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by

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HOLOMORPHIC 2-VECTOR BUNDLES ON 2-TORI WITH ALGEBRAIC DIMENSION ZERO

Vasile Brînzănescu and Paul Flondor

It is a classical result that on compact complex surfaces the topological complex 2-vector bundles are well-determined by their Chern classes c_1 and c_2, which can be arbitrary chosen (Wu [10]). It is known (see Schwarzenberger [7]) that on a projective surface every topological 2-vector bundle has a holomorphic structure iff its first Chern class c_1 is of the form $c_1(L)$ with L a holomorphic line bundle. In the analytic nonprojective case the result is no longer true, as for holomorphic vector bundles one has restrictions on c_2 (see Elencwajg-Forster [3], Prop.4.3 or Prop.1 in this paper).

This paper is concerned with the study of holomorphic structures on 2-vector bundles on tori with algebraic dimension zero.

In the first section we show that, for a 2-torus X with algebraic dimension $a(X)=0$ and the Neron-Severi group $NS(X)=0$, a topological vector bundle with Chern class c_2 ($c_1=0$ in this case) has a holomorphic structure iff $c_2 \geq 0$.

In the second section of the paper we prove that the quadratic intersection form on the Neron-Severi group NS(X) of a 2-torus X, with algebraic dimension $a(X)=0$, is negative definite (it is a classical result of Kodaira [4] that this quadratic form is negative semi-definite).

In the third section we give (by using the above result) necessary and sufficient conditions for an integer Δ to be the discriminant of a filtrable 2-vector bundle E with given first Chern class $c_1(E) \in NS(X)$, in the case of a 2-torus X with $a(X)=0$. Here filtrable means that E has a rank one coherent subsheaf. For algebraic surfaces every holomorphic bundle is filtrable. It is a remarkable fact that on nonalgebraic surfaces nonfiltrable bundles exist (see Elencwajg-Forster [3], Prop.4.9).

We wish to thank Constantin Bănică for introducing us to this subject and for some useful discussions during the preparation of this paper.

1. Two-vector bundles on nonalgebraic compact complex surfaces

Let X be a compact complex surface and let $a(X)=\dim_{\mathbb{C}} \mathbb{C}(X)$ be its algebraic dimension. Let E be a holomorphic vector bundle of rank r on X. E is called filtrable if there exists a filtration

$$0 = F_0 \subset F_1 \subset \ldots \subset F_r = E,$$

with F_i coherent subsheaf of rank i, $i=0,1,\ldots,r$.

For r=2 the vector bundle E is filtrable iff there exists an exact sequence

$$(1) \qquad 0 \longrightarrow L \longrightarrow E \longrightarrow M \otimes \mathcal{J}_Y \longrightarrow 0,$$

where L,M are holomorphic line bundles and Y is a locally complete intersection of codimension 2 in X or empty (see Elencwajg-Forster, [3]).

PROPOSITION 1. Let X be a nonalgebraic compact analytic surface and let $a \in NS(X)$ be fixed. Then, for every holomorphic 2-vector bundle E on X with $c_1(E)=a$, we have

$$c_2(E) \geqslant \min\left\{2\chi(X,\mathcal{O}_X)+\frac{1}{2}(c_1(X).a+a^2), \frac{1}{4}a^2\right\}.$$

Proof. The Riemann-Roch theorem gives

$$\chi(X,E)=2\chi(X,\mathcal{O}_X)+\frac{1}{2}(c_1(X).a+a^2)-c_2(E).$$

If E is nonfiltrable, then

$$H^0(X,E)=0,\ H^2(X,E)\cong H^0(X,E^*\otimes K_X)^*=0$$

(Elencwajg-Forster, [3]). It follows that

$$\chi(X,E)=-h^1(X,E)\leqslant 0,$$

hence

$$c_2(E)\geqslant 2\chi(X,\mathcal{O}_X)+\frac{1}{2}(c_1(X).a+a^2).$$

If E is filtrable we get from the exact sequence (1) that

$$a=c_1(L)+c_1(M)$$
$$c_2(E)=c_1(L)c_1(M)+\deg Y.$$

It follows that the discriminant

$$\Delta(E)=c_1(E)^2-4c_2(E)=(c_1(L)-c_1(M))^2-4\deg Y=$$
$$=c_1(L\otimes M^{-1})^2-4\deg Y.$$

