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by

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It has been suggested by E.G. Effros ([5]) that a study of inductive limits of C^*-algebras of the form $\mathcal{C}(X) \otimes F$ (with F finite-dimensional C^*-algebra), as a generalization of A.F.-algebras, may be of interest.

Indeed, such algebras are, for instance, those considered by J.Bunce and J.Deddens (in [3]).

The present paper contains general results concerning inductive limits of algebras of the form $\mathcal{C}(X) \otimes F$ as well as a structure theorem for a certain class of inductive limits of C^*-algebras of the form $\mathcal{C}(T^2) \otimes M_n$.

For each positive integer $p \geq 1$, let $\mathcal{C}^*(S(p))$ be the C^*-algebra generated by all weighted shifts (with respect to some fixed orthonormal basis $(e_m)_{m \geq 0}$ of the Hilbert space $\mathcal{H}$) of period p.

Given a strictly increasing sequence of positive integers (n_k), with n_k dividing n_{k+1} for all $k \geq 1$, the <u>Bunce-Deddens algebra</u>, denoted in this paper by $\mathcal{A}((n_k))$, is $\bigcup_k \nu(\mathcal{C}^*(S(n_k)))$ (see [3]), where $\nu : B(\mathcal{H}) \longrightarrow B(\mathcal{H})/K(\mathcal{H})$ is the canonical surjection onto the Calkin algebra.

It follows from ([2], proof of Theorem 2.2) that, for $p \geq 2$, $\nu(\mathcal{C}^*(S(p)))$ is $*$-isomorphic to $M_p(\mathcal{C}(\mathbb{T})) \equiv$ the algebra of all $p \times p$ matrices whose entries are continuous functions on the unit circle $\mathbb{T}$. Under this isomorphism, if S is the weighted shift given by $Se_m = d_{m+1}e_{m+1}$ $(m \geq 0)$, where $d_{m+p} = d_m$ for all $m \geq 0$, then $\nu(S)_{1,p}(z) = d_p \cdot z$, $\nu(S)_{i+1,i}(z) = d_i \cdot 1$ for $1 \leq i \leq p-1$ and $\nu(S)_{i,j} = 0$ for all other i,j.

Using these $*$-isomorphisms, each $\mathcal{A}((n_k))$ is $*$-isomorphic to the inductive limit of:

$$\mathcal{C}(\mathbb{T}) \otimes M_{n_1} \xrightarrow{\Psi_1} \mathcal{C}(\mathbb{T}) \otimes M_{n_2} \xrightarrow{\Psi_2} \cdots$$

where for each k, Ψ_k is a *particular* isometric $*$-homomorphism which is compatible with the covering map $\mathbb{T} \ni z \longmapsto z^{n_{k+1}/n_k} \in \mathbb{T}$ (in the sense of Definition 2.2.below).

We shall now state the main result of this paper.

We fix two strictly increasing sequences of positive integers (p_k), (q_k), with p_k (resp. q_k) dividing p_{k+1} (resp. q_{k+1}) for all $k \geq 1$.

We consider systems of the form:

$$\mathcal{C}(\mathbb{T}^2) \otimes M_{p_1q_1} \xrightarrow{\Lambda_1} \mathcal{C}(\mathbb{T}^2) \otimes M_{p_2q_2} \xrightarrow{\Lambda_2} \cdots$$

where for each k, Λ_k is an arbitrary isometric $*$-homomorphism compatible with the covering map $\mathbb{T}^2 \ni (z_1, z_2) \longmapsto (z_1^{p_{k+1}/p_k}, z_2^{q_{k+1}/q_k}) \in \mathbb{T}^2$. Denote by $\mathcal{A}((\Lambda_k))$ the corresponding inductive limit. Note that there is a large freedom in choosing the Λ_k's. A natural question arises: what C^*-algebras $\mathcal{A}((\Lambda_k))$ can one get (up to $*$-isomorphism) in this way ? and the answer we give is contained in the following:

Theorem

Each C^*-algebra $\mathcal{A}((\Lambda_k))$ is $*$-isomorphic to the (spatial) C^*-tensor product $\mathcal{A}((p_k)) \otimes \mathcal{A}((q_k))$.

The paper is divided into three sections.

Section 1 deals with preliminaries.

Section 2 considers $*$-homomorphisms compatible with a covering map; their local structure is given and, under some additional assumptions, also their global form is established. It is shown that there is a canonical map, which is a bijection, between the set of all $*$-homomorphisms compatible with a fixed covering map and the set of all homomorphisms between certain continuous fields of finite-dimensional C^*-algebras.

Section 3 contains the proof of the above mentioned theorem. The main step in the proof is a lemma which asserts that if $\Phi, \Psi : \mathcal{C}(T^2) \otimes M_n \longrightarrow \mathcal{C}(T^2) \otimes M_{npq}$ are isometric $*$-homomorphisms compatible with the covering map $T^2 \ni (z_1, z_2) \longmapsto (z_1^p, z_2^q) \in T^2$ $(p,q \in \mathbb{N})$ then $\Phi(.)=u\Psi(.)u^*$ for some unitary $u \in \mathcal{C}(T^2) \otimes M_{npq}$.

$\square$

§1.

In the sequel, for a finite-dimensional C^*-algebra $A=\bigoplus_{i=1}^{n} M_{k_i}$, each M_{k_i} will be embedded in A, in the canonical way ($M_{k_i} \subset A$, $1 \leq i \leq n$). Moreover, if X is a compact metric space, we shall often identify, in the canonical way, $\mathcal{C}(X) \otimes A = \mathcal{C}(X,A)$. For $f \in \mathcal{C}(X) \otimes A$, we define $f_{k_i} \in \mathcal{C}(X) \otimes M_{k_i}$, $1 \leq i \leq n$, by:

$$f(x)=\bigoplus_{i=1}^{n} f_{k_i}(x) \in \bigoplus_{i=1}^{n} M_{k_i} \; (=A), \qquad x \in X.$$

The group of all unitaries of any unital C^*-algebra B will be denoted by U(B). If $B=M_m$, then we put $U(m):=U(M_m)$.

A space T will be called (simbolically $T \in AR$) an absolute retract, or an AR-space, iff T is a compact metrizable space and whenever T is embedded as a subspace of a metrizable space V, there is a retraction of V onto T (see, for instance [1]).

The following result will be necessary in proving Proposition 2.5.:

1.1. Theorem ([1], p.100)

Let T be a compact metrizable space. $T \in AR$ if and only if T is homeomorphic to a retract of the Hilbert cube Q^ω ($Q^\omega := \prod_{k=1} [0,\frac{1}{k}]$). □

Let C, D be unital C^*-algebras. Two $*$-homomorphisms $\phi, \psi : C \longrightarrow D$ are said to be inner equivalent if, there is a unitary $u \in D$ such that $\phi(.)=u\psi(.)u^*$.

§ 2.

Let us give the definition (probably well-known) of a homomorphism of continuous fields of C^*-algebras.

2.1. <u>Definition</u>. Let T be a topological space and let $\mathcal{E}_i = ((E_i(t))_{t\in T}, \Gamma_i)$ ($i=1,2$) be two continuous fields of C^*-algebras ([6]). Then $\mathcal{L} = (\mathcal{L}_t)_{t\in T}$ is said to be a <u>homomorphism from $\mathcal{E}_1$ to $\mathcal{E}_2$</u> iff: 1° every $\mathcal{L}_t$ is a $*$-homomorphism of C^*-algebras from $E_1(t)$ to $E_2(t)$; 2° $\mathcal{L}$ takes Γ_1 into Γ_2 (If we consider fields of unital C^*-algebras, each $\mathcal{L}_t$ is taken unital).

□

Let X,Y be two compact metric spaces, let A,B be two finite-dimensional C^*-algebras and let $\varphi : X \longrightarrow Y$ be a p-fold covering map ($p \in \mathbb{N}$). We denote by $\{x_1(y), x_2(y), \ldots, x_p(y)\}$ the fibre of φ at y, $y \in Y$. For every $f \in \mathcal{C}(X) \otimes A$ and every $y \in Y$, introduce the notation:

$$\bigoplus_{\varphi(x)=y} f(x) = \bigoplus_{i=1} f(x_i(y))$$

We now define two continuous fields of C^*-algebras: $\mathcal{E}_1 = ((E_1(y))_{y\in Y}, \Gamma_1)$ given by

$$E_1(y) = \bigoplus_1^p A \quad , \; y \in Y$$

$$\Gamma_1 = \{ Y \ni y \longmapsto \bigoplus_{\varphi(x)=y} f(x) \mid f \in \mathcal{C}(X) \otimes A \}$$

and $\mathcal{E}_2$, the constant field on Y, of fibre B.

