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by

Dan VOICULESCU

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Dan Voiculescu

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Department of Mathematics, National Institute for Scientific and Technical Creation, Bd.Păcii 220, 79622 Bucharest, Romania.

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By Dan Voiculescu

The non-commuting random variables in the title can be illustrated by the following example.

Let G be the (non-commutative) free group on two generators and let u_j $(j=1,2)$ be the unitaries in $\ell^2(G)$ corresponding to left translation by the generators and consider $\xi \in \ell^2(G)$ the function $\xi(g) = \delta_{g,e}$. Let further X_j be operators of the form $X_j = \varphi_j(u_j)$. The operators X_j may be viewed as "random variables" with moments $\langle X_j^k \xi, \xi \rangle$, or equivalently with distributions given by the analytic functionals μ_j where $\mu_j(f) = \langle f(X_j)\xi, \xi \rangle$. With these conventions, the distribution of $X_1 + X_2$ depends only on the distributions of X_1 and X_2 and the aim of the present paper is to explicitate this relationship. This may be viewed as a non-commutative analogue of the addition of independent random-variables. Indeed if G in the above example is replaced by the abelian free group on two generators, then we have precisely the usual situation of independent random variables for which addition means convolution of the distributions.

In [11], where we began the study of this kind of non-commutative independence of random variables, we constructed a certain functor from real Hilbert spaces and contractions to operator algebras with specified trace states and unital completely positive maps. This functor may be regarded as the analogue in our non-commutative framework of the functor which associates to a real Hilbert space the L^∞-algebra with respect to the gaussian measure on it with the trace state corresponding to this measure. Related to this we also obtained the corresponding central limit theorem, the limit distribution being a certain modified arcsine law.

In the commutative situation, addition of random variables corresponds to convolution of distributions, which in turn corresponds to addition of the logarithms of their Fourier-transforms (near the origin). In the present paper we exhibit the analogue of the logarithm of the Fourier transform for our noncommutative situation. It is obtained in a rather different way. For the distribution μ of a random variable consider its Cauchy transform

$$G(z) = \int \frac{d\mu(\zeta)}{z-\zeta}$$

$$G(z) = z^{-1} + m_1 z^{-2} + m_2 z^{-3} + \ldots.$$

where the m_j are the moments of the random variable. Then there is a series

$$\mathcal{R}_\mu(z) = R_1(\mu) + R_2(\mu)\, z + R_3(\mu) z^2 + \ldots.$$

such that for

$$K(z)=z+\mathcal{R}_{\mu}(z^{-1})$$

we have

$$K\left(\frac{1}{G(z)}\right)\doteq z$$

The series $\mathcal{R}_{\mu}(z)$ is the analogue of the logarithm of the Fourier transform. Thus if μ_1, μ_2, μ_3 are the distributions of X_1, X_2 and X_1+X_2 in the example given at the beginning, then we have

$$\mathcal{R}_{\mu_3}(z) = \mathcal{R}_{\mu_1}(z)+\mathcal{R}_{\mu_2}(z)$$

This gives then the possibility of computing the Cauchy transform of μ_3 and then μ_3 itself.

If our non-commutative random variables are self-adjoint then their distributions are probability measures on $\mathbb{R}$. We study also the analogue of infinitely divisible measures and of semigroups. We prove that infinite divisibility of μ in the present framework is equivalent to the requirement that $\mathcal{R}_{\mu}(z)$ have real coefficients and an analytic continuation to the upper half plane such that

$$\operatorname{Im} z > 0 \Longrightarrow \operatorname{Im} \mathcal{R}_{\mu}(z) \geq 0$$

For a semigroup of measures $(\mu_t)_{t \geq 0}$ let $G(z,t)$ be the Cauchy-transform of μ_t. Then $G(z,t)$ satisfies a quasi-linear equation of the conservation law type

$$\frac{\partial G}{\partial t}(z,t)+\frac{\partial G}{\partial z}(z,t)\,\varphi(G(z,t)) = 0$$

where $\varphi = \mathcal{R}_{\mu_1}$. This equation may be interpreted as a system of two quasilinear equations for μ_t and its Hilbert-transform.

The paper has four sections.

Section 1 is devoted to preliminary material.

In section 2 we obtain our main result about the series $\mathcal{R}_\mu$ which gives the solution of the addition problem.

Section 3 contains various results about the addition we are studying in the self-adjoint case i.e. the distributions are probability measures on $\mathbb{R}$.

Section 4 deals with infinite divisibility and semi-groups.

§ 1

This section contains preliminaries about free families of non-commutative random-variables. Reduced C^*-algebraic free products, though conceptually related, are not included here, their role being reduced to a minimum in the present paper.

We begin by recalling, in a slightly adapted version, facts from § 4 of [10].

Let (A,φ) be a unital algebra over $\mathbb{C}$ together with a specified state φ (i.e. a linear functional $\varphi: A \longrightarrow \mathbb{C}$ such that $\varphi(1)=1$). An element $a \in A$ will be viewed as a "random variable" the "distribution" of which is the functional $\mu_a: \mathbb{C}[X] \longrightarrow \mathbb{C}$ given by $\mu_a(1)=1, \mu_a(X^n)= \varphi(a^n)$. If A is a Banach algebra and φ is continuous then μ_a extends to an analytic functional such that $\mu_a(f)= \varphi(f(a))$ where f is a holomorphic function on $\mathbb{C}$. In case A is a C^*-algebra and φ is a C^*-algebraic state (i.e. φ is also positive) then for selfadjoint a, the distribution μ_a is a probability measure on $\mathbb{R}$ with compact support.

1.1. Definition. ([11]). Let (A,φ) be a unital algebra over $\mathbb{C}$ with specified state φ and let $1 \in A_\iota \subset A (\iota \in I)$ be subalgebras. The family $(A_\iota)_{\iota \in I}$ will be called free if

$$\varphi(a_1 a_2 \ldots a_n) = 0$$

whenever $a_j \in A_{i_j}$ with $i_1 \neq i_2 \neq \ldots \neq i_n$ and $0 = \varphi(a_j)$ for $1 \leq j \leq n$. A family of subsets $X_\iota \subset A$ (elements $a_\iota \in A$) will be called free if the family of subalgebras A_ι generated by $\{1\} \cup X_\iota$ (respectively $\{1, a_\iota\}$) is free.

1.2. Proposition ([11]). If $\{a,b\}$ is a free pair of elements of (A, φ) then μ_{a+b} depends only on μ_a and μ_b. There are universal polynomials with integer coefficients $P_n(x_1,\ldots,x_n,y_1,\ldots,y_n)$ such that, assigning degree j to x_j and y_j, we have

(i) P_n is homogeneous of degree n in the x and y variables taken together;

(ii) $\mu_{a+b}(X^n) = P_n(\mu_a(X),\ldots,\mu_a(X^n), \mu_b(X),\ldots,\mu_b(X^n))$;

(iii) $P_n(x_1,\ldots,x_n,y_1,\ldots,y_n) = P_n(y_1,\ldots,y_n,x_1,\ldots,x_n)$;

(iv) $\sum = \{\xi : \mathbb{C}[X] \to \mathbb{C} \mid \xi(1) = 1, \ \xi \text{ linear}\}$ is an abelian group for the operation

$$(\xi \boxplus \eta)(X^n) = P_n(\xi(X),\ldots,\xi(X^n), \eta(X),\ldots,\eta(X^n))$$

1.3. Proposition. There are universal polynomials $R_n(x_1,\ldots,x_n)$, such that considering x_j as having degree j, we have

(i) R_n is homogeneous of degree n;

(ii) $R_n(x_1,\ldots,x_n) = x_n + \tilde{R}_n(x_1,\ldots,x_{n-1})$

(iii) $R_n((\xi \boxplus \eta)(X),\ldots,(\xi \boxplus \eta)(X^n)) =$
$= R_n(\xi(X),\ldots,\xi(X^n)) + R_n(\eta(X),\ldots,\eta(X^n))$.

The polynomials R_n satisfying (i),(ii),(iii) are unique.

Proof. Excepting the uniqueness statement the proposition coincides with Proposition 4.4 of [11].