By a classical result of Kodaira [4], on non-algebraic surfaces we have $c_1(L)^2 \leq 0$ for every $L \in \operatorname{Pic} X$, hence

$$\Delta(E) \leq 0,$$

or equivalently

$$c_2(E) \geq \frac{1}{4}a^2.$$

REMARK. If X is a 2-torus we get for a nonfiltrable bundle E that $c_2(E) \geq \frac{1}{2}c_1(E)^2$ and for a filtrable bundle E that $c_2(E) \geq \frac{1}{4}c_1(E)^2$. We do not know if there exist nonfiltrable bundles with

$$\frac{1}{2}c_1(E)^2 \leq c_2(E) < \frac{1}{4}c_1(E)^2, \quad (c_1^2(E) < 0).$$

The examples of nonfiltrable bundles given by Elencwajg-Forster, [3], have the same second Chern class as filtrable bundles.

COROLLARY 2. Let X be a nonalgebraic compact analytic surface. There exist on X topological 2-vector bundles E with holomorphic structure on det E, which do not have any holomorphic structure.

THEOREM 3. Let X be a 2-torus with the Neron-Severi group NS(X)=0 (then a(X)=0).

(i) A topological 2-vector bundle E on X has a holomorphic structure iff $c_2(E) \geq 0$.

(ii) The set of classes of isomorphism of simple, filtrable 2-vector bundles, with fixed second Chern class $c_2 > 0$, carries a natural structure of complex manifold of dimension

Proof. (i) The hypothesis $NS(X)=0$ implies that $c_1(L)=0$ for any $L\in \mathrm{Pic}\, X$. Then, for any holomorphic 2-vector bundle E on X we have

$$c_1(E)=c_1(\det E)=0.$$

Because $\chi(\mathcal{O}_X)=0$ and $c_1(X)=0$ it follows, by the Proposition 1, that $c_2(E)\geqslant 0$ in both cases (filtrable or nonfiltrable bundles).

Now, let E be a topological 2-vector bundle with holomorphic structure on det E and $c_2=c_2(E)\geqslant 0$. It follows that $c_1(E)=c_1(\det E)=0$. The case $c_2=0$ is obvious: by the result of Wu E is (topologically) trivial, hence has a holomorphic structure (the trivial one).

Now suppose that $c_2>0$. We can choose $L\in \mathrm{Pic}\, X$ with $L\not\cong\mathcal{O}_X$ ($\mathrm{Pic}\, X=\mathrm{Pic}_o\, X$ is a 2-torus). Let us take a locally complete intersection Y of codimension 2 in X with $\deg Y=c_2>0$, and consider the extensions

$$0\to L\to E'\to \mathcal{J}_Y\to 0.$$

By using the isomorphism $Hom(\mathcal{J}_Y,L)\cong L$ and the local duality isomorphism

$$Ext^1_{\mathcal{O}_X}(\mathcal{J}_Y,L)\cong H^o(Y,\mathcal{O}_Y),$$

the exact sequence of small terms in the Ext-spectral sequence becomes

$$0\to H^1(X,L)\to Ext^1_{\mathcal{O}_X}(\mathcal{J}_Y,L)\to H^o(Y,\mathcal{O}_Y)\to H^2(X,L).$$

But $H^2(X,L)\cong H^o(X,L^*)^*$. Since $L^*\not\cong\mathcal{O}_X$ it follows that $H^o(X,L^*)=0$.

Indeed, if $H^{o}(X,L^{*})\neq 0$ there exists a divisor D on X such that $L^{*}\cong \mathcal{O}_X(D)$. But X being a Kähler manifold with $\rho(X)=rk\ NS(X)=0$, it has no divisors, hence $L^{*}\cong \mathcal{O}_X$, contradiction.