2.2. <u>Definition</u>. A $*$-homomorphism $\Phi : \mathcal{C}(X) \otimes A \longrightarrow \mathcal{C}(Y) \otimes B$

is said to be φ-compatible, or compatible with φ, iff:

$$\phi(g\circ\varphi\otimes 1_A)=g\otimes 1_B\ , \quad g\in\mathcal{C}(Y)\ .$$

Let $\mathcal{M}$ be the set of all φ-compatible $*$-homomorphisms from $\mathcal{C}(X)\otimes A$ to $\mathcal{C}(Y)\otimes B$ and let $\mathcal{N}$ be the set of all homomorphisms from $\mathcal{E}_1$ to $\mathcal{E}_2$. We consider the map $\mathcal{F}:\mathcal{M}\longrightarrow\mathcal{N}$ given by:

$$\mathcal{F}(\phi)=R=(R_t)_{t\in Y}$$

$$R_y(\bigoplus_{\varphi(x)=y} f(x))=\phi(f)(y)$$

where $\phi\in\mathcal{M}$, $y\in Y$, $f\in\mathcal{C}(X)\otimes A$.

2.3. Proposition

$\mathcal{F}$ is a bijection.

Proof: For any $f\in\mathcal{C}(X)\otimes A$, the map $Y\ni y\longmapsto R_y(\bigoplus_{\varphi(x)=y} f(x))\in B$ is continuous (because $\phi(f)\in\mathcal{C}(Y)\otimes B$) and therefore the definition of $\mathcal{F}$ is consistent.

On the other hand, for a given $R=(R_t)_{t\in Y}\in\mathcal{N}$ the formula:

$$\phi(f)(y)=R_y(\bigoplus_{\varphi(x)=y} f(x)),\quad f\in\mathcal{C}(X)\otimes A,\ y\in Y$$

defines a $*$-homomorphism $\phi:\mathcal{C}(X)\otimes A\longrightarrow\mathcal{C}(Y)\otimes B$ (the fact that R takes Γ_1 into $\mathcal{C}(Y)\otimes B$ implies the continuity of the map $\phi(f)$, for any $f\in\mathcal{C}(X)\otimes A$).

But ϕ is φ-compatible because for every $g\in\mathcal{C}(Y)$ and every $y\in Y$ one gets:

$$\phi(g\circ\varphi \otimes 1_A)(y)=R_y(\bigoplus_{\varphi(x)=y} g(\varphi(x)).1_A)=g(y).1_B$$

We have: $\phi\in\mathfrak{M}$ and $\mathcal{F}(\phi)=R$.

Since the injectivity of $\mathcal{F}$ is obvious, the proof is complete. $\square$

2.4. Proposition

Let $\phi:\mathcal{C}(X)\otimes A\longrightarrow\mathcal{C}(Y)\otimes B$ be a φ-compatible $*$-homomorphism, where $\varphi:X\longrightarrow Y$ is a p-fold covering map ($p\in\mathbb{N}$).

Then, for each $y'\in Y$ there exist: $V\in\mathcal{V}(y')$, $U_i\in\mathcal{V}(x_i')$ (where $\{x_1',\dots,x_p'\}$ is the fibre of φ at y') such that $U_i\cap U_j=\emptyset$ if $1\leqslant i, j\leqslant p$, $i\neq j$, homeomorphisms $z_i:V\longrightarrow U_i$ with the property that $\varphi\circ z_i=id_V$ $(1\leqslant i\leqslant p)$, $*$-homomorphisms $\psi_1,\dots,\psi_p:A\longrightarrow B$ and $u\in\mathcal{C}(V,\mathcal{U}(B))$ such that:

$$\phi(f)(y)=u(y)(\bigoplus_{i=1}^{p}\psi_i(f(z_i(y))))u(y)^*$$

for all $f\in\mathcal{C}(X)\otimes A$ and all $y\in V$.

Proof

It is enough to consider the case when $A=\bigoplus_{i=1}^{n} M_{k_i}$, $B=M_l$.

For every $y\in Y$, $\mathcal{C}(X)\otimes A\ni f\longmapsto\phi(f)(y)\in M_l(=B)$ is a unital, finite-dimensional $*$-representation, so that it is a direct sum of irreducible $*$-representations. Therefore for any $f\in\mathcal{C}(X)\otimes A$ and any $y\in Y$:

$$(1)\begin{cases}\phi(f)(y)=u_1(y)(\bigoplus_{i=1}^{q'(y)} f_{r_i'(y)}(z_i'(y)))u_1(y)^* \\ \text{where } u_1(y)\in\mathcal{U}(B), q'(y)\in\mathbb{N}, r_i'(y)\in\{k_1,\dots,k_n\}, z_i'(y)\in X \\ \text{for } y\in Y \text{ and } 1\leqslant i\leqslant q'(y).\end{cases}$$

Using (1) and the hypothesis, it is easy to deduce that $z_1'(y),\dots,z_{q'}'(y)$ belong to the fibre of φ at y, for all $y\in Y$.

For each $y\in Y$, let $q(y)$, $r_1(y)<r_2(y)<\dots<r_{q(y)}(y)$ (where $\{r_1(y),\dots,r_{q(y)}(y)\}\subset\{k_1,\dots,k_n\}$) be the positive integers and let $X_y^{(1)},\dots,X_y^{(q(y))}$ be the subsets of the fibre of φ at y, determined by the condition that:

$$\bigoplus_{i=1}^{q(y)}(\mathscr{C}(X_y^{(i)})\otimes M_{r_i(y)})\ni\bigoplus_{i=1}^{q(y)} f_{r_i(y)}\big|X_y^{(i)}\xmapsto{\phi_y}\phi(f)(y)\in M_1,$$

$$f\in\mathscr{C}(X)\otimes A$$

defines an isometric, unital $*$-homomorphism ϕ_y.

We denote: $q:=q(y')$, $r_i:=r_i(y')$, $s_j(y):=|X_y^{(j)}|$, for $1\leq i\leq q$, $y\in Y$, $1\leq j\leq q(y)$. (Here, for a finite set $F(\subset X)$, $|F|$ is its cardinal number).

The proof will be given in several steps.

Step 1. We shall prove that:

(*) the map $q(.):Y\longrightarrow\mathbb{N}$ is constant on a neighborhood of y'

If this is not true, then $(\exists)$ $(y_t^{(1)})_{t\in\mathbb{N}}$ in Y, $\lim_t y_t^{(1)}=y'$, $(\exists)$ positive integers $q'\neq q$, $r_1'<r_2'<\dots<r_{q'}'$, (where $\{r_1',\dots,r_{q'}'\}\subset\{k_1,\dots,k_n\}$) such that $q(y_t^{(1)})=q'$, $r_i(y_t^{(1)})=r_i'$ $(t\in\mathbb{N},\ 1\leq i\leq q')$.

We consider two cases:

Case (a): $(\exists)$ $r\in\{r_1',\dots,r_{q'}'\}\setminus\{r_1,\dots,r_q\}$.

If $e:=1_{\mathscr{C}(X)}\otimes 1_r\in\mathscr{C}(X)\otimes M_r$ $(\subset\mathscr{C}(X)\otimes A)$, then, using the fact that ϕ_y is isometric $(y\in Y)$, we obtain:

$$\lim_t\Big\|\phi_{y_t^{(1)}}\Big(\bigoplus_{i=1}^{q'}e_{r_i'}\Big|X_{y_t^{(1)}}^{(i)}\Big)\Big\|=\lim_t\Big\|\bigoplus_{i=1}^{q'}e_{r_i'}\Big|X_{y_t^{(1)}}^{(i)}\Big\|=1=$$

$$=\Big\|\phi_{y'}\Big(\bigoplus_{i=1}^{q}e_{r_i}\Big|X_{y'}^{(i)}\Big)\Big\|=0$$

a contradiction.

Case (b): $(\exists)\ r' \in \{r_1,\dots,r_q\} \setminus \{r'_1,\dots,r'_{q'}\}$.

If $e' := 1_{\mathscr{C}(X)} \otimes 1_{r'} \in \mathscr{C}(X) \otimes M_{r'} (\subset \mathscr{C}(X) \otimes A)$, we have:

$$0=\lim_t \left\| \phi_{y_t^{(1)}} \left(\bigoplus_{q=1}^{q'} e'_{r'_i} \middle| X_{y_t^{(i)}}^{(i)} \right) \right\| = \left\| \phi_{y'} \left(\bigoplus_{i=1}^{q} e'_{r_i} \middle| X_{y'}^{(i)} \right) \right\| =$$

$$= \left\| \bigoplus_{i=1}^{q} e'_{r_i} \middle| X_{y'}^{(i)} \right\| = 1$$

a contradiction.

The proof of Step 1 is complete.

Let $\tilde{V} \in \mathcal{V}(y')$ be a neighborhood given by (*).

