$$

$$q_i = q'_{g(i)} \quad \text{for every} \quad i \in [p],$$

$$j(r(g;q'_1,q'_2,\ldots,q'_p)+1_c) = j' \quad \text{and}$$

$$u(r(g;q'_1,q'_2,\ldots,q'_p)+1_c) = r'(g;q'_1,q'_2,\ldots,q'_p)u'.$$

Therefore

$$mq = m_1 m_2 \ldots m_n q_1 q_2 \ldots q_p$$

$$m'q' = m'_1 m'_2 \ldots m'_n q'_1 q'_2 \ldots q'_p$$

and the isomorphism $f+g \in N(n+p,n+p)$ fulfils the neccessary conditions to show that $(i,t,m)(j,u,q) \sim (i',t',m')(j',u',q')$:

$$i(1_{r(m)}+j)(r(f+g;m'_1,m'_2,\ldots,m'_n,q'_1,q'_2,\ldots,q'_p)+1_c) =$$
$$= i(r(f;m'_1,m'_2,\ldots,m'_n)+1_b)(1_{r(m')}+j(r(g;q'_1,q'_2,\ldots,q'_p)+1_c)) =$$
$$= i'(1_{r(m')}+j') \quad \text{and}$$

$$\langle t(1_{r(m)}+j), 0_{r(m)}+u\rangle(r(f+g;m'_1,m'_2,\ldots,m'_n,q'_1,q'_2,\ldots,q'_p)+1_c) =$$
$$= \langle t(r(f;m'_1,m'_2,\ldots,m'_n)+1_b)(1_{r(m')}+j(r(g;q'_1,q'_2,\ldots,q'_p)+1_c)),$$
$$0_{r(m')}+u(r(g;q'_1,q'_2,\ldots,q'_p)+1_c)\rangle =$$
$$= \langle r'(f;m'_1,m'_2,\ldots m'_n)t'(1_{r(m')}+j'), 0_{r(m')}+r'(g;q'_1,q'_2,\ldots,q'_p)u'\rangle =$$
$$= r'(f+g;m'_1,m'_2,\ldots,m'_n,q'_1,q'_2,\ldots,q'_p)\langle t'(1_{r(m')}+j'), 0_{r(m')}+u'\rangle.$$

Let us suppose $(i,t,m) \sim (i',t',m')$ with the same notations as in definition 2.4 and $(j,u,q) \sim (j',u',q')$ in $Fl_{M,T}(c,b)$ with notation similar to the above one. The isomorphism $f+g \in N(n+p,n+p)$

fulfills the neccessary conditions to show that $\langle (i,t,m),(j,u,q)\rangle \sim$

$$\sim \langle (i',t',m'),(j',u',q')\rangle : \langle i(1_{r(m)}+0_{r(q)}+1_b), 0_{r(m)}+j\rangle$$

$$(r(f+g;m'_1,\ldots,m'_n,q'_1,\ldots,q'_p)+1_b) =$$

$$=\langle i(r(f;m'_1,\ldots,m'_n)+1_b)(1_{r(m')}+0_{r(q')}+1_b), 0_{r(m')}+j(r(g;q'_1,\ldots,q'_p)+1_b)\rangle =$$

$$=\langle i'(1_{r(m')}+0_{r(q')}+1_b), 0_{r(m')}+j'\rangle \text{ and}$$

$$\langle t(1_{r(m)}+0_{r(q)}+1_b), 0_{r(m)}+u\rangle (r(f+g;m'_1,\ldots,m'_n,q'_1,\ldots,q'_p)+1_b) =$$

$$=\langle r'(f;m'_1,\ldots,m'_n)t'(1_{r(m')}+0_{r(q')}+1_b), 0_{r(m')}+r'(g;q'_1,\ldots,q'_p)u'\rangle =$$

$$= r'(f+g;m'_1,\ldots,m'_n,q'_1,\ldots,q'_p)\langle t'(1_{r(m')}+0_{r(q')}+1_b), 0_{r(m')}+u'\rangle .$$

Let us suppose $(i,t,m)\sim(i',t',m')$ in $Fl_{M,T}(a,ab)$, i.e. there exist $n\geq 1$, an isomorphism $f\in N(n,n)$ and $m_1,m_2,\ldots,m_n,m'_1,m'_2,\ldots,m'_n$ such that

$$m = m_1 m_2 \ldots m_n$$

$$m' = m'_1 m'_2 \ldots m'_n$$

$$m_i = m'_{f(i)} \quad \text{for each } i\in[n]$$

$$i(r(f;m'_1,m'_2,\ldots,m'_n)+1_{ab}) = i' \text{ and}$$

$$t(r(f;m'_1,m'_2,\ldots,m'_n)+1_{ab}) = r'(f;m'_1,m'_2,\ldots,m'_n)t'.$$

The following calculation proves that $(i,t,m)^{\dagger}\sim(i',t',m')^{\dagger}$:

$$(i(\overset{a}{S}_{r(m)}+1_b))^{\dagger}(r(f;m'_1,m'_2,\ldots,m'_n)+1_b) =$$

$$=(i(\overset{a}{S}_{r(m)}+1_b)(1_a+r(f;m'_1,m'_2,\ldots,m'_n)+1_b))^{\dagger} =$$

$$=(i(r(f;m'_1,m'_2,\ldots,m'_n)+1_{ab})(\overset{a}{S}_{r(m')}+1_b))^{\dagger}=(i'(\overset{a}{S}_{r(m')}+1_b))^{\dagger} \text{ and}$$

$$t(\overset{a}{S}_{r(m)}+1_b)\langle (i(\overset{a}{S}_{r(m)}+1_b))^{\dagger}, 1_{r(m)b}\rangle (r(f;m'_1,m'_2,\ldots,m'_n)+1_b) =$$

$$=t(\overset{a}{S}_{r(m)}+1_b)(1_a+r(f;m'_1,m'_2,\ldots,m'_n)+1_b)\langle (i'(\overset{a}{S}_{r(m')}+1_b))^{\dagger}, 1_{r(m')b}\rangle =$$

$$=t(r(f;m'_1,m'_2,\ldots,m'_n)+1_{ab})(\overset{a}{S}_{r(m')}+1_b)\langle (i'(\overset{a}{S}_{r(m')}+1_b))^{\dagger}, 1_{r(m')b}\rangle =$$

$$=r'(f;m'_1,m'_2,\ldots,m'_n)t'(\overset{a}{S}_{r(m')}+1_b)\langle (i'(\overset{a}{S}_{r(m')}+1_b))^{\dagger}, 1_{r(m')b}\rangle .$$

Let $CFl_{M,T}$ be the *quotient* of $Fl_{M,T}$ by $\sim$, $CSt: T \longrightarrow CFl_{M,T}$ is structural functor and $C_M: M \longrightarrow CFl_{M,T}$ the standard interpretation of M in $CFl_{M,T}$, i.e. the composition of I_M by canonical T-modules with iterate morphism $C : Fl_{M,T} \longrightarrow Fl_{M,T}/\sim$.

3. Commutative T-modules

3.1. *Definition*. A T-module $H : T \longrightarrow Q$ is said to be *commutative* if

$$H(S^a_b)(\alpha+\beta) = (\beta+\alpha)H(S^c_d)$$

for every $\alpha \in Q(a,c)$ and $\beta \in Q(b,d)$.

The T-module T with structural functor 1_T is commutative.

3.2. *Proposition.* A T-module with iterate $H : T \longrightarrow Q$ is commutative if and only if

$$H(S^b_a)\langle\alpha,\beta\rangle = \langle\beta,\alpha\rangle$$

for every $\alpha \in Q(b,c)$ and $\beta \in Q(a,c)$.

Proof. Let us suppose that Q is commutative, $\alpha \in Q(b,c)$ and $\beta \in Q(a,c)$. As $\langle\alpha,\beta\rangle = (\alpha+\beta)H(\langle 1_c,1_c\rangle)$ we deduce

$$H(S^b_a)\langle\alpha,\beta\rangle = H(S^b_a)(\alpha+\beta)H(\langle 1_c,1_c\rangle) =$$

$$=(\beta+\alpha)H(S^c_c\langle 1_c,1_c\rangle) = (\beta+\alpha)H(\langle 1_c,1_c\rangle) = \langle\beta,\alpha\rangle.$$

Let us prove the other implication. If $\alpha \in Q(a,c)$ and $\beta \in Q(b,d)$ then

$$H(S^a_b)(\alpha+\beta) = H(S^a_b)\langle\alpha H(1_c+0_d), \beta H(0_c+1_d)\rangle =$$

$$= \langle\beta H(0_c+1_d), \alpha H(1_c+0_d)\rangle =$$

$$=\langle\beta H((1_d+0_c)S^c_d), \alpha H((0_c+1_d)S^c_d)\rangle =$$

$$=\langle\beta H(1_d+0_c), \alpha H(0_c+1_d)\rangle H(S^c_d) =$$

$$= (\beta+\alpha)H(S^c_d) .$$

3.3. <u>Proposition</u>. The T-module $CFl_{M,T}$ is commutative.

<u>Proof</u>. To show that $CFl_{M,T}$ is commutative, it is enough to prove that $(i,t,m)\in Fl_{M,T}(a,c)$ and $(i',t',m')\in Fl_{M,T}(b,c)$ imply

$$St(S_a^b)\langle (i',t',m'),(i,t,m)\rangle \sim \langle (i,t,m),(i',t',m')\rangle \ .$$

Let us recall that

$$\langle (i,t,m),(i',t',m')\rangle = (\langle i(1_{r(m)}+0_{r(m')}+1_c),i'(0_{r(m)}+1_{r(m')c})\rangle,$$
$$\langle t(1_{r(m)}+0_{r(m')}+1_c),t'(0_{r(m)}+1_{r(m')c})\rangle, mm').$$

It is easy to see that

$$St(S_a^b)\langle (i',t',m'),(i,t,m)\rangle =$$
$$= (\langle i\ (0_{r(m')}+1_{r(m)c}),i'(1_{r(m')}+0_{r(m)}+1_c)\rangle,$$
$$\langle t'(1_{r(m')}+0_{r(m)}+1_c),t(0_{r(m')}+1_{r(m)c})\rangle \ , \ m'm)$$

Let $f\in N(2,2)$ be the isomorphism defined by $f(1)=2$ and $f(2)=1$. As $r(f;m,m')=S^{r(m)}_{r(m')}$ and $r'(f;m,m')=S^{r'(m)}_{r'(m')}$ it follows that

$$\langle i(0_{r(m')}+1_{r(m)c}),i'(1_{r(m')}+0_{r(m)}+1_c)\rangle(r(f;m,m')+1_c)$$
$$= \langle i(1_{r(m)}+0_{r(m')}+1_c),i'(0_{r(m)}+1_{r(m')c})\rangle \quad \text{and}$$
$$\langle t'(1_{r(m')}+0_{r(m)}+1_c),t(0_{r(m')}+1_{r(m)c})\rangle(r(f,m,m')+1_c)$$
$$=r'(f;m,m')\langle t(1_{r(m)}+0_{r(m')}+1_c),t'(0_{r(m)}+1_{r(m')c})\rangle.$$

Then

$$CSt(S_a^b)\langle C(i',t',m'),C(i,t,m)\rangle = \langle C(i,t,m),C(i',t',m')\rangle$$

therefore $CFl_{M,T}$ is commutative.