Now take an element in $Ext^1_{\mathcal{O}_X}(\mathcal{J}_Y,L)$ which has the element 1 of

$$H^{o}(X,\ Ext^1)\cong H^{o}(Y,\mathcal{O}_Y)$$

as its image. By Serre [6], the corresponding extension

$$0\longrightarrow L\longrightarrow E'\longrightarrow \mathcal{J}_Y\longrightarrow 0$$

gives a holomorphic bundle E' with

$$c_2(E')=\deg Y=c_2.$$

Then E' is topologically isomorphic with E and the proof of (i) is finished.

(ii) We use essentially [3], Th.2.2. The simple, filtrable 2-vector bundles, with fixed second Chern class $c_2>0$, are given by the extensions

$$0\longrightarrow L\longrightarrow E\longrightarrow \mathcal{J}_Y\otimes M\longrightarrow 0,$$

where $L\not\cong M$ and Y is a locally complete intersection of dimension zero and length c_2. The extension is uniquely determined by E. As $H^1(X,L\otimes M^{*})=0$ (it follows from Riemann-Roch theorem, since $H^{o}(X,L\otimes M^{*})=H^2(X,L\otimes M^{*})=0$) it follows that

$$\dim Ext^1_{\mathcal{O}_X}(\mathcal{J}_Y\otimes M,L)=c_2.$$

Then we obtain the parametrization of the set of classes of isomorphism of simple, filtrable 2-vector bundles, with fixed Chern classes $c_1=0$, $c_2>0$ as follows: Consider the complex manifold

$$S = \mathrm{Pic}_o X \times \mathrm{Pic}_o X \times \mathbb{H},$$

where $\mathbb{H}$ is the Douady-space of locally complete intersections of dimension zero and length c_2 in X (by a result of Fogarty this is nonsingular). From [1] we obtain on S a vector bundle $\mathcal{E}$ of rank c_2 such that for any $s=(L,M,Y)\in S$ we have

$$\mathcal{E}_s/_{m_s}\mathcal{E}_s \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{J}_Y\otimes M, L).$$

Then the desired moduli-space will be an open set in the projective space $\mathbb{P}(\mathcal{E})$.

Let us conclude this section with the following remark. Let X be a 2-torus with NS(X)=0 and let us consider the holomorphic 2-vector bundles on X with $c_2=0$. In this case the vector bundles are topologically trivial and filtrable (even on a 2-torus X with $NS(X)\neq 0$); see Elencwajg-Forster [3]. These bundles may be decomposable, hence of the form $L\oplus M$, with $L,M\in \mathrm{Pic}\,X=\mathrm{Pic}_o X$, or indecomposable and then they are nontrivial extensions of the form

$$0\to L\to E\to L\to 0$$

(if $M\neq L$ then $H^1(X,M^*\otimes L)=0$!). In the first case the pair (L,M) is uniquely determined up to the order and, in the second case, the extension is uniquely determined by E. It is clear how these two sets of vector bundles can be parametrized.

2. The intersection form on the Neron-Severi group of a 2-torus

A complex 2-torus X is isomorphic with $\mathbb{C}^2/\Gamma$, where Γ is a lattice of rank 4 in $\mathbb{C}^2$. One has a natural isomorphism

$$H^2(X,\mathbb{Z})=\mathrm{Alt}^2_{\mathbb{Z}}(\Gamma,\mathbb{Z})$$

of $H^2(X,\mathbb{Z})$ with the space of alternating integer-valued 2-forms on Γ. Let

$$H(\mathbb{C}^2,\Gamma)=\{H \mid H \text{ hermitian form on } \mathbb{C}^2 \text{ with } \mathrm{Im}H(\Gamma\times\Gamma)\subset\mathbb{Z}\}.$$

Since the imaginary part ImH of a hermitian form H is an alternating 2-form which determines completely H, we may consider $H(\mathbb{C}^2,\Gamma)$ as a subgroup of $\mathrm{Alt}^2_{\mathbb{Z}}(\Gamma,\mathbb{Z})\cong H^2(X,\mathbb{Z})$. With this identification one has by the theorem of Appell-Humbert (Mumford [5])

$$NS(X)=H(\mathbb{C}^2,\Gamma).$$

Modulo an analytic isomorphism of the 2- torus X, we can take Γ be the lattice generated by the column vectors of the matrix

$$P=\begin{pmatrix} 1 & 0 & p_1+ip_2 & r_1+ir_2 \\ 0 & 1 & q_1+iq_2 & s_1+is_2 \end{pmatrix}=(I_2,B).$$

P is called the period matrix. We have

$$B_1=\mathrm{Re}B=\begin{pmatrix} p_1 & r_1 \\ q_1 & s_1 \end{pmatrix},\quad B_2=\mathrm{Im}B=\begin{pmatrix} p_2 & r_2 \\ q_2 & s_2 \end{pmatrix}$$

and we can choose B such that $D=\det B_2>0$.