We denote $X_{y'}^{(i)} = \{z_{i,1}(y'),\dots,z_{i,s_i}(y')\}$, $1 \leq i \leq q$. We choose $U_{i,s} \in \mathcal{V}(z_{i,s}(y'))$, $1 \leq i \leq q$, $1 \leq s \leq s_i$, $V_o \in \mathcal{V}(y')$ such that every map $\varphi_{i,s} : U_{i,s} \longrightarrow V_o$, given by $\varphi_{i,s}(x) := \varphi(x)$ $(x \in U_{i,s})$ is a homeomorphism and $U_{i,s} \cap U_{i,s'} = \emptyset$ for any $1 \leq i \leq q$, $1 \leq s$, $s' \leq s_i$, $s \neq s'$ (it is possible since φ is a p-fold covering map). Using smaller neighborhoods, if necessary, we may suppose that $V_o = \tilde{V}$.

Step 2. We prove that:

$\{(r_j(y), s_j(y)) \mid 1 \leq j \leq q\} = \{(r_j, s_j) \mid 1 \leq j \leq q\}$ on a neighborhood of y' (smaller then $\tilde{V}$).

If this is not true, then $(\exists)$ $(y_t^{(2)})_{t \in N}$ in $\tilde{V}$, $\lim_t y_t^{(2)} = y'$ $(\exists)$ positive integers $\tilde{r}_1 < \tilde{r}_2 < \dots < \tilde{r}_q$, $\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_q$ such that for any $1 \leq i \leq q$ and any $t \in \mathbb{N}$:

$$r_i(y_t^{(2)}) = \tilde{r}_i \in \{k_1,\dots,k_n\},\quad s_i(y_t^{(2)}) = \tilde{s}_i$$

(1)

$$\{(\tilde{r}_j, \tilde{s}_j) \mid 1 \leq j \leq q\} \neq \{(r_j, s_j) \mid 1 \leq j \leq q\}.$$

Using the fact that ϕ_y is isometric $(y \in Y)$, we obtain:

(2) $$\lim_t \left\| \bigoplus_{i=1}^{q} f_{r_i} \,|\, X^{(i)}_{y_t^{(2)}} \right\| = \left\| \bigoplus_{i=1}^{q} f_{r_i} \,|\, X^{(i)}_{,} \right\| \quad , \quad f \in \mathcal{C}(X) \otimes A .$$

If we denote:

$$X^{(i)}_{y_t^{(2)}} = \{ v_t^{(i)}(1), \ldots, v_t^{(i)}(\tilde{s}_i) \} \quad (\subset \varphi^{-}(\{y_t^{(2)}\})), \ 1 \leq i \leq q, \ t \in N$$

we may suppose that:

$$\lim_t v_t^{(i)}(s) := v^{(i)}(s) \in \varphi^{-1}(\{y'\}), \quad 1 \leq i \leq q, \quad 1 \leq s \leq \tilde{s}_i .$$

Let us observe that :

$$v^{(i)}(s) \neq v^{(i)}(s')$$

if $1 \leq i \leq q$, $1 \leq s, s' \leq \tilde{s}_i$, $s \neq s'$ (Let us suppose that $(\exists)$ $1 \leq i_o \leq q$ $(\exists)$ $1 \leq s_o, s'_o \leq \tilde{s}_{i_o}$, $s_o \neq s'_o$ such that $x_o := v^{(i_o)}(s_o) = v^{(i_o)}(s'_o)$. $(\in \varphi^{-1}(\{y'\}))$. Since φ is a covering map, $(\exists)$ $U_{x_o} \in \mathcal{V}(x_o)$, $(\exists)$ $V_{y'} \in \mathcal{V}(y')$ such that the map $\varphi_o : U_{x_o} \longrightarrow V_{y'}$, given by $\varphi_o(x) := \varphi(x)$, $x \in U_{x_o}$, is a homeomorphism. Then $(\exists)$ $t_o = t_o(U_{x_o}) \in N$ such that $v_t^{(i_o)}(s_o)$, $v_t^{(i_o)}(s'_o) \in U_{x_o}$ if $t \geq t_o$; because $\varphi_o(v_t^{(i_o)}(s_o)) = \varphi_o(v_t^{(i_o)}(s'_o))$ $(= y_t^{(2)})$ for $t \geq t_o$; it follows that $v_t^{(i_o)}(s_o) = v_t^{(i_o)}(s'_o)$ for $t \geq t_o$, a contradiction).

We denote:

$$X^{(i)} := \{ v^{(i)}(1), \ldots, v^{(i)}(\tilde{s}_i) \} \quad , \quad 1 \leq i \leq q .$$

Then, (2) can be written:

(3) $$\max_{1\leq i\leq q} \left\| f_{\tilde{r}_i} \middle| X^{(i)} \right\| = \max_{1\leq i\leq q} \left\| f_{r_i} \middle| X^{(i)}_{y'} \right\| , \quad f \in \mathcal{C}(X) \otimes A$$

Using (1), one can find a map $f \in \mathcal{C}(X) \otimes A$ which contradicts (3).

Thus, the proof of Step 2 is complete. We may suppose that $r_i(y)=r_i$, $s_i(y)=s_i$ for $1\leq i\leq q$, $y \in \tilde{V}$.

Step 3. We prove that:

for arbitrary $U'_{i,s} \in \mathcal{V}(z_{i,s}(y'))$, $1\leq i\leq q$, $1\leq s\leq s_i$, $(\exists)$ $V' \in \mathcal{V}(y')$, $V' \subset \tilde{V}$ such that: $X^{(i)}_y \cap U'_{i,s} \neq \emptyset$ if $y \in V'$ and $1\leq i\leq q$, $1\leq s\leq s_i$

If this is not true, then $(\exists)$ $1\leq i'\leq q$, $(\exists)$ $x' \in X^{(i')}_{y'}$, $(\exists)$ $W \in \mathcal{V}(x')$, $(\exists)$ $(y^{(3)}_t)_{t\in \mathbb{N}}$ in $\tilde{V}$ such that: $\lim_t y^{(3)}_t = y'$ and $X^{(i')}_{y^{(3)}_t} \cap W=\emptyset$ for $t \in \mathbb{N}$. Let $f_1 \in \mathcal{C}(X) \otimes A$ be such that $(f_1)_{r_{i'}}(x') = 1_{r_{i'}} \in M_{r_{i'}}$ $(\subset A)$, $\operatorname{supp} f_1 \subset W$, $f_1(X) \subset M_{r_{i'}}$ $(\subset A)$. Then, we have:

$$\phi_{y'}\left(\bigoplus_{i=1}^{q} (f_1)_{r_i} \middle| X^{(i)}_{y'}\right) = \lim_t \phi_{y^{(3)}_t}\left(\bigoplus_{i=1}^{q} (f_1)_{r_i} \middle| X^{(i)}_{y^{(3)}_t}\right) = 0$$

and since $\phi_{y'}$ is isometric:

$$(f_1)_{r_{i'}}(x')=0$$

a contradiction.

Thus, the proof of Step 3 is complete.

We may suppose that $X^{(i)}_y \cap U_{i,s} \neq \emptyset$ if $y \in \tilde{V}$ and $1\leq i\leq q$, $1\leq s\leq s_i$.

If we denote for $1\leq i\leq q$, $1\leq s\leq s_i$, $y \in \tilde{V}$:

$$\{z_{i,s}(y)\} = X_y^{(i)} \cap U_{i,s} \quad .$$

we have:

$$X_y^{(i)} = \{z_{i,1}(y),\dots,z_{i,s_i}(y)\} .$$

It is plain that $z_{i,s}:\tilde{V} \longrightarrow U_{i,s}$ is a homeomorphism and $\varphi \circ z_{i,s} =$ $= id_{\tilde{V}}$ if $1 \leq i \leq q$, $1 \leq s \leq s_i$.

Using (1) and the previous remarks, one gets for any $f \in \mathcal{C}(X) \otimes A$ and any $y \in \tilde{V}$:

$$\Phi(f)(y) = u_2(y)\Big(\bigoplus_{i=1}^{q}\Big(\bigoplus_{s=1}^{s_i}(f_{r_i}(z_{i,s}(y)) \otimes 1_{p_{i,s}(y)})\Big)\Big)u_2(y)^*$$

where $u_2(y) \in U(1)$, $p_{i,s}(y) \in \mathbb{N}$ for $1 \leq i \leq q$, $1 \leq s \leq s_i$.

Step 4. We shall prove that:

all the maps $p_{i,s}(\cdot)$ are constant on a neighborhood of y' (smaller then $\tilde{V}$) .

