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A GENERAL THEORY OF DUAL OTIMIZATION PROBLEMS

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A general theory of dual optimization problems

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We construct a unified theory of dual optimization problems, which encompasses, as particular cases, the known dual problems. For each concept of dual problem, we define first an "unperturbational" version, from which we deduce, via a certain scheme, the corresponding "perturbational" version. We generalize simultaneously the Lagrangian and surrogate cases, using coupling functionals. We study the connections between dual problems and define Lagrangian functionals for them. We study the class of perturbation functionals which can be written as the upper sum of the primal objective functional h and a functional with values not depending on h.

§1. Introduction

Let F be a set, G a subset of F (assumed non-empty, throughout the sequel), called the constraint set and $h:F \to \overline{R}=[-\infty,+\infty]$ a functional, called the objective functional. We shall consider the (global, scalar) primal infimization problem

$$(P)=(P_{G,h}) \qquad \alpha=\alpha_{G,h}=\inf h(G), \tag{1.1}$$

and we shall assume that $\alpha<+\infty$, or, equivalently, that

$$G\cap \operatorname{dom} h\neq\emptyset, \tag{1.2}$$

where $\emptyset$ denotes the empty set and

$$\operatorname{dom} h=\{y\in F \mid h(y)<+\infty\}; \tag{1.3}$$

the number α is called the value of problem (P).

Assumption (1.2) is no restriction of the generality, since it excludes only the trivial cases when either $G=\emptyset$ (throughout the paper we shall adopt the usual conventions $\inf\emptyset=+\infty$, $\sup\emptyset=-\infty$), which we have already excluded above, or $G\neq\emptyset$ and $h|_G\equiv+\infty$. Moreover, we shall say that two optimization problems (P_1) and (P_2) are equivalent, and we shall write $(P_1)\sim(P_2)$, if $\alpha_1=\alpha_2$, where α_i is the value of (P_i) $(i=1,2)$; thus, if we do not assume (1.2), then every problem (1.1) with $\alpha=+\infty$ is equivalent to the trivial problem $(P_{\emptyset,h})$. We shall make no further assumptions on F, G and h.

By a dual problem to (P) we shall mean any supremization problem of the form

$$(Q)=(Q^{G,h}) \qquad \beta=\beta^{G,h}=\sup \lambda(W), \tag{1.4}$$

where $W=W^{G,h}$ is a set (assumed non-empty, without loss of generality) called dual constraint set and $\lambda=\lambda^{G,h}:W\to\overline{R}$ is a functional, called dual objective functional; note that here the inequality $\alpha\geq\beta$ need not hold (see remark 2.3 below). We shall call $\{(P),(Q)\}$ a primal-dual pair (of optimization problems).

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$\lambda = \lambda^{G,h}$), etc., will mean that α, β, etc. (may) depend on G,h (and possibly on other arguments). This specification of G,h, etc. will be necessary (see e.g. definition 2.1); moreover, often we shall use $\alpha, \tilde{\alpha}$, etc. with respect to different G's, h's, etc., in the same formula (see e.g. (3.9), (3.66), etc.). There will be some exceptions, e.g., W or the set of perturbations $X = X_{G,h}$, for which we shall often omit the specification of G and h, but we shall use, when necessary, notations such as $V_{W\gamma}$, $V_{W\Delta}$, $\tilde{W}$, etc. (see e.g. (2.35), (2.51), (3.6)).

The dual problems to $(P_{G,h})$, which have been considered until the present, are rather diversified. Thus, the concept of "Lagrangian dual optimization problem", introduced by Rockafellar [17] (see also [9]) and extended in [33], involves an embedding of $(P_{G,h})$ into a family of "perturbed optimization problems" $(P^x_{G,h})_{x \in X}$, with the aid of a perturbation functional p, and a conjugation with respect to an arbitrary coupling functional, in the sense of [15], [16] (therefore, we have used in [32] the term "perturbational conjugate dual" problem). In a different direction, generalizing the usual "surrogate dual problem" ([5], [12], [8]) and "the quasi-convex dual" [2], "pseudo-dual" [25] and "semi-dual" ([29], [30]) problems to

$$(P) \qquad \alpha = \inf_{\substack{y \in R^n \\ u(y) \geqslant x_1}} h(y), \qquad (1.5)$$

where $u: R^n \to R^m$, $x_1 \in R^m$, we have introduced, in [32], two concepts of dual problems: the first one involves only a one-parameter family of "surrogate constraint sets" $\Delta_{G,w} \subset F$ $(w \in W)$, while the second one involves a perturbation functional $p: F \times X \to \overline{R}$ and a family of sets $\tilde{\Delta}_{(F,x_0),w} \subset F \times X$ $(w \in W)$, where $x_0 \in X$ satisfies (1.6) below and $W \subset \overline{R}^X$ is the constraint set of the dual problem (we have used in [32], respectively, the terms "surrogate dual" and "perturbational extended surrogate dual" problems, but here we shall omit the word "extended"); we recall that $\overline{R}^X$ denotes the family of all functionals $f: X \to \overline{R}$. Some connections between the above dual problems have been shown in [32]. For some further dual problems to $(P_{G,h})$, see remarks 2.9 f), 3.6 c), 3.9 d) and 3.13 b) below.

The aim of the present paper is to construct a unified theory of dual optimization problems to problem $(P_{G,h})$, which encompasses, as particular cases, the known dual problems to $(P_{G,h})$. In the sequel, we shall concentrate only on the various concepts of dual problems (1.4) and on the connections between them; therefore, we shall not consider here conditions for the absence of a duality gap (with a few exceptions, such as remarks 2.9 d), f), 3.9 c) etc.), solutions of primal or dual problems, etc.

In the theory of dual optimization problems, which we shall construct in this paper, for each concept of dual problem we shall first define an "unperturbational" version (i.e., without assuming any perturbation of $(P_{G,h})$), and then we shall deduce from it, with the aid of a certain scheme, the corresponding "perturbational" version (i.e. assuming a perturbation of $(P_{G,h})$, by a functional $p:F\times X\rightarrow\overline{R}$, where $X=X_{G,h}$ is a set); this extends the situation of the "surrogate dual" and "perturbational extended surrogate dual problems" of [32], mentioned above. Another feature of our approach will be a simultaneous generalization of the Lagrangian and surrogate cases, with the aid of coupling functionals $\varphi,\tilde{\varphi}$.

Firstly, in §2, without assuming a perturbation of $(P_{G,h})$, for any dual objective set $W=W^{G,h}$ and any coupling functional $\varphi=\varphi_G:F\times W\rightarrow$ with values $\varphi_G(y,w)\in\overline{R}$ not depending on h, we shall define a dual objective functional $\lambda=\lambda_{W\varphi}^{G,h}$, working directly with the objective functional h and with φ_G. Any dual problem $(Q_{W\varphi}^{G,h})$ $\beta=\sup\lambda(W)$, defined in this way, will be called a "$(W\varphi)$-dual problem" to $(P_{G,h})$. In the particular case when W does not depend on h (we shall write then $W=W_G$, since W^G denotes the family of all mappings $u:G\rightarrow W$), the $(W\varphi)$-dual objective functional will turn out to be the negative of the conjugate of h, with respect to $-\varphi_G$. Next, we shall define directly the "unperturbational Lagrangian dual" and the "unperturbational surrogate dual" problems to $(P_{G,h})$ and we shall show that they are $(W\varphi)$-dual problems for suitable particular choices of $\varphi=\varphi_G$. In the converse direction, it will turn out that if $\varphi=\varphi_G$ satisfies a simple condition (namely, (2.37)), then the $(W\varphi)$-dual problem to $(P_{G,h})$ becomes a particular case of the unperturbational Lagrangian dual problem to $(P_{G,h})$, with a suitably modified dual constraint set V; in particular, we shall show that this conclusion holds for the unperturbational surrogate dual problem (since the corresponding φ_G satisfies condition (2.37)).

Next, in §3, we shall consider $(P_{G,h})$ embedded into a family of optimization problems $(P_{G,h}^{x})_{x\in X}$, where $X=X_{G,h}$ is an arbitrary set, with the aid of a perturbation functional, i.e., a functional $p=p_{G,h}:F\times X\rightarrow\overline{R}$ for which there exists $x_o=x_o^{G,h,p}\in X$ such that

$$p(y,x_o)=h(y)\dotplus\chi_G(y)\qquad(y\in F),\tag{1.6}$$

where $\chi_G:F\rightarrow\overline{R}$ is the "indicator functional" of the set G, defined by

$$\chi_G(y)=\begin{cases}0 & \text{for } y\in G\\ +\infty & \text{for } y\in F\setminus G,\end{cases}\tag{1.7}$$

and where $\dotplus$ denotes the "upper addition" on $\overline{R}$. We recall that the "upper addition" $\dotplus$ and the "lower addition" $\underset{\cdot}{+}$ on $\overline{R}$ are defined ([15], [16]) by

$$a\dotplus b=a\underset{\cdot}{+}b=a+b \text{ if } R\cap\{a,b\}\neq\emptyset \text{ or } a=b=\pm\infty;\tag{1.8}$$

$$a\dotplus b=+\infty,\ a\underset{\cdot}{+}b=-\infty, \text{if } a=-b=\pm\infty,\tag{1.9}$$

refer to [16].

We shall show that, for any perturbation (X,p,x_0) of $(P_{G,h})$, each unperturbational dual problem (Q_1) $\beta_1=\sup \lambda_1(W_1)$, to $(P_{G,h})$; where $W_1\subset \overline{R}^F$ and $\lambda_1:W_1\to\overline{R}$, induces, via a certain scheme, a corresponding perturbational dual problem $(Q^{G,h}_{pW\lambda})$ to $(P_{G,h})$, with suitable $W=W^{G,h}\subset\overline{R}^X$. Applying this scheme, in particular, to any $(W\varphi)$-dual problem, to the unperturbational Lagrangian dual problem and to any unperturbational surrogate dual problem, we shall arrive, respectively at the concepts of "perturbational $(W\tilde{\varphi})$-dual", where $\tilde{\varphi}=\tilde{\varphi}_{(F,x_0)}:(F\times X)\times(0,W)\to\overline{R}$ is any coupling functional, with values $\tilde{\varphi}_{(F,x_0)}((y,x),(0,w))\in\overline{R}$ not depending on p, "perturbational Lagrangian dual" and "perturbational surrogate dual" problems to $(P_{G,h})$ (with respect to the perturbation functional p); furthermore, we shall show that a certain dual problem to $(P_{G,h})$, introduced by Gould [6] and Tind and Wolsey [36], is a perturbational dual $(Q^{G,h}_{pW\lambda})$, corresponding to a suitable unperturbational dual to $(P_{G,h})$. We shall give various relations between the above classes of dual problems (among which, in particular, some complements to the results of [32], e.g., some new relations between unperturbational surrogate and perturbational surrogate dual problems). We shall obtain some further relations between them, for a large class of perturbation functionals $p=p_{G,h}:F\times X\to\overline{R}$ which we shall introduce here, namely, the p's that can be written as the upper sum of h and of a term $\pi_G:F\times X\to\overline{R}$, with values $\pi_G(y,x)\in\overline{R}$ not depending on h (we shall call them "h-separated" perturbation functionals p). For example, the "natural perturbation functionals" $p=p^n$ of (3.29) and, in particular, the "standard perturbation functional $p=p^s$ of (3.38), considered in [30], [32], are h-separated, with π_G's which are convenient for computations. Let us also note that, in the particular case when X is a linear (or, a locally convex) space, one often has $x_0=0$ for these p's. For such an X, we shall denote by $X^{\#}$ (respectively, X^*) the linear space of all linear (respectively, continuous linear) functionals $w:X\to R$; if S is a linear subspace of X, we shall denote $S^{\perp}=\{w\in X^{\#} \mid w(s)=0\ (s\in S)\}$. For any sets F and X, we shall use the canonical embedding $R^F\times\overline{R}^X\subset\overline{R}^{F\times X}$, given by

$$(v,w)(y,x)=v(y)+w(x) \qquad (v\in R^F, w\in\overline{R}^X, y\in F, x\in X). \qquad (1.10)$$

We shall also express some of our results in terms of equivalence of primal-dual pairs. We recall that two primal-dual pairs $\{(P_1),\ (Q_1)\}$ and $\{(P_2),\ (Q_2)\}$ are said [30] to be equivalent, and we write $\{(P_1),(Q_1)\}\sim\{(P_2),\ (Q_2)\}$, if $(P_1)\sim(P_2)$ and $(Q_1)\sim(Q_2)$ (in the sense mentioned above, after formula (1.3)).