If $\overline{R}_n$ are some other polymials satisfying (i)-(iii), then there are polynomials $H_n(x_1,\ldots,x_n)$, homogeneous of degree n such that

$$R_n = H_n(R_1,\ldots,R_n)$$

and

$$H_n(x_1,\ldots,x_n) = x_n + \tilde{H}_n(x_1,\ldots,x_{n-1})$$

Then (iii) implies H_n is linear, so that

$H_n(x_1,\ldots,x_n) = x_n$.

Q.E.D.

For a linear functional $\xi : \mathbb{C}[X] \longrightarrow \mathbb{C}$ with $\xi(1)=1$, we shall write $R_n(\xi)$ for $R_n(\xi(X),\ldots,\xi(X^n))$.

Note that if ξ, η extend to analytic functionals, then $\xi \boxplus \eta$ also extends to an analytic functional. Indeed if ξ, η are analytic then it is easily seen that we can find C^*-algebras with specified states (A_j, φ_j) $(j=1,2)$ and elements $a_j \in A_j$ such that $\xi = \mu_{a_1}$, $\eta = \mu_{a_2}$ and forming the reduced free product $(A_1, \varphi_1) * (A_2, \varphi_2)$ (see [11] § 1) we have $\xi \boxplus \eta = \mu_{\sigma_1(a_1) + \sigma_2(a_2)}$ which is an analytic functional.

Similarly, considering self-adjoint elements it is easily seen that if ξ, η extend to compactly supported probability measures on $\mathbb{R}$ then $\xi \boxplus \eta$ also extend to

<u>a compactly supported probability measure on</u>

<u>The operation $\boxplus$ for analytic functionals or compactly supported probability measures on $\mathbb{R}$ will be called free convolution, abbreviated F-convolution.</u>

Let us also recall some facts about a certain extension of the Cuntz-algebra O_n (see [5], [10], [9], [7]). As in ([7]) this extension may be realized on the full Fock-space

$$\mathcal{T}(H_n) = \mathbb{C}\,1 \oplus \bigoplus_{m\geq 1} H_n^{\otimes m}$$

where H_n is an n-dimensional complex Hilbert space with an orthonormal basis $e_1,\dots,e_n$. The algebra we are interested in, is the C^*-algebra $\mathcal{E}_n$ generated by the left creation operators ℓ_j $(1\leq j\leq n)$

$$\ell_j \xi = e_j \otimes \xi,\quad \xi\in\mathcal{T}(H_n)$$

Consider also the state ε_n on $\mathcal{E}_n$ given by

$$\varepsilon_n(X) = \langle X1, 1\rangle = \mathrm{Tr}(X(I - \ell_1\ell_1^* - \dots - \ell_n\ell_n^*)$$

As explained in ([11] § 2 or [2]) the pair $(\mathcal{E}_n, \varepsilon_n)$ may be viewed as the reduced free product of n copies of $(\mathcal{E}_1, \varepsilon_1)$. This implies that <u>a family</u> $a_1,\dots,a_n$ <u>where</u> $a_j\in C^*(\ell_j)\subset\mathcal{E}_n$ <u>is free in</u> $(\mathcal{E}_n, \varepsilon_n)$.

§ 2.

In this section we obtain our main result concerning the addition problem. We first exhibit certain Toeplitz operators for which the addition takes a particularly simple form. We solve the addition problem by showing how one finds such a Toeplitz operator, given its distribution.

We use throughout the definitions and notations introduced in § 1.

2.1. Lemma. *In $(\mathcal{E}_n, \varepsilon_n)$ the "random variable"*

$$X=(\ell_1^*+\ldots+\ell_n^*)+\alpha I+\sum_{k=1}^{\infty}\sum_{\substack{1\le i_j\le n\\ 1\le j\le k}}\alpha_{i_1\ldots i_k}\ell_{i_1}\cdots\ell_{i_k}$$

where only a finite number of $\alpha_{i_1\ldots i_k}$ are non-zero, has the same distribution as

$$Y=\ell_1^*+\alpha I+\sum_{k=1}^{\infty}\Big(\sum_{\substack{1\le i_j\le n\\ 1\le j\le k}}\alpha_{i_1\ldots i_k}\Big)\ell_1^k$$

Proof. We have to show that $\varepsilon_n(X^m)=\varepsilon_n(Y^m)$ for all $m>0$. This can be seen as follows. Note first that $\varepsilon_n(\ell_{j_1}\cdots\ell_{j_p}\ell_{i_1}^*\cdots\ell_{i_q}^*)=0$ whenever $p+q>0$ and $\varepsilon_n(I)=1$. With this in mind, after expanding X^m and Y^m our assertion follows from the following remark. Consider $\ell_1^{\omega_1}\cdots\ell_1^{\omega_m}$ where the symbols ω_j designate either the exponent 1 or the symbol $*$ of the adjoint. Then, replacing each $\ell_1^{\omega_j}$ by some ℓ_{k_j} in case $\omega_j=1$ and by $(\ell_1^*+\cdots+\ell_n^*)$ in case $\omega_j=*$, it is easily seen that $\varepsilon_n(\ell_1^{\omega_1}\cdots\ell_n^{\omega_m})$ is left unchanged by these replacements.

Q.E.D.

2.2.Corollary. Consider in $(\mathcal{E}_2, \varepsilon_2)$ the "random variables"

$$X_1 = \ell_1^* + \sum_{k=0}^{\infty} \alpha_{k+1} \ell_1^k$$

$$X_2 = \ell_2^* + \sum_{k=0}^{\infty} \beta_{k+1} \ell_2^k$$

$$X_3 = \ell_1^* + \sum_{k=0}^{\infty} (\alpha_{k+1} + \beta_{k+1}) \ell_1^k$$

where only a finite number of α_k, β_k are non-zero. Then the pair X_1, X_2 is free and X_1+X_2 has the same distribution as X_3.

2.3.Proposition. Consider in $(\mathcal{E}_1, \varepsilon_1)$ the "random variable"

$$X = \ell_1^* + \sum_{k=0}^{\infty} \alpha_{k+1} \ell_1^k$$

where only a finite number of α_k's are non-zero. Then we have

$$R_n(\mu_X) = \alpha_n \qquad (n \geqslant 1)$$

where R_n is the polynomial defined in Proposition 1.3.

Proof. Expanding $\varepsilon_1(X^n)$ it is easily seen that

$$\varepsilon_1(X^n) = E_n(\alpha_1, \ldots, \alpha_n)$$

where E_n is homogeneous of degree n when α_j is assigned degree j and

$$E_n(\alpha_1, \ldots, \alpha_n) = \alpha_n + \tilde{E}_n(\alpha_1, \ldots, \alpha_n)$$

This implies $R_n(\mu_X) = F_n(\alpha_1, \ldots, \alpha_n)$ where $F_n(\alpha_1, \ldots, \alpha_n) = \alpha_n + \tilde{F}_n(\alpha_1, \ldots, \alpha_{n-1})$ is homogeneous of degree n. By Corollary 2.2 F_n is a linear function of $\alpha_1, \ldots, \alpha_n$ so that $R_n(\mu_X) = \alpha_n$

Q.E.D.

2.4. Remark. The finiteness assumptions for the sums appearing in Lemma 2.1, Corollary 2.2 and Proposition 2.3 are inessential. In fact one may replace $(\mathcal{E}_n, \varepsilon_n)$ by $(\tilde{\mathcal{E}}_n, \tilde{\varepsilon}_n)$ where $\tilde{\mathcal{E}}_n$ is the algebra of formal series of the form

$$S = \beta I + \sum_{\substack{p \geq 0,\, 0 \leq q \leq N \\ p+q \geq 1}} \beta_{i_1 \ldots i_p j_1 \ldots j_q} \, \ell_{i_1} \cdots \ell_{i_p} \ell^*_{j_1} \cdots \ell^*_{j_q}$$

and $\tilde{\varepsilon}_n(S) = \beta$. Then in $(\tilde{\mathcal{E}}_n, \tilde{\varepsilon}_n)$ the statements of 2.1, 2.2, 2.3, hold without any finiteness assumption. Indeed the m-th moment of

$$X = (\ell^*_1 + \ldots + \ell^*_n) + \alpha I + \sum_{k=1}^{\infty} \sum_{\substack{1 \leq i_j \leq n \\ 1 \leq j \leq k}} \alpha_{i_1 \ldots i_k} \ell_{i_1} \cdots \ell_{i_k}$$

depends only on α and $\alpha_{i_1 \ldots i_k}$ with $k \leq m$.