Consider the complex vector space $\mathbb{C}^2$ as the real vector space $\mathbb{R}^4$ with the complex structure given by the matrix

$$J=\begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}$$

and take on $\mathbb{R}^4$ also the complex structure given by the matrix

$$J_B=\begin{pmatrix} -B_1B_2^{-1} & -B_2-B_1B_2^{-1}B_1 \\ B_2^{-1} & B_2^{-1}B_1 \end{pmatrix}.$$

Let $f:\mathbb{R}^4 \longrightarrow \mathbb{C}^2$ be the map given by the matrix

$$F=\begin{pmatrix} I_2 & B_1 \\ 0 & B_2 \end{pmatrix}.$$

Then $FJ_B=JF$ and since $f(\mathbb{Z}^4)=\Gamma$ the map f extends to an analytic isomorphism between the topological standard torus $\mathbb{R}^4/\mathbb{Z}^4$, with the complex structure given by the matrix J_B, and the complex torus $X=\mathbb{C}^2/\Gamma$.

Now, the Appell-Humbert theorem can be reformulated and we have

$$NS(X)\cong\left\{A=\begin{pmatrix} A_1 & A_2 \\ -A_2^t & A_3 \end{pmatrix}\in M_4(\mathbb{Z}) \;\middle|\; \begin{matrix} A \text{ skew-symmetric and} \\ B^tA_1B+A_2^tB-B^tA_2+A_3=0 \end{matrix}\right\}$$

(see Selder[8]). The condition

$$B^tA_1B+A_2^tB-B^tA_2+A_3=0$$

express the fact that A is the imaginary part ImH of a hermitian form H on $\mathbb{C}^2$. The matrix of the hermitian form in the canonical basis of $\mathbb{C}^2$ is the hermitian matrix

$$H_A=(A_1B_1-A_2)B_2^{-1}+iA_1.$$

The algebraic dimension of the torus X is given by

$$a(X)=\max\{\text{rank } H_A \mid H_A \text{ positive semi-definite}\}$$

(see Elencwajg-Forster [3]).

Every $A\in NS(X)$ is the first Chern class of a line bundle $L\in \text{Pic } X$ $(A=c_1(L))$. If we identify the group $H^2(X,\mathbb{Z})$ with $\text{Alt}^2_{\mathbb{Z}}(\Gamma,\mathbb{Z})$ then the cup-product on $H^2(X,\mathbb{Z})$ becomes the exterior product of 2-forms (see Mumford [5]). The intersection form on the Neron-Severi group is given by the formula

$$c_1(L)c_1(L')=\alpha\delta'+\alpha'\delta-\beta\gamma'-\beta'\gamma-\theta\tau'-\theta'\tau,$$

where $c_1(L)=A$, $c_1(L')=A'$

$$A=\begin{pmatrix} A_1 & A_2 \\ -A_2^t & A_3 \end{pmatrix};\quad A_1=\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix},\quad A_2=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},\quad A_3=\begin{pmatrix} 0 & \tau \\ -\tau & 0 \end{pmatrix}\in M_2(\mathbb{Z})$$

and similarly for A'. For the quadratic intersection form we get

$$c_1(L)^2=2(\alpha\delta-\beta\gamma-\theta\tau).$$

<u>LEMMA 4.</u> Let $A\in NS(X)$ be the first Chern class of a line bundle $L\in \text{Pic } X$. Then, with the above notations, we have

$$c_1(L)^2=2D.\det H_A.$$

<u>Proof</u>. By direct computation. If $A\in NS(X)$ then A satisfies the condition

$$B^tA_1B+A_2^tB-B^tA_2+A_3=0$$

or, equivalently

$$(2)\quad \begin{cases} \theta\,(p_1s_1-q_1r_1-p_2s_2+q_2r_2)+\alpha r_1+\gamma s_1-\beta p_1-\delta q_1+\tau=0 \\ \theta\,(p_1s_2-q_1r_2+p_2s_1-r_1q_2)+\alpha r_2+\gamma s_2-\beta p_2-\delta q_2=0\ . \end{cases}$$

