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by

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0. Introduction

By singularity we mean any germ $(\mathcal{X},o)$ of analytic space; throughout this paper all singularities are assumed to have dimension ≥ 2. By algebraisation of a singularity $(\mathcal{X},o)$ we mean a pair (X,o) where X is an affine complex algebraic variety and $o \in X$ is a closed point such that $(\mathcal{X},o) \simeq (X^{an},o)$ as analytic germs. By [9] any complete intersection isolated singularity has an algebraisation (X,o); one can of course assume X normal. It was conjectured by J.Kollàr [7] that any hypersurface isolated singularity of dimension 2 has an algebraisation (X,o) with $Cl(X)=0$. In §1 of this paper we will prove that at least one can "kill moduli" in $Cl(X)$, more precisely:

COROLLARY 1. Any complete intersection isolated singularity $(\mathcal{X},o)$ has an algebraisation (X,o) such that $Cl(X)$ is finitely generated.

Note that in the above corollary the divisor class group $Cl(\mathcal{O}_{\mathfrak{X},o})$ is far from being finitely generated in general; for instance the divisor classes on the vertex of a cone over a smooth irrational complete intersection curve in projective space depend on g moduli, where g is the genus of the curve. On the other hand by a theorem of Grothendieck [5] one has $Cl(\mathcal{O}_{\mathfrak{X},o})=0$ for any complete intersection isolated singularity $(\mathfrak{X},o)$ of dimension $\geqslant 4$. This together with our Corollary 1 implies that any complete intersection isolated singularity of dimension $\geqslant 4$ has an algebraisation (X,o) such that $Cl(X)=0$.

In §2 we take a closer look at the group $Cl(X)$ appearing in Corollary 1 in the 2-dimensional case. We first associate to any closed embedding $Y \subset \mathbb{A}^n$ of a normal surface Y with finitely generated class group $Cl(Y)$ two decompositions of $Cl(Y)$/torsion into finite sets

$$Cl(Y)/\text{torsion} = \bigcup_r S_r = \bigcup_d F_d$$

where roughly speaking the S_r's are the sets of classes of fixed length $=r$ with respect to some canonical euclidian metric and the F_d's are sets of classes of curves of fixed degree$=d$ (see §2 for the precise definitions). For any r and d we can form the sums

$$\sigma_{rd} = \sum_{\alpha \in F_d \cap S_r} \alpha$$

We will prove that these sums can be killed, which may be interpreted as a symmetry property of Cl:

COROLLARY 2. One can choose X in Corollary 1 and an embedding $X \subset \mathbb{A}^n$ such that all σ_{rd} vanish.

The method of proof of the above statements is to "move" inside sufficiently large linear subspaces contained in the contact orbit of the singularity and to consider the monodromy produced by this movement.

We are indebted to J.Kollar for his letter [7] which was the starting point of this investigation.

1. Killing moduli.

For any algebraic variety X let $h^{po}(X)$ denote the Hodge number $h^{po}(\overline{X})$ where $\overline{X}$ is some smooth projective model of the function field of X; since h^{po} are birational invariants of smooth projective varieties, the definition above is correct. We will prove the following:

THEOREM. Any complete intersection isolated singularity $(\mathcal{X},o)$ has an algebraisation (X,o) with $h^{po}(X)=0$ for $1 \leqslant p \leqslant \dim(X)-1$.

REMARK. If X is a normal algebraic variety then $h^{1o}(X)=0$ iff $Cl(X)$ is finitely generated. Indeed, $Cl(X) \simeq Cl(U)$ where U is a Zariski open subset of a smooth projective variety $\overline{X}$. Now $h^{1o}(\overline{X})=0$ iff $Pic^o(\overline{X})=0$ hence (by the Neron-Severi theorem) iff $Pic(\overline{X})$ is finitely generated and we are done. In particular Corollary 1 from §0 follows from the above Theorem.

The rest of this § is devoted to the proof of the Theorem. The key point will be a variation on an argument from [1].