If this is not true, then $(\exists)$ $1 \leq \bar{i} \leq q$; $1 \leq \bar{s} \leq s_{\bar{i}}$, $(\exists)$ $(y_t^{(4)})_{t \in \mathbb{N}}$ in $\tilde{V}$, $(\exists)$ $p'_{i,s} \in \mathbb{N}$ such that:

$$\lim_t y_t^{(4)} = y' \ , \ \lim_t p_{i,s}(y_t) = p'_{i,s} \qquad (1 \leq i \leq q,\ 1 \leq s \leq s_i)$$

$$p'_{\bar{i},\bar{s}} \neq p_{\bar{i},\bar{s}}(y') \ .$$

Let $h \in \mathcal{C}(X) \otimes A$ be such that: $h_{r_{\bar{i}}}(z_{\bar{i},\bar{s}}(y')) = 1_{r_{\bar{i}}} \in M_{r_{\bar{i}}}$ $(\subset A)$, $h(X) \subset M_{r_{\bar{i}}}(\subset A)$,

$h_{r_{\bar{i}}}(x)=0$ for $x \in X_{y'}^{(4)} \setminus \{z_{\bar{i},\bar{s}}(y')\}$. Then we have:

$$\lim_t tr_\ell(\Phi(h)(y_t^{(4)}))=tr_\ell(\Phi(h)(y'))$$

(tr_λ denotes the usual trace on M_λ , $\lambda \in \mathbb{N}$) and on the other hand:

$$\lim_t tr_\ell(\Phi(h)(y_t^{(4)}))=\lim_t\Big(\sum_{i=1}^{q}\Big(\sum_{s=1}^{s_i} p_{i,s}(y_t^{(4)})\, tr_{r_i}(h_{r_i}(z_{i,s}(y_t^{(4)})))\Big)\Big)=$$

$$=\sum_{i=1}^{q}\Big(\sum_{s=1}^{s_i} p'_{i,s}\, tr_{r_i}(h_{r_i}(z_{i,s}(y')))\Big)=p'_{\bar{i},\bar{s}}\, r_{\bar{i}} \neq p_{\bar{i},\bar{s}}(y')\cdot r_{\bar{i}}=$$

$$=\sum_{i=1}^{q}\Big(\sum_{s=1}^{s_i} p_{i,s}(y')\, tr_{r_i}(h_{r_i}(z_{i,s}(y')))\Big)=tr_\ell(\Phi(h)(y'))$$

a contradiction.

The proof of Step 4 is complete.

Let us denote $p_{i,s}:=p_{i,s}(y')\,(\in \mathbb{N})$, $1\leq i\leq q$, $1\leq s\leq s_i$. Then we may suppose that for any $f \in \mathcal{C}(X) \otimes A$ and any $y \in \tilde{V}$, we have:

$$\Phi(f)(y)=u_2(y)\Big(\bigoplus_{i=1}^{q}\Big(\bigoplus_{s=1}^{s_i}(f_{r_i}(z_{i,s}(y))\otimes 1_{p_{i,s}})\Big)\Big)u_2(y)^*$$

We denote $G:=U(l)$, $S:=\bigoplus_{i=1}^{q}\Big(\bigoplus_{s=1}^{s_i}(1_{r_i}\otimes U(p_{i,s}))\Big)$, $G/S:=\{gS \mid g \in G\}$ and $\Pi: G \longrightarrow G/S$, the canonical map. Then, G/S will be embedded into the set of all unital, isometric, $*$-homomorphisms from $\bigoplus_{i=1}^{q}\Big(\bigoplus_{1}^{s_i} M_{r_i}\Big)$ to M_l , by the formula:

$$\Pi(g)\Big(\bigoplus_{i=1}^{q}\Big(\bigoplus_{s=1}^{s_i} a_{i,s}\Big)\Big):=g\Big(\bigoplus_{i=1}^{q}\Big(\bigoplus_{s=1}^{s_i}(a_{i,s}\otimes 1_{p_{i,s}})\Big)\Big)g^*$$

where $g \in G$ and $a_{i,s} \in M_{r_i}$, $1 \leq i \leq q$, $1 \leq s \leq s_i$.

The previous results show that we can define a continuous map $\Theta : \tilde{V} \longrightarrow G/S$, by the formula:

$$\Theta(y)\Big(\bigoplus_{i=1}^{q}\Big(\bigoplus_{s=1}^{s_i} f_{r_i}(z_{i,s}(y))\Big)\Big) := \phi(f)(y),$$
$$y \in \tilde{V},\ f \in \mathcal{C}(X) \otimes A.$$

Step 5. We prove that:

there is a neighborhood $V_1 \in \mathcal{V}(y')$, $V_1 \subset \tilde{V}$, and there is a continuous map $\tilde{u} : V_1 \to G$ such that the diagram:

$$\begin{tikzcd}
G \arrow[r, "\pi"] & G/S \\
 & V_1 \arrow[u, "\Theta_{|V_1}"'] \arrow[ul, dashed, "\tilde{u}"]
\end{tikzcd}$$

commutes.

Since G is a compact Lie group and S is a closed subgroup it is not difficult to see that $(\exists)$ a neighborhood U of $\Theta(y')$, in G/S, and a continuous map $\gamma : U \longrightarrow G$ such that $\pi \circ \gamma = id_U$. Let $V_1 \subset \tilde{V}$ be a neighborhood of y' in Y such that $\Theta(V_1) \subset U$ (Θ is continuous). We define $\tilde{u} : V_1 \longrightarrow G$ by $\tilde{u} := \gamma \circ \Theta_{|V_1}$

It is plain that $\tilde{u}$ is continuous and $\pi \circ \tilde{u} = \Theta_{|V_1}$

Thus, the proof of Step 5 is complete.

The proof of the proposition follows from:

$$\Theta_{|V_1}(y)\Big(\bigoplus_{i=1}^{q}\Big(\bigoplus_{s=1}^{s_i} f_{r_i}(z_{i,s}(y))\Big)\Big) = (\pi \circ \tilde{u})(y)\Big(\bigoplus_{i=1}^{q}\Big(\bigoplus_{s=1}^{s_i} f_{r_i}(z_{i,s}(y))\Big)\Big)$$

where $y \in V_1$ and $f \in \mathcal{C}(X) \otimes A$.

$\square$

2.5. <u>Proposition</u>

Let $\varphi : X \longrightarrow Y$ be a p-fold covering map ($p \in \mathbb{N}$), where X is a compact metric space and assume Y is an AR-space. Then, there exists a partition $(U_i)_{i=1}^{p}$ of X into clopen sets and there exist homeomorphisms $z_i : Y \longrightarrow U_i$ satisfying $\varphi \circ z_i = id_Y$ $(1 \leq i \leq p)$ such that:

if $\Phi : \mathcal{C}(X) \otimes A \longrightarrow \mathcal{C}(Y) \otimes B$ is a φ-compatible $*$-homomorphism then, there are $u \in \mathcal{C}(Y, \mathcal{U}(B))$ and $*$-homomorphisms $\Psi_1, \ldots, \Psi_p : A \longrightarrow B$ such that

$$\Phi(f)(y) = u(y)\Big(\bigoplus_{i=1}^{p} \Psi_i(f(z_i(y)))\Big)u(y)^*$$

for all $f \in \mathcal{C}(X) \otimes A$ and all $y \in Y$.

<u>Proof</u>

It will be enough to consider the case when $A = \bigoplus_{i=1}^{n} M_{k_i}$, $B = M_l$.

The proof will be given in two steps.

<u>Step 1</u>

We prove the proposition under the additional assumption that $Y = Q^{\omega}$ (=the Hilbert cube).

Since $Y (= Q^{\omega})$ is a connected, locally path-connected, simply connected topological space, there is a homeomorphism $H : X \to Y \times \{1, 2, \ldots, p\}$ such that the diagram:

$$\begin{tikzcd} X \arrow[r, "H"] \arrow[d, "\varphi"'] & Y \times \{1,2,\ldots,p\} \arrow[dl, "\psi"] \\ Y & \end{tikzcd}$$

commutes, where $\{1,2,\ldots,p\}$ is a discrete topological space and ψ denotes the canonical projection (see, for instance [7]).

For each $1\leq i\leq p$ we define: $U_i := H^{-1}(Y \times \{i\})$, the homeomorphism $h_i : Y \longrightarrow Y \times \{i\}$ given by $h_i(y)=(y,i)$, $y \in Y$ and $z_i : Y \longrightarrow U_i$, $z_i := H^{-1} \circ h_i$.