Recall that

$$D=\det B_2=p_2s_2-q_2r_2$$

and denote

$$D_1=\det B_1=p_1s_1-q_1r_1.$$

Computing $\det H_A$ we have:

$$\begin{aligned} D^2\det H_A=&\theta^2(q_1s_2-s_1q_2)(p_1r_2-r_1p_2)+\theta(q_1s_2-s_1q_2)(\gamma r_2-\delta p_2)+ \\ &+\theta(p_1r_2-r_1p_2)(\beta q_2-\alpha s_2)+(\gamma r_2-\delta p_2)(\beta q_2-\alpha s_2)- \\ &-\theta^2(p_2s_1-q_1r_2)(r_1q_2-p_1s_2)-\theta(r_1q_2-p_1s_2)(\alpha r_2-\beta p_2)- \\ &-\theta(p_2s_1-q_1r_2)(\delta q_2-\gamma s_2)-(\alpha r_2-\beta p_2)(\delta q_2-\gamma s_2)-\theta^2D^2. \end{aligned}$$

Looking on the powers of θ and using the formulas (2) we may futher compute:

$$\begin{aligned} D^2\det H_A=&\theta^2D(D_1-D)+\theta D(-\delta q_1+\gamma s_1-\beta p_1+\alpha r_1)+D(\alpha\delta-\beta\gamma)= \\ &=D\Big(\theta^2(D_1-D)+\theta(-\theta(D_1-D)-\tau)+\alpha\delta-\beta\gamma\Big)= \\ &=D(\alpha\delta-\beta\gamma-\theta\tau). \end{aligned}$$

Finally we get

$$2D\ \det H_A=2(\alpha\delta-\beta\gamma-\theta\tau)=c_1(L)^2.$$

<u>THEOREM 5.</u> Let X be a 2-torus with algebraic dimension $a(X)=0$. Then the quadratic intersection form $c_1(L)^2$ on the Neron-Severi group NS(X) is negative definite.

<u>Proof.</u> By the classical result of Kodaira [4] we know that the quadratic intersection form on the group NS(X) is negative semi-definite ($c_1(L)^2 \leq 0$) because X is a nonalgebraic surface.

Let us suppose that there exists $c_1(L)=A\in NS(X)$, $c_1(L)\neq 0$ such that $c_1(L)^2=0$. From the Lemma 4 we get that $\det H_A=0$. The hermitian matrix H_A is unitary similar to a diagonal matrix

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ are the eigenvalues of the matrix H_A. Since $\lambda_1\lambda_2=\det H_A=0$, it follows that $\lambda_1=0$. Because $c_1(L)=A\neq 0$ we get $H_A\neq 0$ and also $\lambda_2\neq 0$. By changing(if necessary) A with $-A$ (and H_A with $-H_A$) we may suppose that $\lambda_2>0$. Then the hermitian matrix H_A is positive semi-definite and thus it follows that $a(X)\geq 1$, contradiction.

<u>REMARK.</u> The statement of the Theorem 5 is no longer true in the case of a 2-torus X with algebraic dimension $a(X)=1$, as the following example shows. We define the 2-torus X by taking

$$B_1=0 \quad \text{and} \quad B_2=\begin{pmatrix} 1 & \sqrt{2} \\ 0 & \sqrt{3} \end{pmatrix}.$$

The equations (2) become

$$-\theta\sqrt{3}+\tau=0,\ \alpha\sqrt{2}+\gamma\sqrt{3}-\beta=0,$$

and the solutions are $\theta=\tau=\alpha=\beta=\gamma=0$ and $\delta\in\mathbb{Z}$, arbitrary. For $\delta<0$ we get an element $A\in NS(X)$ such that $A=c_1(L)\neq 0$ and $c_1(L)^2=0$. The corresponding hermitian matrix H_A has the form

$$H_A=\begin{pmatrix} 0 & 0 \\ 0 & -\frac{\delta}{\sqrt{3}} \end{pmatrix}$$

and is positive semi-definite of rank one. Then the algebraic dimension is $a(X)=1$.