Let's fix some notations. Put $A=\mathbb{C}[t_1,\ldots,t_n]$ = polynomial ring in n variables, $\mathcal{O}=\mathbb{C}\{t_1,\ldots,t_n\}$ = convergent power series ring in n variables. The set of germs of analytic maps $(\mathbb{C}^n,o)\longrightarrow\mathbb{C}^s$ will be identified with $\mathcal{O}^s$. The contact group [9] acting on $\mathcal{O}^s$ will be denoted by $\mathcal{K}$.

Now for any finitely dimensional linear subspace L of $t_1A+\ldots+t_nA$ let d_L be the maximum of the degrees of the polynomials in L and consider the injective $\mathbb{C}$-linear map $e:L\longrightarrow A[t_o]$ defined by $e(F)=$ $=t_o^{d_L}F(t_1/t_o,\ldots,t_n/t_o)$. Let $\mathbb{P}_L$ be the projective space associated to $e(L)$. Clearly $\mathbb{P}_L$ is a linear subsystem of $|\mathcal{O}_{\mathbb{P}^n}(d_L)|$ where $\mathbb{P}^n=\mathrm{Proj}(A[t_o])$. Call L a large linear space if the set-theoretic base locus of the linear system $\mathbb{P}_L$ consists only of the point $o=(1:0:\ldots:0)$ and if the associated rational map $R_L:\mathbb{P}^n\dashrightarrow\mathbb{P}_L$ is generically finite-to-one.

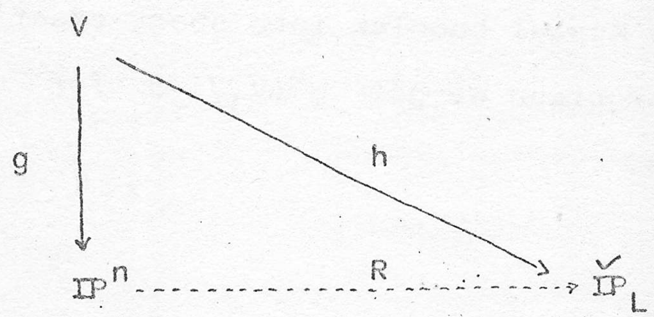
To prove the Theorem note that $\mathcal{O}_{X,o}\simeq\mathcal{O}/(f_1,\ldots,f_s)$ where $(f_1,\ldots,f_s)\in\mathcal{O}^s$ is some finitely determined germ [9]. So the Theorem will be proved if we prove the following lemmas:

LEMMA 1. If $f\in\mathcal{O}^s$ is a finitely determined germ then there exists a large linear space L such that $(\mathcal{K}f)\cap L^s$ contains an open Zariski subset of L^s.

LEMMA 2. If L is a large linear space then there exists an open Zariski subset U of L^s such that for any $(f_1,\dots,f_s)\in U$, we have $h^{p0}(\mathrm{Spec}(A/(f_1,\dots,f_s)))=0$ for $1\leq p\leq n-s-1$.

Proof of lemma 1. Let m be the determination order of f. We may suppose that the components of f are polynomials of degree $\leq m$. Let N be an integer $\geq m+1$, let L_1 be the linear space spaned by $f_1,\dots,f_s$ and L_2 a linear space of homogenous polynomials of degree N such that the corresponding map $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}(L_2)^{\vee}$ is everywhere defined and finite-to-one (for instance let L_2 be spaned by $t_1^N,\dots,t_n^N$). Put $L=L_1\oplus L_2$. It is easy to check that L is a large linear space; on the other hand if w denotes the composition of canonical maps $L^s \longrightarrow L_1^s \longrightarrow \bigwedge^s L_1$ then the complement in L^s of $w^{-1}(0)$ clearly lies inside $\mathcal{K}f$.