Proposition 2.3. shows that the connection between the moments of a "random raviable" and the polynomials R_n in these moments is the same as the connection between the moments of the "random variable" X, considered there, and the numbers α_k. Thus our next task will be to find formulae for the moments of X.

2.5. Lemma. Consider in $(\mathcal{E}_1, \varepsilon_1)$ the "random variable"

$$X = \ell^*_1 + \sum_{k=0}^{\infty} \alpha_{k+1} \ell_1$$

where only a finite number of α_k's are non-zero. Then we have

$$\varepsilon_1(X^m)=\operatorname*{Res.}_{z=0}\frac{1}{m+1}(z^{-1}+\sum_{k=0}^{\infty}\alpha_{k+1}z^k)^{m+1}$$

<u>where</u> $\operatorname*{Res}_{z=0}$ <u>denotes the residue at zero. Moreover the same result holds also in</u> ($\tilde{\mathcal{E}}_1, \tilde{\varepsilon}_1$) <u>without the finiteness condition on the</u> α_k'<u>s</u>.

<u>Proof</u>. We have

$$\varepsilon_1(X^m)\operatorname{Tr}(X^m[\ell_1^*,\ell_1])=$$

$$=\operatorname{tr}(X^m[X,\ell_1])=\frac{1}{m+1}\operatorname{Tr}[X^{m+1},\ell_1].$$

Using the Helton-Howe formula ([8],[3]) with the Jacobian written with respect to $z,\bar{z}$ instead of x,y we have

$$\varepsilon_1(X^m)=$$

$$=\frac{i}{2\pi(m+1)}\int_{|z|\le 1}(-\frac{\partial}{\partial z}(\bar{z}+f(z))^{m+1}\cdot\frac{\partial}{\partial z}z)dz\wedge d\bar{z}=$$

$$=\frac{-1}{2\pi i}\int_{|z|\le 1}(\bar{z}+f(z))^m dz\wedge d\bar{z}$$

where $f(z)=\sum_{k=0}^{\infty}\alpha_{k+1}z^k$.

Since

$$-(\partial+\bar{\partial})(\frac{1}{m+1}(\bar{z}+f(z))^{m+1}dz)=$$

$$=(\bar{z}+f(z))^m dz\wedge d\bar{z}$$

it follows that

$$\varepsilon_1(X^m)=\frac{1}{2\pi i}\int_{|z|=1}\frac{1}{m+1}(\bar{z}+f(z))^{m+1}dz=$$

$$=\frac{1}{2\pi i}\int_{|z|=1}\frac{1}{m+1}(z^{-1}+f(z))^{m+1}dz)=$$

$$=\operatorname*{Res.}_{z=0}\frac{1}{m+1}(z^{-1}+f(z))^{m+1}.$$

For the more general result in $(\tilde{\Sigma}_1, \tilde{\varepsilon}_1)$ note that both $\tilde{\varepsilon}_1(X^m)$ and $\operatorname{Res.}_{z=0} \frac{1}{m+1}(z^{-1}+f(z))^{m+1}$ depend only on $\alpha_1, \dots, \alpha_m$ so that the result in $(\tilde{\Sigma}_1, \tilde{\varepsilon}_1)$ follows from the one already obtained.

Q.E.D.

2.6. Corollary. For the "random variable"

$$X = \ell_1^* + \sum_{k=0}^{\infty} \alpha_{k+1}^* \ell_1^k$$

in $(\tilde{\Sigma}_1, \tilde{\varepsilon}_1)$ we have

$$\tilde{\varepsilon}_1(X^m) = \sum_{n_1+2n_2+\dots+mn_m=m} \frac{m!}{n_1!\dots n_m!(m+1-n_1-\dots-n_m)!} \alpha_1^{n_1}\dots\alpha_m^{n_m}$$

Equivalently, we have

$$X_m = \sum_{\substack{n_1+2n_2+\dots+mn_m=m \\ n_j \geqslant 0}} \frac{m!}{n_1!\dots n_m!(m+1-n_1-\dots-n_m)!} (R_1(x_1))^{n_1}\dots$$
$$\dots(R_m(x_1,\dots,x_m))^{n_m}$$

Proof. By Lemma 2.5 we have

$$\tilde{\varepsilon}_1(X^m) = \operatorname{Res.} \frac{1}{m+1} z^{-m-1}(1+\sum_{k=1}^{\infty} \alpha_k z^k)^{m+1}$$

and the formula for $\tilde{\varepsilon}_1(X^m)$ follows from the multinomial theorem. The second assertion follows from Proposition 2.3, Remark 2.4 and Proposition 1.3.

Q.E.D.

2.7. Remark. The coefficients $\frac{m!}{n_1!\dots n_m!(m+1-n_1-\dots-n_m)!}$ are integral, since the coefficient of $\alpha_1^{n_1}\dots\alpha_m^{n_m}$ in

$\tilde{\mathcal{E}}_1(X^m)$ is clearly and integer. Moreover this actually implies that $R_m(x_1,\ldots,x_m)$ is also a polynomial with integer coefficients.

2.8. Corollary. Let

$$X = \ell_1^* + \sum_{k=0}^{\infty} \alpha_{k+1}\, \ell_1^k \in \mathcal{E}_1$$

and let

$$F(z) = \sum_{k=0}^{\infty} a_k z^k$$

be a power series with radius of convergence ρ. Assume $\rho > \|X\|$ and let $0 < \delta < 1$ be such that $(1-\delta)^{-1} - (1-\delta) < \rho - \|X\|$. Then we have

$$\mathcal{E}_1(F'(X)) = \frac{1}{2\pi i} \int_{|z|=1-\delta} F\Big(z^{-1} + \sum_{k=0}^{\infty} \alpha_{k+1} z^k\Big)\,dz$$

Proof. By the basic facts on Toeplitz operators (see [6]) the power series $\sum_{k=0}^{\infty} \alpha_{k+1} z^k$ converges in the open unit disk to a holomorphic function having a continuous extension $A(z)$ to the closed unit disk. Moreover we have

$$\|X\| = \sup_{|z|=1} \left| \bar{z} + A(z) \right|$$

and

$$\sup_{|z|=1-\delta} \left| z^{-1} + A(z) \right| = \sup_{|z|=1-\delta} \left| (1-\delta)^{-2} \bar{z} + A(z) \right| \leq$$

$$\leq ((1-\delta)^{-1} - (1-\delta)) + \sup_{|z|=1-\delta} \left| \bar{z} + A(z) \right| \leq$$

$$\leq ((1-\delta)^{-1} - (1-\delta)) + \|X\| < \rho$$

since $\bar{z} + A(z)$ is harmonic.

If F is a polynomial, the equality to be established is an immediate consequence of Lemma 2.5. Our assertion for general F is obtained by considering a sequence of polynomials which converge uniformly to F on compact subsets of the open disk of radius ρ.

Q.E.D.

2.9. Theorem. *Consider the formal series*

$$G(z)=z^{-1}+\sum_{j=1}^{\infty} g_j z^{-j-1}$$

Let further

$$H(z)=z+\sum_{j=0}^{\infty} h_j z^{-j}$$

be such that $G(z)H(z)=1$ *and let*

$$K(z)=z+\sum_{j=1}^{\infty} k_j z^{-j+1}$$

be such that $H(K(z))=z$. *Then we have*

$$k_j=R_j(g_1,\ldots,g_j)$$

where R_j *are the polynomials defined in Proposition 1.3.*

Proof. It is easily seen that h_j and k_j depend only on a finite number of g_k's. Hence it will be sufficient to prove the theorem only for a restricted class of power series, provided for every $n \geq 1$, given $g_1,\ldots,g_n$, there is a series $G(z)$ in this class with the given first n coefficients. Thus it will suffice to prove the theorem for

$$G(z)=\varepsilon_1((z-X)^{-1})$$

where

$$X=l_1^*+\sum_{k=0}^{\infty}\alpha_{k+1}\, l_1^{k}\in\mathcal{E}_1 .$$

Indeed, $g_j = \mathcal{E}_1(X^j)$ and $\alpha_1,\dots,\alpha_n$ can be determined so that $g_1,\dots,g_n$ have the prescribed values.