3. Filtrable 2-vector bundles on nonalgebraic 2-tori

In this section we shall use the result of the previous section for the study of the existence of holomorphic filtrable structures on topological 2-vector bundles on 2-tori with algebraic dimension zero.

Let X be a nonalgebraic 2-torus and let $G=NS(X)$ be the Neron-Severi group of X. If $a\in G$ then we denote by $G_a=a+2G$, the class of a modulo the subgroup 2G. Let ϕ denotes the quadratic intersection form on G and let m_a be the integer

(3) $$m_a:=\max_{x\in G_a}\Phi(x).$$

<u>THEOREM 6.</u> Let X be a 2-torus with $a(X)=0$ and let $G=NS(X)$ be the Neron-Severi group of X. Let $a\in G$ be a fixed element. Then an integer Δ is the discriminant of a filtrable 2-vector bundle E with $c_1(E)=a$ iff it satisfies the conditions

(4) $$\Delta\leq m_a,\quad \Delta\equiv m_a \pmod 4.$$

Proof. Let E be a filtrable 2-vector bundle on X with $c_1(E)=a$ and let

$$0 \to L \to E \to M \otimes \mathcal{J}_Y \to 0$$

be an associated extension. Let us denote

$$c_2(E)=b,\ c_1(L)=u,\ c_1(M)=v;$$

then we get

$$u+v=a, \quad uv+\deg Y=b,$$

hence

$$\Delta(E)=a^2-4b=(u-v)^2-4\deg Y.$$

For $x=u-v=a-2v\in G_a$,
we have

$$\Delta(E)\leq \max_{x\in G_a} \Phi(x)=m_a\leq 0.$$

For any $x', x\in G_a$ we have $x'=x+2t$, $t\in G$, and

$$\Phi(x')=x'^2=x^2+4xt+4t^2=\Phi(x)+4(xt+t^2).$$

It follows that $\Phi(x')\equiv\Phi(x) \pmod 4$ and we get

$$\Delta(E)\equiv m_a \pmod 4.$$

Conversely, let $x_o=a-2t_o\in G_a$ such that $\Phi(x_o)=m_a$.

We take $u=a-t_o$, $v=t_o$ and we have $u+v=a$. There exist line bundles $L,M\in \mathrm{Pic}\, X$ such that $c_1(L)=u$ and $c_1(M)=v$ and we take Y a locally complete intersection of codimension 2 in X such that

$$\deg Y = (m_a-\Delta)/4;$$

for $\Delta=m_a$ Y is empty. If $u\neq v$ then

$$c_1(L^*\otimes M)=v-u\neq 0$$

and $L^*\otimes M\not\cong \mathcal{O}_X$. If $u=v$, by tensoring (if necessary) M with an element of $\mathrm{Pic}_o X$, we can suppose that $L^*\otimes M\not\cong \mathcal{O}_X$. In any case since a torus with algebraic dimension zero has no divisors, we have that

$$H^2(X,M^*\otimes L)\cong H^o(X,L^*\otimes M)^*=0.$$

As in the proof of Theorem 3 (i) it follows that there exists an extension

$$0\to L\to E\to M\otimes \mathcal{J}_Y\to 0,$$

with E a 2-vector bundle. For E we have

$$c_1(E)=c_1(L)+c_1(M)=a$$
$$\Delta(E)=c_1(E)^2-4c_2(E)=(u-v)^2-4(m_a-\Delta)/4=$$
$$=\Phi(x_o)-m_a+\Delta=\Delta.$$

<u>REMARK.</u> It is well known that for a nonalgebraic 2-torus X with algebraic dimension $a(X)=0$ the rank of the Neron-Severi group, $\rho(X)=\mathrm{rk}\, NS(X)$, takes the values 0,1,2

and 3 (see, for example, [3]). In the case $\rho(X)=0$ the bound m_a is zero and we have again the filtrable part of the Theorem 3(i), but in this case also for nonfiltrable bundles we have $\Delta\leq 0$.