Proof of lemma 2. By Hironaka's resolution there exists a diagram



where g is a birational projective morphism, V is smooth and h is everywhere defined. Put $\mathcal{L}=h^*\mathcal{O}_{\check{\mathbb{P}}_L}(1)$. By Bertini's theorem [3, p.33] there exists a non-empty open Zariski subset D of $(\mathbb{P}_L)^s$ such that for any $(H_1,\dots,H_s)\in D$ the scheme-theoretic intersection

$$S = \bigcap_{i=1}^{s} h^{-1}(H_i)$$

is smooth and connected. Let U be the preimage of D under the projection $L^s \longrightarrow (\mathbb{P}_L)^s$ and let $f_1,\ldots,f_s \in$ $\in L$ be the polynomials corresponding to $H_1,\ldots,H_s$. Then S is a smooth projective model of the function field of $\mathrm{Spec}(A/(f_1,\ldots,f_s))$. Let's prove that $h^{po}(S)=0$ for $1 \leq p \leq n-s-1$. We have an exact sequence

$$H^p(V,\mathcal{O}_V) \longrightarrow H^p(S,\mathcal{O}_S) \longrightarrow H^{p+1}(V,J)$$

where J is the ideal sheaf of S on V. Since $h^{po}(V)=$ $=h^{po}(\mathbb{P}^n)=0$ for $p \geq 1$ we only have to prove that $H^q(V,J)=0$ for $q \leq n-s$. Put $E=(\mathcal{L}^{-1})^{\oplus s}$; we have the exact Koszul complex

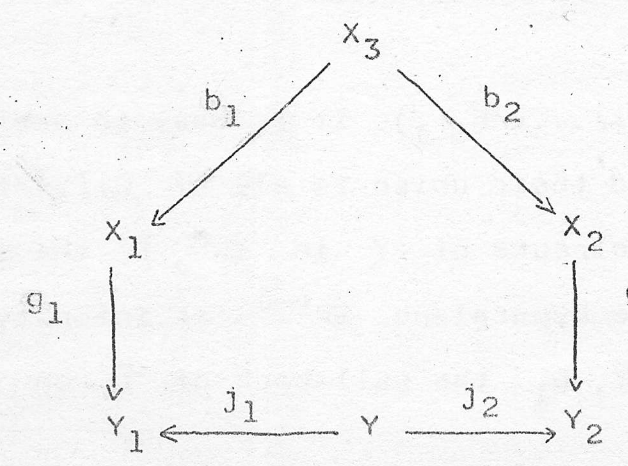
$$0 \longrightarrow \bigwedge^s E \longrightarrow \ldots \longrightarrow \bigwedge^2 E \longrightarrow E \longrightarrow J \longrightarrow 0$$

Since $\bigwedge^i E$ are direct sums of negative powers of $\mathcal{L}$ we have by the Grauert-Riemenschneider vanishing theorem [4] that $H^q(V,\bigwedge^i E)=0$ for any $i=1,..,s$ and $q=0,\ldots,n-1$. Decomposing the Koszul complex into short exact sequences and applying induction we get $H^q(V,J)=0$ for $q \leq n-s$ and we are done.

2. Killing σ_{rd}.

In this § we suppose $\dim \mathcal{X}=2$. We begin with some general constructions.

First we will show that for any noemal algebraic affine surface Y whose class group $Cl(Y)$ is finitely generated, there exists a "canonical" positive definite $\mathbb{Q}$-bilinear form ψ_Y on $Cl(Y)\otimes\mathbb{Q}$. Indeed take an embedding $j_1 : Y \longrightarrow Y_1$ of Y into a normal projective surface Y_1 such that $Y_1 \setminus Y$ is the support of an ample Cartier divisor D_1 and take a desingularization $g_1 : X_1 \longrightarrow Y_1$. We will define the bilinear form on $Cl(Y)\otimes\mathbb{Q}$ in terms of Y_1 and X_1 and then remark it actually depends only on Y. Since $h^{10}(X_1)=0$ the intersection form φ on $Pic(X_1)\otimes\mathbb{Q}$ is nondegenerate. Let M_1 be the kernel of the surjection $Pic(X_1)\otimes\mathbb{Q} \longrightarrow Cl(Y)\otimes\mathbb{Q}$. Since M_1 contains an element x with $\varphi(x,x)>0$ (take for instance $x=g_1^*D_1\otimes 1$) it follows by the Hodge index theorem that $Pic(X_1)\otimes\mathbb{Q} = M_1 \oplus M_1^{\perp}$ and φ is negative definite on $M_1^{\perp}$. Identifying $Cl(Y)\otimes\mathbb{Q}$ with $M_1^{\perp}$ we define ψ_Y to be $-\varphi$ restricted to $M_1^{\perp}$. To check independence of ψ_Y on Y_1 and X_1 take another compactification $j_2 : Y \longrightarrow Y_2$ and a desingularization $g_2 : X_2 \longrightarrow Y_2$ and let M_2 be the corresponding kernel. There exists a smooth projective surface X_3 and birational morphisms b_1 and b_2 making the following diagram commute:



