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by

Dan Tudor VUZA

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Dan Tudor VUZA)

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The National Institute for Scientifc and Tehnical Creation
Department of Mathematics, Bd.Pacii 220, 79622 Bucharest, Romania

IDEAL PROPERTIES OF ORDER BOUNDED OPERATORS ON ORDERED BANACH SPACES WHICH ARE NOT BANACH LATTICES

by Dan Tudor VUZA-Department of Mathematics, The National Institute for Scientific and Technical Creation, Bd. Pacii 220, 79622 Bucharest, Romania.

0. Introduction

In 1979, P. Dodds and D. H. Fremlin published their famous result asserting that if E,F are Banach lattices such that E' and F have order continuous norms and if U, V:E $\longrightarrow$F are linear operators such that $0 \leq U \leq V$ then the compacity of V implies the compacity of U.

Since then, many theorems of Dodds-Fremlin type for various classes of operators were given. In particular, the problem of the inclusion of the order ideal generated by an order bounded operator into the closed algebraic ideal generated by it was considered by several authors (C. D. Aliprantis and O. Burkinshaw [1], N. J. Kalton and P. Saab [6], H. Leinfelder [7], B. de Pagter [10], D. Vuza [17], etc.). All this was done assuming that the operators act between Banach lattices (or more generally, vector lattices endowed with locally solid topologies). The present paper considers a variant of the same problem in the case when the operators are defined on an ordered Banach space which might not be a Banach lattice (i.e. it is not a lattice and/or its topology is not locally solid). The importance of this case is due to the existence of an order relation on the Sobolev spaces.

The methods used here are based on principal modules theory. This theory was developed by the author during the years 1980-1981 in a series of papers circulated as INCREST preprints (see [12], [13]). Some applications of it were presented at the First Romanian-GDR Seminar on Banach space theory held in Bucureşti, 1981 (see [14]) and at the International Conference on operator algebras and ideals held in Leipzig, 1983 (see [16]).

The theory of principal modules provides an unified framework for the results in the area of Dodds-Fremlin type theorems. Besides its applications we shall present here, we refer the reader to [14] for an application to perfect M-tensor products and to [16] for a proof of Schep's theorem on kernel operators based on principal modules theory as well for various applications to approximable operators and to the characterization of the band generated by the finite rank operators.

1. Preliminaries

1_M will always denote the identity map of a set M.

For an ordered vector space E we shall use the standard notations:

$$E_+ = \{x \mid x \in E,\ x \geq 0\},$$
$$[x,y] = \{z \mid z \in E,\ x \leq z \leq y\}.$$

We say that the positive cone E_+ in the ordered vector space E is generating if $E=E_+-E_+$. If E is an ordered normed vector space we say that E_+ is b-strict if there is $a>0$ such that for every $x\in E$ there is $y\in E$ verifying $-y\leq x\leq y$ and $\|y\|\leq a\|x\|$. Every b-strict cone is generating. By corollary 1.28 of [11], if the positive cone of an ordered Banach space is closed and generating, it is also b-strict.

The vector space of all linear maps between two ordered vector spaces is ordered in the usual way: $U\geq 0$ if $U(E_+)\subset F_+$.

For E,F vector lattices with F order complete we denote by $L_r(E,F)$ the vector lattice of all order bounded linear maps $U:E \longrightarrow F$.

If E,F are Banach spaces, L(E,F) will be the Banach space of all linear continuous maps $U:E \longrightarrow F$. The dual of a Banach space E will be denoted by E'.

A set M in a Banach lattice E is called L-bounded if for every $\varepsilon>0$ there is $y\in E_+$ such that $\|(|x|-y)_+\|\leq\varepsilon$ for every $x\in M$. Every compact set is L-bounded; the solid convex hull of an L-bounded set is L-bounded. If E is a Banach space and F a Banach lattice, LW(E,F) will be the space of all linear maps $U:E \longrightarrow F$ which carry the unit ball of E into an L-bounded subset of F. Every compact linear map from E to F is in LW(E,F).

Let E be an Archimedian vector lattice. The center of E is the set of all linear maps $U:E \longrightarrow F$ for which there is $a>0$ such that $|U(x)|\leq a|x|$ for every $x\in E$. Denote by $\mathcal{C}(E)$ the center of E; it is a subalgebra of the algebra of all linear maps on E and a vector lattice having 1_E as strong order unit. The modulus $|U|$ of $U\in\mathcal{C}(E)$ is given by

$$|U|(x) = |U(x)|$$

for every $x\in E_+$.

Let E,F be Archimedian vector lattices. The tensor product $E\overline{\otimes}F$ in the sense of D. H. Fremlin ([3]) is an Archimedian vector lattice and there is a canonical Riesz bimorphism $\psi:E\times F \longrightarrow E\overline{\otimes}F$; we use the notation $x\otimes y$ for $\psi(x,y)$. The couple $(E\overline{\otimes}F, \psi)$ is universal in the following sense: for every Archimedian vector lattice G and every Riesz bimorphism $\varphi:E\times F \longrightarrow G$ there is an unique Riesz morphism $\overline{\varphi}:E\overline{\otimes}F \longrightarrow G$ such that $\varphi=\overline{\varphi}\psi$. We recall that the linear map $U:E \longrightarrow F$ is a Riesz morphism if $|U(x)|=U(|x|)$ for every $x\in E$; the bilinear map

$\varphi: E \times F \to G$ is a Riesz bimorphism if $|\varphi(x,y)| = \varphi(|x|,|y|)$ for every $x \in E$, $y \in F$. The canonical morphism ψ induces an injective map from the algebraic tensor product $E \otimes F$ into $E \bar{\otimes} F$; we shall identify $E \otimes F$ with its image in $E \bar{\otimes} F$.