A useful tool for studying dual problems will be the following general concept of Lagrangian functional:

Definition 1.1. We shall call Lagrangian functional (or, briefly, Lagrangian) of the primal-dual pair $\{(P_{G,h}), (Q^{G,h})\}$, any functional $L=L^{G,h}:F\times W\to\overline{R}$, such that

$$\lambda(w)=\inf_{y\in F} L(y,w) \qquad (w\in W). \qquad (1.11)$$

Remark 1.1. a) By (1.4) and (1.11), we have

$$\beta=\sup_{w\in W}\inf_{y\in F} L(y,w). \qquad (1.12)$$

b) For many dual problems (Q), the explicit form of $\lambda:W\to\overline{R}$ will suggest a natural choice of a Lagrangian L, which we shall call the Lagrangian of the pair $\{(P),(Q)\}$. In particular, let us mention that for perturbational surrogate dual problems no Lagrangian functional L has been given in [32], but we shall introduce one in the present paper (see (3.133) below).

c) We shall also be interested in the particular cases when the Lagrangian $L=L^{G,h}$ satisfies

$$h(y)\dot{+}\chi_G(y)\geq L(y,w) \qquad (y\in F,\ w\in W), \qquad (1.13)$$

whence, by (1.1), (1.7) and (1.12),

$$\alpha=\inf_{y\in F}\{h(y)\dot{+}\chi_G(y)\}\geq\inf_{y\in F}\sup_{w\in W} L(y,w)\geq\sup_{w\in W}\inf_{y\in F} L(y,w)=\beta; \qquad (1.14)$$

we shall give some simple conditions for (1.13), (1.14) to hold.

Moreover, the particular cases when

$$h(y)\dot{+}\chi_G(y)=\sup_{w\in W} L(y,w) \qquad (y\in F), \qquad (1.15)$$

are also of interest; however, (1.15) holds only under rather strong assumptions on h (see e.g. [18]), which we shall not consider here.

Another useful tool is the following: If X and W are two sets, and $\varphi:X\times W\to\overline{R}$ is an arbitrary functional (called, in [15], [16], coupling functional), then we shall call Fenchel-Moreau conjugation (with respect to φ) the operator $f\in\overline{R}^X\to f^{c(\varphi)}\in\overline{R}^W$ defined ([15], [16]) by

$$f^{c(\varphi)}(w)=\sup_{x\in X}\{\varphi(x,w)\dot{+}-f(x)\} \qquad (w\in W). \qquad (1.16)$$

As has been observed in [34], remark 2.2 and Addendum, the general case of arbitrary X, W and φ is equivalent to the particular case of $n:X\times V\to\overline{R}$, the "natural" coupling functional, defined by

$$n(x,v)=v(x) \qquad (x\in X,\ v\in V). \qquad (1.17)$$

Indeed, given any X, W and $\varphi:X\times W\to\overline{R}$, for each $w\in W$ let us define $v=v_{w\varphi}\in\overline{R}^X$ by

$$v_{w\varphi}(x)=-\varphi(x,w) \qquad (x\in X) \qquad (1.18)$$

(we take $-\varphi$ instead of φ, in (1.18), for later convenience) and define $V=V_{W\varphi}\subset\overline{R}^X$ by

$$V_{W\varphi}=\{v_{w\varphi}\mid w\in W\}. \qquad (1.19)$$

Then, the mapping $w\to v_{w\varphi}$ need not be one-to-one, but the conjugations $c(-\varphi):\overline{R}^X\to\overline{R}^W$ and $c(n):\overline{R}^X\to\overline{R}^V$ satisfy

$$f^{c(n)}(v_{w\varphi})=\sup_{x\in X}\{v_{w\varphi}(x)\dot{+}-f(x)\}=$$

$$=\sup\{-\varphi(x,w)\dot{+}-f(x)\}=f^{c(-\varphi)}(w) \qquad (w\in W), \qquad (1.20)$$

dual optimization problems based on conjugation, this observation has the following two consequences: 1°. It is no restriction of generality to consider only $W \subset \overline{R}^X$ and the natural coupling functional $n: X \times W \to \overline{R}$ (as has been done, e.g., in [4], for certain dual problems), since the results on $(X, V_W, c(n))$ imply corresponding results on arbitrary $(X, W, c(-\varphi))$. 2°. For a given set X, the problem of choosing a suitable set W and a coupling functional $\varphi: X \times W \to \overline{R}$, is equivalent to the problem of choosing a suitable $W \subset \overline{R}^X$.

According to [34], theorem 3.1, the Fenchel - Moreau conjugation operators $c = c(\varphi): \overline{R}^X \to \overline{R}^W$ are characterized by the following two properties (the second one of which may be called the property of "anti-additivity"):

$$(\inf_{i \in I} f_i)^c = \sup_{i \in I} f_i^c \qquad (\{f_i\}_{i \in I} \subset \overline{R}^X), \tag{1.21}$$

$$(f \dot{+} d)^c = f^c \dot{+} -d \qquad (f \in \overline{R}^X,\ d \in \overline{R}); \tag{1.22}$$

we recall that the structures on $\overline{R}^X$, occurring in (1.21), (1.22) and in the sequel, are defined pointwise (i.e., $(\inf_{i \in I} f_i)(x) = \inf_{i \in I} f_i(x)$ for all $x \in X$, etc.) and that, for simplicity, the same notation is used for the elements of $\overline{R}$ and the constant functionals on any set, with values in $\overline{R}$. In remark 3.9 d) we shall also consider operators $c: \overline{R}^X \to \overline{R}^W$ satisfying (1.21), but not (1.22), and we shall call them, by abuse of language, "non-anti-additive conjugations".

Let us also recall that the <u>dual conjugation</u> $h \in \overline{R}^W \to h^{c(\varphi)'} \in \overline{R}^X$ to (1.16) is defined ([15], [16]) by

$$h^{c(\varphi)'}(x) = \sup_{w \in W} \{\varphi(x, w) \dot{+} -h(w)\} \qquad (x \in X); \tag{1.23}$$

finally, the <u>second conjugation</u> with respect to φ ([15], [16]) is the operator $f \in \overline{R}^X \to f^{c(\varphi)c(\varphi)'} = (f^{c(\varphi)})^{c(\varphi)'} \in \overline{R}^X$, i.e.,

$$f^{c(\varphi)c(\varphi)'}(x) = \sup_{w \in W} \{\varphi(x, w) \dot{+} -f^{c(\varphi)}(w)\} \qquad (x \in X). \tag{1.24}$$

§2. <u>Unpertubational dual problems</u>

As mentioned in §1, by "unperturbational dual problems" we shall mean dual optimization problems defined directly, i.e., without assuming a perturbation of the primal problem.

2.1. <u>$(W\gamma)$-dual problems</u>

<u>Definition 2.1.</u> Let F, C, h and $(P) = (P_{G,h})$ be as in §1, let $W = W^{G,h}$ be a set and let $\gamma = \gamma_G: F \times W \to \overline{R}$ be a coupling functional, with values $\gamma_G(y, w) \in \overline{R}$ not depending on h. We define the $(W\gamma)$-<u>dual to</u> $(P) = (P_{G,h})$ as the supremization problem

$$(Q) = (Q_{W\gamma}^{G,h}) \qquad \beta = \beta_{W\gamma}^{G,h} = \sup \lambda(W);\ \lambda(w) = \lambda_{W\gamma}^{G,h}(w) = \inf_{y \in F} \{h(y) \dot{+} \gamma_G(y, w)\} \qquad (w \in W). \tag{2.1}$$

<u>Remark 2.1.</u> a) For a given dual objective set $W = W^{G,h}$, problem

$(P)=(P_{G,h})$ has an infinity of $(W\gamma)$-dual problems (in general, differe coupling functionals $\gamma=\gamma_G:F\times W\to\overline{R}$ yield different $(W\gamma)$-dual problems $(Q_{W\gamma}^{G,h})$).

b) The assumption that the numbers $\gamma_G(y,w)\in\overline{R}$ do not depend on h means that if $h_1,h_2:F\to\overline{R}$ are such that $W^{G,h_1}\cap W^{G,h_2}\neq\emptyset$, and if $\gamma_{G,h_1}=\gamma_G:F\times W^{G,h_1}\to\overline{R}$, $\gamma_{G,h_2}=\gamma_G:F\times W^{G,h_2}\to\overline{R}$ are defined, then

$$\gamma_{G,h_1}(y,w)=\gamma_{G,h_2}(y,w) \qquad (y\in F,\ w\in W^{G,h_1}\cap W^{G,h_2}).$$

Note also that we write $\gamma=\gamma_G$ by an abuse of notation, since γ_G is defined on $F\times W^{G,h}$; we use this notation in order to emphasize that the values $\gamma_G(y,w)\in\overline{R}$ are independent on h. Similar remarks are also valid for the notations λ^G of (2.63), π_G of (3.20), etc. For some examples of dual objective sets $W=W^{G,h}$ depending on both G and h, see remark 2.9 e) below.

c) One cannot omit in definition 2.1 the assumption that the numbers $\gamma_G(y,w)\in\overline{R}$ do not depend on h, since otherwise, for each h satisfying $h(y)\in R$ $(y\in F)$, every dual problem (1.4) would be a $(W\gamma)$-dual problem (2.1), with $\gamma=\gamma_{G,h}:F\times W\to\overline{R}$ defined, for example, by

$$\gamma_{G,h}(y,w)=\lambda(w)-h(y) \qquad (y\in F,\ w\in W). \qquad (2.2)$$

On the other hand, we shall show that many unperturbational and perturbational dual problems (1.4) are $(W\gamma)$-dual problems, with the numbers $\gamma_G(y,w)\in\overline{R}$ not depending on h.

d) If $W=W_G$ does not depend on h (e.g., if F is a linear space, G is a linear subspace of F and $W=G^{\perp}$), then, by (2.1), (1.16) and [16], formula (2.1), we have

$$\lambda_{W\gamma}^{G,h}(w)=-h^{c(-\gamma)}(w) \qquad (w\in W), \qquad (2.3)$$

so in this case $(W\gamma)$-duality may be called "unperturbational conjugate duality". One can also express conveniently (2.3) with the aid of Lindberg conjugation [11], which is equivalent to the Fenchel-Moreau conjugation (1.16) (see [35]), but here we shall use only (1.16).

Definition 2.2. Under the assumptions of definition 2.1, we define the $(W\gamma)$-Lagrangian $L=L_{W\gamma}^{G,h}:F\times W\to\overline{R}$ of $\{(P_{G,h}),(Q_{W\gamma}^{G,h})\}$ by

$$L_{W\gamma}^{G,h}(y,w)=h(y)\dot{+}\gamma_G(y,w) \qquad (y\in F,\ w\in W). \qquad (2.4)$$

Remark 2.2. a) By (2.3) and (2.4), we have (1.11), (1.12) for $L_{W\gamma}^{G,h}$, $\lambda_{W\gamma}^{G,h}$, $\beta_{W\gamma}^{G,h}$, so $L_{W\gamma}^{G,h}$ is indeed a Lagrangian. Moreover, since $L_{W\gamma}^{G,h}(y,w)=+\infty$ $(y\in F\setminus\operatorname{dom} h,\ w\in W)$, we have

$$\lambda_{W\gamma}^{G,h}(w)=\inf_{y\in\operatorname{dom} h} L_{W\gamma}^{G,h}(y,w) \qquad (w\in W). \qquad (2.5)$$

b) If, in addition,

$$\chi_G(y)\geq\gamma_G(y,w) \qquad (y\in F,\ w\in W), \qquad (2.6)$$

then, by (2.4) and a remark of [16], p.117, we have (1.13), (1.14) for $L_{W\gamma}^{G,h}$. Clearly, (2.6) is equivalent to

$$0\geq\gamma_G(g,w) \qquad (g\in G,\ w\in W). \qquad (2.6')$$

2.2. Unperturbational Lagrangian dual problems

In this section, for simplicity, we shall work with dual objective sets

$$W=W^{G,h}\subset \bar{R}^F, \qquad (2.7)$$

with the mention that definition 2.3 and all results of this section can be extended to any pair (W,φ), where $W=W^{G,h}$ is an arbitrary set and $\varphi:F\times W\to\bar{R}$ is any coupling functional, replacing $w(y)$ by $\varphi(y,w)$ (whence $c(n)$ of (2.10) by $c(-\varphi)$); indeed, this follows from (2.10) and the remark made after (1.20). Also, we shall assume that $G\cap \mathrm{dom}\, w\neq\emptyset$ $(w\in W)$, or, equivalently, that

$$\inf w(G)<+\infty \qquad (w\in W); \qquad (2.8)$$

in remark 2.3 f) we shall show that, essentially, this is no restriction of the generality.