Using the fact that b_i are both compositions of blowing ups one immediately identifies the quadratic linear space $M_i^{\perp}$ with the quadratic linear space $N_i^{\perp}$ where $N_i = \mathrm{Ker}(\mathrm{Pic}(X_3)\otimes\mathbb{Q} \longrightarrow \mathrm{Cl}(Y_i)\otimes\mathbb{Q} \longrightarrow \mathrm{Cl}(Y)\otimes\mathbb{Q})$. Finally it is easy to see that $N_1=N_2$ hence $N_1^{\perp}=N_2^{\perp}$ and we are done.

So, for any normal algebraic affine surface Y whose class group is finitely generated we have a canonical decomposition

$$\mathrm{Cl}(Y)/\mathrm{torsion} = \bigcup_r S_r$$

where $S_r=\{\alpha ; \psi_Y(\alpha,\alpha)=r\}$ are obviously finite.

Now for any closed embedding of Y above into an affine space $\mathbb{A}^n$ one can associate another decomposition

$$\mathrm{Cl}(Y)/\mathrm{torsion} = \bigcup_d F_d$$

(which will depend on the embedding) as follows. It makes sense to speak of the degree $\deg(C)$ of a curve C on Y: it is the degree of its projective closure $\overline{C}$ in $\mathbb{P}^n \supset \mathbb{A}^n$. Define $\overline{F}_d \subset \mathrm{Cl}(Y)/\mathrm{torsion}$ by

$$\overline{F}_d = \{\alpha = \mathrm{cl}(C);\ C = \text{irreducible curve on } Y \text{ with } \deg(C)=d\}$$

and put $F_d=\overline{F}_d \setminus (\overline{F}_1 \cup \overline{F}_2 \cup \ldots \cup \overline{F}_{d-1})$. It is easy to see that F_d are finite sets and their union is all of $\mathrm{Cl}(Y)/\mathrm{torsion}$. Indeed let $\overline{Y}$ be the closure of Y in $\mathbb{P}^n$, $\overline{D}$ the intersection of $\overline{Y}$ with the hyperplane $\mathbb{P}^{n-1}$ at infinity, Y_1 the normalization of $\overline{Y}$, D_1 the pull-back of $\overline{D}$ on Y_1,

$g:X_1 \longrightarrow Y_1$ a desingularization and $D=g^*D_1$. By [6, p.172] there is a very ample divisor on X_1 of the form $H=$ $=kD+\sum a_iE_i$ where $k \geqslant 1$ and E_i are irreducible curves contracted by g. In particular the image of H in $Cl(Y)$ is zero. Since any divisor R on X_1 may be writen as $R\sim C-mH$ where C is an irreducible curve and m is an integer we get that any class in $Cl(Y)$ may be represented by an irreducible curve. To see that F_d are finite note that for any irreducible curve G on X_1 we have $(G.H)=k\cdot \deg(u_*(G))+\sum a_i(G.E_i)$ where $u:X_1 \longrightarrow \mathbb{P}^n$ is the canonical morphism. Furthermore for any i, $(G.E_i)\leqslant$ $\leqslant \deg(u_*(G))$. It follows that the strict transforms on X_1 of irreducible curves on Y of bounded degree have still bounded degree with respect to H, consequently by the theory of the Chow variety there are finitely many of them up to algebraic equivalence and hence up to linear equivalence, since X_1 is a regular surface.

Our result in this § is the following:

PROPOSITION. Let L be a large linear space (see §1). There exists a Zariski open subset U of L^s, a member $f \in U$ and a representation

$$\rho : \pi = \pi_1(U,f) \longrightarrow O(Cl(X)/\text{torsion}, \psi_X)$$

where $X=\operatorname{Spec}(A/(f_1,\dots,f_s))$, $f=(f_1,\dots,f_s)$ such that if $Cl(X)/\text{torsion} = \bigcup F_d$ is the decomposition associated to the embedding $X \subset \operatorname{Spec}(A)$ then:

1. F_d is globally invariant under π for any $d \geqslant 1$.
2. The group of invariants $(Cl(X)/\text{torsion})^{\pi}$ vanishes.