2. Principal modules

We collect here the basic definitions and results we shall need from principal modules theory (see [12] and [13]).

A lattice-ordered algebra with unit is an Archimedian vector lattice A with a strong unit e endowed with a bilinear multiplication which is a Riesz bimorphism and admits e as algebraic unit. On every lattice-ordered algebra with unit e we can give a norm by

$$\|x\| = \inf \{a \mid a \in \mathbb{R}_+,\ |x| \leq ae\}.$$

By a lattice-ordered subalgebra of the lattice-ordered algebra A we shall mean a subalgebra which is also a vector lattice.

Let A be a lattice-ordered algebra with unit. By an A-module we shall mean a vector lattice E which is an algebraic module over A such that the map $(a,x) \mapsto ax$ (from $A \times E$ into E) is a Riesz bimorphism.

The center $\mathscr{C}(E)$ of an Archimedian vector lattice E is a lattice-ordered algebra with unit; the map $(U,x) \mapsto U(x)$ defines a structure of $\mathscr{C}(E)$-module on E.

A principal A-module E is an A-module endowed with a locally solid topology such that for every $x \in E$ the set $\{ax \mid a \in A\}$ is dense in the principal order ideal generated by x. This is equivalent to require the set $\{ax \mid a \in [0,e]\}$ to be dense in $[0,x]$ for every $x \in E_+$ or to require the set $\{ax \mid a \in [-e,e]\}$ to be dense in $[-|x|,|x|]$ for every $x \in E$.

THEOREM 2.1.([12]) Let E be an Archimedian A-module endowed with a locally solid topology. Then E is principal if and only if for every neighborhood V of 0 and every x_1, $x_2 \in E$ such that $x_1 \wedge x_2 = 0$ there are a_1, $a_2 \in A$ such that $a_1 \wedge a_2 = 0$ and $x_i - a_i x_i \in V$, $i=1,2$.

If E is an order complete vector lattice and x_1, x_2 E, $x_1 \wedge x_2 = 0$ we can find band projections P_1, P_2 such that $P_1 \wedge P_2 = 0$ and $x_i - P_i(x_i) = 0$, $i=1,2$. Hence, by theorem 2.1, E is a principal $\mathscr{C}(E)$-module for every locally solid topology on E.

If A,B are lattice-ordered algebras with units e_1, e_2, by the universality property of $A \bar{\otimes} B$ there is an unique structure of lattice-ordered algebra on $A \bar{\otimes} B$ such that $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$ for $a_1, a_2 \in A$, $b_1, b_2 \in B$; its unit is $e_1 \otimes e_2$.

Suppose that E is an Archimedian A-module and also a B-module. Then we can give an unique structure of $A \bar{\otimes} B$-module on E such that $(a \otimes b)x = a(bx)$ for every $a \in A$, $b \in B$, $x \in E$.

Let E be an A-module and let F be a B-module. Suppose that E and F are principal, that F is order complete and its topology is order continuous. Denote by $L'_r(E,F)$ the vector lattice of all continuous linear maps in $L_r(E,F)$. Define structures of A-module and B-module on $L'_r(E,F)$ by

$$(aU)(x) = U(ax),$$
$$(bU)(x) = bU(x).$$

Then $L'_r(E,F)$ becomes an $A\bar{\otimes}B$-module. The solid strong topology on $L'_r(E,F)$ has as a basis of neighborhoods of 0 the sets $\{U \mid U\in L'_r(E,F),\ |U|(x)\in V\}$ for every $x\in E_+$ and every neighborhood V of 0 in F.

THEOREM 2.2.([13]) With respect to the solid strong topology, $L'_r(E,F)$ is a principal $A\bar{\otimes}B$-module.

3. Additional results about A-modules

It is well known that if B is a subalgebra of a lattice-ordered algebra A and $e\in B$ then its closure $\bar{B}$ is a lattice-ordered subalgebra of A.

DEFINITION 3.1. Let E be an A-module endowed with a locally solid topology and let B be a subalgebra of A. We say that E is B-principal if for every x_1, $x_2\in E$ such that $x_1\wedge x_2=0$ and every neighborhood V of 0 in E there are b_1, b_2 B such that $x_1-bx_1\in V$ and $bx_2\in V$.

THEOREM 3.1. Let E be an Archimedian A-module endowed with a locally solid topology and let B be a subalgebra of A containing e. Then E is B-principal if and only if E is a principal $\bar{B}$-module.

PROOF. We shall apply theorem 2.1. Let E be B-principal and let V be a neighborhood of 0. There is a solid neighborhood W of 0 such that $W+W\subset V$. Take x_1, $x_2\in E$ such that $x_1\wedge x_2=0$. By the hypothesis there is $b\in B$ such that $x_1-bx_1\in W$ and $bx_2\in W$. Put $b_1=b$, $b_2=e-b$. From

$$|x_i-|b_i|\,x_i| = ||x_i|-|b_ix_i|| \leq |x_i-b_ix_i|$$

it follows that $x_i-|b_i|\,x_i\in W$. We have

$$(|b_1|\wedge|b_2|)x_1\leq|b_2|x_1 = |b_2x_1|$$

hence $(|b_1|\wedge|b_2|)x_1\in W$; similarly $(|b_1|\wedge|b_2|)x_2\in W$. Consequently, if $c_i= =|b_i|-|b_1|\wedge|b_2|$ then $c_i\in\bar{B}$, $c_1\wedge c_2=0$ and

$$|x_i-c_ix_i|\leq|x_i-|b_i|\,x_i|+(|b_1|\wedge|b_2|)x_i;$$

therefore $x_i-c_ix_i\in W+W\subset V$. As V is arbitrary, we have obtained that E is $\bar{B}$-principal.