<u>Definition 2.3.</u> Let $W=W^{G,h}\subset\bar{R}^F$ (satisfying (2.8)). The <u>unperturbational Lagrangian dual to problem</u> $(P)=(P_{G,h})$, <u>with respect to</u> W, or, briefly, the (W)-<u>dual</u> to $(P_{G,h})$, is defined as the supremization problem

$$(Q)=(Q_W^{G,h}) \qquad \beta=\beta_W^{G,h}=\sup\lambda(W);\ \lambda(w)=\lambda_W^{G,h}(w)=$$
$$=\inf_{y\in F}\{h(y)\dot{+}-w(y)\}\dot{+}\inf w(G) \qquad (w\in W). \qquad (2.9)$$

<u>Remark 2.3.</u> a) By (2.9), (1.16), (1.17) and [16], formula (2.1), we have

$$\lambda_W^{G,h}(w)=-h^{c(n)}(w)\dot{+}\inf w(G) \qquad (w\in W). \qquad (2.10)$$

b) Since $h(y)\dot{+}-w(y)=+\infty$ $(y\in F\setminus\mathrm{dom}\,h,\ w\in W)$, we have

$$\lambda_W^{G,h}(w)=\inf_{y\in\mathrm{dom}\,h}\{h(y)\dot{+}-w(y)\}\dot{+}\inf w(G) \qquad (w\in W). \qquad (2.11)$$

c) In the particular case when $G=\{y_0\}$ (a singleton), $\{(P_{G,h}),(Q_W^{G,h})\}$ becomes the "local" [30] primal-dual pair

$$(P)=(P_{\{y_0\},h}) \qquad \alpha=\alpha_{\{y_0\},h}=\inf h(\{y_0\})=h(y_0) \qquad (2.12)$$

$$(Q)=(Q_W^{\{y_0\},h}) \qquad \beta=\beta_W^{\{y_0\},h}=\sup\lambda(W);\ \lambda(w)=\lambda_W^{G,h}(w)=$$
$$=\inf_{y\in F}\{h(y)\dot{+}-w(y)\}\dot{+}w(y_0) \qquad (w\in W); \qquad (2.13)$$

this observation will be useful in the sequel.

d) In the particular case when $G=F$, $(P_{G,h})$ becomes the "unconstrained" problem $(P_{F,h})$ and, if $G=F$ is a linear space and $W\subset F^{\#}$, then

$$\inf w(G)=\inf w(F)=-\infty \quad \text{for } w\neq 0$$
$$=0 \quad \text{for } w=0, \qquad (2.14)$$

whence, by (2.9),

$$\lambda_W^{G,h}(w)=-\infty \quad \text{for } w\neq 0$$
$$=\inf h(F) \quad \text{for } w=0. \qquad (2.15)$$

Thus, in the particular case of unconstrained problems on a linear space F and of $W \subset F^{\#}$, problem $(Q_W^{F,h})$ of (2.9) is rather trivial. This may be one of the reasons for which problem $(Q_W^{G,h})$ has been used only rarely (e.g., in [20], for $W=F^*$).

c) Since

$$\lambda_W^{G,h}(w)=-\infty \qquad (w\in W,\ \inf w(G)=-\infty), \tag{2.16}$$

we have

$$\beta = \sup \lambda_{W'}^{G,h}(W') \tag{2.17}$$

where

$$W'=W'_{W,G}=\{w\in W \mid \inf w(G)>-\infty\},\ \lambda_{W'}^{G,h}=\lambda_W^{G,h}|_{W'}. \tag{2.18}$$

Thus, <u>if</u> $W'\neq\emptyset$ (see §1), then one can replace $(Q_W^{G,h})$ by the equivalent problem $(Q_{W'}^{G,h})$ of (2.17), (2.18). However, the assumption $W'\neq\emptyset$ need not be satisfied. For example, if F is a linear space, $G=F$ and $W\subset F^{\#}\setminus\{0\}$, then, by (2.14), we have $W'=\emptyset$. More generally, <u>if</u> F <u>is a linear space and</u> G <u>a linear manifold</u> in F, say $G=y_0+S$, where S is a linear subspace of F and $y_0\in F$, <u>and if</u> $W\subset F^{\#}$, then

$$\begin{aligned} \inf w(G) &= w(y_0) \quad \text{for } w\in W\cap S^{\perp} \\ &= -\infty \quad \text{for } w\notin W\cap S^{\perp}, \end{aligned} \tag{2.19}$$

whence, by (2.18) and (2.9), we obtain

$$W'=W\cap S^{\perp}, \tag{2.20}$$

$$\lambda_{W'}^{G,h}(w)=\inf_{y\in F}\{h(y)-w(y)\}+w(y_0)=w(y_0)-h^{c(n)}(w) \qquad (w\in W'). \tag{2.21}$$

Here we may have $W\cap S^{\perp}=\emptyset$, for suitable $W\subset F^{\#}\subset R^F$, and then the assumption $W'\neq\emptyset$ (with W' of (2.18)) is not satisfied. Similar remarks can be made for the more general case when G is a convex cone in F, with vertex $y_0\in F$ (using [20] the polar of $G-y_0$). Let us also mention that these formulae have some computational advantages, as well as further theoretical interest (see e.g. remark 3.7 b) below).

f) If we do not assume (2.8) in definition 2.3, then

$$\lambda_W^{G,h}(w)=-\infty \qquad (w\in W,\ \inf w(G)=+\infty). \tag{2.22}$$

Indeed, if $\inf w(G)=+\infty$, then $w(g)=+\infty$ ($g\in G\neq\emptyset$), whence, by (1.2),

$$h(g)\dot{+}-w(g)=-\infty \qquad (g\in G\cap \operatorname{dom} h\neq\emptyset);$$

thus,

$$\inf_{y\in \operatorname{dom} h}\{h(y)\dot{+}-w(y)\}\leq \inf_{g\in G\cap \operatorname{dom} h}\{h(g)\dot{+}-w(g)\}=-\infty,$$

and hence, by (2.11), we obtain (2.22). Now, by (2.22), there holds

$$\beta=\sup \lambda_{W''}^{G,h}(W''), \tag{2.23}$$

where

$$W''=\{w\in W \mid \inf w(G)<+\infty\},\ \lambda_{W''}^{G,h}=\lambda_W^{G,h}|_{W''}. \tag{2.24}$$

Hence, if we omit the cases when $W''=\emptyset$ (i.e., when $w|_G=+\infty$ for all $w\in W$), then we can replace $(Q_W^{G,h})$ by the equivalent problem

$(Q^{G,h}_{W''})$ of (2.23), (2.24). Thus, essentially, the assumption (2.8) is no restriction of the generality.

Let us recall ([16], corollary 4 b) and lemma 3 c)) that for any set E and any $f:E\to\overline{R}$ and $a,b,c\in\overline{R}$ we have

$$\inf_{y\in E} f(y)\dot{+}c \leq \inf_{y\in E}\{f(y)\dot{+}c\}, \tag{2.25}$$

$$(a\dot{+}b)\dot{+}c\leq a\dot{+}(b\dot{+}c), \tag{2.26}$$

and ([16], p.117), if $a\in R$ or $c\in R$, then we have equality in (2.26). We shall need the additional observation that if $a=c=-\infty$, then, clearly, both sides of (2.25) and (2.26) are $-\infty$. Thus, there holds

<u>Lemma 2.1. If E is any set, $f:E\to\overline{R}$, $b\in\overline{R}$ and $a,c\in R\cup\{-\infty\}$, then</u>

$$\inf_{y\in E} f(y)\dot{+}c=\inf_{y\in E}\{f(y)\dot{+}c\}, \tag{2.27}$$

$$(a\dot{+}b)\dot{+}c=a\dot{+}(b\dot{+}c). \tag{2.28}$$

<u>Definition 2.4.</u> If $W=W^{G,h}\subset\overline{\overline{R}}^F$ satisfies (2.8), the (W)-<u>Lagrangian</u> of $\{(P_{G,h}),(Q^{G,h}_W)\}$ <u>is the</u> functional $L=L^{G,h}_W:F\times W\to\overline{R}$ defined by

$$L^{G,h}_W(y,w)=\{h(y)\dot{+}-w(y)\}\dot{+}\inf w(G) \qquad (y\in F,\ w\in W). \tag{2.29}$$

<u>Remark 2.4.</u> a) By (2.9), (2.8), (2.27) and (2.29), we have (1.11), (1.12) for $L^{G,h}_W$, $\lambda^{G,h}_W$ and $\beta^{G,h}_W$, so $L^{G,h}_W$ is indeed a Lagrangian. Moreover, by (2.11), (2.27) for $E=\operatorname{dom} h$, $f=h\dot{+}-w$, $c=\inf w(G)$, and (2.29), there holds

$$\lambda^{G,h}_W(w)=\inf_{y\in\operatorname{dom} h} L^{G,h}_W(y,w) \qquad (w\in W). \tag{2.30}$$

b) By [16], formula (2.1) and corollary 3 c), the obvious relations $w(g)\geq\inf w(G)$ $(g\in G)$ are equivalent to

$$0\geq -w(g)\dot{+}\inf w(G) \qquad (g\in G,\ w\in W), \tag{2.31}$$

and thus also to

$$\chi_G(y)\geq -w(y)\dot{+}\inf w(G) \qquad (y\in F,\ w\in W). \tag{2.31'}$$

Hence, by a remark of [16], p.117 and by (2.26), we obtain, even when (2.8) is not assumed,

$$h(y)\dot{+}\chi_G(y)\geq h(y)\dot{+}\{-w(y)\dot{+}\inf w(G)\}\geq$$
$$\geq\{h(y)\dot{+}-w(y)\}\dot{+}\inf w(G)=L^{G,h}_W(y,w) \qquad (y\in F,\ w\in W), \tag{2.32}$$

so $L^{G,h}_W$ satisfies (1.13), (1.14).

<u>Theorem 2.1. If G and $W=W^{G,h}\subset\overline{\overline{R}}^F$ satisfy (2.8), then for $\gamma=\gamma_G:F\times W\to\overline{R}$ defined by</u>

$$\gamma_G(y,w)=-w(y)\dot{+}\inf w(G) \qquad (y\in F,\ w\in W), \tag{2.33}$$

<u>we have</u>

$$L^{G,h}_W(y,w)=L^{G,h}_{W\gamma}(y,w) \qquad (y\in\operatorname{dom} h,\ w\in W), \tag{2.34}$$

$$\lambda^{G,h}_W(w)=\lambda^{G,h}_{W\gamma}(w) \qquad (w\in W), \tag{2.35}$$

<u>and hence the (W)-dual to (P) coincides with the $(W\gamma)$-dual to (P).</u>

<u>Proof.</u> By (2.29), (1.3), (2.8), (2.28) with $a=h(y)$, $b=-w(y)$,

$c=\inf w(G)$, (2.33) and (2.4), we have

$$L_W^{G,h}(y,w)=\{h(y)\dotplus -w(y)\}\dotplus\inf w(G)=$$
$$=h(y)\dotplus\{-w(y)\dotplus\inf w(G)\}=L_{W\gamma}^{G,h}(y,w) \quad (y\in\mathrm{dom}\, h,\ w\in W). \quad (2.36)$$

Finally, by (2.30), (2.36) and (2.5), there holds

$$\lambda_W^{G,h}(w)=\inf_{y\in\mathrm{dom}\, h} L_W^{G,h}(y,w)=\inf_{y\in\mathrm{dom}\, h} L_{W\gamma}^{G,h}(y,w)=\lambda_{W\gamma}^{G,h}(w) \quad (w\in W).$$

<u>Remark 2.5.</u> Theorem 2.1 shows that, under the assumptions of definitions 2.3, 2.4, there exists $\gamma=\gamma_G$ such that (2.34), (2.35) hold, so <u>every unperturbational Lagrangian dual problem $(Q_W^{G,h})$ is a $(W\gamma)$-dual problem,</u> with the same W and suitable γ_G. In the converse direction, theorem 2.1 shows that, for $W\subset\bar{R}^F$ <u>satisfying</u> (2.8), <u>every</u> $(W\gamma)$-<u>dual problem with γ of the form</u> (2.33), <u>is an unperturbational Lagrangian dual problem,</u> namely, $(Q_W^{G,h})$. Similar remarks are also valid for some of the subsequent results (e.g., theorems 2.3 and 3.1 below).

<u>Theorem 2.2. Let $W=W^{G,h}$ be any set and assume that $\gamma_G:F\times W\rightarrow\bar{R}$ is a coupling functional, with values $\gamma_G(y,w)\in\bar{R}$ not depending on</u> h, <u>such that</u>

$$\sup_{g\in G}\gamma_G(g,w)=0 \qquad (w\in W). \qquad (2.37)$$

<u>Then, for $V\subset\bar{R}^F$ defined by</u>

$$V=V_{W\gamma}=\{v_{W\gamma}\,|\,w\in W\}, \qquad (2.38)$$

<u>where</u>

$$v_{W\gamma}(y)=-\gamma_G(y,w) \qquad (w\in W), \qquad (2.39)$$

<u>we have</u>

$$\inf v_{W\gamma}(G)=0 \qquad (w\in W), \qquad (2.40)$$
$$L_{W\gamma}^{G,h}(y,w)=L_V^{G,h}(y,v_{W\gamma}) \qquad (y\in F,\ w\in W), \qquad (2.41)$$
$$\lambda_{W\gamma}^{G,h}(w)=\lambda_V^{G,h}(v_{W\gamma}) \qquad (w\in W), \qquad (2.42)$$
$$(Q_{W\gamma}^{G,h})\sim(Q_V^{G,h}). \qquad (2.43)$$

<u>Proof.</u> By (2.39) and (2.37), we have (2.40), whence, by (2.29) for W and w replaced by V and $v_{W\gamma}$, we get

$$L_V^{G,h}(y,v_{W\gamma})=\{h(y)\dotplus -v_{W\gamma}(y)\}\dotplus\inf v_{W\gamma}(G)=$$
$$=h(y)\dotplus\gamma_G(y,w)=L_{W\gamma}^{G,h}(y,w) \qquad (y\in F, w\in W),$$

i.e. (2.41). Hence, by (1.11), we obtain (2.42), (2.43).