In the above statement $O(Cl(X)/\text{torsion}, \psi_X)$ denotes the orthogonal group of the lattice $Cl(X)/\text{torsion}$ with respect to the restriction of ψ_X. In particular if $\bigcup S_r$ is the decomposition of the lattice into sets of vectors of fixed length then each S_r is globally invariant under π. This together with the above proposition gives the vanishing of all σ_{rd} in Corollary 2 from §0.

Proof of the Proposition. We shall use the notations from the proof of Lemma 2, §1. Let P be a sufficiently general $(s+1)$-dimensional linear subspace of L, let $\mathbb{P}_P \subset \mathbb{P}_L$ be the projective space associated to it and let B be the base locus of $\mathbb{P}_P$ on V. By Bertini's theorem [3, p.33] again, B is smooth connected. Let $b: W \longrightarrow V$ be the blowing up of V along B and let F be the exceptional locus of b. The rational map $V \dashrightarrow \check{\mathbb{P}}_P$ lifts then to a morphism $W \longrightarrow \check{\mathbb{P}}_P$. Let $\lambda \in \mathbb{P}_P$ be a generic point of $\check{\mathbb{P}}_P$ (in Weil's sense) over the common field of definition of our varieties and morphisms, write λ as an intersection of s hyperplanes in $\check{\mathbb{P}}_P$ and lift these hyperplanes to s hyperplanes in $\check{\mathbb{P}}_L$ corresponding to polynomials $f_1, \ldots, f_s \in L$; put $f = (f_1, \ldots, f_s)$. Now if U is as in the proof of Lemma 2 and $S \subset V$ corresponds to f then the construction from Lemma 2 clearly yelds a monodromy representation $\theta : \pi_1(U, f) \longrightarrow O(H^2(S, \mathbb{Z}), \varphi)$. We claim that if $\eta : \pi' = \pi_1(U', \lambda) \longrightarrow O(H^2(S, \mathbb{Z}), \varphi)$ is the monodromy representation defined by the family $W \longrightarrow \check{\mathbb{P}}_P$ (where U' is the Zariski open subset of $\check{\mathbb{P}}_P$ above which $W \longrightarrow \check{\mathbb{P}}_P$ is smooth) then $Im(\theta) \supseteq Im(\eta)$; in particular any π-invariant element is π'-invariant. Indeed put $Y=w^{-1}(0)\subset L^s$ where $w:L^s \longrightarrow \bigwedge^s L$ is the canonical map. There is an obvious morphism $(U\setminus Y)\cap P^s \longrightarrow U'$ which is a locally-trivial fibration with connected fibres hence the map $\pi_1((U\setminus Y)\cap P^s, f) \longrightarrow \pi_1(U', \lambda)$ is surjective and we are done.

Now since S is regular, $Pic(S)=Im(H^1(S,\mathcal{O}^*)\longrightarrow H^2(S,\mathbb{Z}))$ and since f is generic it follows by the theory in [10] that the above subgroup of $H^2(S,\mathbb{Z})$ is a π-submodule hence η induces a representation $\bar{\eta}:\pi \longrightarrow O(Pic(S),\varphi)$. To show that $\bar{\eta}$ induces a representation ρ as in the statement of the proposition it is sufficient to see that $M_o=Ker(Pic(S)\longrightarrow Cl(X))$ is a π-submodule. Let T be the projective closure of X in $\mathbb{P}^n$ and $H=\mathbb{P}^{n-1}\cap T$ where $\mathbb{P}^{n-1}=\mathbb{P}^n\setminus \mathbb{A}^n$. Let Z_i be the irreducible components of the exceptional locus of $g:V\longrightarrow \mathbb{P}^n$ and E_{ij} the irreducible components of $Z_i\cap S$. Let $q:S\longrightarrow T$ be the restriction of g to S. Then M_o is spanned by $cl(q^*H)$ and $cl(E_{ij})$. Now $cl(q^*H)$ clearly is π-invariant because it is the pull-back on S of $cl(b^*g^*\mathcal{O}(1))$. Consequently for any $\gamma\in\pi$ we must have

$$(cl(q^*H).\gamma cl(E_{ij}))=(\gamma cl(q^*H).\gamma cl(E_{ij}))=(q^*H.E_{ij})=0$$