3.4. <u>Proposition</u>. Let $H : T \longrightarrow Q$ be a commutative T-module. If $g\in N(n,n)$ is an isomorphism and $\alpha_i\in Q(a^i,b^i)$ for every $i\in[n]$ then

$$H(1_{S^*}(g;a^1,a^2,\dots,a^n))(\alpha_1+\alpha_2+\dots+\alpha_n) =$$

$$=(\alpha_{g(1)}+\alpha_{g(2)}+\dots+\alpha_{g(n)})H(1_{S^*}(g;b^1,b^2,\dots b^n)).$$

Proof. We regard the above identity as a property of the isomorphism g and we prove that all the isomorphisms from N having this property form a subcategory of N which is closed under sum.

If $g,f\in N(n,n)$ are isomorphisms with the above property then

$$H(1_{S^*}(gf;a^1,a^2,\dots,a^n))(\alpha_1+\alpha_2+\dots+\alpha_n) =$$

$$=H(1_{S^*}(g;a^{f(1)},a^{f(2)},\dots,a^{f(n)}))(\alpha_{f(1)}+\alpha_{f(2)}+\dots+\alpha_{f(n)})$$

$$H(1_{S^*}(f;b^1,b^2,\dots,b^n)) =$$

$$=(\alpha_{f(g(1))}+\dots+\alpha_{f(g(n))})H(1_{S^*}(g;b^{f(1)},b^{f(2)},\dots,b^{f(n)}))$$

$$H(1_{S^*}(f;b^1,b^2,\dots,b^n)) =$$

$$=(\alpha_{(gf)(1)}+\alpha_{(gf)(2)}+\dots+\alpha_{(gf)(n)})H(1_{S^*}(gf;b^1,b^2,\dots,b^n)).$$

If $g\in N(n,n)$ and $f\in N(p,p)$ are isomorphisms with the above property and $\alpha_i\in Q(a^i,b^i)$ for every $i\in[n+p]$ then

$$H(1_{S^*}(g+f;a^1,a^2,\dots,a^{n+p}))(\alpha_1+\alpha_2+\dots+\alpha_{n+p}) =$$

$$=H(1_{S^*}(g;a^1,a^2,\dots,a^n))(\alpha_1+\alpha_2+\dots+\alpha_n)+$$

$$H(1_{S^*}(f;a^{n+1},a^{n+2},\dots,a^{n+p}))(\alpha_{n+1}+\alpha_{n+2}+\dots+\alpha_{n+p}) =$$

$$=(\alpha_{g(1)}+\alpha_{g(2)}+\dots+\alpha_{g(n)})H(1_{S^*}(g;b^1,b^2,\dots,b^n)) +$$

$$+(\alpha_{n+f(1)}+\alpha_{n+f(2)}+\dots+\alpha_{n+f(p)})H(1_{S^*}(f,b^{n+1},b^{n+2},\dots,b^{n+p})) =$$

$$=(\alpha_{(g+f)(1)}+\alpha_{(g+f)(2)}+\dots+\alpha_{(g+f)(n+p)})H(1_{S^*}(g+f)\,;\,b^1,b^2,\dots,b^{n+p})).$$

Let $f\in N(2,2)$ be the isomorphism defined by $f(1) = 2$ and $f(2) = 1$. As Q is commutative, the isomorphism f has the above property. But every subcategory of N containing f and being closed under sum contains all the isomorphisms, therefore every isomorphism has the above property.

4. The free commutative T-module with iterate

Let M be an equidivisible monoid, let $r : M \longrightarrow S^*$ and $r' : M \longrightarrow S^*$ be two monoid morphisms and let T be an S-sorted algebraic theory with iterate.

We recall that if $\alpha \in Q(a,ab)$ is a morphism in a T-module with iterate $H : T \longrightarrow Q$ and if $f \in T(c,a)$ is a isomorphism then

$$(4.1) \qquad H(f)\alpha^{\dagger} = (H(f)\alpha H(f^{-1}+1_b))^{\dagger} .$$

4.1. Theorem. $CFl_{M,T}$ is the free commutative T-module with iterate generated by M.

Proof. We have to prove that for every interpretation I of M in a commutative T-module with iterate $H : T \longrightarrow Q$ there exists a unique T-module with iterate morphism $G : CFl_{M,T} \longrightarrow Q$ such that $C_M G = I$.

As $Fl_{M,T}$ is freely generated by M there exists a unique T-module with iterate morphism $F : Fl_{M,T} \longrightarrow Q$ such that $I_M F = I$.

To apply proposition 2.3 we shall prove that $(i,t,m)\sim(i',t',m')$ in $Fl_{M,T}(a,b)$ implies $F(i,t,m) = F(i',t',m')$.

Let us recall that $F(i,t,m)=H(i)\langle (I(m)H(t))^{\dagger},1_b\rangle$. With the same notations as in definition 2.4, using in turn the above remark and proposition 3.4 we deduce

$$H(r(f;m_1',m_2',\dots,m_n'))(I(m')H(t'))^{\dagger} =$$

$$=(H(r(f;m_1',m_2',\dots,m_n'))I(m')H(t'(r(f;m_1',m_2',\dots,m_n')^{-1}+1_b)))^{\dagger} =$$

$$=(H(1_{S^*}(f;r(m_1'),r(m_2'),\dots,r(m_n')))(I(m_1')+I(m_2')+\dots+I(m_n'))\cdot$$
$$H(t'(r(f;m_1',m_2',\dots,m_n')^{-1}+1_b)))^{\dagger} =$$

$$=((I(m_1)+I(m_2)+\ldots+I(m_n))H(1_{S^x}(f;r'(m_1'),r'(m_2'),\ldots,r'(m_n')))$$
$$H(t'(r(f;m_1',m_2',\ldots,m_n')^{-1}+1_b)))^{\dagger} =$$
$$=(I(m)H(r'(f;m_1',m_2',\ldots,m_n')t'(r(f;m_1',m_2',\ldots,m_n')^{-1}+1_b)))^{\dagger} =$$
$$=(I(m)H(t))^{\dagger} .$$

Therefore

$$F(i',t',m') = H(i')\langle (I(m')H(t'))^{\dagger},1_b\rangle =$$
$$=H(i)\langle H(r(f;m_1',m_2',\ldots,m_n'))(I(m')H(t'))^{\dagger},1_b\rangle =$$
$$=H(i)\langle (I(m)H(t))^{\dagger},1_b\rangle = F(i,t,m).$$

We deduce from proposition 2.3 the existence of a unique T-module with iterate morphism $G : CFl_{M,T} \longrightarrow Q$ such that $CG = F$. It follows that $C_M G = I_M CG = I_M F = I$.

We finish this paper with some identities in a commutative T-module with iterate which are not true in some T-module with iterate.

4.2. <u>Proposition</u>. Let $H : T \longrightarrow Q$ be a commutative T-module with iterate.

a) If $\alpha\in Q(a,b)$, $\beta\in Q(c,d)$, $\gamma\in Q(b,e)$ and $\delta\in Q(d,e)$ then

$$(4.2)\qquad (\alpha+\beta)\langle\gamma,\delta\rangle = \langle\alpha\gamma,\beta\delta\rangle$$

b) If $\alpha\in Q(a,b)$, $\beta\in Q(c,d)$, $\gamma\in Q(b,e)$ and $\delta\in Q(d,f)$ then

$$(4.3)\qquad (\alpha+\beta)(\gamma+\delta) = \alpha\gamma+\beta\delta .$$

<u>Proof</u>. We know that if either β or γ is in the image of H then the above identities are true in every T-module with iterate. The proof of the first identity is

$$(\alpha+\beta)\langle\gamma,\delta\rangle = (\alpha+1_c)(1_b+\beta)\langle\gamma,\delta\rangle =$$
$$=(\alpha+1_c)H(S_b^c)(\beta+1_b)H(S_d^b)\langle\gamma,\delta\rangle =$$
$$=(\alpha+1_c)H(S_b^c)(\beta+1_b)\langle\delta,\gamma\rangle =$$

$$=(\alpha+1_c)H(S_b^c)<\beta\delta,\gamma> = (\alpha+1_c)<\gamma,\beta\delta> =$$

$$=<\alpha\gamma,\beta\delta> .$$

The proof of the second identity is

$$(\alpha+\beta)(\gamma+\delta) = (\alpha+\beta)<\gamma H(1_e+0_f),\delta H(0_e+1_f)> =$$

$$=<\alpha\gamma H(1_e+0_f),\beta\delta H(0_e+1_f)> = \alpha\gamma+\beta\delta .$$

4.3. Proposition. Let $H : T \longrightarrow Q$ a commutative T-module with iterate. If $\alpha \in Q(a,bc)$, $\gamma \in Q(d,c)$ and $\beta \in Q(b,c)$ then

$$(4.4) \qquad <\alpha,\gamma H(0_b+1_c)><\beta,1_c> = <\alpha<\beta,1_c>,\gamma> .$$

Proof. We know that in every T-module with iterate

$$<\gamma H(0_b+1_c),\alpha><\beta,1_c> = <\gamma,\alpha<\beta,1_c>>$$

therefore

$$<\alpha,\gamma H(0_b+1_c)><\beta,1_c> = H(S_a^d)<\gamma H(0_b+1_c),\alpha><\beta,1_c> =$$

$$=H(S_a^d)<\gamma,\alpha<\beta,1_c>> = <\alpha<\beta,1_c>,\gamma> .$$

The next proof is based on (4.1) and the following property of every T-module with iterate:

$$(4.5) \qquad H(1_a+0_b)<\alpha,\beta>^{\dagger} = (\alpha H(S_a^b+1_c)<(\beta H(S_a^b+1_c))^{\dagger},1_{ac}>)^{\dagger}$$

for every $\alpha \in Q(a,abc)$ and $\beta \in Q(b,abc)$.

4.4 Proposition. Let $H : T \longrightarrow Q$ be a commutative T-module with iterate. If $\alpha \in Q(a,abc)$ and $\beta \in Q(b,abc)$ then

$$(4.6) \qquad H(0_a+1_b)<\alpha,\beta>^{\dagger} = (\beta<\alpha^{\dagger},1_{bc}>)^{\dagger} .$$

Proof. $H(0_a+1_b)<\alpha,\beta>^{\dagger} = H(1_b+0_a)H(S_b^a)<\alpha,\beta>^{\dagger} =$

$$=H(1_b+0_a)(H(S_b^a)<\alpha,\beta>H(S_a^b+1_c))^{\dagger} =$$

$$=H(1_b+0_a)<\beta H(S_a^b+1_c),\alpha H(S_a^b+1_c)>^{\dagger} = (\beta<\alpha^{\dagger},1_{bc}>)^{\dagger} .$$

4.5. Proposition. Let $H : T \longrightarrow Q$ be a commutative T-module with iterate. If $\alpha \in Q(a,ac)$ and $\beta \in Q(b,ac)$ then

(4.7) $$(\langle\alpha,\beta\rangle H(1_a+0_b+1_c))^\dagger = \langle H(1_a+0_c),\beta\rangle\langle\alpha^+,1_c\rangle$$

<u>Proof.</u> As by (4.1)

$$(\langle\alpha,\beta\rangle H(1_a+0_b+1_c))^\dagger = H(S^b_a)(H(S^a_b)\langle\alpha,\beta\rangle H(1_a+0_b+1_c)H(S^b_a+1_c))^\dagger =$$

$$= H(S^b_a)(\langle\beta,\alpha\rangle H(0_b+1_{ac}))^\dagger = H(S^b_a)\langle\beta H(0_b+1_{ac}),\alpha H(0_b+1_{ac})\rangle^\dagger$$

we deduce using in turn axioms I3 and I2 and (4.4) that

$$\langle(\langle\alpha,\beta\rangle H(1_a+0_b+1_c))^\dagger,1_c\rangle = H(S^b_a+1_c)\langle\langle\beta H(0_b+1_{ac}),\alpha H(0_b+1_{ac})\rangle^\dagger,1_c\rangle =$$

$$= H(S^b_a+1_c)\langle(\beta H(0_b+1_{ac}))^\dagger,1_{ac}\rangle\langle\alpha^\dagger,1_c\rangle =$$

$$= \langle H(1_a+0_c),\beta,H(0_a+1_c)\rangle\langle\alpha^+,1_c\rangle =$$

$$= \langle\langle H(1_a+0_c),\beta\rangle\langle\alpha^+,1_c\rangle,1_c\rangle .$$

As tupling by 1_c is an injection we obtain (4.7).