<u>Remark 2.6.</u> a) Theorem 2.2 shows that if (2.37) holds, then $(W\gamma)$-duality is a "particular case of" Lagrangian (V-) duality, for a suitable modification V of W. The family $-V\subset\bar{R}^F$, with V of (2.38), (2.39), may be compared to a "penalty system" in the sense of [3]: (2.37) is stronger than (ii) and (iv), but we do not require (i) and (iii), of [3], p.32. In the sequel we shall give some properties of our concrete γ_G's (e.g. of (2.33), (2.48), (3.22), etc.) and hence of the corresponding families -V.

b) If $W=W_G$ does not depend on h, then one can also give the fol lowing alternative proof of (2.42), (2.43): As above, we have (2.40), whence, by (2.3), (1.20) (for $X=F$, $\varphi=\gamma_G$) and (2.10), we obtain

$$\lambda_{W\gamma}^{G,h}(w)=-h^{c(-\gamma)}(w)=-h^{c(n)}(v_{W\gamma})=\lambda_V^{G,h}(v_{W\gamma}) \qquad (w\in W).$$

c) If $\inf w(G)\in R$ $(w\in W)$, then γ_G of (2.33) satisfies (2.37) and hence, by theorems 2.1 and 2.2, we obtain, for γ_G (2.33),

$$L_W^{G,h}(y,w)=L_V^{G,h}(y,v_{W\gamma}) \qquad (y\in F,\ w\in W),$$

$$\lambda_W^{G,h}(w)=\lambda_V^{G,h}(v_{W\gamma}) \qquad (w\in W),$$

where, in the right hand sides, we have (2.40).

2.3. <u>Unperturbational surrogate dual problems</u>

We recall that the <u>unperturbational surrogate dual problems</u> (i.e., the "surrogate dual problems", in the sense of [32]), are defined as follows: If $W=W^{G,h}$ is a set and $\Delta_{G,w}\subset F$ $(w\in W)$ is a family of ("surrogate constraint") sets, the $(W\Delta)$-<u>dual to</u> (P) of (1.1) is defined [32] as the supremization problem

$$(Q_{W\Delta})=(Q_{W\Delta}^{G,h}) \quad \beta=\beta_{W\Delta}^{G,h}=\sup\lambda(W);\ \lambda(w)=\lambda_{W\Delta}^{G,h}(w)=$$
$$=\inf h(\Delta_{G,w})=\inf(h\dot{+}\chi_{\Delta_{G,w}})(F) \qquad (w\in W). \quad (2.44)$$

In (2.44) and in the sequel <u>we shall assume</u>, without any special mention, <u>that, for each fixed</u> $w\in W^{G,h}$, <u>the set</u> $\Delta_{G,w}\subset F$ <u>does not depend on</u> h.

<u>Remark 2.7.</u> Actually, in [32] it has been assumed that $W=W^{G,h}$ is a family of functionals $w:X\to R$ on a set X, but this assumption can be omitted.

Let us recal that the $(W\Delta)$-<u>Lagrangian of</u> $\{(P_{G,h}),(Q_{W\Delta}^{G,h})\}$ is the functional $L=L_{W\Delta}^{G,h}:F\times W\to\overline{R}$ defined [32] by

$$L_{W\Delta}^{G,h}(y,w)=h(y)\dot{+}\chi_{\Delta_{G,w}}(y) \qquad (y\in F,\ w\in W). \quad (2.45)$$

<u>Remark 2.8.</u> By (2.44) and (2.45), <u>we have</u> (1.11), (1.12) <u>for</u> $L_{W\Delta}^{G,h}$, $\lambda_{W\Delta}^{G,h}$, $\beta_{W\Delta}^{G,h}$. Also, as has been observed in [32], <u>if</u>

$$G\subset\Delta_{G,w} \qquad (w\in W), \quad (2.46)$$

<u>then we have</u> (1.13), (1.14) <u>for</u> $L_{W\Delta}^{G,h}$. However, there are some natural unperturbational surrogate dual problems for which (1.13), (1.14) do not hold, as shown by the following example ([32], example 1.6), with applications in approximation theory (see [19],[26]): Let F be a locally convex space, $W=F^*$ and

$$\Delta_{G,w}^{\sigma s}=\{y\in F\mid w(y)=\inf w(G)\} \qquad (w\in F^*). \quad (2.47)$$

Then each $\Delta_{G,w}^{\sigma s}$ is a support hyperplane of G, generated by w, and thus, when G is not a singleton, (2.46) does not hold and (1.13), (1.14) need not be satisfied; in fact, in [19],[26], rather strong assumptions have been made on G and h, in order to ensure that

$\alpha \geqslant \beta^{\sigma s}_{G,h}$, even with F^* replaced by $W \subset F^*$ of (2.61) below.

Theorem 2.3. Let F, G, $h \in \overline{\gamma}(F_{G,h})$ be as in §1, let $W=W^{G,h}$ be a set, and let $\Delta_{G,w} \subset F$ $(w \in W)$. Then for $\gamma=\gamma_G : F \times W \to \overline{R}$ defined by

$$\gamma_G(y,w)=\chi_{\Delta_{G,w}}(y) \qquad (y \in F,\ w \in W), \qquad (2.48)$$

we have

$$L^{G,h}_{W\Delta}(y,w)=L^{G,h}_{W\gamma}(y,w) \qquad (y \in F,\ w \in W), \qquad (2.49)$$

$$\lambda^{G,h}_{W\Delta}(w)=\lambda^{G,h}_{W\gamma}(w) \qquad (w \in W), \qquad (2.50)$$

and hence the $(W\Delta)$-dual to (P) coincides with the $(W\gamma)$-dual to (P).

Proof. By (2.45), (2.48) and (2.4), we have

$$L^{G,h}_{W\Delta}(y,w)=h(y)\dot{+}\chi_{\Delta_{G,w}}(y)=L^{G,h}_{W\gamma}(y,w) \quad (y \in F,\ w \in W),$$

whence, by remarks 2.2 a) and 2.8, we obtain (2.50).

Let us show now some relations between unperturbational surrogate dual and unperturbational Lagrangian dual problems.

Theorem 2.4. If (2.46) holds, then for

$$V=V_{W\Delta}=\{-\chi_{\Delta_{G,w}} \mid w \in W\} \subset \overline{R}^F \qquad (2.51)$$

we have

$$L^{G,h}_{W\Delta}(y,w)=L^{G,h}_{V}(y,-\chi_{\Delta_{G,w}}) \qquad (y \in F,\ w \in W), \qquad (2.52)$$

$$\lambda^{G,h}_{W\Delta}(w)=\lambda^{G,h}_{V}(-\chi_{\Delta_{G,w}}) \qquad (w \in W), \qquad (2.53)$$

and hence

$$(Q^{G,h}_{W\Delta}) \sim (Q^{G,h}_{V}). \qquad (2.54)$$

Proof. By (2.46), we have $\inf_{g \in G}(-\chi_{\Delta_{G,w}}(g))=0$ and hence, by (2.45), (2.51) and (2.29), we obtain

$$L^{G,h}_{W\Delta}(y,w)=h(y)\dot{+}\chi_{\Delta_{G,w}}(y)=L^{G,h}_{V}(y,-\chi_{\Delta_{G,w}}) \quad (y \in F, w \in W).$$

Hence, by (1.11), there follow (2.53) and (2.54).

Remark 2.9. a) The functionals $v=-\chi_{\Delta_{G,w}} \in V$ assume only the value 0 and $-\infty$.

b) Let us also mention the following alternative proof of (2.52): For $\gamma_G : F \times W \to \overline{R}$ defined by (2.48), condition (2.46) is equivalent to (2.37), whence, by theorems 2.3 and 2.2, we obtain

$$L^{G,h}_{W\Delta}(y,w)=L^{G,h}_{W\gamma}(y,w)=L^{G,h}_{V}(y,-\chi_{\Delta_{G,w}}) \quad (y \in F,\ w \in W).$$

c) Theorem 2.4 shows that if (2.46) holds, then surrogate $(W\Delta-)$ duality is a "particular case of" Lagrangian (V-) duality, for a suitable modification V of W. However, we have seen that $\alpha \geqslant \beta$ holds for Lagrangian duality, but need not hold for surrogate duality, so if

(2.46) is not assumed, then surrogate duality is not a "particular case of" Lagrangian duality. Let us also note that if (2.46) does not hold, say $w\in W$ and $g_o\in G\setminus\Delta_{G,w}$, then $\inf_{g\in G}(-\chi_{\Delta_{G,w}}(g))\leqslant-\chi_{\Delta_{G,w}}(g_o)=-\infty$, whence $L_V^{G,h}(y,-\chi_{\Delta_{G,w}})=-\infty$.

d) In the opposite direction, we have only the following remark: If $W\subset\overline{R}^F$ satisfies (2.8) and if we define, as in [32],

$$\Delta_{G,w}^{\gamma s}=\{y\in F|w(y)\in w(G)\}\cup\{y\in F|w(y)>\inf w(G)\} \qquad (w\in W), \qquad (2.55)$$

$$\Delta_{G,w}^{\delta s}=\{y\in F|w(y)\geqslant\inf w(G)\}=\{y\in F|0\geqslant-w(y)\dot{+}\inf w(G)\} \ (w\in W) \qquad (2.56)$$

(for the last equality, see (2.31)), then $G\subset\Delta_{G,w}^{\gamma s}\subset\Delta_{G,w}^{\delta s}$ $(w\in W)$, whence, by (2.36),

$$h(y)\dot{+}\chi_G(y)\geqslant h(y)\dot{+}\chi_{\Delta_{G,w}^{\gamma s}}(y)=L_{W\Delta^{\gamma s}}^{G,h}(y,w)\geqslant$$

$$\geqslant h(y)\dot{+}\chi_{\Delta_{G,w}^{\delta s}}(y)=L_{W\Delta^{\delta s}}^{G,h}(y,w)\geqslant$$

$$\geqslant h(y)\dot{+}\{-w(y)\dot{+}\inf w(G)\}=L_W^{G,h}(y,w) \quad (y\in F, w\in W); \qquad (2.57)$$

hence, by $\alpha=\inf_{y\in F}\{h(y)\dot{+}\chi_G(y)\}$ and (1.11), we obtain

$$\alpha\geqslant\lambda_{W\Delta^{\gamma s}}^{G,h}(w)\geqslant\lambda_{W\Delta^{\delta s}}^{G,h}(w)\geqslant\lambda_W^{G,h}(w) \qquad (w\in W), \qquad (2.58)$$

and thus

$$\alpha\geqslant\beta_{W\Delta^{\gamma s}}^{G,h}\geqslant\beta_{W\Delta^{\delta s}}^{G,h}\geqslant\beta_W^{G,h}. \qquad (2.59)$$

Consequently, <u>if</u> $\alpha=\beta_W^{G,h}$, then $\alpha=\beta_{W\Delta^{\gamma s}}^{G,h}=\beta_{W\Delta^{\delta s}}^{G,h}$.

e) Let us mention <u>some examples of problems</u> $(Q_{W\Delta}^{G,h})$, <u>in which</u> $W=W^{G,h}$ <u>depends on both</u> G <u>and</u> h. In [19] we have assumed that there exists $y'\in F$ satisfying $h(y')<\inf h(G)$, and, using separation, we have arrived at problem $(Q_{W\Delta^{\gamma s}})$ of (2.44), with $\Delta_{G,w}^{\gamma s}$ of (2.47) and with

$$W=W^{G,h,y'}=\{w\in F^*|w(y')<\inf w(G)\}\neq\emptyset, \qquad (2.60)$$

$$W=W^{G,h}=\{w\in F^*|\sup w(A_\alpha(h))\leqslant\inf w(G)\}\neq\emptyset, \qquad (2.61)$$

where $A_\alpha(h)=\{y\in F|h(y)<\inf h(G)\}$. Similarly, in [26],[27], we have arrived at problem $(Q_{W\Delta^{\delta s}}^{G,h})$, with F a locally convex space, $\Delta_{G,w}^{\delta s}\subset F$ $(w\in F^*)$ of (2.56), and $W=W^{G,h,y'}$ of (2.60), as well as at some other problems $(Q_{W\Delta}^{G,h})$ with W depending on G,h and y' (of course, in these cases one can define an equivalent dual problem $(Q_o^{G,h})$, enlarging $W^{G,h}$ to $W_o=F^*$, which does not depend on G or h, and extending $\lambda^{G,h}$ to $\lambda_o^{G,h}:F^*\to R$ by putting $\lambda_o^{G,h}(w)=-\infty$ for all $w\in F^*\setminus W^{G,h}$; however, in this way one would loose the separation properties occurring in (2.60)