Let $|\zeta| > \|X\|$. We shall use Corollary 2.7 with $F(z) = \ln(\zeta - z) = \ln\zeta + \ln(1-\zeta^{-1}z)$ where $\ln\zeta$ has been chosen arbitrarily among the possible values, while $\ln(1-\zeta^{-1}z)$ is defined for $|z|<|\zeta|$ and equals 0 at the origin. Thus, if $0 < \delta < 1$ is such that $(1-\delta)^{-1}-(1-\delta) < |\zeta| - \|X\|$ then we have

$$\mathcal{E}_1((X-\zeta)^{-1}) = \frac{1}{2\pi i}\int_{|z|=1-\delta} \ln(\zeta - (z^{-1} + \sum_{k=0}^{\infty} \alpha_{k+1} z^k))dz$$

Let $0<\beta<1$ be such that $z^{-1}+\sum_{k=0}^{\infty}\alpha_{k+1}z^k$ be an analytic isomorphism of the disk $\{z \mid |z|<\beta\}$ onto some neighborhood V of ∞. Assuming $|\zeta| > \|X\| + \beta^{-1} - \beta$ we choose $\delta > 1-\beta$ and consider $\Gamma \subset V$ the image of the circle $\{z \mid |z| = 1-\delta\}$ under this isomorphism. Define K by

$$K(z^{-1}) = z^{-1} + \sum_{k=0}^{\infty}\alpha_{k+1} z^k$$

The inverse of $z \longrightarrow K(z^{-1})$ is then $G(z) = \dfrac{1}{H(z)}$ where $H(z)$ is the inverse of $K(z)$. We have

$$\begin{aligned}
&\mathcal{E}_1((\zeta - X)^{-1}) = \\
&= -\frac{1}{2\pi i}\int_{\Gamma} \ln(\zeta - z)G'(z)dz = \\
&= \frac{1}{2\pi i}\int_{\Gamma} \frac{G(z)}{z-\zeta}\, dz = G(\zeta)
\end{aligned}$$

(the final application of the Cauchy formula being for the part of V boundede by Γ ,which contains ∞ and ζ when $|\zeta|$ is sufficiently big).Thus for

$$K(z)=z+\sum_{k=0}^{\infty}\alpha_{k+1}z^{-k}$$

the corresponding G(z) is

$$\varepsilon_1((\zeta-X)^{-1})=\zeta^{-1}+\sum_{j=1}^{\infty}\varepsilon_1(X^j)\zeta^{-j-1}$$

and since $\alpha_j=R_j(\varepsilon_1(X),\ldots,\varepsilon_1(X^j))$ our assertion follows.

Q.E.D.

2.1o.Definition.If μ is the distribution of some random variable,then the series

$$\mathcal{R}\mu(z)=\sum_{n=0}^{\infty}R_{n+1}(\mu)z^n$$

is called the R-series of μ.

Theorem 2.9 provides a way for computing the R-series of μ from the series

$$G(z)=z^{-1}+\sum_{j=1}^{\infty}\mu(X^n)z^{-n-1}$$

Indeed with the notations of the theorem we then have

$\mathcal{R}\mu(z)=K(z^{-1})-z^{-1}$.This also solves the problem of computing $\mu_1\boxplus\mu_2$ .In fact if G_j,H_j,K_j are the corresponding series for μ_j (j=1,2), then we can find $\mathcal{R}\mu_j(z)$ and hence also

$$\mathcal{R}(\mu_1\boxplus\mu_2)(z)=\mathcal{R}\mu_1(z)+\mathcal{R}\mu_2(z)$$

The K-series of $\mu_1\boxplus\mu_2$ is then

$$K(z)=z+\mathcal{R}(\mu_1\boxplus\mu_2)(z^{-1})$$

so that applying again the theorem we can find

$$G(z)=z^{-1}+\sum_{j=1}^{\infty}(\mu_1\boxplus\mu_2)(X^n)z^{-n-1}$$

In case μ is an analytic functional, the corresponding G-series is the Cauchy-transform

$$G(z)=\int \frac{d\mu(\zeta)}{z-\zeta}$$

In case μ is a compactly supported probability measure on $\mathbb{R}$ the computation of μ from G amounts to solving a moment problem.

§ 3.

This section is a collection of various facts concerning the F-convolution and the R-series of compactly supported probability measures on $\mathbb{R}$. Part of this material serves as a preparation for the next section.

By $\mathcal{P}$ we shall denote the set of probability measures on $\mathbb{R}$ with compact support. For $\mu \in \mathcal{P}$ we shall denote by $M(\mu)$ and $m(\mu)$ the least upper bound and respectively the largest lower bound of the support of μ. Then $d(\mu)=M(\mu)-m(\mu)$ is the diameter of the support of μ. Recall also that

$$R_1(\mu)= \int x \, d\mu(x)$$

$$R_2(\mu)= \int x^2 d\mu(x)-\left(\int x d\mu(x)\right)^2$$

With respect to F-convolution $\mathcal{P}$ is a semigroup and $\mathcal{P}_0=\{\mu \in \mathcal{P} \mid R_1(\mu)=0\}$ is a subsemigroup. Note also that $\mu \boxplus \delta_t$ as for the usual convolution is the corresponding trabslate of μ. $\mathcal{RP}$ and $\mathcal{RP}_0$ will denote the sets of R-series of measures from $\mathcal{P}$ and respectively $\mathcal{P}_0$.

The following two lemmas give estimates for the supports of F-convolutions.

3.1. Lemma. Let $\mu_1, \mu_2 \in \mathcal{P}$. We have

(i) $M(\mu_1)+R_1(\mu_2) \leq M(\mu_1 \boxplus \mu_2)$

$m(\mu_1)+R_1(\mu_2) \geq m(\mu_1 \boxplus \mu_2)$

(ii) $\max(d(\mu_1), d(\mu_2)) \leq d(\mu_1 \boxplus \mu_2) \leq d(\mu_1)+d(\mu_2)$

Proof.(i) Let (A,φ) the von Neumann factor generated by the left regular representation of the free group on two generators G together with its trace. Let $u_j (j=1,2)$ be the canonical unitaries generating A and let $X_j=X_j^*\in(u_j)''$ be such that $\mu_{X_j}=\mu_j$ $(j=1,2)$. Since φ is faithful $M(\mu_j)$ and $M(\mu_1 \boxplus \mu_2)$ coincide with the least upper bounds of the spectra of X_j and respectively X_1+X_2

Let $G_1\subset G$ denote the subgroup generated by the first generator and let $\xi\in \ell^2(G)$ be the function $\xi(g)=$ $=\delta_{g,e}$. Since $(u_1)''$ has no minimal projections we can find $\eta_k\in \ell^2(G_1)\subset \ell^2(G)$ such that $\|\eta_k\|=1, \lim\limits_{k\to\infty}\langle\eta_k,\xi\rangle=0$ and

$$\lim_{k\to\infty}\langle X_1\eta_k,\eta_k\rangle=\sup\sigma(X_1)=M(\mu_1).$$

Since $\eta_k\in\ell^2(G_1)$ we have

$$\langle X_2\eta_k,\eta_k\rangle=\varphi(X_2)=R_1(\mu_2)$$

This gives

$$M(\mu_1\boxplus\mu_2)=\sup\sigma(X_1+X_2)\geqslant\lim_{k\to\infty}\langle(X_1+X_2)\eta_k,\eta_k\rangle=$$

$$=M(\mu_1)+R(\mu_2)$$

For the other inequality it suffices to replace X_j by $-X_j$.

(ii) The lower bound for $d(\mu_1\boxplus\mu_2)$ follows from (i). The upper bound, with the notatkons of (i) is immediate from

$$d(\mu_1\boxplus\mu_2)=\inf_{t\in\mathbb{R}}\|X_1+X_2-tI\|\leq$$

$$\leq\inf_{t\in\mathbb{R}}\|X_1-tI\|+\inf_{t\in\mathbb{R}}\|X_2-tI\|=d(\mu_1)+d(\mu_2)$$

Q.E.D.