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On Context-free Trees

by

Virgil Emil CĂZĂNESCU

Our main theorem gives four different ways of generating the context-free trees.

A) The set of generators contains all the variables and all the trees $\sigma(x_1,x_2,\ldots,x_n)$ where σ is an operation symbol and x_i are distinct variables.

The trees are generated in two ways:

- substitution by trees of the variables of a tree,
- iteration of one operation symbol in one tree.

B) The set of generators is equal to the set of all the variables.

The tree are generated in two ways:

- by application of the usual operations of the algebra of trees,
- substitution by trees of the variables of the result of an iteration of one operation symbol in one tree.

C) The set of generators is equal to the set of all the variables.

The trees are generated in three ways:

- by application of the usual operations of the algebra of trees,
- iteration of one operation symbol in one tree,
- substitution into a tree of one operation symbol defined with the aid of another tree.

D) The set of generators contains all the variables and all the trees $\sigma(x_1,x_2,\ldots,x_n)$ where σ is an operation symbol and

x_i are distinct variables.

The trees are generated in one way:

- substitution by trees of the variables of the result of an iteration of one operation symbol in one tree.

Our main theorem gives the theoretical ground for defining finite expressions for context-free trees. Perhaps it is not uninteresting to study the computation rules for such expressions.

§ 1. Rationally closed subtheories

For each nonnegative integer n let $[n]$ denote the set $\{1,2,\dots,n\}$.

If S is a set then S^* is the free monoid generated by S. The length of $u \in S^*$ is denoted by $|u|$ and u itself will be denoted by

$$u = u_1 u_2 \dots u_{|u|}$$

where $u_i \in S$ for all $i \in [|u|]$.

Let $\mathcal{T}$ be an S-sorted algebraic theory, i.e. $\mathcal{T}$ is a cathegory such that

a) the class of objects, denoted $|\mathcal{T}|$, is the set S^*,

b) each $u \in S^*$ is the coproduct of u_i for $i \in [|u|]$.

The empty word of S^*, denoted λ, is an initial object of $\mathcal{T}$. For each $u \in S^*$, we set

$$0_u : \lambda \longrightarrow u$$

the unique morphism form λ to u.

For each $u \in S^*$, let

$$x^u_1, x^u_2, \dots, x^u_{|u|}$$

be the distinguished morphisms of the coproduct u. If $|u| = 1$ then $x^u_1 = 1_u$ where 1_u is the identity morphism of u. For each $v \in S^*$ and each family $\{f_i : u_i \longrightarrow v \mid i \in [|u|]\}$ we denote by

$$\langle f_1, f_2, \ldots, f_{|u|} \rangle : u \longrightarrow v$$

the unique morphism such that

$$x_i^u \langle f_1, f_2, \ldots, f_{|u|} \rangle = f_i$$

for all $i \in [|u|]$.

For each $u, v \in S^*$ the function

$$\langle \ ,\ldots, \ \rangle : \prod_{i \in [|u|]} \mathcal{T}(u_i, v) \longrightarrow \mathcal{T}(u,v)$$

where $\mathcal{T}(u,v)$ is the set of morphisms from u to v, will be called <u>tupling operation</u>.

We recall some usual notations and properties in an S-sorted algebraic theory.

If $u, v \in S^*$ we denote

$$x_{(1)}^{uv} = \langle x_1^{uv}, x_2^{uv}, \ldots, x_{|u|}^{uv} \rangle$$

and

$$x_{(2)}^{uv} = \langle x_{|u|+1}^{uv}, x_{|u|+2}^{uv}, \ldots, x_{|uv|}^{uv} \rangle$$

If $u, v, w \in S^*$, $\alpha \in \mathcal{T}(u,w)$ and $\beta \in \mathcal{T}(v,w)$ then

$$\langle \alpha, \beta \rangle : uv \longrightarrow w$$

is the unique morphism with the following properties:

$$x_{(1)}^{uv} \langle \alpha, \beta \rangle = \alpha \quad \text{and} \quad x_{(2)}^{uv} \langle \alpha, \beta \rangle = \beta .$$

The morphism $\langle \alpha, \beta \rangle$ is called the source pairing of α and β. We recall that:

$$\langle \alpha, \beta \rangle = \langle x_1^u \alpha, \ldots, x_{|u|}^u \alpha, x_1^v \beta, \ldots, x_{|v|}^v \beta \rangle ,$$

$$\langle \alpha, \beta \rangle \gamma = \langle \alpha\gamma, \beta\gamma \rangle \quad \text{for all } \gamma : w \longrightarrow w',$$

$$\langle \langle \alpha, \beta \rangle, \gamma \rangle = \langle \alpha, \langle \beta, \gamma \rangle \rangle \quad \text{for all } \gamma : v' \longrightarrow w.$$

If $\alpha : u \longrightarrow v$ and $\beta : u' \longrightarrow v'$ then

$$\alpha + \beta : uu' \longrightarrow vv'$$

is the unique morphism with the following properties

$$x^{uu'}_{(1)}(\alpha+\beta) = x^{vv'}_{(1)}$$

and

$$x^{uu'}_{(2)}(\alpha+\beta) = x^{vv'}_{(2)}$$

A subcategory of $\mathcal{T}$ is called <u>an algebraic subtheory</u> of $\mathcal{T}$ if it has same class of objects, contains all the distinguished morphisms and is closed under tupling operation.

If $\mathcal{T}_1$ is an algebraic subtheory of $\mathcal{T}$ then the family $\{\mathcal{T}_1(s,u) \mid (s,u) \in S \times S^*\}$ has the following properties:

(S1) $x^u_i \in \mathcal{T}_1(u_i,u)$ for all $u \in S^*$ and $i \in [|u|]$,

(S2) $\left\{ \begin{array}{l} \text{for all } s \in S,\ u,v \in S^*,\ f \in \mathcal{T}_1(s,u) \text{ and all family} \\ \{f_i \in \mathcal{T}_1(u_i,v) \mid i \in [|u|]\},\ f\langle f_1,f_2,\ldots,f_{|u|}\rangle \in \mathcal{T}_1(s,v). \end{array} \right.$

Conversely, if for each $s \in S$ and $u \in S^*$, the subsets $\mathcal{T}_1(s,u) \subseteq \mathcal{T}(s,u)$ form a family with the properties (S1) and (S2) then, defining for each $u,v \in S^*$.

$$\mathcal{T}_1(u,v) = \{\langle f_1,f_2,\ldots,f_{|u|}\rangle \mid f_i \in \mathcal{T}_1(u_i,u) \text{ for all } i \in [|u|]\}$$

the family $\{\mathcal{T}_1(u,v) \mid u,v \in S^*\}$ forms an algebraic subtheory of $\mathcal{T}$.

An <u>S-sorted algebraic theory</u> is said to be <u>ordered</u> if for each $u,v \in S^*$, $\mathcal{T}(u,v)$ is a partially ordered set with a least element $\perp_{uv}$ such that

a) the composition of morphisms is monotonic,

b) for all $u,v,w \in S^*$ and all $f \in \mathcal{T}(v,w)$

$$\perp_{uv} f = \perp_{uw}$$

c) the tupling operation is monotonic.

All the cartesian products are ordered componentwise.

The tupling operation of an ordered S-sorted algebraic theory is an isomorphism of partially ordered sets, therefore

$$\perp_{uv} = \langle \perp_{u_1 v}, \perp_{u_2 v}, \ldots, \perp_{u_{|u|} v} \rangle$$

for all $u, v \in S^*$.

A rationally closed S-sorted algebraic theory $\mathcal{T}$ is an ordered S-sorted algebraic theory equipped with a function

$$^{\dagger} : \mathcal{T}(u, uv) \longrightarrow \mathcal{T}(u, v)$$

for all $u, v \in S^*$. $\alpha^{\dagger}$ is called the iterate of or the least solution for $\alpha : u \longrightarrow uv$ and must satisfy the following conditions for all $\eta : u \longrightarrow v$ and $\zeta : v \longrightarrow w$:

1) $\alpha \langle \alpha^{\dagger}, 1_v \rangle = \alpha^{\dagger}$,

2) if $\alpha \langle \eta, 1_v \rangle \leq \eta$ then $\alpha^{\dagger} \leq \eta$,

3) $(\alpha (1_u + \zeta))^{\dagger} = \alpha^{\dagger} \zeta$.

Note that $\alpha^{\dagger}$ is the least solution for x in the equation

$$\alpha \langle x, 1_v \rangle = x.$$

In the sequel $\mathcal{T}$ will be a rationally closed S-sorted algebraic theory.

1. Remarks

a) $(\alpha x^{uv}_{(2)})^{\dagger} = \alpha$ for all $\alpha : u \longrightarrow v$,

b) $(x^{uu}_{(2)})^{\dagger} = 1_u$

c) $(x^{uv}_{(1)})^{\dagger} = \perp_{uv}$

Proof.

a) $(\alpha x^{uv}_{(2)})^{\dagger} = \alpha x^{uv}_{(2)} \langle (\alpha x^{uv}_{(2)})^{\dagger}, 1_v \rangle = \alpha 1_v = \alpha$.

b) From a) with $\alpha = 1_u$.

c) It follows from

$$x^{uv}_{(1)} \langle \perp_{uv}, 1_v \rangle = \perp_{uv}$$

that $(x^{uv}_{(1)})^{\dagger} \leq \bot_{uv}$, hence $(x^{uv}_{(1)})^{\dagger} = \bot_{uv}$

2. Proposition. Let $\alpha : u \longrightarrow uv$ and $\tau : v \longrightarrow w$ be morphisms in $\mathcal{T}$. The morphism $\alpha^{\dagger}\tau$ is the least solution for x in the equation

$$\alpha \langle x, \tau \rangle = x.$$

Proof. The above equation is equivalent to

$$\alpha(1_u + \tau)\langle x, 1_w \rangle = x$$

therefore the least solution is

$$(\alpha(1_u + \tau))^{\dagger} = \alpha^{\dagger}\tau \quad \square$$

3. Proposition. For all $\alpha : u \longrightarrow uvw$ and $\beta : v \longrightarrow uvw$

$$\langle \alpha, \beta \rangle^{\dagger} = \langle \alpha^{\dagger}\langle \gamma^{\dagger}, 1_w \rangle, \gamma^{\dagger} \rangle$$

where

$$\gamma = \beta\langle \alpha^{\dagger}, 1_{vw} \rangle .$$

Proof. Let $\delta = \langle \alpha^{\dagger}\langle \gamma^{\dagger}, 1_w \rangle, \gamma^{\dagger} \rangle$. It follows that

$$\langle \delta, 1_w \rangle = \langle \alpha^{\dagger}\langle \gamma^{\dagger}, 1_w \rangle, \langle \gamma^{\dagger}, 1_w \rangle\rangle =$$

$$= \langle \alpha^{\dagger}, 1_{vw} \rangle \langle \gamma^{\dagger}, 1_w \rangle .$$