Conversely, suppose that E is $\bar{B}$-principal. Let x_1, $x_2\in E$ be such that $x_1\wedge x_2=0$ and let V be a neighborhood of 0 in E. There is a solid neighborhood W of 0 such that $W+W\subset V$. By the hypothesis there are c_1, $c_2\in\bar{B}$ such that $c_1\wedge c_2=$

$=0$ and $x_i - c_i x_i \in W$, $i=1,2$; we may assume that $c_i \in [0,e]$. Let $\varepsilon > 0$ be such that $\varepsilon(x_1+x_2) \in W$. There is $b \in B$ with $\|c_1 - b\| \leq \varepsilon$. We have

$$|x_1 - bx_1| \leq |x_1 - c_1 x_1| + |c_1 - b|\, x_1,$$
$$bx_2 \leq |c_1 x_2| + |b - c_1|\, x_2 \leq |c_1(x_2 - c_2 x_2)| + |c_1 c_2 x_2| + |b - c_1|\, x_2 \leq$$
$$\leq |x_2 - c_2 x_2| + |b - c_1|\, x_2,$$

hence $x_1 - bx_1 \in V$ and $bx_2 \in V$. As V is arbitrary, we have obtained that E is B-principal.

THEOREM 3.2. Let E be an Archimedian A-module endowed with a locally solid topology and let B be a subalgebra of A containing e. Consider a vector subspace F of E and a vector subspace F_0 dense in F with the following properties:

i) $BF_0 \subset F$.

ii) For every $x \in F_0$ and every neighborhood V of 0 in E there is $b \in B \cap [0,e]$ such that $bx \geqslant 0$ and $(e-b)x_+ \in V$.

Then the following are true:

i) The closure $\bar{F}$ of F is a vector sublattice of E.

ii) $(\bar{F})_+$ is equal to the closure of $F \cap E_+$.

iii) $\bar{F}$ is a principal $\bar{B}$-module.

PROOF. First we show that for every solid neighborhood V of 0 and every $x \in F_0$ there is $y \in F \cap E_+$ such that $x_+ - y \in V$. Indeed there is $b \in B \cap [0,e]$ such that $bx \geqslant 0$ and $(e-b)x_+ \in V$. As $bx \in F$ and

$$x_+ - bx = x_+ - (bx)_+ = x_+ - bx_+ \in V$$

the result follows.

From the above assertion, i) follows at once. To prove ii), take any $x \in (\bar{F})_+$. Ley V be a neighborhood of 0 and let W be a solid neighborhood of 0 such that $W+W \subset V$. As $x \in \bar{F}$, there is $y \in F_0$ such that $x-y \in W$. From

$$|x - y_+| = |x_+ - y_+| \leq |x-y|$$

it follows that $x - y_+ \in W$. There is also $z \in F \cap E_+$ such that $y_+ - z \in W$. It follows th $x - z \in V$; as V is arbitrary, x belongs to the closure of $F \cap E_+$.

By i), $\bar{F}$ is a $\bar{B}$-module. To prove it is principal it is enough, according to theorem 3.2, to show that $\bar{F}$ is B-principal. Let $x_1, x_2 \in \bar{F}$ be such that $x_1 \wedge x_2$ $=0$ and let V be a neighborhood of 0. Consider a solid neighborhood W of 0 such that $W+W+W \subset V$. There are $y_1, y_2 \in F_0$ such that $x_i - y_i \in W$, $i=1,2$. As

$$|y_1 \wedge y_2| = |y_1 \wedge y_2 - x_1 \wedge x_2| \leq |y_1 - x_1| + |y_2 - x_2|$$

we have that $y_1 \wedge y_2 \in W+W$. There is $b \in B \cap [0,e]$ such that

$$b(y_1 - y_2) \geqslant 0, \quad (e-b)(y_1 - y_2)_+ \in W.$$

We have

$$y_1-(y_1-y_2)_+ = y_1\wedge y_2 \in W+W.$$

Therefore

$$(e-b)y_1 = (e-b)(y_1-(y_1-y_2)_+)+(e-b)(y_1-y_2)_+ \in W+W+W\subset V.$$

On the other side

$$by_2 = by -b(y_1-y_2) = by_1-(b(y_1-y_2))_+ = by_1-b(y_1-y_2)_+ = \\ = b(y_1-(y_1-y_2)_+)\in W+W\subset V.$$

As V is arbitrary we have obtained that $\bar{F}$ is principal.

4. B-pairs

DEFINITION 4.1. By a B-pair we shall mean a couple (E,G) formed by a Banach lattice E and a Banach space G ordered by a b-strict cone together with a positive continuous linear map $J:G \longrightarrow E$.

DEFINITION 4.2. Let (E,G) be a B-pair. A map $U\in\mathcal{C}(E)$ is called G-central if there is $V\in L(G,G)$ such that $UJ=JV$.

We denote by $\mathcal{C}_G(E)$ the set of all G-central maps; $\mathcal{C}_G(E)$ is a subalgebra of $\mathcal{C}(E)$ containing 1_E.

DEFINITION 4.3. We say that a B-pair (E,G) is principal if the $\mathcal{C}(E)$-module E is $\mathcal{C}_G(E)$-principal.

If (E,G) is a B-pair and F is an order complete Banach lattice we let $LW_r(E,G,F)$ be the subset of all $U\in L_r(E,F)$ such that $|U|J\in LW(E,F)$.

PROPOSITION 4.1. $LW_r(E,G,F)$ is an order ideal in $L_r(E,F)$.

PROOF. Let M be the solid hull of $J(B_G)$, B_G being the closed unit ball of G. The assertion will be proved if we show that $LW_r(E,G,F)$ coincides with the set of all $U\in L_r(E,F)$ such that $|U|(M)$ is L-bounded.