(2.61)). Similarly, one can give examples of unperturbational Lagrangian dual problems $(Q_W^{G,h})$, with W depending on both G and h.

f) Finally, let us mention an example of an unperturbational dual problem $(Q^{G,h})$, with $W=W^h$ not depending on G and the values of $\lambda=\lambda^G$ not depending on h (whence $(Q^{G,h})$ is not a $(W\gamma)$-dual problem to $(P_{G,h})$, for any $\gamma_G: F\times W\to\overline{R}$), which will be used in §3. Namely, let F be a linear space, let

$$W=W^h=\{w\in F^\# \mid w\leq h\}\neq\emptyset \tag{2.62}$$

(whence $h(0)\geq 0$ and $h(y)>-\infty$ for all $y\in F$), and let

$$(Q)=(Q^{G,h})\quad \beta=\beta^{G,h}=\sup\lambda(W);\ \lambda(w)= =\lambda^G(w)=\inf w(G)\qquad (w\in W). \tag{2.63}$$

Then, similarly to e) above, one can define an equivalent dual problem $(Q_o^{G,h})$, enlarging W^h to $W_o=F^\#$ and extending $\lambda^G: W^h\to R\cup\{-\infty\}$ to $\lambda_o^{G,h}: F^\#\to R\cup\{-\infty\}$ defined by

$$\lambda_o^{G,h}(w)=\lambda^G(w)\dot{+}-\chi_{W^h}(w)=\inf w(G)\ \text{for}\ w\in F^\#,\ w\leq h$$
$$=-\infty \qquad \text{for}\ w\in F^\#\setminus W^h; \tag{2.64}$$

indeed, clearly, $\beta_o^{G,h}=\sup\lambda_o^{G,h}(F^\#)=\sup\lambda^G(W^h)=\beta^{G,h}$. Note also that, by (2.62), we have $\alpha=\inf h(G)\geq\inf w(G)=\lambda(w)$ for all $w\in W$, whence $\alpha\geq\beta^G$. Moreover, by (2.32), (2.62) and (2.63) we have

$$h(y)\dot{+}\chi_G(y)\geq L_W^{G,h}(y,w)=\{h(y)\dot{+}-w(y)\}\dot{+}\inf w(G)\geq$$
$$\geq\inf w(G)=\lambda^G(w)\qquad (y\in F,\ w\in W),$$

whence, by (1.11) for $L_W^{G,h}$ (see remark 2.4 a)),

$$\alpha=\inf_{y\in F}\{h(y)\dot{+}\chi_G(y)\}\geq\inf_{y\in F} L_W^{G,h}(y,w)=\lambda_W^{G,h}(w)\geq\lambda^G(w)\qquad (w\in W),$$

and hence

$$\alpha\geq\beta_W^{G,h}\geq\beta^{G,h}; \tag{2.65}$$

thus, <u>if</u> $\alpha=\beta^{G,h}$, <u>then</u> $\alpha=\beta_W^{G,h}$. Similar remarks can be made for the case when F is a partially ordered linear space and (2.62) is replaced by

$$W=W^h=\{w\in F^\# \mid w\geq 0,\ \ w\leq h\}\neq\emptyset. \tag{2.66}$$

§3. Perturbational dual problems

As mentioned in §1, by "perturbational dual problems" we shall mean dual optimization problems defined with the aid of perturbations of the primal problem. Thus, assume that $(P_{G,h})$ of (1.1) is "embedded into a family of "<u>perturbed</u>" (or, "parametrized") <u>problems</u>

$$(P^x)=(P^x_{G,h})\quad \alpha^x=\alpha^x_{G,h}=\inf_{y\in F} p(y,x)\qquad (x\in X), \tag{3.1}$$

where $X=X_{G,h}$ is a set, called <u>set of perturbations</u> (or, set of parameters) and $p=p_{G,h}: F\times X\to\overline{R}$ is a (coupling) functional, called <u>perturbation functional</u>, such that for some $x_o=x_o^{G,h,p}\in X$ there holds (1.6).

Our next aim is to define, using (X,p,x_o), "perturbational dual problems" to $(P_{G,h})$, with dual objective set

$$W=W^{G,h}\subset\overline{R}^X; \tag{3.2}$$

thus, in these dual problems, we shall couple W with X, rather than with F.

3.1. $(pW\tilde{\lambda})$-dual problems

We shall first give a general scheme of defining, for certain dual problems to a suitable "extended" problem $(\tilde{P})$, equivalent to $(P_{G,h})$ (see (3.3) below), corresponding perturbational dual problems to $(P_{G,h})$.

Consider the optimization problem

$$(\tilde{P})=(\tilde{P}_{F,x_o),p})\qquad \tilde{\alpha}=\tilde{\alpha}_{(F,x_o),p}=\inf p(F,x_o), \tag{3.3}$$

where $p=p_{G,h}:F\times X\to\overline{R}$ and $x_o=x_o^{G,h,p}\in X$ satisfy (1.6) and where

$$(F,x_o)=F\times\{x_o\}=\{(y,x_o)|\,y\in F\}\subset F\times X. \tag{3.4}$$

Since both the objective functional p and the constraint set (F,x_o) of $(\tilde{P})$, are defined in the "extended space" $F\times X$, we have called $(\tilde{P})$, in [32], an "extended problem"; note also that, by (1.6), we have $\tilde{\alpha}=\alpha$, so $(\tilde{P})\sim(P_{G,h})$.

Definition 3.1. Let

$$(\tilde{Q})=(\tilde{Q}^{(F,x_o),p})\qquad \tilde{\beta}=\sup\tilde{\lambda}(\tilde{W}) \tag{3.5}$$

be a dual problem to $(\tilde{P}_{(F,x_o),p})$ of (3.3), with $\tilde{W}\subset\overline{R}^{F\times X}$ of the form

$$\tilde{W}=(0,W)=\{(v,w)\in\overline{R}^F\times W|\,v=0\}=\{(0,w)|\,w\in W\}, \tag{3.6}$$

for some $W=W^{G,h}\subset\overline{R}^X$, and with $\tilde{\lambda}:\tilde{W}\to\overline{R}$. By the $(W\tilde{\lambda})$-dual problem to $(P_{G,h})$, with respect to (X,p,x_o), or, briefly, the $(pW\tilde{\lambda})$-dual problem to $(P_{G,h})$, we shall mean the supremization problem

$$(Q)=(Q^{G,h}_{pW\tilde{\lambda}})\quad \beta=\beta^{G,h}_{pW\tilde{\lambda}}=\sup\lambda(W);\ \lambda(w)=\lambda^{G,h}_{pW\tilde{\lambda}}(w)=\tilde{\lambda}(0,w)\qquad (w\in W). \tag{3.7}$$

Remark 3.1. a) By (3.3) and (3.5)-(3.7), we have

$$\{(P_{G,h}),\ (Q^{G,h}_{pW\tilde{\lambda}})\}\sim\{(\tilde{P}_{(F,x_o),p}),\ (\tilde{Q}^{(F,x_o),p})\}. \tag{3.8}$$

b) Given X,p,x_o and $W\subset\overline{R}^X$, each dual problem (3.5) to $(\tilde{P}_{(F,x_o),p})$ with $\tilde{W}$ of the form (3.6), induces, via (3.7), a perturbational dual problem $(Q^{G,h}_{pW\tilde{\lambda}})$ to $(P_{G,h})$, satisfying (3.8).

Definition 3.2. In the situation of definition 3.1, if $\tilde{L}=\tilde{L}^{(F,x_o),p}:(F\times X)\times\tilde{W}\to\overline{R}$ is a Lagrangian for $\{(\tilde{P}_{(F,x_o),p}),\ (\tilde{Q}^{(F,x_o),p})\}$ of (3.3), (3.5), we define the corresponding $(pW\tilde{\lambda})$-Lagrangian $L=L^{G,h}_{pW\tilde{\lambda}}:F\times W\to\overline{R}$ for $\{(P_{G,h}),\ (Q^{G,h}_{pW\tilde{\lambda}})\}$, by

$$L^{G,h}_{pW\tilde{\lambda}}(y,w)=\inf_{x\in X}\tilde{L}^{(F,x_o),p}((y,x),(0,w)) \qquad (y\in F,\ w\in W). \quad (3.9)$$

Remark 3.2. a) $L^{G,h}_{pW\tilde{\lambda}}$ is indeed a Lagrangian for $\{(P_{G,h}),(Q^{G,h}_{pW\tilde{\lambda}})\}$, since by (3.7), (1.11) (for $\tilde{\lambda},\tilde{L}$) and (3.9), we have

$$\lambda(w)=\tilde{\lambda}(0,w)=\inf_{(y,x)\in F\times X}\tilde{L}((y,x),(0,w))=\inf_{y\in F}L(y,w) \qquad (w\in W).$$

b) If

$$p(y,x)\dot{+}\chi_{(F,x_o)}(y,x)\geqslant\tilde{L}((y,x),(0,w)) \quad (y\in F, x\in X, w\in W), \qquad (3.10)$$

then L satisfies (1.13), (1.14), since by (1.6), (3.10) and (3.9) we have

$$h(y)\dot{+}\chi_G(y)=p(y,x_o)\dot{+}\chi_{(F,x_o)}(y,x_o)\geqslant\tilde{L}((y,x_o),(0,w))\geqslant$$
$$\geqslant\inf_{x\in X}\tilde{L}((y,x),(0,w))=L(y,w) \qquad (y\in F,\ w\in W). \qquad (3.11)$$

Now we shall give a general scheme of defining, <u>for each unpertu</u><u>bational dual problem</u> to $(P_{G,h})$, say

$$(Q_1)=(Q^{G,h}_1) \qquad \beta_1=\beta^{G,h}_1=\sup\lambda_1(W_1), \qquad (3.12)$$

where $W_1=W^{G,h}_1\subset\bar{R}^F$, $\lambda_1=\lambda^{G,h}_1:W_1\to\bar{R}$, and for each Lagrangian functional $L^{G,h}_1:F\times W_1\to\bar{R}$ of $\{(P_{G,h}),(Q^{G,h}_1)\}$, a corresponding unperturbational dual problem (3.5) to the "extended problem" $(\tilde{P}_{(F,x_o),p})$ of (3.3) and a Lagrangian $\tilde{L}^{(F,x_o),p}$ of $\{(\tilde{P}_{(F,x_o),p},(\tilde{Q}^{(F,x_o),p})\}$, and hence, applying definitions 3.1 and 3.2, a corresponding perturbational dual problem $(Q^{G,h}_{pW\tilde{\lambda}})$ to $(P_{G,h})$ and a Lagrangian $L^{G,h}_{pW\tilde{\lambda}}:F\times W\to\bar{R}$.

Definition 3.3. For any unperturbational dual problem $(Q^{G,h}_1)$ to $(P_{G,h})$, as in (3.12), and any (X,p,x_o) (satisfying (1.6)), define $\tilde{W}=(0,W^{G,h})\subset\bar{R}^{F\times X}$ (with $W^{G,h}\subset\bar{R}^X$), by replacing formally $w_1\in\bar{R}^F$, $y\in F$, G and h by $(0,w)\in(0,\bar{R}^X)$, $(y,x)\in F\times X$, (F,x_o) and p respectively, in the formula which defines $W_1=W^{G,h}_1$. Furthermore, define $\tilde{\lambda}:\tilde{W}\to\bar{R}$ and $\tilde{L}^{(F,x_o),p}:(F\times X)\times\tilde{W}\to\bar{R}$ similarly, replacing also $w_1\in W_1$, in $\lambda^{G,h}_1$ and $L^{G,h}_1$, by $(0,w)\in\tilde{W}$; also, an arbitrary coupling functional $\varphi_G:F\times W\to\bar{R}$, respectively, an arbitrary family of sets $\Delta_{G,w}\subset F$ $(w\in W)$, should be replaced by an arbitrary coupling functional $\tilde{\varphi}_{(F,x_o)}:(F\times X)\times\tilde{W}\to\bar{R}$, respectively, by an arbitrary family $\tilde{\Delta}_{(F,x_o),(0,w)}\subset F\times X$ $(w\in W)$. Then, $(Q^{G,h}_{pW\tilde{\lambda}})$ of (3.7) and $L^{G,h}_{pW\tilde{\lambda}}$ of (3.9), obtained in this way, are called <u>the perturbational dual problem to</u> $(P_{G,h})$ and <u>the Lagrangian of</u> $\{(P_{G,h}),(Q^{G,h}_{pW\tilde{\lambda}})\}$, <u>corresponding to</u> $(Q^{G,h}_1)$ <u>and</u> $L^{G,h}_1$, respectively.