The equalities

$$\langle \alpha, \beta \rangle \langle \delta, 1_w \rangle = \langle \alpha\langle \alpha^{\dagger}, 1_{vw} \rangle\langle \gamma^{\dagger}, 1_w \rangle, \beta\langle \alpha^{\dagger}, 1_{vw} \rangle\langle \gamma^{\dagger}, 1_w \rangle\rangle =$$

$$= \langle \alpha^{\dagger}\langle \gamma^{\dagger}, 1_w \rangle, \gamma\langle \gamma^{\dagger}, 1_w \rangle\rangle =$$

$$= \langle \alpha^{\dagger}\langle \gamma^{\dagger}, 1_w \rangle, \gamma^{\dagger} \rangle =$$

$$= \delta$$

shows that

$$\langle \alpha, \beta \rangle^{\dagger} \leq \delta$$

Let $x = x^{uv}_{(1)}\langle \alpha, \beta \rangle^{\dagger}$ and $y = x^{uv}_{(2)}\langle \alpha, \beta \rangle^{\dagger}$. It follows from

$$\langle \alpha, \beta \rangle^{\dagger} = \langle \alpha, \beta \rangle \langle \langle \alpha, \beta \rangle^{\dagger}, 1_w \rangle$$

that

$$\langle x,y\rangle = \langle \alpha\langle x,y, 1_w\rangle, \beta\langle x,y, 1_w\rangle\rangle$$

therefore

$$x = \alpha\langle x, \langle y, 1_w\rangle\rangle$$

and

$$y = \beta\langle x, \langle y, 1_w\rangle\rangle .$$

We deduce from proposition 2 that

$$\alpha^\dagger\langle y, 1_w\rangle \leq x.$$

From

$$\begin{aligned}\gamma\langle y, 1_w\rangle &= \beta\langle \alpha^\dagger, 1_{vw}\rangle\langle y, 1_w\rangle = \\ &= \beta\langle \alpha^\dagger\langle y, 1_w\rangle, \langle y, 1_w\rangle\rangle \leq \\ &\leq \beta\langle x, \langle y, 1_w\rangle\rangle = y\end{aligned}$$

it follows that

$$\gamma^\dagger \leq y$$

and $\alpha^\dagger\langle\gamma^\dagger, 1_w\rangle \leq \alpha^\dagger\langle y, 1_w\rangle \leq x.$

Therefore,

$$\delta \leq \langle x,y\rangle = \langle\alpha, \beta\rangle^\dagger$$

From $\delta = \langle\alpha, \beta\rangle^\dagger$ we obtain

$$\langle\alpha, \beta\rangle^\dagger = \langle\alpha^\dagger\langle\gamma^\dagger, 1_w\rangle, \gamma^\dagger\rangle \quad \square$$

4. <u>Corollary</u>. For all $\alpha : u \longrightarrow uvw$ and $\beta : v \longrightarrow vw$

$$\langle\alpha, \beta x_{(2)}^{u(vw)}\rangle^\dagger = \langle\alpha^\dagger\langle\beta^\dagger, 1_w\rangle, \beta^\dagger\rangle \quad \square$$

5. <u>Lemma.</u> Let $\alpha : u \longrightarrow uv$ and $i : w \longrightarrow u$ be morphisms.

a) If $j : u \longrightarrow w$ and $ji \leq 1_u$ then

$$(i\alpha(j + 1_v))^\dagger \leq i\alpha^\dagger$$

b) If i is an isomorphism then

$$(i\,\alpha(i^{-1} + 1_v))^{\dagger} = i\,\alpha^{\dagger} .$$

Proof. a) It follows from

$$i\alpha(j+1_v)\langle i\,\alpha^{\dagger}, 1_v\rangle = i\alpha\langle ji\alpha^{\dagger}, 1_v\rangle \leq i\alpha\langle\alpha^{\dagger}, 1_v\rangle = i\alpha^{\dagger}$$

that

$$(i\alpha(j + 1_v))^{\dagger} \leq i\alpha^{\dagger} .$$

b) From a) we infer

$$(i\alpha(i^{-1} + 1_v))^{\dagger} \leq i\alpha^{\dagger}$$

and

$$(i^{-1}(i\alpha(i^{-1} + 1_v))(i + 1_v))^{\dagger} \leq i^{-1}(i\alpha(i^{-1} + 1_v))^{\dagger}$$

therefore

$$i\alpha^{\dagger} = i(i^{-1}(i\alpha(i^{-1}+1_v))(i+1_v))^{\dagger} \leq (i\alpha(i^{-1} + 1_v))^{\dagger} \square$$

6. Proposition. If $\alpha : u \longrightarrow uvw$, $\beta : v \longrightarrow uvw$

$$\delta = \beta\langle x^{vuw}_{(2)}, x^{vuw}_{(1)}, x^{vuw}_{(3)}\rangle$$

and $$\gamma = \alpha\langle x^{vuw}_{(2)}, x^{vuw}_{(1)}, x^{vuw}_{(3)}\rangle\langle\delta^{\dagger}, 1_{uw}\rangle$$

then $$\langle\alpha, \beta\rangle^{\dagger} = \langle\gamma^{\dagger}, \delta^{\dagger}\langle\gamma^{\dagger}, 1_w\rangle\rangle .$$

Proof. We notice that

$$i = \langle x^{uv}_{(2)}, x^{uv}_{(1)}\rangle : vu \longrightarrow uv$$

is an isomorphism and $i^{-1} = \langle x^{vu}_{(2)}, x^{vu}_{(1)}\rangle$.

As $i^{-1} + 1_w = \langle x^{vuw}_{(2)}, x^{vuw}_{(1)}, x^{vuw}_{(3)}\rangle$ it follows from lemma that

$$i\langle\alpha, \beta\rangle^{\dagger} = (i\langle\alpha, \beta\rangle\langle x^{vuw}_{(2)}, x^{vuw}_{(1)}, x^{vuw}_{(3)}\rangle)^{\dagger} =$$

$$= \langle\delta, \alpha\langle x^{vuw}_{(2)}, x^{vuw}_{(1)}, x^{vuw}_{(3)}\rangle\rangle^{\dagger} .$$

Proposition 3 implies

$$\langle\alpha, \beta\rangle^{\dagger} = i^{-1}\langle\delta^{\dagger}\langle\gamma^{\dagger}, 1_w\rangle, \gamma^{\dagger}\rangle = \langle\gamma^{\dagger}, \delta^{\dagger}\langle\gamma^{\dagger}, 1_w\rangle\rangle \square$$

7. <u>Proposition</u>. Let n be a nonnegative integer and for $i \in [n]$

$$\alpha_i : w^i \longrightarrow w^i v .$$

If for all $i \in [n]$

$$x_i = \langle x^{w^1 \dots w^n v}_{(i)} , x^{w^1 \dots w^n v}_{(n+1)} \rangle$$

then

$$\langle \alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n \rangle^{\dagger} = \langle \alpha_1^{\dagger}, \alpha_2^{\dagger}, \dots, \alpha_n^{\dagger} \rangle .$$

<u>Proof</u>. If $n = 2$ then

$$\langle \alpha_1 x_1, \alpha_2 x_2 \rangle^{\dagger} = \langle (\alpha_1 x_1)^{\dagger} \langle \gamma^{\dagger}, 1_v \rangle , \gamma^{\dagger} \rangle$$

where

$$\gamma = \alpha_2 x_2 \langle (\alpha_1 x_1)^{\dagger} , 1_{w^2 v} \rangle = \alpha_2 .$$

But

$$(\alpha_1 x_1)^{\dagger} = (\alpha_1 (1_{w^1} + x^{w^2 v}_{(2)}))^{\dagger} = \alpha_1^{\dagger} \; x^{w^2 v}_{(2)}$$

therefore

$$\langle \alpha_1 x_1, \alpha_2 x_2 \rangle^{\dagger} = \langle \alpha_1^{\dagger} \, x^{w^2 v}_{(2)} \langle \alpha_2^{\dagger} , 1_v \rangle , \alpha_2^{\dagger} \rangle =$$

$$= \langle \alpha_1^{\dagger} , \alpha_2^{\dagger} \rangle .$$

If $\beta = \langle x^{w^1 \dots w^n v}_{(1)} , \dots , x^{w^1 \dots w^n v}_{(n-1)} , x^{w^1 \dots w^n v}_{(n+1)} \rangle$

and for $i \in [n-1]$

$$y_i = \langle x^{w^1 \dots w^{n-1} v}_{(i)} , x^{w^1 \dots w^{n-1} v}_{(n)} \rangle$$

then $y_i \beta = x_i$.

Therefore by induction

$$\langle \alpha_1 x_1, \dots, \alpha_n x_n \rangle^{\dagger} = \langle \langle \alpha_1 y_1, \dots, \alpha_{n-1} y_{n-1} \rangle \beta , \alpha_n x_n \rangle^{\dagger} =$$

$$= \langle \langle \alpha_1 y_1, \dots, \alpha_{n-1} y_{n-1} \rangle^{\dagger} , \alpha_n^{\dagger} \rangle =$$

$$= \langle \alpha_1^{\dagger}, \dots, \alpha_{n-1}^{\dagger} , \alpha_n^{\dagger} \rangle \; \square$$

8. <u>Proposition</u>. Let $\mathcal{A}(s,u) \subseteq \mathcal{T}(s,u)$ for all $(s,u) \in S \times S^*$ such that

(S3) for all $(s,v) \in S \times S^*$ if $\alpha \in \mathcal{A}(s,sv)$ then $\alpha^\dagger \in \mathcal{A}(s,v)$

and

$$\text{(S4)} \begin{cases} \text{for all } s,s' \in S \text{ and all } v \in S^*, \\ \text{if } \beta \in \mathcal{A}(s,s'v) \text{ and } \alpha \in \mathcal{A}(s',s'v) \text{ then } \beta\langle\alpha^\dagger, 1_v\rangle \in \mathcal{A}(s,v). \end{cases}$$

If $\beta : u \longrightarrow uv$ and

$$(\forall i \in [|u|])(x_i^u \beta \in \mathcal{A}(u_i, uv))$$

then

a) $(\forall i \in [|u|])\ (x_i^u \beta^\dagger \in \mathcal{A}(u_i, v))$

b) if $s \in S$ and $\gamma \in \mathcal{A}(s,uv)$ then $\gamma\langle\beta^\dagger, 1_v\rangle \in \mathcal{A}(s,u)$.

<u>Proof</u>. Induction by $|u|$. For $|u| = 1$ the conclusion follows directly from the hypothesis.