Let us consider two neighborhoods $U=\overline{U} \in \mathcal{V}(y_1)$, $V=\overline{V} \in \mathcal{V}(y_2)$, given by Proposition 2.4, where $y_1, y_2 \in Y$. It is not difficult to see that there are positive integers q, $r_1 < r_2 < \ldots < r_q$ ($\{r_1,\ldots,r_q\} \subset \{k_1,\ldots,k_n\}$), $s_1,\ldots,s_q$, $p_{i,s}$, $[i,s]$ ($1\leq i\leq q$, $1\leq s\leq s_i$), where each $[i,s] \in \{1,2,\ldots,p\}$ and $[i,s] \neq [i,s']$ for all i, whenever $s\neq s'$, such that, if U and V are small enough:

$$(1)\quad \begin{cases} \Phi(f)(y)=u_U(y)\left(\bigoplus_{i=1}^{q}\left(\bigoplus_{s=1}^{s_i}\left(f_{r_i}(z_{[i,s]}(y)) \otimes 1_{p_{i,s}}\right)\right)\right)u_U(y)^* \\ \text{for } f \in \mathcal{C}(X) \otimes A,\ y \in U, \text{ where } u_U \in \mathcal{C}(U, \mathcal{U}(B)) \end{cases}$$

and:

$$(2)\quad \begin{cases} \Phi(f)(y)=u_V(y)\left(\bigoplus_{i=1}^{q}\left(\bigoplus_{s=1}^{s_i}\left(f_{r_i}(z_{[i,s]}(y)) \otimes 1_{p_{i,s}}\right)\right)\right)u_V(y)^* \\ \text{for } f \in \mathcal{C}(X) \otimes A,\ y \in V, \text{ where } u_V \in \mathcal{C}(V, \mathcal{U}(B)). \end{cases}$$

(Note that $Y=Q^{\omega}$ is a connected space).

Using (1) and (2), one gets:

$$(u_U)|_{U\cap V} = ((u_V)|_{U\cap V})\cdot w$$

where $w \in \mathcal{C}\left(U\cap V, \bigoplus_{i=1}^{2}\left(\bigoplus_{s=1}^{s_i}\left(1_{r_i} \otimes \mathcal{U}(p_{i,s})\right)\right)\right)$.

Let us suppose that:

$U \cap V$ is a retract of V.

Using this assumption, it follows that there exists $\tilde{w} \in \mathcal{C}\left(V, \bigoplus_{i=1}^{2}\left(\bigoplus_{s=1}^{s_i}\left(1_{r_i} \otimes \mathcal{U}(p_{i,s})\right)\right)\right)$ such that:

$\tilde{w}|_{U\cap V}=w$

If we define:

$$u_{U\cup V}(y) := \begin{cases} u_U(y), & \text{if } y \in U \\ u_V(y)\cdot \tilde{w}(y), & \text{if } y \in V \end{cases}$$

we obtain a map $u_{U\cup V} \in \mathcal{C}(U\cup V, U(B))$ such that:

$$\phi(f)(y) = u_{U\cup V}(y)\Big(\bigoplus_{i=1}^{2}\Big(\bigoplus_{s=1}^{s_i}(f_{r_i}(z_{[i,s]}y))\otimes 1_{p_{i,s}}\Big)\Big)u_{U\cup V}(y)^*$$

for $f \in \mathcal{C}(X) \otimes A$, $y \in U \cup V$.

Using Proposition 2.4. and the fact that $Y=Q^{\omega}$ is a compact space, we can find a closed covering of Y, denoted by $\{V_j\}_{j=1}^{m}$ such that on each V_j a formula as in the relations (1) or (2) holds and each V_j is a product of closed intervals $\prod_{k=1}^{\infty} I_k^{(j)}$, such that: $I_k^{(j)} = [0, \frac{1}{k}]$ for $k \geq 1$, $k \notin \Lambda$, where Λ is a finite subset of $\mathbb{N}$. Using a rafiniment of $\{V_j\}_{j=1}^{m}$, if necessary, we may also suppose that: $\emptyset \neq (\bigcup_{j=1}^{t} V_j) \cap V_{t+1}$ and $(\bigcup_{j=1}^{t} V_j) \cap V_{t+1}$ is a retract of V_{t+1}, for each $1 \leq t \leq m-1$ $(m \geq 2)$.

The proof in the case $Y=Q^{\omega}$ now follows using the previous discussion for $\bigcup_{j=1}^{t} V_j$ and V_{t+1}, $1 \leq t \leq m-1$.

Step 2

Let $\varphi' : X' \longrightarrow Y'$ be a p-fold covering map, where X' is a compact metric space and $Y' \subseteq Q^{\omega}$ is a retract of Q^{ω}. Let $\phi' : \mathcal{C}(X') \otimes A \longrightarrow \mathcal{C}(Y') \otimes B$ be a φ'-compatible $*$-homomorphism. We shall prove the proposition for φ' and ϕ'.

It is obvious that Y' is a connected, locally path-connected, simply connected space; as in Step 1, there exists a partition $(U_i')_{i=1}^{p}$ of X' into clopen sets and there exist homeomorphisms $z_i' : Y' \to U_i'$ satisfying $\varphi' \circ z_i' = id_{Y'}$, $(1 \leq i \leq p)$.

Using Proposition 2.4 and the fact that Y' is a connected space it is not difficult to see that there are positive integers $q', r'_1 < r'_2 < \dots < r'_{q'}$ $(\{r'_1, \dots, r'_{q'}\} \subset \{k_1, k_2, \dots, k_n\})$, $s'_1, \dots, s'_{q'}$, $p'_{i,s}$, $\langle i,s \rangle$ $(1 \leq i \leq q', 1 \leq s \leq s'_i)$ (where each $\langle i,s \rangle \in \{1,2,\dots,p\}$ and $\langle i,s \rangle \neq \langle i,s' \rangle$ for $s \neq s'$) such that:

$$\phi'(f')(y') = u(y')\Big(\bigoplus_{i=1}^{q'}\Big(\bigoplus_{s=1}^{s'_i}\big(f'_{r'_i}(z'_{\langle i,s \rangle}(y')) \otimes 1_{p'_{i,s}}\big)\Big)\Big)u(y')^*$$

for $f' \in \mathcal{C}(X') \otimes A$, $y' \in Y'$, where $u(y') \in U(l)$. We denote $G := U(l)$, $S := \bigoplus_{i=1}^{q'} \Big(\bigoplus_{s=1}^{s'_i} (1_{r'_i} \otimes U(p'_{i,s}))\Big)$, $G/S := \{gS \mid g \in G\}$. Let $\pi : G \longrightarrow G/S$ be the canonical map. G/S will be embedded in the set of all unital, isometric $*$-homomorphisms from $\bigoplus_{i=1}^{q'} \Big(\bigoplus_{1}^{s'_i} M_{r'_i}\Big)$ to M_l by the formula:

$$\pi(g)\Big(\bigoplus_{i=1}^{q'}\Big(\bigoplus_{s=1}^{s'_i} b_{i,s}\Big)\Big) = g\Big(\bigoplus_{i=1}^{q'}\Big(\bigoplus_{s=1}^{s'_i}(b_{i,s} \otimes 1_{p'_{i,s}})\Big)\Big)g^*$$

where $g \in G$ and $b_{i,s} \in M_{r'_i}$, $1 \leq i \leq q'$, $1 \leq s \leq s'_i$. Then, we can define a continuous map $\theta' : Y' \longrightarrow G/S$ by the formula:

$$\theta'(y')\Big(\bigoplus_{i=1}^{q'}\Big(\bigoplus_{s=1}^{s'_i} f'_{r'_i}(z'_{\langle i,s \rangle}(y'))\Big)\Big) := \phi'(f')(y')$$

for $y' \in Y'$, $f' \in \mathcal{C}(X') \otimes A$.

Let $r : Q^{\omega} \longrightarrow Y'$ be a retraction. Using Step 1, it follows that $(\exists)$ a continuous map $u_1 : Q^{\omega} \longrightarrow G$ such that the diagram:

$$\begin{tikzcd}
G \arrow[r, "\pi"] & G/S' \\
 & Y' \arrow[u, "\theta'"'] \\
 & Q^{\omega} \arrow[u, "r"'] \arrow[uul, dashed, "u_1"]
\end{tikzcd}$$

commutes. Then, if we denote by $i_{Y'}$ the inclusion of Y' into Q^{ω}, and $u' := u_1 \circ i_{Y'}$, we obtain the following commutative diagram:

$$\begin{tikzcd}
G \arrow[r, "\pi"] & G/S' \\
 & Y' \arrow[u, "\theta'"'] \arrow[ul, "u'"]
\end{tikzcd}$$

Since u' is continuous, Step 2 is complete.

The proof of the proposition now follows from Theorem 1.1., Step 1 and Step 2.

$\square$

§3.

In this section we shall give the proof of the theorem stated in the introduction.

For any $1\leq k\leq p$ and any $1\leq l\leq q$ (where p and q are positive integers) we define the map:

$$[0,2\pi]^2 \ni (t_1,t_2) \xmapsto{z_{k,l}} \left(\exp\left(i\cdot\frac{t_1+(k-1)2\pi}{p}\right), \exp\left(i\cdot\frac{t_2+(l-1)2\pi}{q}\right)\right) \in \mathbb{T}^2$$

When working with elements of the form $\bigoplus\limits_{(k,l)} f(z_{k,l}(t_1,t_2))$ (where $f\in \mathcal{C}(\mathbb{T}^2)\otimes M_n$, $t_1,t_2\in[0,2\pi]$, $1\leq k\leq p$, $1\leq l\leq q$) it will be convenient to assume that the set of indices (k,l) is endowed with the lexicographical order.