Of course if $|U|(M)$ is L-bounded then $U\in LW_r(E,G,F)$. Conversely, let $U\in LW_r(E,G,F)$. As G_+ is b-strict there is $a>0$ such that for every $x\in B_G$ there is $y\in G$ with $-y\leq x\leq y$ and $\|y\|\leq a$. Let $\varepsilon>0$. As $|U|(J(aB_G))$ is L-bounded there is $z\in F_+$ such that $\|(||U|(J(x))|-z)_+\|\leq\varepsilon$ for every $x\in aB_G$. Consider $x\in M$; there is $u\in B_G$ such that $|x|\leq |J(u)|$. As $u\in B_G$ there is $v\in aB_G$ for which $-v\leq u\leq v$; it follows that $|J(u)|\leq J(v)$. From

$$||U|(x)|\leq |U|(|x|)\leq |U|(|J(u)|)\leq |U|(J(v))$$

and

$$\|(|U|(J(v))-z)_+\|\leq\varepsilon$$

we have that $\|(||U|(x)|-z)_+\|\leq\varepsilon$ for every $x\in M$. Hence $|U|(M)$ is L-bounded.

Let E,F be Banach spaces. A bilateral ideal of L(E,F) is a vector subspace $\mathcal{J}$ of L(E,F) with the property that $WUV\in\mathcal{J}$ whenever $U\in\mathcal{J}$, $V\in L(E,E)$ and $W\in L(F,F)$.

We recall that the dual of a Banach lattice has order continuous norm if and only if for every $f\in E'_+$ and every $\varepsilon>0$ there is $y\in E_+$ such that $f((x-y)_+)\leq\varepsilon$ for every $x\in E$ with $\|x\|\leq 1$. Similarly, E has order continuous norm if for every $x\in E_+$ and every $\varepsilon>0$ there is $g\in E'_+$ such that $(f-g)_+(x)\leq\varepsilon$ for every $f\in E'$ with $\|f\|\leq 1$ (for the proof of these assertions see [2]).

THEOREM 4.1. Let (E,G) be a principal B-pair such that E' has order continuous norm. Let F be a Banach lattice with order continuous norm. Consider a closed bilateral ideal $\mathcal{J}$ in L(G,F). Then the set of all $U\in LW_r(E,G,F)$ such that $UJ\in\mathcal{J}$ is a band in $LW_r(E,G,F)$.

The proof of the theorem will rely on theorem 2.2 and on the following lemma:

LEMMA 4.1. Let E,F be Banach lattices such that E' and F have order continuous norms and let M be a bounded solid subset of E. Consider the solid seminorm p_M on $L_r(E,F)$ given by

$$p_M(U)=\sup\{\||U|(x)\|\mid x\in M\}.$$

Let $U\in L_r(E,F)_+$ be such that U(M) is L-bounded. Then the solid strong topology is stronger on $[-U,U]$ than the topology defined by p_M.

PROOF. It suffices to prove that for every $\varepsilon>0$ there is $x\in E_+$ such that $p_M(V)\leq\|V(x)\|+\varepsilon$ whenever $V\in[0,U]$ (because if $V_1,V_2\in[-U,U]$ then $|V_1-V_2|\in[0,2U]$ and we may apply the above inequality to $|V_1-V_2|$).

Without loss of generality we may assume that M is contained in the unit ball of E. Let $\varepsilon>0$. As U(M) is L-bounded there is $y\in F_+$ such that $\|(U(x)-y)_+\|\leq\varepsilon/3$ for every $x\in M\cap E_+$ with $\|x\|\leq 1$. As F has order continuous norm there is $f_\varepsilon\in F'_+$ such that $(f-f_\varepsilon)_+(y)\leq\varepsilon/3$ for every $f\in F'$ with $\|f\|\leq 1$. As E' has order continuous norm there is $x_\varepsilon\in E_+$ such that $U'(f_\varepsilon)((x-x_\varepsilon)_+)\leq\varepsilon/3$ for every $x\in E$ with $\|x\|\leq 1$. Let $V\in[0,U]$ and let $x\in M\cap E_+$, $f\in F'_+$, $\|f\|\leq 1$. We have

$$f(V(x))=(f-f_\varepsilon)(V(x))+f_\varepsilon(V(x-x_\varepsilon))+f\ (V(x_\varepsilon))\leq$$
$$\leq(f-f_\varepsilon)_+(U(x))+U'(f_\varepsilon)((x-x_\varepsilon)_+)+f_\varepsilon(V(x_\varepsilon)).$$

But

$$(f-f_\varepsilon)_+(U(x))=(f-f_\varepsilon)_+(U(x)-y)+(f-f_\varepsilon)_+(y)\leq$$
$$\leq f((U(x)-y)_+)+(f-f_\varepsilon)(y)\leq 2\varepsilon/3.$$

Hence

$$f(V(x))\leq\|V(\|f_\varepsilon\|x_\varepsilon)+\varepsilon.$$

As

$$p_M(V) = \sup\{f(V(x)) \mid x \in M \cap E_+,\ f \in F'_+,\ \|f\| \leq 1\}$$

the result follows.

PROOF OF THEOREM 4.1. Let B be the set of all $U \in LW_r(E,G,F)$ such that $UJ \in \mathcal{J}$. Clearly B is a vector subspace. We prove first that B is an order ideal of $LW_r(E,G,F)$. Let $V, U \in LW_r(E,G,F)$ be such that $|V| \leq |U|$ and $U \in B$. By theorems 2.2 and 3.1, $L_r(E,F)$ is a principal $\overline{\mathcal{C}_G(E)\overline{\otimes}\mathcal{C}(F)}$-module. Hence there is a net $(a_\delta) \subset \overline{\mathcal{C}_G(E)}\overline{\otimes}\mathcal{C}(F)$ such that $a_\delta \in [-1_E \otimes 1_F, 1_E \otimes 1_F]$ and $a_\delta U \longrightarrow V$ in the solid strong topology; clearly $a_\delta U \in [-|U|, |U|]$.