Let β: su $\longrightarrow$ suv be a morphism such that

$s \in S$, $u,v \in S^*$, $x_1^{su}\beta \in \mathcal{A}(s,suv)$ and $x_{i+1}^{su}\beta \in \mathcal{A}(u_i, suv)$ for all $i \in [|u|]$. Since

$$\beta = \langle x_1^{su}\beta, x_{(2)}^{su}\beta\rangle$$

proposition 3 implies

$$\beta^\dagger = \langle (x_1^{su}\beta)^\dagger\langle\delta^\dagger, 1_v\rangle, \delta^\dagger\rangle$$

where

$$\delta = \langle x_2^{su}\beta\langle(x_1^{su}\beta)^\dagger, 1_{uv}\rangle, \ldots, x_{1+|u|}^{su}\beta\langle(x_1^{su}\beta)^\dagger, 1_{uv}\rangle\rangle.$$

For each $i \in [|u|]$, (S4) implies

$$x_i^u\delta = x_{i+1}^{su}\beta\langle(x_1^{su}\beta)^\dagger, 1_{uv}\rangle \in \mathcal{A}(u_i, uv).$$

Therefore by the induction hypothesis and (S3)

$$x_i^u \delta^\dagger \in \mathcal{A}(u_i, v) \text{ for all } i \in [|u|]$$

and

$$(x_1^{su}\beta)^\dagger \langle \delta^\dagger, 1_v \rangle \in \mathcal{A}(s, v).$$

Therefore

$$x_1^{su}\beta^\dagger \in \mathcal{A}(s,v)$$

and for each $i \in [|u|]$.

$$x_{i+1}^{su}\beta^\dagger \in \mathcal{A}(u_i, v).$$

If $s' \in S$ and $\gamma \in \mathcal{A}(s', suv)$ then

$$\gamma \langle \beta^\dagger, 1_v \rangle = \gamma \langle (x_1^{su}\beta)^\dagger \langle \delta^\dagger, 1_v \rangle, \langle \delta^\dagger, 1_v \rangle\rangle =$$

$$= \gamma \langle (x_1^{su}\beta)^\dagger, 1_{uv} \rangle \langle \delta^\dagger, 1_v \rangle \in \mathcal{A}(s'v) \square$$

$\mathcal{T}_1$ is a <u>rationally closed subtheory</u> of $\mathcal{T}$ if $\mathcal{T}_1$ is a algebraic subtheory of $\mathcal{T}$ and $\mathcal{T}_1$ is closed under iteration.

If $\mathcal{T}_1$ is a rationally closed subtheory of $\mathcal{T}$ then the family

$$\mathcal{A}(s,u) = \mathcal{T}_1(s,u) \text{ with } (s,u) \in S \times S^*$$

has properties S1, S2, S3, S4,

$$\text{(S5)} \quad \begin{cases} \text{if } s, s' \in S,\ u \in S^*, \\ \alpha \in \mathcal{A}(s', su) \text{ and } \beta \in \mathcal{A}(s,u) \text{ then } \alpha\langle \beta, 1_u \rangle \in \mathcal{A}(s', u) \end{cases}$$

and

$$\text{(S6)} \quad \begin{cases} \text{if } s \in S,\ u, v, w \in S^* \text{ and} \\ \alpha \in \mathcal{A}(s, uv) \text{ then } \alpha\langle x_{(1)}^{uwv}, x_{(3)}^{uwv} \rangle \in \mathcal{A}(s, uwv). \end{cases}$$

9. <u>Proposition</u>. Let $\mathcal{A}(s,u) \subseteq \mathcal{T}(s,u)$ for all $(s,u) \in S \times S^*$ and

$$\mathcal{T}_1(u,v) = \{ \langle f_1, f_2, \ldots, f_{|u|} \rangle_u \mid f_i \in \mathcal{A}(u_i, v) \text{ for } i \in [|u|] \}$$

for all $u, v \in S^*$. Each of the following conditions

(SA) S1, S2 and S3

(SB) S1, S2 and S4

(SC) S1, S3, S5 and S6

(SD) S1, S4 and S6

is sufficient for $\mathcal{T}_1$ to be a rationally closed subtheory of $\mathcal{T}$.

Proof. We first prove the equivalence of conditions SA, SB, SC and SD.

S1 and S2 imply S5 : with the same notations as in S5 we remark that

$$\alpha \langle \beta, 1_u \rangle = \alpha \langle \beta, x_1^u, x_2^u, \ldots, x_{|u|}^u \rangle .$$

S1 and S2 imply S6 : with the same notations as in S6 we remark that

$$\alpha \langle x_{(1)}^{uwv}, x_{(3)}^{uwv} \rangle = \alpha \langle x_1^{uwv}, \ldots, x_{|u|}^{uwv}, x_{|uw|+1}^{uwv}, \ldots, x_{|uw|+|v|}^{uwv} \rangle .$$

S1 and S4 imply S3 : with the same notations as in S3 we remark that

$$\alpha^\dagger = x_1^{sv} \langle \alpha^\dagger, 1_v \rangle .$$

S3 and S5 imply S4 : obvious.

S4 and S6 imply S5 : with the same notations as in S5 we remark that

$$\alpha \langle \beta, 1_u \rangle = \alpha \langle (\beta\, x_{(2)}^{su})^\dagger \alpha, 1_v \rangle$$

S5 and S6 imply S2 : with the same notations as in S2 we remark that

$f \langle f_1, \ldots, f_{|u|} \rangle =$

$$= (f\, x_{(1)}^{uv}) \langle f_1\, x_{(2)}^{(u_2 \ldots u_{|u|})v}, 1_{u_2 \ldots u_{|u|} v} \rangle \ldots \langle f_{|u|}, 1_v \rangle .$$

The conclusion follows from S1, S2, S3, S4 and proposition 8 □

The last proposition may be used to obtain the least rationally closed subtheory of $\mathcal{T}$ which includes a given set of morphisms from $\mathcal{T}$. Of course, each given morphism $f : u \longrightarrow v$ is replaced by the set of its components $\{x_i^u f \mid i \in [|u|]\}$ then one of the four conditions of proposition 9 may be used.

Condition SD seems to be preferable because if $\mathcal{A}$ has properties S1 and S6 then the least family of subsets which contains $\mathcal{A}$ and has property S4 will have properties S1 and S6 too. For S6, the proof is by induction with respect to the number of applications of rule S4. Indeed for $s, s' \in S$, $u, v, w \in S^*$, $\alpha : s' \longrightarrow s'uv$ and $\beta : s \longrightarrow s'uv$ we notice that

$$\beta \langle \alpha^{\dagger}, 1_{uv} \rangle \langle x_{(1)}^{uwv}, x_{(3)}^{uwv} \rangle =$$

$$= \beta \langle \alpha^{\dagger} \langle x_{(1)}^{uwv}, x_{(3)}^{uwv} \rangle, \langle x_{(1)}^{uwv}, x_{(3)}^{uwv} \rangle \rangle =$$

$$= \beta (1_{s'} + \langle x_{(1)}^{uwv}, x_{(3)}^{uwv} \rangle) \langle (\alpha (1_{s'} + \langle x_{(1)}^{uwv}, x_{(3)}^{uwv} \rangle))^{\dagger}, 1_{uwv} \rangle =$$

$$= (\beta \langle x_{(1)}^{(s'u)wv}, x_{(3)}^{(s'u)wv} \rangle) \langle (\alpha \langle x_{(1)}^{(s'u)wv}, x_{(3)}^{(s'u)wv} \rangle)^{\dagger}, 1_{uwv} \rangle .$$

This remark may be used when $\mathcal{A}(s,t) = \mathcal{T}_1(s,u)$ where $\mathcal{T}_1$ is an algebraic subtheory of $\mathcal{T}$ because in this case $\mathcal{A}$ has properties S1 and S6.

Another way to obtain the least rationally closed subtheory of $\mathcal{T}$ which includes a given set of morphisms, is given in the following proposition which seems to be close to the main result of [[4]].

10. Proposition. Assume $\mathcal{A}(s,u) \subseteq \mathcal{T}(s,u)$ for all $(s,u) \in S \times S^*$. We assume that $\mathcal{A}$ is closed under right composition with morphisms from the least algebraic subtheory of $\mathcal{T}$. Let

$$\mathcal{R}(s,u) = \{ x_1^{sv} \alpha^{\dagger} \mid \alpha \in \mathcal{T}(sv, svu),\ (\forall i \in [|sv|])(x_i^{sv} \alpha \in \mathcal{A}((sv)_i, svu) \}$$

for all $(s,u) \in S \times S^*$. The family $\mathcal{R}$ contains $\mathcal{A}$ and has properties S2 and S3.

<u>Proof</u>. If $(s,u) \in S \times S^*$ and $\alpha \in \mathcal{A}(s,u)$ it follows from $\alpha = (\alpha\, x^{su}_{(2)})^{\dagger}$ that $\alpha \in \mathcal{R}(s,u)$, therefore $\mathcal{R}$ contains $\mathcal{A}$.

S2. Assume that $\beta \in \mathcal{R}(s,u)$ where $(s,u) \in S \times S^*$. Then there exists $\alpha \in \mathcal{T}(sw, swu)$ such that $w \in S^*$,

$$\beta = x^{sw}_1\, \alpha^{\dagger}$$

and $x^{sw}_i\, \alpha \in \mathcal{A}((sw)_i,\, swu)$ for all $i \in [|sw|]$. For each $i \in [|u|]$ assume that $\beta_i \in \mathcal{R}(u_i, v)$ where $v \in S^*$. Then for all $i \in [|u|]$ there exist

$$\alpha_i \in \mathcal{T}(u_i w^i,\, u_i w^i v) \text{ such that } w^i \in S^*,$$

$$\beta_i = x^{u_i w^i}_1\, \alpha_i^{\dagger}$$

and $x^{u_i w^i}_j\, \alpha_i \in \mathcal{A}((u_i w^i)_j,\, u_i w^i v)$ for all $j \in [|u_i w^i|]$.

Let $r = u_1 w^1 u_2 w^2 \ldots u_{|u|} w^{|u|}$,

$$\gamma = \langle x^r_{(1)}, x^r_{(3)}, \ldots, x^r_{(2|u|-1)} \rangle : u \longrightarrow r$$

and for each $i \in [|u|]$

$$\gamma_i = \langle x^r_{(2i-1)}, x^r_{(2i)} \rangle : u_i w^i \longrightarrow r.$$

Let

$$\delta = \langle \alpha(1_{sw} + \gamma + 0_v), \alpha_1(0_{sw} + \gamma_1 + 1_v), \ldots, \alpha_{|u|}(0_{sw} + \gamma_{|u|} + 1_v) \rangle .$$

We notice that $x^{swr}_i\, \delta \in \mathcal{A}((swr)_i,\, swrv)$ for all $i \in [|srw|]$. and that

$$\delta^{\dagger} = (\alpha(1_{sw} + \gamma + 0_v))^{\dagger} \langle \varepsilon^{\dagger}, 1_v \rangle, \varepsilon^{\dagger} \rangle$$

where

$$\varepsilon = \langle \alpha_1(0_{sw} + \gamma_1 + 1_v), \ldots, \alpha_{|u|}(0_{sw} + \gamma_{|u|} + 1_v) \rangle \langle \alpha^{\dagger}(\gamma + 0_v), 1_{rv} \rangle =$$

$$= \langle \alpha_1(\gamma_1 + 1_v), \ldots, \alpha_{|u|}(\gamma_{|u|} + 1_v) \rangle .$$

It follows from proposition 7 that

$$\varepsilon^{\dagger} = \langle \alpha_1^{\dagger}, \alpha^{\dagger}_2, \ldots, \alpha^{\dagger}_{|u|} \rangle$$

therefore

$$x_1^{swr} \delta^{\dagger} = x_1^{sw} \alpha^{\dagger} (\gamma + 0_v) \langle \alpha_1^{\dagger}, \ldots, \alpha^{\dagger}_{|u|}, 1_v \rangle =$$

$$= \beta \gamma \langle \alpha_1^{\dagger}, \ldots, \alpha^{\dagger}_{|u|} \rangle$$

$$= \beta \langle x^r_{(1)} \langle \alpha^{\dagger}_1, \ldots, \alpha^{\dagger}_{|u|} \rangle, \ldots, x^r_{(2|u|-1)} \langle \alpha^{\dagger}_1, \ldots, \alpha^{\dagger}_{|u|} \rangle =$$

$$= \beta \langle x_1^{u_1 w^1} \alpha^{\dagger}_1, \ldots, x_1^{u_{|u|} w^{|u|}} \alpha^{\dagger}_{|u|} \rangle =$$

$$= \beta \langle \beta_1, \beta_2, \ldots, \beta_{|u|} \rangle \in \mathcal{R}(s,v).$$

S3. Assume that $\beta \in \mathcal{R}(s,sv)$ where $s \in S$ and $v \in S^*$. Then there exists $\alpha \in \mathcal{T}(sw, swsv)$ such that $w \in S^*$,

$$\beta = x_1^{sw} \alpha^{\dagger}$$

and $x_i^{sw} \alpha \in \mathcal{R}((sw)_i, swsv)$ for all $i \in [|sw|]$.