3.1. Lemma

Let $\phi : \mathcal{C}(\mathbb{T}^2)\otimes M_n \longrightarrow \mathcal{C}(\mathbb{T}^2)\otimes M_{npq}$ be an isometric $*$-homomorphism compatible with the covering map $\mathbb{T}^2\ni(z_1,z_2)\longmapsto (z_1^p,z_2^q)\in\mathbb{T}^2$, where $p,q\in\mathbb{N}$. Then, there exists $u_\phi\in \mathcal{C}([0,2\pi]^2, U(npq))$ such that:

$$\phi(f)(\exp(it_1),\exp(it_2)) = u_\phi(t_1,t_2)\cdot\left(\bigoplus_{(k,l)} f(z_{k,l}(t_1,t_2))\right)\cdot u_\phi(t_1,t_2)^*$$

for $f\in\mathcal{C}(\mathbb{T}^2)\otimes M_n$; $t_1,t_2\in[0,2\pi]$.

Proof

Using Proposition 2.4. and the fact that ϕ is isometric one can easily get:

$$(1)\quad \begin{cases} \Phi(f)(\exp(it_1),\exp(it_2)) = u(t_1,t_2)\Big(\bigoplus\limits_{(k,l)} f(z_{k,l}(t_1,t_2))\Big)u(t_1,t_2)^* \\ \text{for } f\in \mathcal{C}(\mathbb{T}^2)\otimes M_n \text{ and } t_1,t_2\in[0,2\pi]^2, \text{ where } u(t_1,t_2)\in \\ \in U(npq) \text{ for all } t_1,t_2 \end{cases}$$

Let σ be the canonical map from $G:=U(npq)$ onto $G/S:=$ $:=\{gS \mid g\in G\}$, where $S:=\bigoplus\limits_{1}^{pq}\mathbb{T}.1_n$. We embed G/S into the set of all isometric, unital $*$-homomorphisms from $\bigoplus\limits_{1}^{pq} M_n$ to M_{npq} as in the proof of Proposition 2.4.

Using (1), we can define a continuous map $\Theta:[0,2\pi]^2\to G/S$, by the formula:

$$\Theta(t_1,t_2)\Big(\bigoplus_{(k,l)} f(z_{k,l}(t_1,t_2))\Big) := \Phi(f)(\exp(it_1),\exp(it_2)).$$

for $(t_1,t_2)\in[0,2\pi]^2$, $f\in\mathcal{C}(\mathbb{T}^2)\otimes M_n$.

Since $[0,2\pi]^2$ is an AR-space, using Proposition 2.5 one gets a continuous map $u_\Phi:[0,2\pi]^2\longrightarrow G$ such that the diagram:

$$\begin{tikzcd} G \arrow[r, "\sigma"] & G/S \\ & {[0,2\pi]^2} \arrow[u, "\Theta"'] \arrow[ul, dashed, "u_\Phi"] \end{tikzcd}$$

commutes. The proof is complete. $\square$

3.2. <u>Lemma</u>

If $\Phi,\Psi:\mathcal{C}(\mathbb{T}^2)\otimes M_n\longrightarrow\mathcal{C}(\mathbb{T}^2)\otimes M_{npq}$ are isometric $*$-homomorphisms compatible with the covering map $\mathbb{T}^2\ni(z_1,z_2)\longmapsto(z_1^p,z_2^q)\in\mathbb{T}^2$, where $p,q\in\mathbb{N}$, then they are inner equivalent.

Proof

By Lemma 3.1, $(\exists)\ u_\Phi, u_\Psi \in \mathcal{C}([0,2\pi]^2, U(npq))$ such that:

$$\Phi(f)(\exp(it_1),\exp(it_2)) = u_\Phi(t_1,t_2)\Big(\bigoplus_{(k,\ell)} f(z_{k,\ell}(t_1,t_2))\Big)u_\Phi(t_1,t_2)^*$$

$$\Psi(f)(\exp(it_1),\exp(it_2)) = u_\Psi(t_1,t_2)\Big(\bigoplus_{(k,\ell)} f(z_{k,\ell}(t_1,t_2))\Big)u_\Psi(t_1,t_2)^*$$

for $f \in \mathcal{C}(\mathbb{T}^2)\otimes M_n$, $t_1,t_2 \in [0,2\pi]$.

We define $G := Z_p \times Z_q$, where $Z_p = Z/pZ$, $Z_q = Z/qZ$. Let $\{e_g\}_{g\in G}$ be the canonical orthonormal basis of $l^2(G)$. We consider the following isomorphisms of groups G: $(h_1,h_2) \xmapsto{s_1} (h_1-1,h_2) \in G$, G; $(h_1,h_2) \xmapsto{s_2} (h_1,h_2-1) \in G$ ($1 \in Z_p$ (resp. $1 \in Z_q$) denotes the image of $1 \in Z$ by the canonical map from Z onto Z_p (resp. by the canonical map from Z onto Z_q)). We also define the operators $S_1, S_2 \in B(l^2(G))$ by $S_1(e_g) = e_{s_1^{-1}(g)}$, $S_2(e_g) = e_{s_2^{-1}(g)}$, $g \in G$.

We consider the bijection $\gamma : \{1,2,\dots,pq\} \longrightarrow G$ obtained identifying the matrix:

$$\begin{bmatrix} 1 & 2 & \dots & q \\ q+1 & q+2 & & 2q \\ \vdots & & & \\ (p-1)q+1 & (p-1)q+2 & \dots & pq \end{bmatrix}$$

(with entries in Z) with the matrix:

$$\begin{bmatrix} (p-1,q-1) & \cdots & (p-1,1) & (p-1,0) \\ \vdots & & \vdots & \vdots \\ (2,q-1) & & (2,1) & (2,0) \\ (1,q-1) & & (1,1) & (1,0) \\ (0,q-1) & \cdots & (0,1) & (0,0) \end{bmatrix}$$

(with entries in $G=Z_p \times Z_q$). γ induces in an obvious way an identification between the canonical orthonormal basis of $\mathbb{C}^{pq}$ and $l^2(G)$, which induces an identification $M_{pq} = B(l^2(G))$. We denote $\lambda_i := \gamma^{-1} \circ s_i \circ \gamma$, $i=1,2$, and let $\{e^{g,l}\}_{g,l \in C}$ be the canonical set of matrix units for $B(l^2(G))$ (i.e. $e^{g,l} = \langle \cdot, e_l \rangle e_g$, $g,l \in G$).

Using the relations:

$$\phi(f)(\exp(i\cdot 0), \cdot) = \phi(f)(\exp(i\cdot 2\pi), \cdot)$$

$$\phi(f)(\cdot, \exp(i\cdot 0)) = \phi(f)(\cdot, \exp(i\cdot 2\pi)),$$

$f \in \mathcal{C}(\mathbb{T}^2) \otimes M_n$

one can obtain:

$$(1) \begin{cases} u_\phi(2\pi, \cdot)^* u_\phi(0, \cdot) = (1_{\mathcal{C}([0,2\pi], M_n)} \otimes S_1) \cdot \Delta_{\phi,1} \\ u_\phi(\cdot, 2\pi)^* u_\phi(\cdot, 0) = (1_{\mathcal{C}([0,2\pi], M_n)} \otimes S_2) \cdot \Delta_{\phi,2} \\ \text{with } \Delta_{\phi,1} = \bigoplus_{k=1}^{pq} b'_k \cdot 1_n,\ \Delta_{\phi,2} = \bigoplus_{k=1}^{pq} c'_k \cdot 1_n \text{ and} \\ b'_k, c'_k \in \mathcal{C}([0,2\pi], \mathbb{T}),\ 1 \le k \le pq. \end{cases}$$

(We use the obvious identifications)

In the same way one can define $\Delta_{\Psi,1}$, $\Delta_{\Psi,2}$. Then one gets:

$$(2)\begin{cases} u_{\Phi}(2\pi,\cdot)^{*}u_{\Phi}(0,\cdot)=u_{\Psi}(2\pi,\cdot)^{*}u_{\Psi}(0,\cdot)\left(\bigoplus_{k=1}^{pq}b_k\cdot 1_n\right)\\ \text{where } b_k\in\mathcal{C}([0,2\pi],\mathbb{T}),\ 1\le k\le pq\end{cases}$$

$$(3)\begin{cases} u_{\Phi}(\cdot,2\pi)^{*}u_{\Phi}(\cdot,0)=u_{\Psi}(\cdot,2\pi)^{*}u_{\Psi}(\cdot,0)\cdot\left(\bigoplus_{k=1}^{pq}c_k\cdot 1_n\right)\\ \text{where } c_k\in\mathcal{C}([0,2\pi],\mathbb{T}),\ 1\le k\le pq\end{cases}$$

For a scalar valued path g avoiding 0 we denote by $A(g)$ the angle sweeped by g with respect to the point 0 (see [4]). If in addition g is closed then $A(g)=2\pi\cdot W(g)$ (where $W(g)$ is the winding number of g).