Let B_G be the closed unit ball of G and let M be the solid hull of $J(B_G)$ in E. By the proof of proposition 4.1, $|U|(M)$ is L-bounded. By lemma 4.1 it follows that $p_M(a_\delta U - V) \longrightarrow 0$. Therefore, for every $\varepsilon > 0$ there is $a \in \overline{\mathcal{C}_G(E)}\overline{\otimes}\mathcal{C}(F)$ such that $p_M(aU-V) \leq \varepsilon/2$. It is known (see[3]) that $\overline{\mathcal{C}_G(E)}\otimes\mathcal{C}(F)$ is dense in $\overline{\mathcal{C}_G(E)}\overline{\otimes}\mathcal{C}(F)$; as $\mathcal{C}_G(E)$ is dense in $\overline{\mathcal{C}_G(E)}$ it follows that $\mathcal{C}_G(E)\otimes\mathcal{C}(F)$ is dense in $\overline{\mathcal{C}_G(E)}\overline{\otimes}\mathcal{C}(F)$. Hence there is $b \in \mathcal{C}_G(E)\otimes\mathcal{C}(F)$ with $\|b-a\| \leq \varepsilon(2p_M(U)+1)^{-1}$. Thus

$$p_M(bU-V) \leq p_M((b-a)U) + p_M(aU-V) \leq \|b-a\| p_M(U) + p_M(aU-V) \leq \varepsilon;$$

as $B_G \subset M$ the above inequality implies

$$\|(bU)J - VJ\| \leq \varepsilon.$$

As $\mathcal{J}$ is a bilateral ideal we have $(bU)J \in \mathcal{J}$; as $\mathcal{J}$ is closed and ε is arbitrary, $VJ \in \mathcal{J}$.

Now let (U_δ) be a net in B and $U \in LW_r(E,G,F)$ be such that $0 \leq U_\delta \uparrow U$. As F has order continuous norm, $U_\delta \longrightarrow U$ in the solid strong topology. By lemma 4.1, $p_M(U_\delta - U) \longrightarrow 0$; hence $\|U_\delta J - UJ\| \longrightarrow 0$. As $U_\delta J \in \mathcal{J}$ and $\mathcal{J}$ is closed it follows that $UJ \in \mathcal{J}$. Therefore B is a band.

COROLLARY 4.1. Let (E,G) and F be as in theorem 4.1 and let $U,V: E \longrightarrow F$ be such that $0 \leq U \leq V$. If VJ is compact then UJ is also compact.

If E,F are Banach spaces, a map $U \in L(E,F)$ will be called approximable if it lies in the uniform closure of all finite-rank continuous linear maps. Every approximable map is compact; the set of all approximable maps is a bilateral ideal.

COROLLARY 4.2. Let (E,G) and F be as in theorem 4.1 and let $U,V: E \longrightarrow F$ be such that $0 \leq U \leq V$. If VJ is approximable then UJ is also approximable.

We pass now to some examples of B-pairs. In a first place, we consider the case of the Sobolev spaces. Let Ω be a domain in $\mathbb{R}^n$. The Sobolev space $W^{k,p}(\Omega)$ is the space of all p-integrable functions on Ω having p-integrable derivatives (in the sense of distributions) of all orders $\leq k$; the norm is defined by

$$\|f\| = \Big(\sum_{0 \leq k_1+\dots+k_n \leq k} \int \Big|\frac{\partial^{k_1+\dots+k_n}}{\partial x_1^{k_1}\dots\partial x_n^{k_n}} f(t)\Big|^p dt\Big)^{1/p}.$$

On the Sobolev spaces the following order relation is given: $f\geq 0$ if $f(t)\geq 0$ for almost every $t\in\Omega$. In this way, a structure of ordered Banach space with closed cone is obtained; however, these sapces are far from Banach lattices (for instance, the order intervals are not norm bounded). The most important cases when the positive cone is generating are the following:

a) $k=1$, in which case $W^{1,p}(\Omega)$ is a vector lattice (though not a Banach lattice).

b) Ω is bounded, has a smooth boundary and $kp>n$. In this case (see[5]), every function in $W^{k,p}(\Omega)$ is bounded; as the constant functions belong to the space, it follows that the positive cone is generating.

Consider a space $W^{k,p}(\Omega)$ with generating cone. Suppose that $1<p<\infty$. Then the couple $(L^p(\Omega), W^{k,p}(\Omega))$ together with the inclusion map $J:W^{k,p}(\Omega)\longrightarrow L^p(\Omega)$ form a B-pair satisfying the hypothesis of theorem 4.1. Indeed, $L^p(\Omega)$ has order continuous dual; to see that the pair is principal, note that the multiplication by a C^∞-function with compact support in Ω defines a $W^{k,p}(\Omega)$-central map on $L^p(\Omega)$. Now if $\varepsilon>0$ is given and $f_1,f_2\in L^p(\Omega)$ are such that $f_1\wedge f_2=0$, there are compact disjoint subsets K_1, K_2 of Ω such that $\int_{\Omega\setminus K_i}|f_i(t)|^p\,dt\leq\varepsilon$. We can find a C^∞-function φ with compact support in Ω such that φ equals 1 on K_1 and 0 on K_2. Then

$$\|f_1-\varphi f_1\|\leq\varepsilon,\quad \|\varphi f_2\|\leq\varepsilon$$

(the norm being taken in $L^p(\Omega)$). The assertion is proved.