Let

$$\gamma = \alpha \langle x_1^{swv}, x_{(2)}^{swv}, x_1^{swv}, x_{(3)}^{swv} \rangle .$$

It is sufficient to prove that $\beta^{\dagger} = x_1^{sw} \gamma^{\dagger}$ and for this we shall show that

$$\langle \beta^{\dagger}, x^{sw}_{(2)} \alpha^{\dagger} \langle \beta^{\dagger}, 1_v \rangle \rangle$$

is the least solution for x in the equation

$$\gamma \langle x, 1_v \rangle = x.$$

We first prove that it is a solution

$$\gamma \langle \langle \beta^{\dagger}, x^{sw}_{(2)} \alpha^{\dagger} \langle \beta^{\dagger}, 1_v \rangle \rangle, 1_v \rangle =$$

$$= \alpha \langle \beta^{\dagger}, x^{sw}_{(2)} \alpha^{\dagger} \cancel{\langle \beta^{\dagger}, 1_v \rangle}, \beta^{\dagger}, 1_v \rangle =$$

$$= \alpha \langle \beta, x^{sw}_{(2)} \alpha^{\dagger}, 1_{sv} \rangle \langle \beta^{\dagger}, 1_v \rangle =$$

$$= \alpha \langle \alpha^{\dagger}, 1_{sv} \rangle \langle \beta^{\dagger}, 1_v \rangle =$$

$$= \alpha^{\dagger} \langle \beta^{\dagger}, 1_v \rangle =$$

$$= \langle \beta, x^{sw}_{(2)} \alpha^{\dagger} \rangle \langle \beta^{\dagger}, 1_v \rangle =$$

$$= \langle \beta^{\dagger}, x^{sw}_{(2)} \alpha^{\dagger} \langle \beta^{\dagger}, 1_v \rangle\rangle .$$

Assume $a : s \longrightarrow v$, $b : w \longrightarrow v$ and

$$\gamma \langle \langle a,b \rangle , 1_v \rangle = \langle a,b \rangle .$$

It follows that

$$\alpha \langle\langle a,b \rangle , \langle a, 1_v \rangle\rangle = \langle a,b \rangle ,$$

therefore, ussing proposition 2, we obtain

$$\alpha^{\dagger} \langle a, 1_v \rangle \leq \langle a,b \rangle$$

Since

$$\beta \langle a, 1_v \rangle = x^{sw}_{1} \alpha^{\dagger} \langle a, 1_v \rangle \leq a$$

it follows that $\beta^{\dagger} \leq a$.

Since $\beta^{\dagger} \leq a$ and

$$x^{sw}_{(2)} \alpha^{\dagger} \langle \beta^{\dagger}, 1_v \rangle \leq x^{sw}_{(2)} \alpha^{\dagger} \langle a, 1_v \rangle \leq b$$

it follows that $\langle \beta^{\dagger}, x^{sw}_{(2)} \alpha^{\dagger} \langle \beta^{\dagger}, 1_v \rangle\rangle$ is the least solution for x in the equation $\gamma \langle x, 1_v \rangle = x$ □

§ 2. The rationally closed subtheory of context-free trees

If $\mathcal{T}$ and $\mathcal{T}'$ are S-worted algebraic theories, a morphism $F : \mathcal{T} \longrightarrow \mathcal{T}'$ is a functor satisfying the following conditions:

a) $F(u) = u$ for all $u \in S^*$,

b) $F(x^u_i) = x^u_i$ for all $u \in S^*$ and $i \in [|u|]$.

We notice that if $\alpha : u \longrightarrow v$ and $\beta : w \longrightarrow v$ are morphisms in $\mathcal{T}$ then $F(\langle \alpha , \beta \rangle) = \langle F(\alpha), F(\beta) \rangle$.

Let Σ be a set and let $a : \Sigma \longrightarrow S \times S^*$ be a function. If $\mathcal{T}$ is an S-sorted algebraic theory, $h : \Sigma \longrightarrow \mathcal{T}$ is an ranked alphabet map iff $h(\sigma) \in \mathcal{T}(a(\sigma))$ for all $\sigma \in \Sigma$. Let T_Σ denote the free S-sorted algebraic theory generated by Σ and $I_\Sigma : \Sigma \longrightarrow T_\Sigma$ be its ranked alphabet map, therefore for each S-sorted algebraic theory $\mathcal{T}$ and for each ranked alphabet map $h : \Sigma \longrightarrow \mathcal{T}$ there exists a unique theory morphism $\bar{h} : T_\Sigma \longrightarrow \mathcal{T}$ such that $I_\Sigma \bar{h} = h$, i.e. $\bar{h}(I_\Sigma(\sigma)) = h(\sigma)$ for all $\sigma \in \Sigma$.

Let $\mathcal{T}$ and $\mathcal{T}'$ be ordered S-sorted algebraic theories. A morphism of ordered S-sorted algebraic theories is a theory morphism $F : \mathcal{T} \longrightarrow \mathcal{T}'$ such that for all $u,v \in S^*$ the restriction of F to $\mathcal{T}(u,v)$ is a monotonic and strict ($F(\perp_{uv}) = \perp_{uv}$) function.

An ω-continuous S-sorted algebraic theory $\mathcal{T}$ is an ordered S-sorted algebraic theory satisfying the following conditions:

a) for each $u,v \in S^*$, $\mathcal{T}(u,v)$ is ω-complete, i.e., each ω-chain has a least upper bound,

b) the composition of morphisms is ω-continuous, i.e., the composition preserves least upper bounds of ω-chains.

Any ω-continuous S-sorted algebraic theory is a rationally closed S-sorted algebraic theory.

Let $\mathcal{T}$ and $\mathcal{T}'$ be ω-continuous S-sorted algebraic theories. A morphism of ω-continuous S-sorted algebraic theories is an ordered theory morphism $F : \mathcal{T} \longrightarrow \mathcal{T}'$ such that for all $u,v \in S^*$ the restriction of F to $\mathcal{T}(u,v)$ is an ω-continuous function. We consider the order-properties of the set of ω-continuous theory morphisms from $\mathcal{T}$ to $\mathcal{T}'$. The ordering is the natural componentwise ordering : for $F, G : \mathcal{T} \longrightarrow \mathcal{T}'$,

$$F \leq G \text{ iff } F(\alpha) \leq G(\alpha) \text{ in } \mathcal{T}'(u,v), \quad \alpha \in \mathcal{T}(u,v).$$

This ordering is ω-complete. Indeed if $\{F_n\}_{n\in\omega}$ is an ω-chain and for $u,v \in S^*$ and $\alpha : u \longrightarrow v$,

$$F(\alpha) = \bigvee\{F_n(\alpha) \mid n \in \omega\}$$

then F is an ω-continuous theory morphism.

Let CT_Σ denote the free ω-continuous S-sorted algebraic theory and let $J_\Sigma : \Sigma \longrightarrow CT_\Sigma$ its ranked alphabet map. Without loss of generality we assume that T_Σ is an algebraic subtheory of CT_Σ and that I_Σ is the corestriction of J_Σ.

For each ranked alphabet map $f : \Sigma \longrightarrow \mathcal{T}$, where $\mathcal{T}$ is an ω-continuous S-sorted algebraic theory, we denote by $f^{\#} : CT_\Sigma \longrightarrow \mathcal{T}$ the unique ω-continuous theory morphism such that $J_\Sigma f^{\#} = f$. If we order componentwise the set of ranked alphabet maps from Σ to $\mathcal{T}$ then the application "$\#$" is an isomorphism of partially ordered sets.

We define an ω-continuous $S\times S^*$-sorted algebraic theory $\overline{T}$ which is used for solving systems of context-free equations. Letters p,q and r will denote elements of $(S\times S^*)^*$.

For all p let

$$\Sigma_p = \{\sigma_1^p, \sigma_2^p, \ldots, \sigma_{|p|}^p\}$$

and let

$$a_p : \Sigma_p \longrightarrow S\times S^*$$

be the function defined by

$$a_p(\sigma_i^p) = p_i$$

for all $i \in [|p|]$.

For all p, q let $\overline{T}(p,q)$ be the set of ranked alphabet map from Σ_p to CT_{Σ_q}. The set $\overline{T}(p,q)$ is ordered componentwise, i.e., for all $f, g \in \overline{T}(p,q)$: $f \leq g$ iff $f(\sigma_i^p) \leq g(\sigma_i^q)$ in $CT_{\Sigma_q}(p_i)$ for all $i \in [|p|]$. This ordering is ω-complete and $\overline{T}(p,q)$ has a least element $\perp_{pq}$.

The composition of morphisms is defined by

$$fg = fg^{\#}$$

where $f \in \overline{T}(p,q)$ and $g \in \overline{T}(g,r)$. The composition is associative and for each p the morphism $1_p = J_{\Sigma_p}$ is an identity. The composition is ω-continuous and $\perp_{pq}\, g = \perp_{pr}$ for all $g \in \overline{T}(q,r)$.

For all p and $i \in [|p|]$ let

$$y_i^p : p_i \longrightarrow p$$

be the morphism defined by

$$y_i^p(\sigma_1^{p_i}) = J_{\Sigma_p}(\sigma_i^p).$$

If $|p| = 1$ then $y_1^p = 1_p$.

The morphisms y_i^p will be the distinguished morphisms of the theory $\overline{T}$. If $i \in [|p|]$ and $\beta_i : p_i \longrightarrow q$ then the morphism

$$\langle \beta_1, \beta_2, \ldots, \beta_{|p|} \rangle : p \longrightarrow q$$

defined by

$$\langle \beta_1, \ldots, \beta_{|p|} \rangle \;(\sigma_i^p) = \beta_i(\sigma_1^{p_i}), \qquad i \in [|p|]$$

is the unique morphism from p to q such that

$$y_i^p \langle \beta_1, \ldots, \beta_{|p|} \rangle = \beta_i$$

for all $i \in [|p|]$. It is easy to see that the tupling operation is monotonic.