In the following we shall give explicit formulas for the bounds m_a in the other cases. Next consider the case $\rho(X)=1$.

COROLLARY 7. With the same notations as in the Theorem 6 let $\rho(X)=1$ and let d be the discriminant of the quadratic form Φ. Then we have:

(i) If $a\in 2G$, then an integer Δ is the discriminant of a filtrable 2-vector bundle E with $c_1(E)=a$ iff

$$\Delta \leq 0, \quad \Delta \equiv 0 \pmod 4.$$

(ii) If $a\notin 2G$, the same result holds iff

$$\Delta \leq d, \quad \Delta \equiv d \pmod 4.$$

Proof. It is sufficient to compute the bound m_a. The case (i) is obvious. In the case (ii) let L_o be a line bundle on X such that $c_1(L_o)\in G$ is a basis of G. Then we have:

$$d=c_1^2(L_o)<0 \quad \text{and} \quad \Phi(x)=\ell^2 d,$$

where $x=\ell c_1(L_o)$ ($\ell\in\mathbb{Z}$) is an arbitrary element of G. For $x\in G_a$ we have $\ell=2k+1$, so $\Phi(x)=(2k+1)^2 d$ and m_a is clearly d.

Let X be a nonalgebraic 2-torus with $a(X)=0$ and $\rho(X)=2$. By the Theorem 5 the quadratic form $f=-\Phi$ is positive definite on $G=NS(X)$ and we have a reduced form for it (see, for example, [9]). This means that there exists a basis $\{e_1,e_2\}$

in G such that

$$\Phi(u)=\eta x^2+2\mu xy+\xi y^2;$$

for any $u=xe_1+ye_2\in G$ $(x,y\in\mathbb{Z})$, where

$$(5)\qquad 0\leq 2\mu\leq -\eta\leq -\xi.$$

COROLLARY 8. With the same notations as in the Theorem 6 let $\rho(X)=2$ and e_1, e_2, η, μ, ξ as above. Let $a=a_1e_1+a_2e_2\in G$. Then the conclusion of the Theorem holds with:

i) $m_a=0$ if $a_1\equiv 0 \pmod 2$ and $a_2\equiv 0 \pmod 2$;

ii) $m_a=\eta$ if $a_1\equiv 1 \pmod 2$ and $a_2\equiv 0 \pmod 2$;

iii) $m_a=\xi$ if $a_1\equiv 0 \pmod 2$ and $a_2\equiv 1 \pmod 2$;

iv) $m_a=\eta+2\mu+\xi$ if $a_1\equiv 1 \pmod 2$ and $a_2\equiv 1 \pmod 2$;

Proof. The case i) is obvious. We shall prove the case iii). The proofs of the cases ii) and iv), being similar to the proof of case iii), are left to the reader.

If $u\in G_a$, then $u=2xe_1+(2y+1)e_2$ with $x,y\in\mathbb{Z}$. It follows that

$$\Phi(u)=\xi+4(\eta x^2+2\mu xy+\xi y^2+\mu x+\xi y),$$

and it would be sufficient to prove that

$$(6)\qquad -\eta x^2-2\mu xy-\xi y^2-\mu x-\xi y\geq 0,$$

for any $x,y\in\mathbb{Z}$. The ellipse

$$-\eta x^2-2\mu xy-\xi y^2-\mu x-\xi y=0$$

has the center in the point $(0,-\frac{1}{2})$ and does not intersect the lines $y=1$ and $y=-2$ as we can see using the conditions (5). It follows that the ellipse is situated in the region $-2<y<1$.

The intersection with the lines $y=0$ and $y=-1$ are the points $(0,0)$, $(-\mu/\eta,0)$, respectively $(0,-1)$, $(\mu/\eta,-1)$. Since $|\mu/\eta|\leq\frac{1}{2}$, it follows that there are no points with integer coordinates inside the ellipse, and thus the inequality (6) is true.

In the case $a(X)=0$, $\rho(X)=3$ the computation of the bound m_a can be done by using the Minkowski reduced form of the quadratic intersection form Φ, but the computation is tedious and we shall omit it.

<u>REMARK</u>. It seems to be quite difficult to obtain similar results in the case of nonfiltrable bundles.

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