3.1. Lemma. Let $\mu_1, \dots, \mu_m \in \mathcal{P}_0$. We have

$$\left(\sum_{j=1}^{m} R_2(\mu_j)\right)^{1/2} \leq N(\mu_1 \boxplus \cdots \boxplus \mu_m) \leq \max_{1 \leq j \leq m} N(\mu_j) + 2\left(\sum_{j=1}^{m} R_2(\mu_j)\right)^{1/2}$$

where $N(\mu) = \max(|M(\mu)|, |m(\mu)|)$.

Proof. The lower bound for $N(\mu_1 \boxplus \dots \boxplus \mu_m)$ follows immediately from the obvious inequalities

$$N(\mu) \geqslant \left(\int x^2 d\mu(x)\right)^{1/2} \geqslant (R_2(\mu))^{1/2}$$

For the upper bound we consider G the free group on m generators $g_1, \dots, g_m$ and the subsets F_j $(1 \leq j \leq m)$ consisting of words of the form $g_{j_1}^{k_1} g_{j_2}^{k_2} \dots$ with $j_1 \neq j_2 \neq \dots$, $j_1 = j$, $k_1 \neq 0$. Let (A, φ) be the von Neumann factor generated by the left regular representation of G and φ its trace. Let further $u_1, \dots, u_m$ be the canonical unitaries generating A and let $X_j \in (u_j)''$, $X_j = X_j^*$ be such that $\mu_{X_j} = \mu_j$. Then we have $N(\mu_j) = \|X_j\|$ and $N(\mu_1 \boxplus \dots \boxplus \mu_m) = \|X_1 + \dots + X_m\|$. Let P_j be the orthogonal projection of $\ell^2(G)$ onto $\ell^2(F_j) \subset \ell^2(G)$ and let $\xi \in \ell^2(G)$ be the function $\xi(g) = \delta_{g,e}$. Since

$$\langle X_j \xi, \xi \rangle = \varphi(X_j) = R_1(\mu_j) = 0$$

we have

$$\| P_j X_j (I - P_j) \| = \| X_j \xi \| = (R_2(\mu_j))^{1/2}$$

and

$$(I - P_j) X_j (I - P_j) = 0$$

Remark that $j\neq k \Longrightarrow P_jP_k=0$ so that

$$\left\|\sum_{j=1}^{m} P_jX_j(I-P_j)\right\| \leq \left(\sum_{j=1}^{m}\| P_jX_j(I-P_j)\|^2\right)^{1/2}$$

$$\left\|\sum_{j=1}^{m} P_jX_jP_j\right\| \leq \max_{1\leq j\leq m}\|P_jX_jP_j\|$$

Hence we have

$$\left\|\sum_{j=1}^{m} X_j\right\| \leq 2\left\|\sum_{j=1}^{m} P_jX_j(I-P_j)\right\| + \left\|\sum_{j=1}^{m} P_jX_jP_j\right\| \leq$$

$$\leq 2\left(\sum_{j=1}^{m} R_2(\mu_j)\right)^{1/2} + \max_{1\leq j\leq m} N(\mu_j)$$

Q.E.D.

We pass now to the R-series of measures in $\mathcal{P}$. The following lemma gives a characterization of the R-series of these measures, obtained as an easy consequence of the theory of the Nevanlinna-Pick problem (see for instance [1]).

3.3. <u>Lemma</u>. <u>Let</u> $\varphi(z)=\sum_{n=0}^{\infty} a_{n+1}z^n$ <u>be a power series. The following conditions are equivalent</u>

(i) $\varphi\in\mathcal{RP}$

(ii) φ <u>is convergent for</u> $|z|<C$, $C>0$, $\varphi(\bar z)=\overline{\varphi(z)}$ <u>and there is</u> $C_1\geq C^{-1}$ <u>such that for any</u> $z_1,\dots,z_m\in\{z \mid |z|\geq C_1, \operatorname{Im} z\geq 0\}$ <u>and</u> $\xi_1,\dots,\xi_m$ <u>we have</u>

$$\sum_{1\leq j,k\leq m}\frac{z_j-\bar z_k}{z_j-\bar z_k+\varphi(z_j^{-1})-\varphi(\bar z_k^{-1})}\,\xi_j\bar\xi_k \geq 0$$

<u>Proof</u>. (i) $\Longrightarrow$ (ii). Let $\mu\in\mathcal{P}$, let $G(z)$ be the Cauchy-transform of μ and consider as in Theorem 2.9 the series $H(z)$ and $K(z)$. Assume $K(z)=z+\varphi(z^{-1})$. Since $G(\bar z)=\overline{G(z)}$ is convergent in

some neighborhood of ∞ we clearly have that $\varphi(\bar{z})=\overline{\varphi(z)}$ is convergent in some neighborhood of 0. Remark that

$\operatorname{Im} z>0 \Rightarrow \operatorname{Im} G(z)<0 \Leftrightarrow \operatorname{Im} H(z)>0.$

Hence by the theory of the Nevanlinna-Pick problem we have

$$\sum_{1\le j,k\le m} \frac{H(\zeta_j)-\overline{H(\zeta_k)}}{\zeta_j-\bar{\zeta}_k}\xi_j\bar{\xi}_k \geqslant 0$$

for any $\zeta_1,\dots,\zeta_m$ with $\operatorname{Im}\zeta_j>0$ and $\xi_1,\dots,\xi_m\in\mathbb{C}$.

If C_1 is such that $\{z \mid |z|\geqslant C_1, \operatorname{Im} z>0\}\subset\{H(z) \mid z\in\mathbb{C}, \operatorname{Im} z>0\}$ the inequality in (ii) follows.

(ii) $\Rightarrow$ (i) Consider $K(z)=z+\varphi(\frac{1}{z})$ and let $H(z)$ be such that $K(H(z))=z$ in some neighborhood of ∞. By the theory of the Nevanlinna-Pick problem there exists a holomorphic function H_1 defined in the upper half plane such that

$$\operatorname{Im} z>0 \Rightarrow \operatorname{Im} H_1(z)>0$$

and which takes the same values as H in a neighborhood of ∞. This proves H_1 is actually the analytic continuation of H to the upper half plane. Then $G(z)=\dfrac{1}{H(z)}$ has the expansion at ∞ of the form

$$G(z)=z^{-1}+g_1z^{-2}+\dots$$

and since $\operatorname{Im} z>0 \Rightarrow \operatorname{Im} G(z)<0$, $G(\bar{z})=\overline{G(z)}$ and $G(z)$ is holomorphic in some neighborhood of ∞, we infer that G is the Cauchy-transform of a measure in $\mathcal{P}$.

Q.E.D.

The last topic in this section are certain <u>affine endomorphisms of</u> $(\mathcal{P},\boxplus)$.

Consider first a formal power series $\psi(z)=z+$ $+\sum_{n=2}^{\infty} a_n z^n$ and let $\eta(z)=\frac{1}{\psi(z)}-\frac{1}{z}$ and $\varphi(z)$ the series such that $\psi(\varphi(z))=z$. Then with the notations of Theorem 2.9 the map at the level of G-series $G \longrightarrow \psi \circ G$ corresponds at the level of R-series, where $R(z)=K(z^{-1})-z^{-1}$ to $R \longrightarrow R\circ\varphi+\eta$. If the coefficients of the G series are the moments of a random variable, then we have obtained an affine endomorphism of the semigroup of distributions of random variables.

Let now G be the Cauchy transform of some measure $\mu \in \mathcal{P}$. Assume ψ is holomorphic in some neighborhood of $(\mathbb{C}\setminus\mathbb{R})\cup\{0\}$ and moreover

$$\psi(\bar{z})=\overline{\psi(z)},\quad \psi(0)=0,\quad \psi'(0)=1$$

$$\operatorname{Im} z>0 \Longrightarrow \operatorname{Im}\psi(z)>0$$

Then $\psi\circ G$ is also the Cauchy transform of a measure in $\mathcal{P}$, which we shall denote $T(\psi)\mu$. Indeed

$$\operatorname{Im} z>0 \Longrightarrow \operatorname{Im}(\psi\circ G)(z)<0$$

$$(\psi\circ G)(\bar{z})=\overline{(\psi\circ G)(z)}$$

and $\psi\circ G$ has the corresponding behaviour at ∞. Thus $\mu \longrightarrow T(\psi)\mu$ is an affine endomorphism of $(\mathcal{P},\boxplus)$ i.e. we have

$$T(\psi)(\mu_1\boxplus\mu_2)\boxplus T(\psi)(\mu_3)=$$

$$T(\psi)(\mu_1)\boxplus T(\psi)(\mu_2\boxplus\mu_3).$$

Remark also that

$$T(\psi_1)\circ T(\psi_2)=T(\psi_1\circ\psi_2)$$

§ 4.