Let us suppose that we have found maps $X_k, X_k \in \mathcal{C}([0,2\pi],\mathbb{T})$, $1\le k\le pq$, such that:

(4) $$X_k(0)=Y_k(0)\quad,\ 1\le k\le pq.$$

(5) $$X_k(2\pi)=b_{\lambda_1(k)}(0)\cdot Y_{\lambda_1(k)}(0)\ ,\ 1\le k\le pq.$$

(6) $$Y_k(2\pi)=c_{\lambda_2(k)}(0)\cdot X_{\lambda_2(k)}(0),\ 1\le k\le pq$$

(7) $$A\left(\frac{X_k\cdot b_{\lambda_1(k)}Y_{\lambda_1(k)}}{c_{\lambda_2(k)}X_{\lambda_2(k)}\cdot Y_k}\right)=0\quad,\ 1\le k\le pq.$$

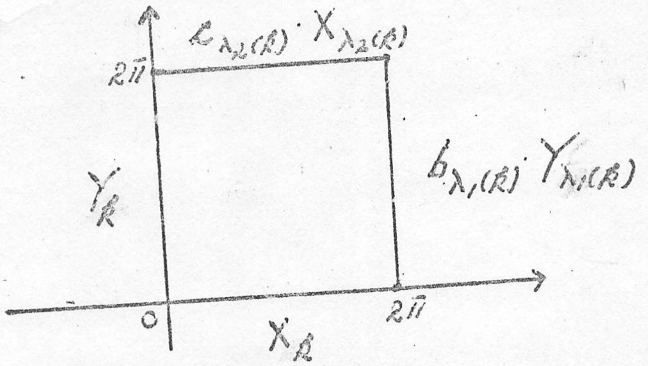
(a way in which one can find such maps will be given later).

Let us suppose also that we have proved that (4), (5) and (6) imply:

(8) $b_{\lambda_1(k)}(2\pi)\, Y_{\lambda_1(k)}(2\pi) = c_{\lambda_2(k)}(2\pi)\, X_{\lambda_2(k)}(2\pi)$, $1 \leq k \leq pq$

(the proof will be given later).

But (4), (5), (6) and (8) allow us to define, for any $1 \leq k \leq pq$, a map Z_k on the boundary ∂ of the set $[0,2\pi]^2 \subset \mathbb{R}^2$, as in the following picture:



i.e.:

$Z_k(t_1,0)=X_k(t_1)$, $\quad 0 \leq t_1 \leq 2\pi$

$Z_k(2\pi,t_2)=b_{\lambda_1(k)}(t_2)\cdot Y_{\lambda_1(k)}(t_2)$, $\quad 0 \leq t_2 \leq 2\pi$.

$Z_k(t_1,2\pi)=c_{\lambda_2(k)}(t_1)\cdot X_{\lambda_2(k)}(t_1)$, $\quad 0 \leq t_1 \leq 2\pi$.

$Z_k(0,t_2)=Y_k(t_2)$, $\quad 0 \leq t_2 \leq 2\pi$.

It is clear that $Z_k \in \mathcal{C}(\partial, T)$, and because $W(Z_k)=0$ (see (7)), $1 \leq k \leq pq$, it follows that each Z_k has a continuous extension up to $[0,2\pi]^2$, also denoted by Z_k, $Z_k : [0,2\pi]^2 \longrightarrow T$.

We define $U : [0,2\pi]^2 \longrightarrow U(npq)$ by:

$$U(t_1,t_2) = u_\varphi(t_1,t_2)\cdot\Big(\bigoplus_{k=1}^{pq} Z_k(t_1,t_2)\cdot 1_n\Big)\cdot u_\varphi(t_1,t_2)^*$$

$t_1,t_2 \in [0,2\pi]$.

We show that if we put for any $t_1,t_2 \in [0,2\pi]$:

$$U(\exp(it_1),\exp(it_2)):=U(t_1,t_2)$$

then: $U \in \mathcal{U}(\mathcal{C}(T^2)\otimes M_{npq})$ and $\Phi(.)=U\Psi(.)U^*$.

Indeed, let us prove the first statement (the second one will be then obvious). For any $t_2 \in [0,2\pi]$, we have:

$$U(0,t_2)=U(2\pi,t_2) \Leftrightarrow (u_\Phi(2\pi,t_2)^* u_\Phi(0,t_2))\cdot\Big(\bigoplus_{k=1}^{pq} Z_k(0,t_2)\cdot 1_n\Big)\cdot$$
$$\cdot(u_\Psi(2\pi,t_2)^* u_\Psi(0,t_2))^* = \bigoplus_{k=1}^{pq} Z_k(2\pi,t_2)\cdot 1_n \Leftrightarrow \text{(by (2))}$$
$$(u_\Psi(2\pi,t_2)^* u_\Psi(0,t_2))\cdot\Big(\bigoplus_{k=1}^{pq} b_k(t_2)\cdot Z_k(0,t_2)\cdot 1_n\Big)\cdot(u_\Psi(2\pi,t_2)^* u_\Psi(0,t_2))^* =$$
$$= \bigoplus_{k=1}^{pq} Z_k(2\pi,t_2)\cdot 1_n \Leftrightarrow \text{(by the analogous of (11))}$$
$$(1_n\otimes S_1)\Big(\sum_{k=1}^{pq} b_k(t_2)Z_k(0,t_2)\cdot 1_n\otimes e^{\gamma(k),\gamma(k)}\Big)\cdot(1_n\otimes S_1)^* = \sum_{k=1}^{pq} Z_k(2\pi,t_2)\cdot 1_n\otimes e^{\gamma(k),\gamma(k)}$$
$$\Leftrightarrow \quad b_{\lambda_1(k)}(t_2)\cdot Z_{\lambda_1(k)}(0,t_2)=Z_k(2\pi,t_2), \quad 1\le k\le pq$$

which is obvious.

In a similar way, one can prove that for any $t_1 \in [0,2\pi]$ we have:

$$U(t_1,0)=U(t_1,2\pi)$$

Now, we shall show that if $X_k(0)$, $X_k(2\pi)$, $Y_k(0)$, $Y_k(2\pi) \in \mathbb{T}$, $1\le k\le pq$, satisfy (4), (5) and (6), then they also satisfy (8).

Let us introduce the notations:

$$X(0)=\bigoplus_{k=1}^{pq} X_k(0)\cdot 1_n \quad , \quad X(2\pi)=\bigoplus_{k=1}^{pq} X_k(2\pi)\cdot 1_n$$
$$Y(0)=\bigoplus_{k=1}^{pq} Y_k(0)\cdot 1_n \quad , \quad Y(2\pi)=\bigoplus_{k=1}^{pq} Y_k(2\pi)\cdot 1_n$$

$$B = \bigoplus_{k=1}^{p2} b_k \cdot 1_n \quad , \quad C = \bigoplus_{k=1}^{p2} c_k 1_n$$

$$P_i = 1_n \otimes S_i \quad , \quad i = 1,2$$

Then, (8) may be written:

$$P_1 B(2\pi) \cdot Y(2\pi) P_1^* = P_2 C(2\pi) X(2\pi) P_2^* \Longleftrightarrow (\text{by } (5) \text{ and } (6))$$

$$(9) \quad P_1 B(2\pi) \cdot P_2 C(0) X(0) P_2^* P_1^* = P_2 C(2\pi) \cdot P_1 B(0) Y(0) P_1^* P_2^*$$

On the other hand it is easily seen that:

$$(10) \quad \begin{cases} B = \Delta_{\phi,1}^* \cdot \Delta_{\phi,1} \quad , \quad C = \Delta_{\phi,2}^* \cdot \Delta_{\phi,2} \\ P_1 \Delta_{\phi,1}(0) \cdot \Delta_{\phi,2}(0)^* P_2^* \Delta_{\phi,1}(2\pi)^* P_1^* \cdot P_2 \Delta_{\phi,2}(2\pi) = 1_{np2} \end{cases}$$

We have:

$$P_1 \cdot B(2\pi) \cdot P_2 \cdot C(0) \cdot X(0) P_2^* P_1^* = (\text{by } (10))$$

(11)

$$= P_1 \cdot \Delta_{\phi,1}(2\pi)^* \Delta_{\phi,1}(2\pi) \cdot P_2 \cdot \Delta_{\phi,2}(0)^* \Delta_{\phi,2}(0) \cdot X(0) P_2^* P_1^*$$

and:

$$P_2 \cdot C(2\pi) \cdot P_1 \cdot B(0) \cdot Y(0) P_1^* P_2^* = (\text{by } (10)) = P_2 \cdot \Delta_{\phi,2}(2\pi)^* \Delta_{\phi,2}(2\pi) \cdot$$

$$\cdot P_1 \cdot \Delta_{\phi,1}(0)^* \Delta_{\phi,1}(0) \cdot Y(0) P_1^* P_2^* = (\text{by } (10)) = P_2 \cdot P_1 \Delta_{\phi,1}(0) \Delta_{\phi,2}(0)^* P_2^* \cdot$$

$$\cdot \Delta_{\phi,1}(2\pi)^* P_1^* P_2 \cdot P_2^* P_1 \Delta_{\phi,1}(2\pi) P_2 \Delta_{\phi,2}(0) \Delta_{\phi,1}(0)^* P_1^* \cdot P_1 \cdot$$

$$\Delta_{\psi,1}(0)^*\Delta_{\varphi,1}(0)Y(0)P_1^*P_2^* = (\text{using (4) and the fact that}$$

$$P_1P_2=P_2P_1) = P_1\Delta_{\psi,1}(2\pi)^*\Delta_{\varphi,1}(2\pi)P_2\Delta_{\psi,2}(0)^*\Delta_{\varphi,2}(0)X(0)P_2^*P_1^*$$

Using this fact and (11), we obtain (9).