Remark 3.3. a) By (1.10), in definition 3.3 the expression w(y), where $y\in F$, $w\in W^{G,h}_1\subset\bar{R}^F$, should be replaced by $(0,w)(y,x)=w(x)$, where the expression $w(g)$ (and hence $\inf w(G)$, $\sup w(G)$, etc.) should be replaced by $(0,w)(y,x_0)=w(x_0)$ (where $y\in F$). Consequently, to different unperturbational dual problems $(Q_1^{G,h})\neq(Q_2^{G,h})$ there may correspond the same perturbational dual problem $(Q_{pW\chi}^{G,h})$, by the scheme of definition 3.3; for an example, see remark 3.29 d) below.

b) By a) above, if $W_1^{G,h}\subset\overline{R}^F$ can be arbitrary, for a class of unperturbational dual problems $(Q_1^{G,h})$, then $W^{G,h}\subset\overline{R}^X$ of definition 3.3 can be arbitrary, for the corresponding class of perturbational dual problems $(Q_{pW\chi}^{G,h})$.

c) If F and X are linear spaces and $W_1^{G,h}\subset F^{\#}$ in problem $(Q_1^{G,h})$, then $W^{G,h}\subset X^{\#}$. Indeed, replacing w and y_i by $(0,w)$ and (y_i,x_i) respectively, in the linearity relations $w(\alpha y_1+\beta y_2)=\alpha w(y_1)+\beta w(y_2)$, we obtain $(0,w)(\alpha(y_1,x_1)+\beta(y_2,x_2))=\alpha(0,w)(y_1,x_1)+\beta(0,w)(y_2,x_2)$, whence, by (1.10), $w(\alpha x_1+\beta x_2)=\alpha w(x_1)+\beta w(x_2)$.

In the sequel, we shall apply this general scheme to the unperturbational dual problems considered in §1, and we shall thus obtain various perturbational dual problems.

3.2. $(pW\tilde{\varphi})$-dual problems

Definition 3.4. Assuming (1.6), (3.2) and (3.6), let $\tilde{\varphi}=\tilde{\varphi}_{(F,x_0)}: (F\times X)\times\tilde{W}\to\overline{R}$ be a coupling functional, with values $\tilde{\varphi}_{(F,x_0)}((y,x),(0,w))\in\overline{R}$ not depending on p. We define the $(pW\tilde{\varphi})$-dual to $(P)=(P_{G,h})$, as the supremization problem

$$(Q)=(Q_{pW\tilde{\varphi}}^{G,h})\quad \beta=\beta_{pW\tilde{\varphi}}^{G,h}=\sup\lambda(W);\ \lambda(w)=\lambda_{pW\tilde{\varphi}}^{G,h}(w)=\inf_{(y,x)\in F\times X}\{p(y,x)\dot{+}\tilde{\varphi}_{(F,x_0)}((y,x),(0,w))\}\ (w\in W),\tag{3.13}$$

and we define the $(pW\tilde{\varphi})$-Lagrangian $L=L_{pW\tilde{\varphi}}^{G,h}:F\times W\to\overline{R}$ for $\{(P_{G,h}),(Q_{pW\tilde{\varphi}}^{G,h})\}$, by

$$L_{pW\tilde{\varphi}}^{G,h}(y,w)=\inf_{x\in X}\{p(y,x)\dot{+}\tilde{\varphi}_{(F,x_0)}((y,x),(0,w))\}\quad (y\in F,w\in W).\tag{3.14}$$

Remark 3.4. a) $(Q_{pW\tilde{\varphi}}^{G,h})$ and $L_{pW\tilde{\varphi}}^{G,h}$ are the perturbational dual problem and Lagrangian corresponding to $(Q_{W\varphi}^{G,h})$ and $L_{W\varphi}^{G,h}$ of definitions 2.1 and 2.2, by the scheme of definition 3.3 and remark 3.3. Indeed, if we replace $w\in\overline{R}^F$, $y\in F,G,h,W\subset\overline{R}^F$ and φ_G by $(0,w)\in(0,\overline{R}^X)$, $(y,x)\in F\times X$, (F,x_0), p, $\tilde{W}=(0,W)\subset(0,\overline{R}^X)$ and $\tilde{\varphi}_{(F,x_0)}$ respectively, then $\lambda_{W\varphi}^{G,h}$ and $L_{W\varphi}^{G,h}$ of (2.1), (2.4) will be replaced, respectively, by

$$\lambda_{\tilde{W}\tilde{\varphi}}^{(F,x_0),p}(0,w)=\inf_{(y,x)\in F\times X}\{p(y,x)\dot{+}\tilde{\varphi}_{(F,x_0)}((y,x),(0,w))\}\quad (w\in W),\tag{3.15}$$

$$\tilde{L}^{(F,x_0),p}_{\tilde{W}\tilde{\gamma}}((y,x)(0,w))=p(y,x)\dot{+}\tilde{\gamma}_{(F,x_0)}((y,x),(0,w))\quad(y\in F, x\in X, w\in W), \tag{3.16}$$

whence, by (3.13), (3.14), (3.7) and (3.9), we obtain

$$\lambda^{G,h}_{pW\tilde{\gamma}}(w)=\lambda^{G,h}_{pW\tilde{\lambda}}(w) \qquad (w\in W), \tag{3.17}$$

$$L^{G,h}_{pW\tilde{\gamma}}(y,w)=L^{G,h}_{pW\tilde{\lambda}}(y,w) \qquad (y\in F, w\in W). \tag{3.18}$$

b) The assumption that the values of $\tilde{\gamma}_{(F,x_0)}$ do not depend on p, will often mean that x_0 of (1.6) should not depend on p. This happens, for example, when X is a linear space and $x_0=0$; moreover, this x_0 does not depend on G or h.

c) If $W=W^{G,h}$ does not depend on p, then, under the assumptions of definition 3.4, we have, by (1.16) and [16], formula (2.1),

$$\lambda^{G,h}_{pW\tilde{\gamma}}(w)=-p^{c(-\tilde{\gamma})}(0,w) \qquad (w\in W). \tag{3.19}$$

Now we shall show that, if the values of $\tilde{\gamma}_{(F,x_0)}$ do not depend on h, then for a large class of perturbation functionals $p=p_{G,h}:F\times X\to\bar{R}$, which we shall introduce here, every $(pW\tilde{\gamma})$-dual problem to $(P_{G,h})$ is an (unperturbational) $(W\gamma)$-dual problem to $(P_{G,h})$, with a suitable $\gamma=\gamma_G:F\times W\to\bar{R}$. Reading these results in the reverse order, it will follow that, essentially, every $(W\gamma)$-dual problem to $(P_{G,h})$ is a $(pW\tilde{\gamma})$-dual problem to $(P_{G,h})$, for suitable p and $\tilde{\gamma}$.

Definition 3.5. We shall say that a perturbation functional $p=p_{G,h}:F\times X\to\bar{R}$ for (1.1) is h-separated, if there exists a coupling functional $\pi_G:F\times X\to\bar{R}$, with values $\pi_G(y,x)\in\bar{R}$ not depending on h, such that

$$p(y,x)=p_{G,h}(y,x)=h(y)\dot{+}\pi_G(y,x) \qquad (y\in F, x\in X). \tag{3.20}$$

Remark 3.5. a) By (3.20) and (1.6), we obtain

$$h(y)\dot{+}\pi_G(y,x_0)=h(y)\dot{+}\chi_G(y) \qquad (y\in F),$$

and hence, by a remark of [16], p.116,

$$\pi_G(y,x_0)=\chi_G(y) \qquad (y\in F, h(y)\in R). \tag{3.21}$$

Thus, if $h(y)\in R$, then $\pi_G(y,x_0)$ is either 0 or $+\infty$, but $\pi_G(y,x)$ may also have other values, for $x\neq x_0$; in the sequel we shall also consider the case when all values $\pi_G(y,x)$ are either 0 or $+\infty$ (see e.g. formula (3.30) below).

b) The similarity between formulae (2.4) and (3.20) suggests to try to consider π_G as γ_G or, conversely, γ_G as π_G. More precisely, the following two problems arise in this way: Firstly, given $F,G,h,X=X_{G,h}$ and $\pi_G:F\times X\to\bar{R}$, with values not depending on h and such that $p_{G,h}:F\times X\to\bar{R}$ of (3.20) satisfies (1.6), if we take $W=X$ and $\gamma_G=\pi_G$ (whence $L^{G,h}_{W\gamma}=p_{G,h}$), what is $(Q^{G,h}_{W\gamma})$? By (2.4) and (3.20), for $f:X\to\bar{R}$ of (3.64) below we obtain

$$\lambda^{G,h}_{W\gamma}(x)=\inf_{y\in F}\{h(y)\dot{+}\pi_G(y,x)\}=\inf_{y\in F} p(y,x)=f(x) \qquad (x\in X),$$

whence $(Q^{G,h}_{W\gamma})$ is the optimization problem $\beta=\sup f(X)$. Conversely, given $F,G,h,W=W^{G,h}$ and $\gamma_G:F\times W\to\overline{R}$, with values not depending on h and such that $\gamma_G(y,w_o)=\chi_G(y)$ $(y\in F)$, for some $w_o=w_o^{G,h,\gamma}\in W$, if we take $X=W$, $\pi_G=\gamma_G$ and $p_{G,h}=L^{G,h}_{W\gamma}$ (which satisfies (1.6) with $x_o=w_o$), i.e., if we embed $(P_{G,h})$ into the family of optimization problems

$$(P^W)=(P^W_{G,h}) \qquad \alpha^W=\alpha^W_{G,h}=\inf_{y\in F}\{h(y)\dot{+}\gamma_G(y,w)\}=\lambda^{G,h}_{W\gamma}(w) \qquad (w\in W),$$

then the usual perturbational dual problems $(Q^{G,h}_{pW\chi})$, with respect to the above (X,p,x_o), and even with another dual constraint set $W'\subset\overline{R}^X=\overline{R}^W$, do not seem to be of interest; it is then more convenient to consider, for each $w\in W$, a dual problem $(Q^{G,h}_W)$ to $(P^W_{G,h})$ (e.g., for the γ_G's of (2.33) and (2.48) above).

<u>Theorem 3.1. Let $p=p_{G,h}:F\times X\to\overline{R}$ be an h-separated perturbation functional for problem $(P_{G,h})$ of (1.1) (satisfying (1.6)) and let $W\subset\overline{R}^X$, $\widetilde{W}\subset\overline{R}^{F\times X}$ and $\widetilde{\gamma}_{(F,x_o)}:(F\times X)\times\widetilde{W}\to\overline{R}$ be as in definition 3.4, with the values of $\widetilde{\gamma}_{(F,x_o)}$ not depending on h. Then for $\pi_G:F\times X\to\overline{R}$ as in (3.20) and for $\gamma=\gamma_G:F\times W\to\overline{R}$ defined by</u>

$$\gamma_G(y,w)=\inf_{x\in X}\{\pi_G(y,x)\dot{+}\widetilde{\gamma}_{(F,x_o)}((y,x),(0,w))\} \qquad (y\in F,\ w\in W), \quad (3.22)$$

<u>we have</u>

$$L^{G,h}_{pW\widetilde{\gamma}}(y,w)=L^{G,h}_{W\gamma}(y,w) \qquad (y\in F,\ w\in W), \quad (3.23)$$

$$\lambda^{G,h}_{pW\widetilde{\gamma}}(w)=\lambda^{G,h}_{W\gamma}(w) \qquad (w\in W); \quad (3.24)$$

<u>and hence the $(pW\widetilde{\gamma})$-dual to $(P_{G,h})$ coincides with the $(W\gamma)$-dual to $(P_{G,h})$.</u>

<u>Proof.</u> Since the values of $\widetilde{\gamma}_{(F,x_o)}$ do not depend on h, they do not depend on p (by (1.6)). By (3.14), (3.20), (3.22), (2.4) and [16] formulae (2.3) and (4.7), we get

$$L^{G,h}_{pW\widetilde{\gamma}}(y,w)=\inf_{x\in X}\{h(y)\dot{+}\pi_G(y,x)\dot{+}\widetilde{\gamma}_{(F,x_o)}((y,x),(0,w))\}=$$
$$=h(y)\dot{+}\gamma_G(y,w)=L^{G,h}_{W\gamma}(y,w) \qquad (y\in F,\ w\in W),$$

and hence, by (1.11), we obtain (3.24).

From theorems 3.1 and 2.2, there follows

<u>Corollary 3.1. Under the assumptions of theorem 3.1, if γ_G of</u> (3.22) <u>satisfies</u> (2.37), <u>then for $V\subset\overline{R}^F$ of</u> (2.38), (2.39), <u>we have</u>

$$L^{G,h}_{pW\widetilde{\gamma}}(y,w)=L^{G,h}_V(y,v_{W\gamma}) \qquad (y\in F,\ w\in W), \quad (3.25)$$

$$\lambda^{G,h}_{pW\widetilde{\gamma}}(w)=\lambda^{G,h}_V(v_{W\gamma}) \qquad (w\in W). \quad (3.26)$$

Let us give now some corollaries of theorem 3.1 for various h-separated perturbation functionals p.