This section is devoted to infinitely divisible measures with respect to F-convolution and to F-convolution semigroups.

4.1. Definition. A measure $\mu \in \mathcal{P}$ is called F-infinitely divisible if for every $n \in \mathbb{N}$ there is $\mu_{1/n} \in \mathcal{P}$ such that

$$\mu = \underbrace{\mu_{1/n} \boxplus \cdots \boxplus \mu_{1/n}}_{n\text{-summands}}$$

4.2. Lemma. Let φ be a holomorphic function in some neighborhood of $(\mathbb{C} \setminus \mathbb{R}) \cup \{0\}$ such that

$$\varphi(\bar{z}) = \overline{\varphi(z)}$$

$$\operatorname{Im} z > 0 \Longrightarrow \operatorname{Im} \varphi(z) \geqslant 0$$

Then $\varphi \in \mathcal{R}\mathcal{P}$.

Proof. We shall prove that condition (ii) of Lemma 3.3 is satisfied. We choose C_1 great enough so that φ be defined for $|z| < C_1^{-1}$ and

$$\sup_{|z| < C_1^{-1}} |\varphi'(z)| < \frac{1}{2} C_1^2$$

For $z_1, \ldots, z_m \in \{ z \in \mathbb{C} \mid |z| > C_1, \operatorname{Im} z > 0 \}$ we must prove that

$$B = \left(\frac{z_j - \bar{z}_k}{z_j - \bar{z}_k + \varphi(z_j^{-1}) - \varphi(\bar{z}_k^{-1})} \right)_{1 \leq j,k \leq m} \geqslant 0$$

Because of our choice of C_1 we have

$$\left| \frac{\varphi(z_j^{-1}) - \varphi(\bar{z}_k^{-1})}{z_j - \bar{z}_k} \right| < \frac{1}{2}$$

and hence

$$B = P + \sum_{k=1}^{\infty} \underbrace{\Phi \circledast \cdots \circledast \Phi}_{k\text{-times}}$$

where P is the m×m matrix with all entries equal 1, $\circledast$ denotes the Hadamard product and

$$\Phi = -\left(\frac{\varphi(z_j^{-1}) - \varphi(\bar{z}_k^{-1})}{z_j - \bar{z}_k} \right)_{1 \leq j,k \leq m}$$

Since $\operatorname{Im} z > 0 \Rightarrow \operatorname{Im} \varphi(z^{-1}) < 0$, by Nevanlinna-Pick we have $\Phi \geq 0$. Since the Hadamard product preserves positivity we infer $B \geq 0$, the desired conclusion.

Q.E.D.

4.3. <u>Theorem. A series $\varphi(z) = \sum_{n=0}^{\infty} a_{n+1} z^n$ is the R-series of an F-infinitely divisible measure in $\mathcal{P}$, if and only if φ is the Taylor series of a holomorphic function in some neighborhood of $(\mathbb{C} \setminus \mathbb{R}) \cup \{0\}$ such that</u>

$$\varphi(\bar{z}) = \overline{\varphi(z)}$$

$$\operatorname{Im} z > 0 \Rightarrow \operatorname{Im} \varphi(z) \geq 0$$

<u>Proof</u>. The sufficiency of the condition follows from Lemma 4.2 applied to $t\varphi$ $(t > 0)$.

Concerning the necessity it is immediate that φ is holomorphic in some neighborhhod of 0 and $\varphi(\bar{z}) = \overline{\varphi(z)}$, so what must be proved is that φ has an analytic continuation to $\mathbb{C}\setminus\mathbb{R}$ and that $\operatorname{Im} z > 0 \implies \operatorname{Im}\varphi(z) \geq 0$.

Assume φ is the R-series of an infinitely divisible measure in $\mathcal{P}$, then $t\varphi$ for positive rational t is also in $\mathcal{RP}$. Using the characterisation of $\mathcal{RP}$ in Lemma 3.3 and the density of $\mathbb{Q}$ in $\mathbb{R}$ we infer $t\varphi$ is the R-series of a measure $\mu_t \in \mathcal{P}$ for every $t \geq 0, t \in \mathbb{R}$. Clearly $\mu_0 = \delta_0$ and $\mu_t \boxplus \mu_s = \mu_{t+s}$. Consider $K(z,t)=z+t\,\varphi(z^{-1})$, $H(z,t)$ such that $K(H(z,t),t)=z$ and $G(z,t)=\dfrac{1}{H(z,t)}$ so that $G(z,t)$ is the Cauchy transform of μ_t. It will be also convenient to consider $\tilde{G}(z,t)=G(z^{-1},t)$.

With these preliminaries settled, the rest of the proof, which is rather long, will be divided into 3 steps.

Step 1. The series φ is the Taylor series of a holomorphic function in some neighborhood of $(\mathbb{C}\setminus\mathbb{R})\cup\{0\}$.

Since $G(z,1)$ is the Cauchy-transform of μ_1 we infer that φ is analytic in some neighborhood of 0. Hence there is a neighborhhod of ∞ where $K(z,t)$ for $0 \leq t \leq 1$ are all univalent.

Let $a > 0$ be such that $\operatorname{supp}\mu_1 \subset [-a/2, a/2]$ and hence by Lemma 3.1 $\operatorname{supp}\mu_t \subset [-a,a]$ for $0 \leq t \leq 1$. This implies

$$|G(z,t)| \leq \Big(\inf_{-a\leq x\leq a} |z-x|\Big)^{-1}$$

or equivalently

$$|\tilde{G}(z,t)| \leq \Big(\inf_{-a\leq x\leq a} |z^{-1}-x|\Big)^{-1}$$

for $0 \leq t \leq 1$. Hence the functions $\{\tilde{G}(.,t)\}_{0 \leq t \leq 1}$ are uniformly bounded on compact subsets of $\omega = (\mathbb{C}\setminus\mathbb{R}) \cup (-a^{-1}, a^{-1})$. Since $G(.,t)$ is the inverse of $K(.,t)$ in some neighborhood of ∞, independent of $0 \leq t \leq 1$, it is easily seen that $t \longrightarrow \tilde{G}(.,t)$ is continuous when the $\tilde{G}(.,t)$ are considered with the topology of uniform convergence in some neighborhood of 0. By the uniform boundedness on compact subsets of ω we infer that $t \longrightarrow \tilde{G}(.,t)$ is actually continuous for the ~~xxxxx~~ topology of uniform convergence on compact subsets of ω. For $\zeta_0 \in \mathbb{C}\setminus\mathbb{R}$ let Γ be the segment $[0, \zeta_0]$ and let $K \subset \omega$ be a compact convex neighborhood of Γ. Then, given $\delta > 0$, there is $\varepsilon > 0$ such that

$$|\tilde{G}(\zeta,t) - \zeta| < \delta$$

$$\left|\frac{d}{d\zeta}\tilde{G}(\zeta,t) - 1\right| < \delta$$

for $\zeta_1, \zeta_2 \in K$ and $0 \leq t \leq \varepsilon$. This gives

$$|\tilde{G}(\zeta_1,t) - \tilde{G}(\zeta_2,t) - (\zeta_1 - \zeta_2)| \leq \delta|\zeta_1 - \zeta_2|$$

$$|\tilde{G}(\zeta_1,t) - \tilde{G}(\zeta_2,t)| \geq (1-\delta)|\zeta_1 - \zeta_2|$$

if $\zeta_1, \zeta_2 \in K$ and $0 \leq t \leq \varepsilon$. For $\delta < 1$, $\tilde{G}(.,t)$ is univalent in K if $0 \leq t \leq \varepsilon$. Let $K_1 \subset \mathring{K}$ be another convex compact neighborhood of Γ and assume

$$\delta < \inf_{\substack{\zeta_1 \in K_1 \\ \zeta_2 \in \mathbb{C}\setminus K}} |\zeta_1 - \zeta_2|$$

Then $|\tilde{G}(\zeta,t) - \zeta| < \delta$ for $\zeta \in \partial K$ implies, in view of the fact that $(\tilde{G}(.,t))(\partial K)$ is the boundary of $(\tilde{G}(.,t))(K)$, that $(\tilde{G}(.,t))(K) \supset K_1$ for $0 \leq t \leq \varepsilon$. Hence for $0 \leq t \leq \varepsilon$, the inverse function of $\tilde{G}(.,t)$ which in some neighborhood of 0 is

$$\frac{1}{K(z^{-1},t)} = \frac{z}{1+tz\varphi(z)}$$

has an analytic continuation to some neighborhood of Γ. The point $\zeta_0 \in (\mathbb{C} \setminus \mathbb{R})$ being arbitrary, we infer that φ has an analytic continuation to some neighborhood of $(\mathbb{C} \setminus \mathbb{R}) \cup \{0\}$.