As a second example we shall construct a B-pair (E,G) and a Banach lattice F satisfying the hypothesis of theorem 4.1 together with a positive linear map $U:E\longrightarrow F$ such that UJ is compact but not approximable. In this way it will be proved that corollary 4.2 is not a direct consequence of corollary 4.1.

We briefly recall the construction of A. Szankowski's reflexive Banach lattice without the approximation property (for details see [9]). Let $\mathcal{B}_n$ be the algebra of subsets of $[0,1]$ generated by the 2^n atoms $[(i-1)/2^n, i/2^n)$, $i=1,\dots,2^n$. For every n, let φ_n be the permutation of $\{1,2,\dots,2^n\}$ defined by $\varphi_n(2i)=2i-1$, $\varphi_n(2i-1)=2i$. The map φ_n induces a permutation between the atoms of $\mathcal{B}_n$ and therefore a map (denoted again by φ_n) on $\mathcal{B}_n$.

For every $n\geq 2^6$ a partition Δ_n of $[0,1]$ into M_n disjoint $\mathcal{B}_n$-measurable sets of equal measure is constructed. The Szankowski space E is defined to be the space of equivalence classes of measurable functions on $[0,1]$ such that the norm

$$\|f\| = \Big(\sum_{n=2^6}^{\infty}\sum_{B\in\Delta_n} M_n^{\alpha p}\Big(\int_B |f(t)|^r\,dt\Big)^{p/r}\Big)^{1/p}$$

is finite (α, p and r being certain positive constants).

A subset M in a Banach space E will be called approximable if for every

$\varepsilon>0$ there is $U\in L(E,E)$ such that $\dim U(E)<\infty$ and $\|x-U(x)\|\leq\varepsilon$ for every $x\in M$.

In the Szankowski space E the following nonapproximable compact set is constructed: let E_n be the set of all $\mathcal{B}_n$-measurable functions such that $|f|$ is the characteristic function of $\varphi_n(A)$ for some $A\in\Delta_n$. Then $M=\{0\}\cup\bigcup_{n=2^6}^{\infty}\alpha_n E_n$ is a compact nonapproximable set, α_n being some suitable positive numbers. As the sets E_n are finite, M can be disposed in a sequence converging to 0: $M=\{f_0, f_1,\ldots\}$. Let K be the closed convex hull of $\{f_n/\|f_n\|^{1/2} \mid n\geq 0\}$ and let G be the vector subspace generated by K. Define on G the norm which makes K its unit ball. Then G becomes a Banach space and the inclusion map $J:G\to E$ is compact but not approximable (see the proof of theorem 1.e.4 in [8]). The order relation on E induces an order relation on G. Therefore, if we prove that (E,G) is a principal B-pair we may take F=E and $U=1_E$ in order to obtain our example.

First of all, let us show that G_+ is b-strict. Observe that K is the set of all elements $\sum_{n=0}^{\infty} a_n g_n$ where $a_n\geq 0$, $\sum_{n=0}^{\infty} a_n\leq 1$ and $g_n=f_n/\|f_n\|^{1/2}$: indeed, the set

$$C=\{(a_n)\mid a_n\geq 0,\ \sum_{n=0}^{\infty} a_n\leq 1\}$$

is a closed subset of the compact space $[0,1]^{\mathbb{N}}$; as $g_n\to 0$, the map $(a_n)\mapsto \sum_{n=0}^{\infty} a_n g_n$ from C into E is continuous, so it takes C into a compact convex subset of E. From the construction of the sets E_n it is obvious that for every n there is n' such that $|g_n|=g_{n'}$. Hence, if

$$f=\sum_{n=0}^{\infty} a_n g_n\in K$$

then

$$\pm f\leq\sum_{n=0}^{\infty} a_n|g_n|\in K$$

which proves our assertion.

Now we show that (E,G) is a principal pair. First we prove that the multiplication by the characteristic function of an atom of $\mathcal{B}_n$ defines a G-central map on E. Indeed, from the construction of the sets E_n it is easy to see that $U(g_n)\in G$ for every $n\geq 0$ and that $U(g_n)$ is a convex combination of the g_j's for sufficiently large n; therefore the restriction of U to G is continuous.

Now let $\mathcal{B}=\bigcup_{n=0}^{\infty}\mathcal{B}_n$. The set of all $\mathcal{B}$-measurable functions (that is, the functions which are $\mathcal{B}_n$-measurable for some $n\geq 0$) is dense in E. By the above argument, the multiplication by such a function defines a G-central map on E. Let $\varepsilon>0$ be given and let $h_1, h_2\in E$ be such that $h_1\wedge h_2=0$. We can find $\mathcal{B}$-measurable functions h_1', h_2' such that $\|h_i-h_i'\|\leq\varepsilon$, $i=1,2$; replacing h_1' by $h_1'-h_1'\wedge h_2'$ we may assume that $h_1'\wedge h_2'=0$. Now let φ be the characteristic function of the support of h_1'. We have

$$\|h_1-\varphi h_1\| \leq \|(e-\varphi)(h_1-h_1')\| + \|(e-\varphi)h_1'\| \leq \varepsilon ,$$
$$\|\varphi h_2\| \leq \|\varphi(h_2-h_2')\| + \|\varphi h_2'\| \leq \varepsilon$$

(e being the function identical one). The proof is complete.

5. The case of a reflexive Banach space with closed generating cone

DEFINITION 5.1. Let E be an ordered vector space. A latticial extension of E is a vector lattice $\tilde{E}$ such that E is a vector subspace of $\tilde{E}$ and $E_+ = E \cap \tilde{E}_+$.