$\widetilde{L}_{W\widetilde{\gamma}}^{(F,x_o),p}((y,x)(0,w))=p(y,x)\dot{+}\widetilde{\gamma}_{(F,x_o)}((y,x),(0,w))(y\in F,x\in X,w\in W)$, (3.16)

whence, by (3.13), (3.14), (3.7) and (3.9), we obtain

$$\lambda_{pW\widetilde{\gamma}}^{G,h}(w)=\lambda_{pW\lambda}^{G,h}(w) \qquad (w\in W), \quad (3.17)$$

$$L_{pW\widetilde{\gamma}}^{G,h}(y,w)=L_{pW\lambda}^{G,h}(y,w) \qquad (y\in F, w\in W). \quad (3.18)$$

b) The assumption that the values of $\widetilde{\gamma}_{(F,x_o)}$ do not depend on p, will often mean that x_o of (1.6) should not depend on p. This happens, for example, when X is a linear space and $x_o=0$; moreover, this x_o does not depend on G or h.

c) If $W=W^{G,h}$ does not depend on p, then, under the assumptions of definition 3.4, we have, by (1.16) and [16], formula (2.1),

$$\lambda_{pW\widetilde{\gamma}}^{G,h}(w)=-p^{c(-\widetilde{\gamma})}(0,w) \qquad (w\in W). \quad (3.19)$$

Now we shall show that, if the values of $\widetilde{\gamma}_{(F,x_o)}$ do not depend on h, then for a large class of perturbation functionals $p=p_{G,h}:F\times X\to\overline{R}$, which we shall introduce here, every $(pW\widetilde{\gamma})$-dual problem to $(P_{G,h})$ is an (unperturbational) $(W\gamma)$-dual problem to $(P_{G,h})$, with a suitable $\gamma=\gamma_G:F\times W\to\overline{R}$. Reading these results in the reverse order, it will follow that, essentially, every $(W\gamma)$-dual problem to $(P_{G,h})$ is a $(pW\widetilde{\gamma})$-dual problem to $(P_{G,h})$, for suitable p and $\widetilde{\gamma}$.

<u>Definition 3.5.</u> We shall say that a perturbation functional $p=p_{G,h}:F\times X\to\overline{R}$ for (1.1) is <u>h-separated</u>, if there exists a coupling functional $\pi_G:F\times X\to\overline{R}$, <u>with values</u> $\pi_G(y,x)\in\overline{R}$ not depending on h, such that

$$p(y,x)=p_{G,h}(y,x)=h(y)\dot{+}\pi_G(y,x) \qquad (y\in F, x\in X). \quad (3.20)$$

<u>Remark 3.5.</u> a) By (3.20) and (1.6), we obtain

$$h(y)\dot{+}\pi_G(y,x_o)=h(y)\dot{+}\chi_G(y) \qquad (y\in F),$$

and hence, by a remark of [16], p.116,

$$\pi_G(y,x_o)=\chi_G(y) \qquad (y\in F, h(y)\in R). (3.21)$$

Thus, if $h(y)\in R$, then $\pi_G(y,x_o)$ is either 0 or $+\infty$, but $\pi_G(y,x)$ may also have other values, for $x\neq x_o$; in the sequel we shall also consider the case when all values $\pi_G(y,x)$ are either 0 or $+\infty$ (see e.g. formula (3.30) below).

b) The similarity between formulae (2.4) and (3.20) suggests to try to consider π_G as γ_G or, conversely, γ_G as π_G. More precisely, the following two problems arise in this way: Firstly, given F, G, h, $X=X_{G,h}$ and $\pi_G:F\times X\to\overline{R}$, with values not depending on h and such that $p_{G,h}:F\times X\to\overline{R}$ of (3.20) satisfies (1.6), if we take $W=X$ and $\gamma_G=\pi_G$ (whence $L_{W\gamma}^{G,h}=p_{G,h}$), what is $(Q_{W\gamma}^{G,h})$? By (2.4) and (3.20), for $f:X\to\overline{R}$ of (3.64) below we obtain

$$\lambda_{W\gamma}^{G,h}(x)=\inf_{y\in F}\{h(y)\dot{+}\pi_G(y,x)\}=\inf_{y\in F} p(y,x)=f(x) \qquad (x\in X),$$

whence $(Q_{W\gamma}^{G,h})$ is the optimization problem $\beta=\sup f(X)$. Conversely, given $F, G, h, W=W^{G,h}$ and $\gamma_G: F\times W \to \bar{R}$, with values not depending on h and such that $\gamma_G(y,w_o)=\chi_G(y)$ $(y\in F)$, for some $w_o=w_o^{G,h,\gamma}\in W$, if we take $X=W$, $\pi_G=\gamma_G$ and $p_{G,h}=L_{W\gamma}^{G,h}$ (which satisfies (1.6) with $x_o=w_o$), i.e., if we embed $(P_{G,h})$ into the family of optimization problems

$$(P^W)=(P_{G,h}^W) \quad \alpha^W=\alpha_{G,h}^W=\inf_{y\in F}\{h(y)\dot{+}\gamma_G(y,w)\}=\lambda_{W\gamma}^{G,h}(w) \qquad (w\in W),$$

then the usual perturbational dual problems $(Q_{pW\chi}^{G,h})$, with respect to the above (X,p,x_o), and even with another dual constraint set $W'\subset \bar{R}^X=\bar{R}^W$, do not seem to be of interest; it is then more convenient to consider, for each $w\in W$, a dual problem $(Q_W^{G,h})$ to $(P_{G,h}^W)$ (e.g., for the γ_G's of (2.33) and (2.48) above).

Theorem 3.1. Let $p=p_{G,h}:F\times X\to\bar{R}$ be an h-separated perturbation functional for problem $(P_{G,h})$ of (1.1) (satisfying (1.6)) and let $W\subset\bar{R}^X$, $\tilde{W}\subset\bar{R}^{F\times X}$ and $\tilde{\gamma}_{(F,x_o)}:(F\times X)\times\tilde{W}\to\bar{R}$ be as in definition 3.4, with the values of $\tilde{\gamma}_{(F,x_o)}$ not depending on h. Then for $\pi_G:F\times X\to\bar{R}$ as in (3.20) and for $\gamma=\gamma_G:F\times W\to\bar{R}$ defined by

$$\gamma_G(y,w)=\inf_{x\in X}\{\pi_G(y,x)\dot{+}\tilde{\gamma}_{(F,x_o)}((y,x),(0,w))\} \qquad (y\in F,\ w\in W), \quad (3.22)$$

we have

$$L_{pW\tilde{\gamma}}^{G,h}(y,w)=L_{W\gamma}^{G,h}(y,w) \qquad (y\in F,\ w\in W), \quad (3.23)$$

$$\lambda_{pW\tilde{\gamma}}^{G,h}(w)=\lambda_{W\gamma}^{G,h}(w) \qquad (w\in W); \quad (3.24)$$

and hence the $(pW\tilde{\gamma})$-dual to $(P_{G,h})$ coincides with the $(W\gamma)$-dual to $(P_{G,h})$.

Proof. Since the values of $\tilde{\gamma}_{(F,x_o)}$ do not depend on h, they do not depend on p (by (1.6)). By (3.14), (3.20), (3.22), (2.4) and [16] formulae (2.3) and (4.7), we get

$$L_{pW\tilde{\gamma}}^{G,h}(y,w)=\inf_{x\in X}\{h(y)\dot{+}\pi_G(y,x)\dot{+}\tilde{\gamma}_{(F,x_o)}((y,x),(0,w))\}=$$

$$=h(y)\dot{+}\gamma_G(y,w)=L_{W\gamma}^{G,h}(y,w) \qquad (y\in F,\ w\in W),$$

and hence, by (1.11), we obtain (3.24).

From theorems 3.1 and 2.2, there follows

Corollary 3.1. Under the assumptions of theorem 3.1, if γ_G of (3.22) satisfies (2.37), then for $V\subset\bar{R}^F$ of (2.38), (2.39), we have

$$L_{pW\tilde{\gamma}}^{G,h}(y,w)=L_V^{G,h}(y,v_{W\gamma}) \qquad (y\in F,\ w\in W), \quad (3.25)$$

$$\lambda_{pW\tilde{\gamma}}^{G,h}(w)=\lambda_V^{G,h}(v_{W\gamma}) \qquad (w\in W). \quad (3.26)$$

Let us give now some corollaries of theorem 3.1 for various h-separated perturbation functionals p.

Assume first that problem $(P_{G,h})$ of (1.1) is embedded into a family of perturbed problems

$$(P^x)=(P^x_{G,h}) \quad \alpha^x=\alpha^x_{G,h}=\inf h(\Gamma(x))=\inf(h\dot{+}\chi_{\Gamma(x)})(F) \qquad (x\in X), \quad (3.27)$$

where X is a set, called <u>set of perturbations</u>, and $\Gamma=\Gamma_G:X\to 2^F$ is a multifunction, called <u>constraint multifunction</u>, such that for some $x_o=x_o^{G,\Gamma}\in X$ there holds

$$\Gamma(x_o)=G; \qquad (3.28)$$

throughout the sequel we shall understand, without any special mention that the sets $\Gamma(x)\subset F$ do not depend on h. Then, the "<u>natural perturbation functional</u>" for problem (1.1), associated to Γ, is (see [23], [24]) the functional $p^n=p^n_{\Gamma(x_o),h}:F\times X\to\overline{R}$ defined by

$$p^n(y,x)=h(y)\dot{+}\chi_{\Gamma(x)}(y) \qquad (y\in F,\ x\in X); \quad (3.29)$$

by (3.28), p^n satisfies (1.6). Clearly, p^n is h-separated, with $\pi_G=\pi_{\Gamma(x_o)}:F\times X\to\overline{R}$ defined by

$$\pi_G(y,x)=\chi_{\Gamma(x)}(y) \qquad (y\in F,\ x\in X); \quad (3.30)$$

note that, by (3.28), π_G of (3.30) satisfies (3.21) for all $y\in F$ (i.e. even when $h(y)\notin R$).

Let us recall that the inverse multifunction $\Gamma^{-1}:F\to 2^X$ is defined by

$$\Gamma^{-1}(y)=\{x\in X \mid y\in\Gamma(x)\} \qquad (y\in F). \quad (3.31)$$

Applying theorem 3.1, we obtain

<u>Corollary 3.2. Let</u> $\Gamma:X\to 2^F$ <u>satisfy</u> (3.28) <u>for some</u> $x_o=x_o^G\in X$ <u>which does not depend on</u> Γ, <u>let</u> $W\subset\overline{R}^X$, $\widetilde{W}\subset\overline{R}^{F\times X}$ <u>satisfy</u> (3.6), <u>and assume that the values of</u> $\widetilde{\gamma}_{(F,x_o)}:(F\times X)\times\widetilde{W}\to\overline{R}$ <u>do not depend on</u> h. <u>Then for</u> $p=p^n$ <u>of</u> (3.29) <u>and for</u> $\gamma_G=\gamma_{\Gamma(x_o)}:F\times W\to\overline{R}$ <u>defined by</u>

$$\gamma_G(y,w)=\inf_{x\in\Gamma^{-1}(y)}\widetilde{\gamma}_{(F,x_o)}((y,x),(0,w)) \qquad (y\in F, w\in W), \quad (3.32)$$

<u>we have</u> (3.23), (3.24).

<u>Proof.</u> It is enough to observe that, by (3.30) and (1.7), the right hand side of (3.22) reduces to the right hand side of (3.32).

Let us consider now the primal problem

$$(P)=(P_{u^{-1}(\Omega),h}) \quad \alpha=\alpha_{u^{-1}(\Omega),h}=\inf_{\substack{y\in F\\ u(y)\in\Omega}} h(y), \qquad (3.33)$$

i.e., (1.1) with $G=u^{-1}(\Omega)$, where F is a set, $h:F\to\overline{R}$, X is a linear space, $u:F\to X$ and $\Omega\subset X$, $u(F)\cap\Omega\neq\emptyset$. Then, the multifunction (see e.g. [22])

$$\Gamma(x)=u^{-1}(\Omega+x) \qquad (x\in X) \quad (3.34)$$

satisfies (3.28) with $x_o=0$ and, clearly,

$$\Gamma^{-1}(y)=u(y)-\Omega \qquad (y\in F). \quad (3.35)$$

Hence, applying corollary 3.2, we obtain

<u>Corollary 3.3. If X is a linear space and</u> $G=u^{-1}(\Omega)$, <u>where</u> $u:F\to X$

$\Omega\subset X$, $u(F)\cap\Omega\neq\emptyset$, <u>then for</u> $W\subset\bar{R}^X$, $\tilde{W}$ <u>and</u> $\tilde{\varphi}_{(F,0)}$ <u>as in theorem 3.1</u>, $p=p^n$ <u>of</u> (3.29), <u>with</u> Γ <u>of</u> (3.34), <u>and</u> $\varphi=\varphi_G:F\times W\to\bar{R}$ <u>defined by</u>

$$\varphi_G(y,w)=\inf_{\omega\in\Omega}\tilde{\varphi}_{(F,0)}((y,u(y)-\omega),(0,w))\quad(y\in F,\ w\in W),\qquad(3.36)$$

<u>we have</u> (3.23), (3.24).