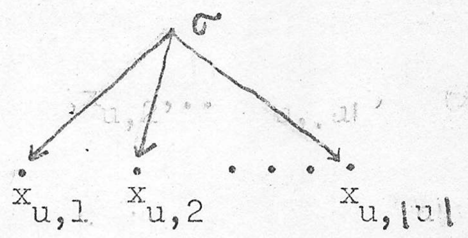
For all p,q let $\overline{T}_1(p,q)$ be the set of randked alphabet maps from Σ_p to CT_{Σ_q} such that $f(\sigma_i^p) \in T_{\Sigma_q}(p_i)$ for all

$i \in [|p|]$. It is easy to see that $\overline{T}_1$ in an algebraic subtheory of $\overline{T}$.

Let Alg be the least rationally closed subtheory of $\overline{T}$ which contains $\overline{T}_1$. We shall give in this special case a better construction of Alg then the general one presented in the first section.

We shall begin with some intuitive explanations and some notation.

For $s \in S$ and $u \in S^*$, the element of $CT_{\Sigma}(s,u)$ are Σ-trees (trees with symbols of operations from Σ) of sort s and with $|u|$ variables $x_{u,1}, x_{u,2}, \ldots, x_{u,|u|}$ of sorts $u_1, u_2, \ldots, u_{|u|}$. The subset $T_{\Sigma}(s,u)$ contains all the total finite trees of the same kind. If $u \in S^*$ and $i \in [|u|]$, the Σ-tree $x_i^u \in T_{\Sigma}(u_i,u)$ is equal to the variable $x_{u,i}$. If $\sigma \in \Sigma$ and $a(\sigma) = (s,u)$ then $I_{\Sigma}(\sigma) = J_{\Sigma}(\sigma)$ is the Σ - tree

$\sigma(x_{u,1}, x_{u,2}, \ldots, x_{u,|u|}) =$ 

If $(s,u) \in S \times S^*$, an element of $\overline{T}((s,u),p)$ may be identified with an element a of $CT_{\Sigma_p}(s,u)$, i.e. with a Σ_p-tree of sort s and with variables $x_{u,1}, x_{u,2}, \ldots, x_{u,|u|}$; but it is perhaps better to think of it as the equality

$$\sigma_1^{(s,u)}(x_{u,1}, x_{u,2}, \ldots, x_{u,|u|}) = a$$

which gives a definition of the operation symbol $\sigma_1^{(s,u)}$ with the aid of the Σ_p-tree a, i.e. in terms of the operation symbols of Σ_p .

A morphism $f \in \overline{T}(p,q)$ gives then a definition of the operation symbols of $\sum_p$ by the operation symbols of $\sum_q$. In this way a system of context-free equations is a morphism

$f \in \overline{T}(p,pq)$ where $\sigma_1^{pq},\dots,\sigma_{|p|}^{pq}$ are unknown operation symbols and $\sigma_{|p|+1}^{pq},\dots,\sigma_{|p|+|q|}^{pq}$ are known operation symbols, because when we calculate $f^{\dagger}$ the operation symbols $\sigma_1^{pq},\dots,\sigma_{|p|}^{pq}$ are identified with operation symbols $\sigma_1^{p},\dots,\sigma_{|p|}^{p}$.

For each p, each $u \in S^*$ and each $i \in [|u|]$ let

$$x_i^{p,u} : (u_i,u) \longrightarrow p$$

be the morphism of $\overline{T}_1$ defined by

$$x_i^{p,u}(\sigma_1^{(u_i,u)}) = x_i^u$$

The morphism $x_i^{p,u}$ shows that the operation $\sigma_1^{(u_i,u)}$ is the i-th projection.

For each $u \in S^*$ and each $t = (s, s_1s_2\dots s_n) \in S \times S^*$ let

$$r = (s_1,u)(s_2,u)\dots(s_n,u)t$$

and let

$$M_t^u : (s,u) \longrightarrow r$$

be the morphism of $\overline{T}_1$ defined by

$$M_t^u(\sigma_1^{(s,u)}) = J_{\sum_r}(\sigma_{n+1}^r)\langle J_{\sum_r}(\sigma_1^r),\dots$$

$$\dots, J_{\sum_r}(\sigma_n^r)\rangle .$$

The above equality shows that

$$\sigma_1^{(s,u)}(x_{u,1}, x_{u,2},\dots,x_{u,|u|}) =$$

$$=\sigma_{n+1}^r(\sigma_1^r(x_{u,1},\dots,x_{u,|u|}),\dots,\sigma_n^r(x_{u,1},\dots,x_{u|u|})).$$

For each $t \in S \times S^*$ and each p we assume $\mathcal{R}(t,p) \subseteq \overline{T}(t,p)$

and we list some conditions on the family $\mathcal{R}$:

(1) $x_i^{p,u} \in \mathcal{R}((u_i,u),p)$ for all $u \in S^*$, $i \in [|u|]$ and p,

(2) $y_i^p \in \mathcal{R}(p_i,p)$ for all p and $i \in [|p|]$,

(3) for each $u \in S^*$, $t = (s,s_1s_2\ldots s_n) \in S \times S^*$ and p if

$$\alpha \in \mathcal{R}(t,p)$$

and $\alpha_i \in \mathcal{R}((s_i,u),p)$ for all $i \in [n]$

then $M_t^u\langle\alpha_1,\alpha_2,\ldots,\alpha_n,\alpha\rangle \in \mathcal{R}((s,u), p)$,

(4) for each $u \in S^*$, $t = (s,s_1s_2\ldots s_n) \in S \times S^*$ and p if

$$\alpha \in \mathcal{R}(t, tp)$$

and $\alpha_i \in \mathcal{R}((s_i,u), p)$ for all $i \in [n]$

then $M_t^u\langle\alpha_1,\alpha_2,\ldots,\alpha_n, \alpha^\dagger\rangle \in \mathcal{R}((s,u), p)$,

(5) for each $u \in S^*$, p and $i \in [|p|]$ if $p_i = (s,s_1s_2\ldots s_n)$ and if

$$\alpha_j \in \mathcal{R}((s_j,u),p) \text{ for all } j \in [n]$$

then $M_{p_i}^u\langle\alpha_1,\alpha_2,\ldots,\alpha_n, y_i^p\rangle \in \mathcal{R}((s,u),p)$,

(6) for each p and each $t \in S \times S^*$

if $\alpha \in \mathrm{R}(t,tp)$ then $\alpha^\dagger \in \mathrm{R}(t,p)$, $\mathcal{R} = \mathrm{R}$

(7) for each p and each $t,t' \in S \times S^*$

if $\alpha \in \mathrm{R}(t',tp)$ and $\beta \in \mathrm{R}(t,p)$

then $\alpha\langle\beta, 1_p\rangle \in \mathrm{R}(t',p)$,

(8) for each $t \in S \times S^*$ and each p,q

if $\alpha \in \mathrm{R}(t,p)$ then $\alpha\, y_{(2)}^{qp} \in \mathrm{R}(t,qp)$.

We first notice that 2 = S1, 6 = S3, 7 = S5 and that S6 implies 8.

Then we give same intuitive explanations, where the element of Ⓡ(t,p) are thought as $\sum_p$ -trees.

Condition 1 says that Ⓡ contains the variables.

Condition 2 says that for each p and $i \in [|p|]$ if $p_i = (s,u)$ then Ⓡ(p_i,p) contains $\sigma_i^p(x_{u,1}, x_{u,2}, \dots, x_{u,|u|})$.

Condition 3 says that Ⓡ is closed under substitution of variables.

Condition 4 says that Ⓡ is closed under substitution of variables in an iteration.

Condition 5 says that Ⓡ is closed under the algebraic operations.

Condition 7 says that Ⓡ is closed under substitution into a tree of an operation symbol defined by another tree.

Theorem. The family

$Alg(t,p)$ where $t \in S \times S^*$ and $p \in (S \times S^*)^*$ is equal to the least family satisfying everyone of the following conditions:

A) 1, 2, 3 and 6,

B) 1, 4 and 5,

C) 1, 5, 6 and 7,

D) 1, 2 and 4.

Proof. I. For each $s \in S$, $u \in S^*$, p and $\alpha \in \overline{T}((s,u),p)$

$$(9) \quad M^u_{(s,u)} \langle x_1^{p,u}, x_2^{p,u}, \dots, x_{|u|}^{p,u}, \alpha \rangle = \alpha.$$

Indeed, if $r = (u_1,u)(u_2,u)\dots(u_{|u|},u)(s,u)$ then

$$(M^u_{(s,u)} \langle x_1^{p,u}, x_2^{p,u}, \dots, x_{|u|}^{p,u}, \alpha \rangle)(\sigma_1^{(s,u)}) =$$

$$= \langle x_1^{p,u}, x_2^{p,u}, \dots, x_{|u|}^{p,u}, \alpha \rangle^{\#} (M^u_{(s,u)}(\sigma_1^{(s,u)})) =$$

$$= \langle x_1^{p,u}, \ldots, x_u^{p,u}, \alpha \rangle^{\#} \left(J_{\Sigma_r}(\sigma^r_{|u|+1}) \langle J_{\Sigma_r}(\sigma_1^r), \ldots, J_{\Sigma_r}(\sigma^r_{|u|}) \rangle\right) =$$

$$= \alpha(\sigma_1^{(s,u)}) \langle x_1^{p,u}(\sigma_1^{(u_1,u)}), x_2^{p,u}(\sigma_1^{(u_2,u)}), \ldots, x_{|u|}^{p,u}(\sigma_{|u|}^{(u_{|u|},u)}) \rangle =$$

$$= \alpha(\sigma_1^{(s,u)}) \langle x_1^u, x_2^u, \ldots, x_{|u|}^u \rangle = \alpha(\sigma_1^{(s,u)}).$$

II. We prove same implications between the previous eight conditions.

a) <u>1 and 5 imply 2</u> : from 9 with y_i^p for α.

b) <u>2 and 4 imply 5</u> : since $y_i^p = (y_{i+1}^{p_i p})^{\dagger}$

c) <u>2 and 3 imply 5</u>: obvious.

d) <u>3 and 6 imply 4</u>: obvious.

e) <u>1 and 4 imply 6</u> : with the same notation as in 6 and with $t = (s,u)$ it follows from 9 that

$$\alpha^{\dagger} = M^u_{(s,u)} \langle x_1^{p,u}, x_2^{p,u}, \ldots, x_{|u|}^{p,u}, \alpha^{\dagger} \rangle .$$

f) <u>4 and 8 imply 3</u> : with the same notation as in 3 it follows from $\alpha = (\alpha\, y^{tp}_{(2)})^{+}$ that

$$M_t^u \langle \alpha_1, \alpha_2, \ldots, \alpha_n, \alpha \rangle = M_t^u \langle \alpha_1, \ldots, \alpha_n, (\alpha\, y^{tp}_{(2)})^{\dagger} \rangle .$$

g) <u>5,7 and 8 imply 3</u> : with the same notation as in 3 it suffices to notice the equality

$$M_t^u \langle \alpha_1, \ldots, \alpha_n, \alpha \rangle = M_t^u \langle \alpha_1\, y^{tp}_{(2)}, \ldots, \alpha_n\, y^{tp}_{(2)}, y_1^{tp} \rangle \langle \alpha, 1_p \rangle .$$

III. It follows from a) and b) that conditions B and D are equivalent.

Let $\mathcal{A}$ be the least family satisfying condition A, $\mathcal{B}$ the least family satisfying condition B and $\mathcal{C}$ the least family satisfying condition C.