Let E be an ordered Banach space and let E be a latticial extension of E. According to §4, a map $U \in \mathscr{C}(\tilde{E})$ will be called E-central if $U(E) \subset E$ and the restriction of U to E is continuous; the subalgebra of all E-central maps will be denoted by $\mathscr{C}_E(\tilde{E})$.

DEFINITION 5.2. We say that an ordered Banach space E has a principal latticial extension if there is a latticial extension E of E and a dense vector subspace E_0 of E with the following property: for every $\varepsilon > 0$ and every $x \in E_0$ there is $U \in \mathscr{C}_E(\tilde{E}) \cap [0, 1_{\tilde{E}}]$ and $y \in E$ such that $\|y\| \leq \varepsilon$, $U(x) \geqslant 0$ and $(1_{\tilde{E}} - U)(x_+) \leq y$.

THEOREM 5.1. Let E be a reflexive Banach space ordered by a closed generting cone and let F be an order continuous Banach lattice. Suppose that E has a principal latticial extension. Let $\mathcal{J}$ be a closed bilateral ideal in L(E,F). Consider $U,V : E \longrightarrow F$ such that $0 \leq U \leq V$ and $V \in LW(E,F) \cap \mathcal{J}$. Then $U \in \mathcal{J}$.

PROOF. We shall construct a Banach lattice H with order continuous dual such that (H,E) will be a principal B-pair with the following property: for every positive $U:E \longrightarrow F$ there is an unique positive $\tilde{U}:H \longrightarrow E$ such that $U = \tilde{U}J$. Thus an application of theorem 4.1 will conclude the proof (we remark that E_+, being closed and generating, is also b-strict).

Let $\tilde{E}$ be a principal latticial extension of E and let K be the convex solid hull of the closed unit ball B_E of E in $\tilde{E}$. Denote by G the vector subspace of E spanned by K; G is an order ideal of E, hence a vector lattice. As $U(G) \subset G$ for every $U \in \mathscr{C}_E(\tilde{E})$, it follows that G is a $\mathscr{C}_E(\tilde{E})$-module. Define the solid seminorm p on G by

$$p(x) = \inf\{a \mid a \in \mathbb{R}_+,\ x \in aK\}.$$

By theorem 3.2 we have that the closure $\bar{E}$ of E with respect to p is a vector sublattice of G, that $(\bar{E})_+$ is equal to the closure of E_+ and that $\bar{E}$ is a principal $\mathscr{C}_E(\tilde{E})$-module. Put $G_0 = \bar{E} \cap p^{-1}(\{0\})$ and let H be the completion of $\frac{\bar{E}}{G_0}$. If $J:E \longrightarrow H$ is the canonical map, (H,E) is a principal B-pair. Every positive (hence continuous) linear map $U:E \longrightarrow F$ is continuous for the restriction of p to E; therefore there is an unique continuous linear $\tilde{U}:H \longrightarrow F$ such that $U = \tilde{U}J$. As H_+ is the closure of $J(E_+)$ it follows that $\tilde{U}$ is positive.

It remains to show that H' has order continuous norm. To this purpose it suffices to prove that for every $\varepsilon>0$ and every linear positive $f:\tilde{E}\to\mathbb{R}$ continuous for p there is $y\in(\tilde{E})_+$ such that $f((|x|-y)_+)\leq\varepsilon$ for every $x\in\tilde{E}$ with $p(x)\leq 1$.

Let $\tilde{E}'$ be the vector lattice of all linear forms on $\tilde{E}$ continuous for p and let $(\tilde{E}')^\times$ be the vector lattice of all order continuous linear forms on $\tilde{E}'$; obviously, $\tilde{E}'\subset L_r(\tilde{E},\mathbb{R})$. As E_+ is closed, the restriction of every $f\in\tilde{E}'$ to E is continuous. Hence the restriction of the map $k:\tilde{E}\to(\tilde{E}')^\times$ given by $k(x)(f)=f(x)$ to E is continuous for $\sigma(E,E')$ and $\sigma((\tilde{E}')^\times,\tilde{E}')$. As B_E is $\sigma(E,E')$-compact it follows that $k(B_E)$ is $\sigma((\tilde{E}')^\times,\tilde{E}')$-compact. Let $\varepsilon>0$ and let $f\in(\tilde{E}')_+$. As $\tilde{E}'$ is order complete, 82E and 82G in [4] imply that there is $\varphi\in(\tilde{E}')^\times_+$ such that $(|k(x)|-\varphi)_+(f)\leq\varepsilon$ for every $x\in B_E$. Examining the proof of 81H in [4] we see that φ can be taken of the form

$$\sum_{i=1}^{n}|k(y_i)|$$

with $y_i\in E$. As E_+ is generating, for every i there is $z_i\in E_+$ such that $-z_i\leq y_i\leq$ $\leq z_i$. Hence $\varphi\leq k(y)$ where $y=\sum_{i=1}^{n}z_i$ By 31C in [4], k is a Riesz morphism. It follows that

$$f((|x|-y)_+) = k((|x|-y)_+)(f) = (|k(x)|-k(y))_+(f)\leq$$
$$\leq(|k(x)|-\varphi)_+(f)$$

for every $x\in B_E$. The set $\{x\,|\,x\in\tilde{E},\ f((|x|-y)_+)\leq\varepsilon\}$ is a closed solid convex set containing B_E; hence it contains every $x\in\tilde{E}$ with $p(x)\leq 1$.

COROLLARY 5.1. Let E and F be as in theorem 5.1 and let $U,V:E\to F$ be such that $0\leq U\leq V$. If V is compact then U is also compact.

COROLLARY 5.2. Let E and F be as in theorem 5.1 and let $U,V:E\to F$ be such that $0\leq U\leq V$. If V is approximable then U is also approximable.