Since by the theory in [10] again, $cl(E_{ij})$ may be represented by an irreducible curve it follows that $\gamma cl(E_{ij})=cl(E_{km})$ for some k and m; in fact it is easy to see that in the above we must have $i=k$

To see that the F_d's are globally invariant take an irreducible curve D on Y of degree d and let D_1 be its proper transform on S; we have $(q^*H.D_1)=d$. By [10] for any $\gamma\in\pi$ one may write $\gamma cl(D_1)=cl(D_2)$ where D_2 is an irreducible curve on S suchthat D_1 and D_2 are algebraically equivalent as cycles on V; in particular $(q^*H.D_2)=d$ and we are done.

Finally suppose $\bar\alpha\in(Cl(X)/\text{torsion})^{\pi}\subseteq(Cl(X)/\text{torsion})^{\pi'}$. Since $Pic(S)\otimes\mathbb{Q}=M\oplus M^{\perp}$ where $M=M_o\otimes\mathbb{Q}$ and $M^{\perp}$ identifies with $Cl(X)\otimes\mathbb{Q}$, it follows that $\bar\alpha$ may be viewed as an element α of $Pic(S)\otimes\mathbb{Q}$ and is π'-invariant with respect to the action η. By Deligne's [2] it follows that $\alpha\in Im(H^2(W,\mathbb{Z})\longrightarrow H^2(S,\mathbb{Z}))$. Since $h^{10}(W)=h^{20}(W)=0$ it follows that $H^2(W,\mathbb{Z})=Pic(W)$ which is spanned by $cl(b^*g^*\mathcal{O}(1))$, $cl(F)$ and $cl(Z_i)$ so α is a linear combination of $cl(q^*H)$, $cl(B)$ and $cl(E_{ij})$. But now we are done because $B\in|\mathcal{O}_S\otimes\mathcal{L}|$ and $\mathcal{L}$ may be expressed again in terms of $g^*\mathcal{O}(1)$ and the Z_i's so we get $\bar\alpha=0$.

Let's close with three remarks:

1. The simplest non-trivial example of monodromy action as in the above Proposition is the following: take $(\mathfrak{X},o)\subset(\mathbb{C}^3,o)$ to be the analytic germ given by $f_m=0$ where f_m is a nondegenerate homogenous polynomial of degree $m=2$ or 3. Then $f_m=0$ is m-determined; take L to be the large linear space of all polynomials $\lambda f_m+\mu f_{m+1}$ where $\lambda,\mu\in\mathbb{C}$ and f_{m+1} is an arbitrary homogenous polynomial of degree $m+1$. Generic singularities $f=0$ with $f\in L$ contain $m(m+1)$ lines $D_1,\dots,D_{m(m+1)}$ through the origin whose union is given by the equations $f_m=f_{m+1}=0$ and which generate the class

class group of the affine surface $\{f=0\} \subset \mathbb{C}^3$. One sees immediately in this example that $D_1+\ldots+D_{m(m+1)}$ is the complete intersection of $f=0$ with the Cartier divisor $f_m=0$ so the class of the above sum vanishes in the class group. The monodromy clearly acts by permuting the lines.

2. Kollàr's conjecture remains open. Note that Kollàr proved his conjecture for certain ratioaal double points using moduli of K3 surfaces. An example of non-rational singularity for which the conjecture holds is provided for instance by $(f_3=0)\subset(\mathbb{C}^3,o)$ where f_3 is a generic homogenous polynomial of degree 3 [1].

3. The monodromy action we introduced in §2 is related with another action which naturally appears in this context, namely with the action of the contact group $\mathcal{K}$. However the connection between the two actions is not an easy direct one; since this aspect is not relevant for our problem, we won't discuss it here.

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