It follows from d) that A implies D, therefore A implies B then for each $t \in S \times S^*$ and each p

$$\mathcal{B}(t,p) \subseteq \mathcal{A}(t,p) .$$

It is easy to show that the family Alg fulfils condition C, therefore for each p and each $t \in S \times S^*$

$$\mathcal{C}(t,p) \subseteq \mathrm{Alg}\,(t,p).$$

IV. We prove that

$$\mathrm{Alg}(t,p) \subseteq \mathcal{B}(t,p)$$

for all p and $t \in S \times S^*$.

It is known that for each $u \in S^*$

$$(\{T_{\Sigma_p}(s,u)\}_{s \in S}, \{\sigma_i\}_{i \in [|p|]})$$

is a free Σ_p-algebra generated by $\{x_i^u \mid i \in [|u|]\}$ where for all $i \in [|p|]$ if $p_i = (s, s_1 s_2 \ldots s_n)$ and $h_j \in T_{\Sigma_p}(s_j,u)$, $j \in [n]$ then

$$\sigma_i(h_1,h_2,\ldots,h_n) = I_{\Sigma_p}(\sigma_i^p)\langle h_1,h_2,\ldots,h_n\rangle$$

Since there exists a natural bijection between $T_{\Sigma_p}(s,u)$ and $\overline{T}_1((s,u),p)$ it follows that

$$(\{\overline{T}_1((s,u),p)\}_{s \in S}, \{\sigma_i\}_{i \in [|p|]})$$

is a free Σ_p-algebra generated by $\{x_i^{p,u} \mid i \in [|u|]\}$ where for all $i \in [|p|]$ if $p_i = (s, s_1 s_2 \ldots s_n)$ and $f_j \in \overline{T}_1((s_j,u),p)$, $j \in [n]$ then

$$\sigma_i(f_1,\ldots,f_n) = f$$

if and only if

$$\sigma_i(f_i(\sigma_1^{(s_1,u)}), f_2(\sigma_1^{(s_2,u)}),\ldots,f_n(\sigma_1^{(s_n,u)})) = f(\sigma_1^{(s,u)}).$$

The following calculation, where $r = (s_1,u)(s_2,u)\ldots(s_n,u)p_i$

$$(M_{p_i}^u\langle f_1,f_2,\ldots,f_n,y_i^p\rangle)(\sigma_1^{(s,u)}) =$$

$$= \langle f_1,f_2,\ldots,f_n, y_i^p\rangle^{\#}(M_{p_i}^u(\sigma_1^{(s,u)})) =$$

$$= \langle f_1, f_2, \ldots, f_n, y_i^p \rangle^{\#} \left(J_{\Sigma_r}(\sigma_{n+1}^r)\langle J_{\Sigma_r}(\sigma_1^r), \ldots, J_{\Sigma_r}(\sigma_n^r)\rangle\right) =$$

$$= y_i^p(\sigma_1^{p_i})\langle f_1(\sigma_1^{(s_1,u)}), \ldots, f_n(\sigma_1^{(s_n,u)})\rangle =$$

$$= I_{\Sigma_p}(\sigma_i^p)\langle f_1(\sigma_1^{(s_1,u)}), \ldots, f_n(\sigma_1^{(s_n,u)})\rangle =$$

$$= \sigma_i(f_1(\sigma_1^{(s_1,u)}), \ldots, f_n(\sigma_1^{(s_n,u)}))$$

shows that

$$\sigma_i(f_1, f_2, \ldots, f_n) = M_{p_i}^u \langle f_1, f_2, \ldots, f_n, y_i^p \rangle .$$

As the family $\mathcal{B}$ fulfils conditions 1 and 5 it follows that

$$\overline{T}_1(t,p) \subseteq \mathcal{B}(t,p)$$

for all $t \in S \times S^*$ and p .

It follows from a) and e) that $\mathcal{B}$ fulfils conditions 2 and 6, i.e. S1 and S3. We shall prove that $\mathcal{B}$ fulfils conditio S2 as well.

We first prove by induction that $\mathcal{B}$ fulfils condition S6, i.e., with $y = \langle y_{(1)}^{qp'r}, y_{(3)}^{qp'r} \rangle$,

$$\gamma \in \mathcal{B}(t', qr) \quad \text{implies} \quad \gamma y \in \mathcal{B}(t', qp'r)$$

for all $t' \in S \times S^*$ and $q, p', r \in (S \times S^*)^*$.

If $\gamma = x_i^{qr,u}$ where $u \in S^*$, $i \in [|u|]$ and $t' = (u_i, u)$ then

$$\gamma y = x_i^{qp'r,u} \in \mathcal{B}(t', qp'r).$$

With the same notation as in 4 where $p = qr$ and $t' = (s,u)$ if

$$\gamma = M_t^u \langle \alpha_1, \alpha_2, \ldots, \alpha_n, \alpha^{\dagger} \rangle$$

then

$$\gamma y = M^u_t \langle \alpha_1 y, \alpha_2 y, \ldots, \alpha_n y, \alpha^{+} y \rangle =$$

$$= M^u_t \langle \alpha_1 y, \alpha_2 y, \ldots, \alpha_n y, (\alpha(1_t + y))^{+} =$$

$$= M^u_t \langle \alpha_1 y, \alpha_2 y, \ldots, \alpha_n y, (\alpha \langle y^{(tq)p'r}_{(1)}, y^{(tq)p'r}_{(3)} \rangle)^{+} \rangle$$

therefore, since by the inductive hypothesis

$$\alpha_i y \in \mathcal{B}((s_i, u), qp'r)$$

and $\alpha \langle y^{(tq)p'r}_{(1)}, y^{(tq)p'r}_{(3)} \rangle \in \mathcal{B}(t, tqp'r)$

it follows that $\gamma y \in \mathcal{B}(t', qp'r)$.

With the same notation as in 5 where $t' = (s,u)$ and $p = qr$ if

$$\gamma = M^u_{p_i} \langle \alpha_1, \alpha_2, \ldots, \alpha_n, y^p_i \rangle$$

then

$$\gamma y = M^u_{p_i} \langle \alpha_1 y, \alpha_2 y, \ldots, \alpha_n y, y^p_i y \rangle =$$

$$= \begin{cases} M^u_{q_i} \langle \alpha_1 y, \alpha_2 y, \ldots, \alpha_n y, y^{qp'r}_i \rangle & \text{if } i \in [|q|] \\ M^u_{r_{i-|q|}} \langle \alpha_1 y, \alpha_2 y, \ldots, \alpha_n y, y^{qp'r}_{|p'|+i} & \text{if } |q| < i \leq |p| \end{cases}$$

therefore it follows from the inductive gypothesis that

$$\gamma y \in \mathcal{B}(t', qp'r).$$

As S6 implies 8, it follows from f) that $\mathcal{B}$ fulfils consition 3.

For technical reasons, we shall prove by induction on γ that $\mathcal{B}$ fulfils a stronger condition than S2, i.e. for each $t' \in S \times S^*$ and each $q, p', r \in (S \quad S^*)^*$ if

$$\gamma \in \mathcal{B}(t', qr)$$

and

$$\gamma_i \in \mathcal{B}(r_i, p') \quad \text{for} \quad i \in [|r|]$$

then

$$\gamma(1_q + \langle \gamma_1, \gamma_2, \ldots, \gamma_{|r|} \rangle) \in \mathcal{B}(t', qp').$$

Let $z = 1_q + \langle \gamma_1, \gamma_2, \ldots, \gamma_r \rangle \cdot$

If $\gamma = x_i^{qr,u}$ where $u \in S^*$, $i \in [|u|]$ and $t' = (u_i, u)$ then

$$\gamma z = x_i^{qp',u} \in \mathcal{B}(t', qp').$$

With the same notation as in 4 where $p = qr$ and $t' = (s,u)$ if

$$\gamma = M_t^u \langle \alpha_1, \alpha_2, \ldots, \alpha_n, \alpha^\dagger \rangle$$

then

$$\gamma z = M_t^u \langle \alpha_1 z, \alpha_2 z, \ldots, \alpha_n z, (\alpha(1_t + z))^\dagger \rangle =$$

$$= M_t^u \langle \alpha_1 z, \alpha_2 z, \ldots, \alpha_n z, (\alpha(1_{tq} + \langle \gamma_1, \gamma_2, \ldots, \gamma_{|r|} \rangle))^\dagger$$

therefore it follows from the inductive hypothesis that

$$\gamma z \in \mathcal{B}(t', qp')$$

With the same notation as in 5 where $p = qr$ and $t' = (s,u)$ if

$$\gamma = M_{p_i}^u \langle \alpha_1, \alpha_2, \ldots, \alpha_n, y_i^p \rangle$$

then

$$\gamma z = M_{p_i}^u \langle \alpha_1 z, \alpha_2 z, \ldots, \alpha_n z, y_i^p z \rangle =$$

$$= \begin{cases} M_{q_i}^u \langle \alpha_1 z, \alpha_2 z, \ldots, \alpha_n z, y_i^{pq'} \rangle & \text{if } i \in [|q|] \\ M_{r_{i-|q|}}^u \langle \alpha_1 z, \alpha_2 z, \ldots, \alpha_n z, \gamma_{i-|q|} y_{(2)}^{qp'} \rangle & \text{if } |q| < i \leq |r| \end{cases}$$

If $i \in [|q|]$ the inductive hypothesis and 5 implies that

$$\gamma z \in \mathcal{B}(t', qp')$$

If $|q| < i \leq |p|$ then

$$\gamma z \in \mathcal{B}(t', qp')$$

by the inductive hypothesis, 8 and 3.

Since $\mathcal{B}$ fulfils conditions S1, S2 and S3 it follows from proposition 9 that

$$Alg(t,p) \subseteq \mathcal{B}(t,p)$$

for all $t \in S \times S^*$ and p.

V. We still have to show that

$$\mathcal{A}(t,p) \subseteq \mathcal{C}(t,p)$$

for all $t \in S \times S^*$ and p. We shall prove that $\mathcal{C}$ fulfils condition A.

It follows from a) that $\mathcal{C}$ fulfils 2.

We shall prove by induction that $\mathcal{C}$ fulfils S6. We shall use the same notation as in the similar proof for $\mathcal{B}$ and we shall omit the identical cases.

If $\gamma = \alpha^{\dagger}$ where $\alpha : t' \longrightarrow t'qr$ then

$$\gamma y = (\alpha(1_{t'} + y))^{\dagger} = (\alpha\langle y_{(1)}^{(t'q)p'r}, y_{(3)}^{(t'q)p'r}\rangle)^{\dagger}$$

therefore it follows from the inductive hypothesis that

$$\gamma y \in \mathcal{C}(t', qp'r)$$

With the same notation as in 7 where $p = qr$ if

$$\gamma = \alpha\langle \beta, 1_p\rangle$$

then

$$\gamma y = \alpha\langle \beta y, y\rangle =$$

$$= (\alpha\langle y_{(1)}^{(tq)p'r}, y_{(3)}^{(tq)p'r}\rangle)\langle \beta y, 1_{qp'r}\rangle$$

therefore it follows from the inductive hypothesis that

$$\gamma y \in \mathcal{C}(t', qp'r).$$

Since S6 implies 8 it follows from g) that $\mathcal{C}$ fulfils 3 □

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