We consider now some examples of spaces satisfying the hypothesis of theorem 5.1. In the first place, take E to be the Sobolev space $W^{k,p}(\Omega)$ such that $1<p<\infty$, $kp>n$ and Ω is bounded and has a smooth boundary; these conditions ensure that E is reflexive and E_+ is closed and generating. Let $\tilde{E}$ be the vector lattice of all functions on Ω; clearly $\tilde{E}$ is a latticial extension of E. To see it is principal, let $\varepsilon>0$ and let $f\in E$ be given. Take g to be the function identic equal to c on Ω, c being a suitable positive constant such that $\|g\|\leq\varepsilon$. As f is continuous on $\overline{\Omega}$ ([5], theorem 5.7.8), the sets $M_1=\{t\,|\,t\in\overline{\Omega},\ f(t)\geq c\}$ and $M_2=\{t\,|\,t\in\overline{\Omega},\ f(t)\leq 0\}$ are disjoint compact subsets of $\mathbb{R}^n$. Therefore there is a C^∞-function φ with compact support on $\mathbb{R}^n$ such that $0\leq\varphi(t)\leq 1$ for $t\in\mathbb{R}^n$, $\varphi(t)=1$ for $t\in M_1$ and $\varphi(t)=0$ for $t\in M_2$. It follows that the map U defined by the multiplication by φ has the properties: $U\in\mathscr{C}_3(\tilde{E})\cap[0,1_{\tilde{E}}]$, $U(f)\geq 0$, $(1_{\tilde{E}}-U)(f_+)\leq g$. Thes proves our assertion.

As a second example take E to be the space $W_0^{k,p}(0,1)$ with $1<p<\infty$. It consists of the closure in $W^{k,p}(0,1)$ of the subspace formed by the C^∞-functions with compact support in $(0,1)$. It can be proved that a function f is in $W_0^{k,p}(0,1$

if and only if f and the derivatives $\frac{d^i f}{dt^i}$ for $1 \leq i \leq k-1$ are absolutely continuous functions on $[0,1]$ vanishing at 0 and 1 and $\frac{d^k f}{dt^k}$ is in $L^p(0,1)$. The space E is ordered by the closed cone of all functions taking only positive values. For $k \geqslant 1$, this cone is not normal (though it is latticial for $k=1$); for $k \geqslant 2$, the cone is neither latticial. Nevertheless, we prove that this cone is generating. Indeed, let $f \in E$ be given and let $a > 0$ and $b > 0$ be such that $f(t) < a < b/4^k k!$ for every $t \in [0,1]$. Define the functions $g_0, \dots, g_k : [0,1/4] \longrightarrow \mathbb{R}$ by

$$g_k(t) = \left(\frac{d^k f}{dt^k}(t)\right)_+ + b,$$

$$g_{i-1}(t) = \int_0^t g_i(s)ds, \quad 1 \leq i \leq k.$$

By induction we have that $\left(\frac{d^i f}{dt^i}(t)\right)_+ \leq g_i(t)$; hence $(f(t))_+ \leq g_0(t)$ for every $t \in [0,1/4]$; on the other side

$$g_0(\tfrac{1}{4}) \geqslant \frac{b}{4^k k!} > a.$$

Clearly, $g_0(0)=0$. In the same way we define the functions $h_0, \dots, h_k : [3/4,1] \longrightarrow \mathbb{R}$ by

$$h_k(t) = \left((-1)^k \frac{d^k f}{dt^k}(t)\right)_+ + b,$$

$$h_{i-1}(t) = \int_t^1 h_i(s)ds, \quad 1 \leq i \leq k.$$

Clearly $(f(t))_+ \leq h_0(t)$ for $t \in [3/4,1]$ and $a < h_0(3/4)$. Hence we may find a C^∞-function φ on $[0,1]$ such that

$$\frac{d^i \varphi}{dt^i}(\tfrac{1}{4}) = \frac{d^i g}{dt^i}0(\tfrac{1}{4}), \quad 0 \leq i \leq k-1,$$

$$\frac{d^i \varphi}{dt^i}(\tfrac{3}{4}) = \frac{d^i h}{dt^i}0(\tfrac{3}{4}), \quad 0 \leq i \leq k-1$$

and $\varphi(t) > a$ for $t \in [1/4,3/4]$. Then the function f_0 given by

$$f_0(t) = g_0(t), \quad t \in [0,\tfrac{1}{4}],$$

$$f_0(t) = \varphi(t), \quad t \in [\tfrac{1}{4},\tfrac{3}{4}]$$

$$f_0(t) = h_0(t), \quad t \in [\tfrac{3}{4},1]$$

is a positive function in E such that $f \leq f_0$.

Now we prove that E has a principal latticial extension; thus, it will provide an example of space satisfying the hypothesis of theorem 5.1. Indeed, the space $\tilde{E}$ of all functions on $[0,1]$ is a latticial extension of E. As we have already mentioned, the subspace E_0 consisting of C^∞-functions with compact support in $(0,1)$ is dense in E. Let $f \in E_0$ be given. There is $g \in E_0$ such that g is identically 1 on the support of f. Let $\varepsilon>0$ be given and let $\eta=\varepsilon/\|g\|$. There is $\varphi\in E_0$ such that $0\leq\varphi(t)\leq 1$ for $t\in[0,1]$, $\varphi(t)=1$ if $f(t)\geq\eta$ and $\varphi(t)=0$ if $f(t)\leq 0$. The map U given by the multiplication by φ is in $\mathcal{C}_E(\tilde{E})$. Clearly $0\leq U\leq 1_{\tilde{E}}$ and $U(f)\geq 0$; on the other side, as $(1-\varphi(t))f(t)\leq\eta g(t)$ for $t\in[0,1]$, it follows that $(1_{\tilde{E}}-U)(f_+)\leq\eta g$. The proof is complete.

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