An immediate consequence of Step 1 is that $H(.,t)$ is injective in some neighborhood of $(\mathbb{C} \setminus \mathbb{R}) \cup \{\infty\}$. Indeed in some neighborhbod of ∞ we have

$$z = H(z,t) + t\,\varphi\left(\frac{1}{H(z,t)}\right)$$

and since φ is defined in some neighborhbod of $(\mathbb{C} \setminus \mathbb{R}) \cup \{0\}$ and $(H(.,t))(\mathbb{C} \setminus \mathbb{R}) \subset \mathbb{C} \setminus \mathbb{R}$ the equality will hold for z in some neighborhood of $(\mathbb{C} \setminus \mathbb{R}) \cup \{\infty\}$ implying the injectivity of $H(.,t)$.

We define Ω_t by

$$\Omega_t = \{H(z,t) \mid z \in \mathbb{C}, \mathrm{Im}\, z > 0\}$$

Clearly $\Omega_t \subset \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$.

Step 2. We have

$$\Omega_t = \{z \mid \mathrm{Im} z > 0, \mathrm{Im}\,(z + t\,\varphi(z^{-1})) > 0\}.$$

Since $\mathrm{Im}\, z > 0 \Rightarrow \mathrm{Im}\, H(z,t) > 0$ and

$$z = H(z,t) + t\,\varphi\left(\frac{1}{H(z,t)}\right)$$

we infer

$\Omega_t \subset \{z \mid \mathrm{Im} z > 0, \mathrm{Im}\,(z+t\,\varphi\,(z^{-1})) > 0\}$.

For the opposite inclusion we shall study the differential equation satisfied by $H(z,t)$.

For z in some neighborhood of ∞, which may be chosen in an uniform way provided t remains in some compact set, we have

$$H(z+t\,\varphi\,(z^{-1}),t)=z$$

and hence

$$\frac{\partial H}{\partial z}(z+t\;\varphi\,(z^{-1}),t)\,\varphi(z^{-1})+\frac{\partial H}{\partial t}\,(z+t\;\varphi\,(z^{-1}),t)\;=0$$

Replacing z by $H(z,t)$, for each $T>0$ there is a neighborhood V_T of ∞ such that

$$\frac{\partial H}{\partial z}(z,t)\;\varphi\left(\frac{1}{H(z,t)}\right)+\;\frac{\partial H}{\partial t}\,(z,t)\;=0$$

for $(z,t) \in V_T \times [0,T]$. Like in Step 1, Lemma 3.1 implies that the supports of the μ_t are in some compact subset of $\mathbb{R}$ while $0 \leq t \leq T$, which easily gives that the $H(.,t)$ for $0 \leq t \leq T$ are uniformly bounded on compact subsets of $\mathbb{C} \setminus \mathbb{R}$, which in turn easily gives the continuity of $[0,T] \ni t \longrightarrow H(.,t)$ where the $H(.,t)$ are considered with the topology of uniform convergence on compact subsets of $\mathbb{C} \setminus \mathbb{R}$. The same conclusion then also holds for $\frac{\partial H}{\partial z}(.,t)$. We infer that in the equality

$$-\int_0^t \frac{\partial H}{\partial z}(z,t)\;\;\varphi\;\left(\frac{1}{H(z,t)}\right)dt=H(z,t)-z$$

both terms are analytic for $z \in \mathbb{C} \setminus \mathbb{R}$ and hence its validity for z in some neighborhood of ∞ implies its validity for $(z,t)\in(\mathbb{C}\setminus\mathbb{R})\times[0,\infty)$. This in turn implies the existence of $\dfrac{\partial H}{\partial t}(z,t)$ for $(z,t)\in(\mathbb{C}\setminus\mathbb{R})\times[0,\infty)$ and we have

$$\frac{\partial H}{\partial z}(z,t)\,\varphi\left(\frac{1}{H(z,t)}\right)+\frac{\partial H}{\partial t}(z,t)=0$$

for $(z,t)\in(\mathbb{C}\setminus\mathbb{R})\times[0,\infty)$. As is well-known ([4]) for such quasilinear equations $H(z,t)$ is constant on the characteristic i.e. on curves $\dfrac{dz}{dt}=\varphi\left(\dfrac{1}{H(z,t)}\right)$ with $z(t)\in\mathbb{C}\setminus\mathbb{R}$ for $0\leq t\leq T$ and which therefore are straight lines. Thus $H(z,t)$ is constant on $\{(z_0+t\,\varphi(z_0^{-1}),t)\,|\,0\leq t\leq T\}$ provided $z_0+t \quad (z_0^{-1})$ does not cross the real line, i.e. provided $\operatorname{Im}(z_0+t\,\varphi(z_0^{-1}))$ doesn't change its sign. So, if $\operatorname{Im} z_0>0$, $\operatorname{Im}(z_0+t\,\varphi(z_0^{-1}))>0$ then $z_0=H(z_0,0)=H(z_0+t\,\varphi(z_0^{-1}),t)$ and hence $z_0\in\Omega_t$, which concludes Step 2.

<u>Step</u> 3. <u>If</u> φ <u>is not a constant then</u> $\bigcap_{t>0}\Omega_t=\emptyset$.

If $z\in\mathbb{C}\setminus\mathbb{R}$, let

$$T(z)=\frac{-\operatorname{Im}\varphi(z^{-1})}{\operatorname{Im} z}\quad\text{so that}$$

$$\Omega_t=\{z\in\mathbb{C}\setminus\mathbb{R}\,|\,\operatorname{Im} z>0,\ T(z)<t^{-1}\}.$$

We have

$$\lim_{z\to\infty}|z|^2T(z)=\lim_{\substack{z\to\infty\\ \operatorname{Im} z\neq0}}\frac{\operatorname{Im}\varphi(z^{-1})}{\operatorname{Im} z^{-1}}=$$

$$=\lim_{\substack{\zeta\to0\\ \operatorname{Im}\zeta\neq0}}\frac{\operatorname{Im}\varphi(\zeta)}{\operatorname{Im}\zeta}=\lim_{\substack{\zeta\to0\\ \operatorname{Im}\zeta\neq0}}\frac{\operatorname{Im}(\varphi(\zeta)-\varphi(\operatorname{Re}\zeta))}{\operatorname{Im}\zeta}=$$

$$= \lim_{\substack{\zeta\to 0\\ \mathrm{Im}\,\zeta\neq 0}} \mathrm{Im}\left(\frac{\varphi(\mathrm{Re}\,\zeta \ \ +i\ \mathrm{Im}\,\zeta)-\varphi(\mathrm{Re}\,\zeta)}{\mathrm{Im}\,\zeta}\right) =$$

$$= \lim_{\substack{\zeta\to 0\\ 0<\theta(\zeta)<1}} \mathrm{Re}\ \varphi'(\mathrm{Re}\,\zeta +i \quad \theta(\zeta)\,\mathrm{Im}\,\zeta\ ) = \mathrm{Re}\ \varphi'(0) = R_2(\mu_1) > 0$$

Hence there is $C>0$ such that that for $z\in \mathbb{C}\setminus\mathbb{R}$ with $|z|\geq C$ we have

$$2^{-1}\ |z|^{-2}R_2(\mu_1\ ) < T(z) < 2\ |z|^{-2}R_2(\mu_1\ ).$$

It follows that

$$\Omega_t\cap\{z\in\mathbb{C}\mid\ |z|\geq C\}\subset\{z\mid\ \mathrm{Im}z>0, |z|>(2^{-1}R_2(\mu_1\ )\ \ t)^{1/2}\}$$

$$\Omega_t\cap\{z\in\mathbb{C}\mid |z|\geq C\}\supset\{z\mid \mathrm{Im}\ z>0, |z|>(2R_2(\mu_1\ )t)^{1/2}\}$$

Since Ω_t is connected and since for large t we have $\Omega_t\ \cap\{z\in\mathbb{C}\mid |z| \ \ =C\}=\emptyset$ we infer that for large t

$$\Omega_t\subset\{z\mid \mathrm{Im}\ z\ >0,\ |z|\ >(2^{-1}R_2(\mu_1)t)^{1/2}\}$$

and hence $\bigcap\limits_{t>0}\Omega_t=\emptyset$, the desired conclusion.