We close the proof showing how one can find maps X_k, Y_k, $1\leq k\leq pq$, with the properties (4), (5), (6) and (7).

First of all, we choose some complex numbers $X_k(0)$, $X_k(2\pi)$, $Y_k(0)$, $Y_k(2\pi)\in T$, $1\leq k\leq pq$, such that (4), (5) and (6) are satisfied. We consider then $X'_k, Y'_k \in \mathcal{C}([0,2\pi],T)$, $1\leq k\leq pq$, such that: $X'_k(0)=X_k(0)$, $X'_k(2\pi)=X_k(2\pi)$, $Y'_k(0)=Y_k(0)$, $Y'_k(2\pi)=Y_k(2\pi)$, $1\leq k\leq pq$.

It is easily seen that:

$$\text{(12)}\quad \begin{cases} A\left(\dfrac{X'_k\cdot b_{\lambda_1(k)}Y'_{\lambda_1(k)}}{c_{\lambda_2(k)}X'_{\lambda_2(k)}\cdot Y'_k}\right) = 2\pi\cdot n_k \\ \text{where } n_k\in Z,\quad 1\leq k\leq pq\,. \end{cases}$$

On the other hand, since: $\det u_\Phi$, $\det u_\Psi \in \mathcal{C}([0,2\pi]^2,T)$, we have:

$$A(\det u_\Phi|\partial) = A(\det u_\Psi|\partial) = 0$$

i.e.:

$$\text{(13)}\quad \begin{aligned} &A(\det u_\Phi(.,0)) + A(\det u_\Phi(2\pi,.)) - \\ &-A(\det u_\Phi(.,2\pi)) - A(\det u_\Phi(0,.)) = 0 \end{aligned}$$

and:

$$\text{(14)}\quad \begin{aligned} &A(\det u_\Psi(.,0)) + A(\det u_\Psi(2\pi,.)) - \\ &-A(\det u_\Psi(.,2\pi)) - A(\det u_\Psi(0,.)) = 0 \end{aligned}$$

Using (2) and (3), one gets:

$$A(\det u_{\phi}(0,.))-A(\det u_{\phi}(2\pi,.))=$$

(15)
$$=A(\det u_{\psi}(0,.))-A(\det u_{\psi}(2\pi,.))+n.\sum_{r=1}^{pq} A(b_k)$$

and :

$$A(\det u_{\phi}(.,0))-A(\det u_{\phi}(.,2\pi))=$$

(16)
$$=A(\det u_{\psi}(.,0))-A(\det u_{\psi}(.,2\pi))+n.\sum_{k=1}^{pq} A(c_k)$$

From (13), (14), (15) and (16) we obtain:

(17) $$\sum_{k=1}^{pq} A(b_k)=\sum_{k=1}^{pq} A(c_k)$$

By virtue of (12) and (17), we have:

$$\sum_{k=1}^{pq} n_k=0$$

Therefore, it is easily seen that there exist $x_1,\dots,x_{pq}$ $y_1,\dots,y_{pq}\in \mathbb{Z}$ such that:

(18) $x_k+y_{\lambda_1(k)}-x_{\lambda_2(k)}-y_k+n_k=0,\quad 1\leq k\leq pq$.

We define $X_k,Y_k\in \mathcal{C}([0,2\pi],T)$ such that for any $1\leq k\leq pq$:

$$X_k\,|\,\{0,2\pi\}=X'_k\,|\,\{0,2\pi\}$$
$$A(X_k)=A(X'_k)+2\pi\cdot x_k$$

$$Y_k\,|\,\{0,2\pi\}=Y'_k\,|\,\{0,2\pi\}$$
$$A(Y_k)=A(Y'_k)+2\pi\cdot y_k\quad .$$

Using these relations, (12) and (18), it follows that:

$$A\left(\frac{X_k \cdot b_{\lambda_1(k)} Y_{\lambda_1(k)}}{c_{\lambda_2(k)} X_{\lambda_2(k)} \cdot Y_k}\right) = 0 \ , \quad 1 \le k \le pq.$$

The proof is complete.

$\square$

3.3. Remark. Any two isometric $*$-homomorphisms from $\mathcal{C}(T) \otimes M_n$ to $\mathcal{C}(T) \otimes M_{np}$ compatible with the covering map $T \ni z \longmapsto z^p \in T$ $(p \in N)$ are inner equivalent. The argument is similar with that given in the proof of Lemma 3.1. and Lemma 3.2. and easier.

Using this fact, one can prove that each $\mathcal{A}((n_k))$ is $*$-isomorphic to the inductive limit of any system:

$$\mathcal{C}(T) \otimes M_{n_1} \xrightarrow{\ \phi_1\ } \mathcal{C}(T) \otimes M_{n_2} \xrightarrow{\ \phi_2\ } \cdots$$

where for each k, ϕ_k is an arbitrary isometric $*$-homomorphism compatible with the covering map $T \ni z \longmapsto z^{n_{k+1}/n_k} \in T$.

$\square$

3.4. We shall now give the proof of the theorem stated in the introduction.

For any k, let $\phi_k^{(1)} : \mathcal{C}(T) \otimes M_{p_k} \longrightarrow \mathcal{C}(T) \otimes M_{p_{k+1}}$ be (resp. $\phi_k^{(2)} : \mathcal{C}(T) \otimes M_{q_k} \longrightarrow \mathcal{C}(T) \otimes M_{q_{k+1}}$) an isometric $*$-homomorphism compatible with the covering map $T \ni z \longmapsto z^{p_{k+1}/p_k} \in T$ (resp. with the covering map $T \ni z \longmapsto z^{q_{k+1}/q_k} \in T$) and let $\theta_k : (\mathcal{C}(T) \otimes M_{p_k}) \otimes (\mathcal{C}(T) \otimes M_{q_k}) \longrightarrow \mathcal{C}(T^2) \otimes M_{p_k q_k}$, be the obvious $*$-isomorphism. For every $k \in N$, let Λ'_k be the $*$-homomorphism given by the condition that the diagram:

$$\begin{array}{ccc}
\mathcal{C}(T^2)\otimes M_{p_k q_k} & \overset{\Lambda'_k}{\dashrightarrow} & \mathcal{C}(T^2)\otimes M_{p_{k+1} q_{k+1}} \\
\Theta_k \Big\uparrow \wr & & \wr \Big\uparrow \Theta_{k+1} \\
(\mathcal{C}(T)\otimes M_{p_k})\otimes(\mathcal{C}(T)\otimes M_{q_k}) & \xrightarrow{\Phi_k^{(1)}\otimes\Phi_k^{(2)}} & (\mathcal{C}(T)\otimes M_{p_{k+1}})\otimes(\mathcal{C}(T)\otimes M_{q_{k+1}})
\end{array}$$

commutes. It is clear that Λ'_k is an isometric $*$-homomorphism compatible with the covering map $\mathbb{T}^2 \ni (z_1,z_2) \longmapsto (z_1^{p_{k+1}/p_k}, z_2^{q_{k+1}/q_k}) \in \mathbb{T}^2$, $k \in \mathbb{N}$. By Lemma 3.2 it follows that Λ_k and Λ'_k are inner equivalent, for all $k \geq 1$. Therefore $\mathcal{A}((\Lambda_k))$ is $*$-isomorphic to the inductive limit of the system:

$$(\mathcal{C}(T)\otimes M_{p_1})\otimes(\mathcal{C}(T)\otimes M_{q_1}) \xrightarrow{\Phi_1^{(1)}\otimes\Phi_1^{(2)}} (\mathcal{C}(T)\otimes M_{p_2})\otimes(\mathcal{C}(T)\otimes M_{q_2}) \xrightarrow{\Phi_2^{(1)}\otimes\Phi_2^{(2)}} \cdots$$

Using this fact and Remark 3.3 we conclude the proof.

$\square$

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$\square$

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