To see that Steps 1-3 imply the assertion of the theorem,remark that by Steps 2 and 3, we have *if φ is not a constant*

$$\{z\in\mathbb{C}\ /\ \mathrm{Im}\ z>0, \mathrm{Im}\ \varphi\ (z^{-1})\ \geq 0\}\subset\bigcap_{t>0}\Omega_t=\emptyset$$

Hence $\mathrm{Im}\,\zeta < \ 0 \Longrightarrow \mathrm{Im}\ \varphi\ (\zeta)<0$ and since $\overline{\varphi(\zeta\ )}=\ \varphi(\bar{\zeta}\ )$ we infer that $\mathrm{Im}\ \zeta>0\Longrightarrow\mathrm{Im}\ \varphi(\zeta)>0$ *if φ is not a constant.*

Q.E.D.

In the course of the proof of Theorem 4.3 we actually studied F-convolution semigroups.We shall briefly discurss some properties of these semigroups.mostly consequences of the preceding proof.

Thus let $[0,\infty) \ni t \longrightarrow \mu_t \in \mathcal{P}$ be an F-convolution semigroup i.e. $\mu_{t+s} = \mu_t \boxplus \mu_s$ and μ_t depends continuously on t when $\mathcal{P}$ is endowed with the weak topology. Using Lemma 3.1 it is easy to see that $\mathcal{R}_{\mu_t} = t\mathcal{R}_{\mu_1}$ and $\mathcal{R}_{\mu_1}$ is characterized by the conditions in Theorem 4.3, If $G(z,t)$ is the Cauchy-transform of μ_t and $\varphi(z) = \mathcal{R}_{\mu_1}(z)$, then $G(z,t)$ satisfies the quasilinear equation

$$\frac{\partial G}{\partial z}(z,t)\,\varphi(G(z,t)) + \frac{\partial G}{\partial t}(z,t) = 0$$

$$0 \le t < \infty, \; z \in \mathbb{C}, \; \mathrm{Im}\, z > 0$$

Moreover $G(.,t)$ is univalent in the upper halfplane and $H(.,t)(\{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}) = \Omega_t = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0,\ \mathrm{Im}(z + t\varphi(z^{-1})) > 0\}$ where $H(z,t) = \dfrac{1}{G(z,t)}$

$H(.,t)$ being a conformal mapping one may derive regularity properties of μ_t by studying the boundary of Ω_t. Since we do not intend to make a detailed study of regularity properties of μ_t here, we shall only present, as an example, some regularity properties of μ_t for large t, obtained in this way.

Assume the support of μ_1 is not reduced to a single point. Then there is t_0 such that if $t \geqslant t_0$, then μ_t is absolutely continuous with respect to Lebesgue measure and its Radon-Nikodym derivative m_t is continuous. Moreover there are $A_t, B_t \in \mathbb{R}$ $(t \geqslant t_0)$ such that

$$(A_t, B_t) = \{ s \in \mathbb{R} \mid m_t(s) \neq 0 \}$$

<u>and</u> m_t <u>is real analytic in</u> (A_t, B_t) <u>and</u>

$$\lim_{\substack{a \to A_t \\ a > A_t}} \frac{m_t(a)}{a - A_t} = +\infty, \quad \lim_{\substack{a \to B_t \\ a < B_t}} \frac{m_t(a)}{B_t - a} = +\infty$$

<u>Also</u>

$$\lim_{t \to \infty} \frac{\| m_t \|_\infty}{t^{1/2}} = (R_2(\mu_1))^{1/2} \cdot \frac{1}{\pi}$$

To see that this is indeed so, we begin by remarking that if $|z|$ and t are sufficiently large, then $\frac{\partial}{\partial z}(z + t\,\varphi(z^{-1})) = 0$ has precisely two solutions $\alpha_j(t)$ $(j=1,2)$ and we have $\alpha_j(t) \in \mathbb{R}$ $(j=1,2)$ and $\alpha_1(t) \cdot \alpha_2(t) < 0$. Indeed, with $\zeta = z^{-1}$ we have

$$\frac{\partial}{\partial z}(z + t\,\varphi(z^{-1})) = 1 - t\,\zeta^2 \varphi'(\zeta)$$

Our assertion follows from $\varphi'(\bar{\zeta}) = \overline{\varphi'(\zeta)}$ and $\varphi'(0) > 0$.

On the other hand if $|z|$ is sufficiently large, then $\frac{\partial^2}{\partial z^2}(z + t\,\varphi(z^{-1})) \neq 0$. This is obvious from

$$\frac{\partial^2}{\partial z^2}(z + t\,\varphi(z^{-1})) = t\,\zeta^3 (2\varphi'(\zeta) + \zeta\varphi''(\zeta))$$

Next, assume $z_n \in \mathbb{C}$, $\mathrm{Im}\, z_n > 0$ $(n \in \mathbb{N})$ are such that $\mathrm{Im}(z_n + t\,\varphi(z_n^{-1})) = 0$ and $\lim_{n \to \infty} z_n = \alpha \in \mathbb{R}$ then if t is sufficiently large, we have $\alpha \in \{\alpha_1(t), \alpha_2(t)\}$. Note that since $\mathrm{Im}(z_n + t\,\varphi(z_n^{-1})) = 0$ and t is large, the proof of Step 3 implies z_n is large. We have

$$t^{-1} = \frac{-\operatorname{Im}\varphi(z_n^{-1})}{\operatorname{Im} z_n} = |z_n|^{-2}\frac{\operatorname{Im}\varphi(z_n^{-1})}{\operatorname{Im} z_n^{-1}} =$$

$$= \lim_{n\to\infty} |z_n|^{-2} \operatorname{Re}\varphi'(\operatorname{Re} z_n^{-1} + i\,\theta_n \operatorname{Im} z_n^{-1}) =$$

$0<\theta_n<1$

$$=\alpha^{-2}\varphi'(\alpha^{-1})$$

which implies $\alpha \in \{\alpha_1(t), \alpha_2(t)\}$.

Using these facts and the proof of Step 3 it is easy to see that

$$\gamma_t = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0, \operatorname{Im}(z+t\,\varphi(z^{-1}))=0\}$$

is a real analytic curve with limit points $\alpha_1(t)$ and $\alpha_2(t)$ and finite length. The boundary of Ω_t equals $\gamma_t \cup [\alpha_1(t), +\infty) \cup (-\infty, \alpha_2(t)] \cup \{\infty\}$ where $\alpha_1(t) > 0$, $\alpha_2(t) < 0$. This implies $G(.,t)$ will have a continuous extension to $\mathbb{R}$ such that $\{s \in \mathbb{R} \mid \operatorname{Im} G(s,t) \neq 0\}$ is an interval (A_t, B_t) and $\operatorname{Im} G(s,t)$ is real analytic for $s \in (A_t, B_t)$. But $m_t(s) = \frac{1}{\pi}\operatorname{Im} G(s,t)$ which gives the desired result about the real analiticity for m_t.

The assertion about the behaviour of m_t at A_t and B_t is a consequence of $\frac{\partial}{\partial z}(z+t\,\varphi(z))\Big|_{z=\alpha_j(t)} = 0$ and

$$\frac{\partial^2}{\partial z^2}(z+t\,\varphi(z))\Big|_{z=\alpha_j(t)} > 0$$

The limit of $\frac{\|m_t\|_\infty}{t^{1/2}}$ is immediate from the proof of Step 3.

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