In the particular case when $X=F$, $u=I_F$ (the identity operator) and $G=\Omega$, the multifunction (3.34) becomes

$$\Gamma(x)=G+x\qquad(x\in F),\qquad(3.37)$$

and hence p^n of (3.29) becomes the <u>"standard perturbation functional"</u> ([30], [32]) $p^s:F\times F\to\bar{R}$ defined by

$$p^s(y,x)=h(y)\dotplus\chi_{G+x}(y)\qquad(y,x\in F).\qquad(3.38)$$

Hence, applying corollary 3.3, we obtain

<u>Corollary 3.4. If</u> F <u>is a linear space and</u> $W\subset\bar{R}^F$, $\tilde{W}=(0,W)\subset\bar{R}^{F\times F}$, $\tilde{\varphi}_{(F,0)}:(F\times F)\times\tilde{W}\to\bar{R}$ <u>are as in theorem 3.1, then for</u> $p=p^s$ <u>of</u> (3.38) <u>and</u> $\varphi=\varphi_G:F\times W\to\bar{R}$ <u>defined by</u>

$$\varphi_G(y,w)=\inf_{g\in G}\tilde{\varphi}_{(F,0)}((y,y-g),\ (0,w))\qquad(y\in F,\ w\in W),\ (3.39)$$

<u>we have</u> (3.23), (3.24).

Reading corollary 3.4 in the reverse order, we obtain

<u>Corollary 3.5. If</u> $F, W=W^{G,h}$ <u>and</u> $\tilde{W}$ <u>are as in corollary 3.4,</u> $\varphi=\varphi_G:F\times W\to\bar{R}$ <u>is a coupling functional, with values not depending on</u> h, <u>and</u> $p=p^s:F\times F\to\bar{R}$ <u>is the standard perturbation functional</u> (3.38), <u>then for any coupling functional</u> $\tilde{\varphi}_{(F,0)}:(F\times F)\times\tilde{W}\to\bar{R}$ <u>satisfying</u> (3.39), <u>we have</u> (3.23), (3.24).

<u>Remark 3.6.</u> a) While the $(pW\tilde{\varphi})$-dual to $(P_{G,h})$ is obtained from the $(\tilde{W}\tilde{\varphi})$-dual to $(\tilde{P}_{(F,x_0),p})$, with $\tilde{W}$ of (3.6), corollary 3.5 shows that, when F is a linear space, the $(W\varphi)$-dual to $(P_{G,h})$ is a particular $(p^sW\tilde{\varphi})$-dual to $(P_{G,h})$, with the same W; thus, one can say that the $(pW\tilde{\varphi})$-dual problems $(Q^{G,h}_{pW\tilde{\varphi}})$ are "extensions" of the $(W\varphi)(=p^sW\tilde{\varphi}))$-dual problems $(Q^{G,h}_{W\varphi})$ (from $W\subset\bar{R}^F$ and $p^s:F\times F\to\bar{R}$ of (3.38), to $W\subset\bar{R}^X$ and $p:F\times X\to\bar{R}$).

b) Reading theorem 3.1 and corollaries 3.2 and 3.3 in the reverse order, one can obtain similar results, which we omit.

c) Finally, let us make some remarks on a dual problem to a particular case of (3.33), considered by Tind and Wolsey [36] and an equivalent dual problem of Gould [6]. Let X be a linear space, partially ordered by a cone $K_+=\{x\in X\mid x\geqslant 0\}$, let $u:F\to X$, $x_1\in X$, and let us consider the primal problem (1.1) with $G=\{y\in F\mid u(y)\geqslant x_1\}$, i.e.,

$$(P)\qquad\alpha=\inf_{\substack{y\in F\\ u(y)\geqslant x_1}} h(y).\qquad(3.40)$$

This is nothing else than problem (3.33) (so $G=u^{-1}(\Omega)$), with

$$\Omega=x_1+K_+=\{x\in X\mid x\geqslant x_1\};\qquad(3.41)$$

d) If we do not assume (3.49) in definition (3.50), then, for any $h\neq+\infty$, we have

$$\lambda_{pW}^{G,h}(w)=-\infty \qquad (w\in W,\ w(x_o)=+\infty).\ (3.58)$$

Indeed, if $w(x_o)=+\infty$, then, by (1.6) and (1.2),

$$\inf_{y\in F,x\in X}\{p(y,x)\dot{+}-w(x)\}\leq\inf_{y\in F}\{p(y,x_o)\dot{+}-w(x_o)\}=$$
$$=\inf_{y\in F}\{h(y)\dot{+}\chi_G(y)\dot{+}-\infty\}=-\infty ,$$

whence, by (3.50), we obtain (3.58). Thus, similarly to remark 2.3 f), considering

$$W'''=\{w\in W|w(x_o)<+\infty\},\ \lambda_{pW'''}^{G,h}=\lambda_{pW}^{G,h}|_{W'''}, \qquad (3.59)$$

we see that the assumption (3.49) is no restriction of the generality.

e) By a) above, we have (1.11), (1.12) for $L_{pW}^{G,h}$, $\lambda_{pW}^{G,h}$ and $\beta_{pW}^{G,h}$, so $L_{pW}^{G,h}$ is indeed a Lagrangian. Furthermore, even when (3.49) is not assumed, from (1.6), (2.26) and [16], pp. 119, 117 we obtain

$$h(y)\dot{+}\chi_G(y)=p(y,x_o)\geq p(y,x_o)\dot{+}\{-w(x_o)\dot{+}w(x_o)\}\geq$$
$$\geq\{p(y,x_o)\dot{+}-w(x_o)\}\dot{+}w(x_o)\geq L_{pW}^{G,h}(y,w) \qquad (y\in F,w\in W),\ (3.60)$$

whence (1.14); the inequality $\alpha\geq\beta_{pW}^{G,h}$ has been also observed in [33].

<u>Proposition 3.1.</u> <u>If $x_o\in X$ of (1.6) does not depend on p, then for $\tilde{\gamma}_{(F,x_o)}:(F\times X)\times\tilde{W}\to\bar{R}$ defined by</u>

$$\tilde{\gamma}_{(F,x_o)}((y,x),(0,w))=-w(x)\dot{+}w(x_o) \qquad (y\in F,x\in X,w\in W),\ (3.61)$$

<u>we have</u>

$$L_{pW}^{G,h}(y,w)=L_{pW\tilde{\gamma}}^{G,h}(y,w) \qquad (y\in F,\ w\in W),\ (3.62)$$

$$\lambda_{pW}^{G,h}(w)=\lambda_{pW\tilde{\gamma}}^{G,h}(w) \qquad (w\in W),\ (3.63)$$

<u>so the (pW)-dual to ($P_{G,h}$) coincides with the ($pW\tilde{\gamma}$)-dual to ($P_{G,h}$).</u>

<u>Proof.</u> By (3.51), (3.49), lemma 2.1, (3.61) and (3.14), we have

$$L_{pW}^{G,h}(y,w)=\inf_{x\in X}\{p(y,x)\dot{+}\{-w(x)\dot{+}w(x_o)\}\}=L_{pW\tilde{\gamma}}^{G,h}(y,w) \qquad (y\in F,w\in W),$$

which proves (3.62). Hence, by (1.11) (see remark 3.7 e)), we obtain (3.63) (alternatively, (3.63) follows also from (3.50), (3.49), lemma 2.1, (3.61) and (3.13)).

<u>Remark 3.8.</u> a) One can also give an alternative proof of proposition 3.1, combining remark 3.7 a), theorem 2.1 (applied to ($\tilde{Q}_{\tilde{W}}^{(F,x_o)},p$) and $\tilde{\gamma}$) and remark 3.4 a).

b) By proposition 3.1, the perturbational Lagrangian dual problems (3.50) with x_o not depending on p, constitute a particular class of ($pW\tilde{\gamma}$)-dual problems.

c) By (1.6), we have $(y,x_o)\in\mathrm{dom}\ p$ if and only if $y\in G\cap\mathrm{dom}\ h$.

ted to p, defined by

$$f(x)=\alpha^{x}=\inf_{y\in F} p(y,x) \qquad (x\in X), \qquad (3.64)$$

and the "local" optimization problem (see (2.12))

$$(P_{\{x_o\},f}) \qquad \alpha_{\{x_o\},f}=\inf f(\{x_o\})=f(x_o). \qquad (3.65)$$

<u>Theorem 3.2. We have</u>

$$\alpha_{G,h}=\alpha_{\{x_o\},f}, \qquad (3.66)$$

$$\lambda_{pW}^{G,h}(w)=\lambda_W^{\{x_o\},f}(w)=-f^{c(n)}(w)\dot{+}w(x_o) \qquad (w\in W), \qquad (3.67)$$

<u>and hence</u>

$$\{(P_{G,h}),(Q_{pW}^{G,h})\}\sim\{(P_{\{x_o\},f}),(Q_W^{\{x_o\},f})\}. \qquad (3.68)$$

<u>Proof.</u> By (1.1), (1.6), (3.64) and (3.65),

$$\alpha_{G,h}=\inf_{y\in F} p(y,x_o)=f(x_o)=\inf f(\{x_o\})=\alpha_{\{x_o\},f}.$$

Furthermore, by (2.13), the <u>unperturbational</u> Lagrangian dual problem to $(P_{\{x_o\},f})$, for the same $W\subset \bar{R}^X$, is

$$(Q)=(Q_W^{\{x_o\},f}) \quad \beta_W^{\{x_o\},f}=\sup \lambda_W^{\{x_o\},f}(W);\ \lambda_W^{\{x_o\},f}(w)=$$

$$=\inf_{x\in X}\{f(x)\dot{+}-w(x)\}\dot{+}w(x_o) \qquad (w\in W). \qquad (3.69)$$

On the other hand, by (3.50) and [16], formula (4.7), for f of (3.64) we have

$$\lambda_{pW}^{G,h}(w)=\inf_{x\in X}\{\inf_{y\in F} p(y,x)\dot{+}-w(x)\}\dot{+}w(x_o)=$$

$$=\inf_{x\in X}\{f(x)\dot{+}-w(x)\}\dot{+}w(x_o)=-f^{c(n)}(w)\dot{+}w(x_o) \quad (w\in W); \qquad (3.70)$$

this has also been observed in [33]. From (3.70) and (3.69) we obtain (3.67) and hence, by (3.66), there follows (3.68).

<u>Remark 3.9.</u> a) In the case when $W\subset R^X$, theorem 3.2 has been shown in [24].

b) The right-hand side of (3.68) (and of similar equivalences, e.g. (3.126), (3.185)), has the advantage that the constraint set $\{x_o\}$ is a singleton. For applications of some results of this type, see [30].

c) From (1.4), (3.67) and (1.24), (1.17), it follows that

$$\beta_{pW}^{G,h}=\sup_{w\in W}\{w(x_o)\dot{+}-f^{c(n)}(w)\}=f^{c(n)c(n)'}(x_o); \qquad (3.71)$$

hence, by (3.66), we have $\alpha=\beta_{pW}^{G,h}$ <u>if and only if</u> $f(x_o)=f^{c(n)c(n)'}(x_o)$. Furthermore, by [33], theorem 4.1, there holds

$$f^{c(n)c(n)'}(x_o)=f_{\mathcal{H}(W+R)}(x_o)=\sup_{\substack{w\in W,\, d\in R\\ w+d\leq f}}\{w(x_o)+d\}, \qquad (3.72)$$

the "(W+R)-convex hull of f at x_o", in the sense of [4] (the notation $f_{\mathcal{K}(W+R)}$ is from [33]); consequently, $\alpha=\beta_{pW}^{G,h}$ <u>if and only if</u> $f(x_o)=f_{\mathcal{K}(W+R)}(x_o)$ (i.e, if and only if f is "(W+R)-convex at x_o", in the sense of [4]). Therefore, $(Q_{pW}^{G,h})$ of (3.50) may be also called "the (W+R)-<u>convex dual</u>" problem to $(P_{G,h})$; in the particular case when $F=R^n$, $X=R^m$ and $W=X^*$, the term "convex dual" problem is used in [2].

d) Assume now that X is a linear space and let us consider the dual problem (3.43), with $W=W^f\subset X^{\#}$ of (3.46), (3.45). Similarly to remark 2.9 f), one can define an equivalent dual problem $(Q_o^{G,h})$, enlarging W^f to $W_o=X^{\#}$ and extending $\lambda^{x_1}:W^f\to R$ to $\lambda_o^{\{x_1\},f}:W_o\to R\cup\{-\infty\}$ defined by

$$\lambda_o^{\{x_1\},f}(w)=\lambda^{x_1}(w)\dot{+}-\chi_{W^f}(w)=w(x_1)\quad \text{for } w\in W_o,\ w\leq f$$
$$=-\infty \text{ for } w\in W_o\setminus W^f. \tag{3.73}$$

Then, by (3.43), (3.73), and $x_1=x_o$ (of (1.6)),

$$\beta_o^{G,h}=\sup\lambda_o^{\{x_1\},f}(W_o)=\sup_{\substack{w\in W_o\\ w\leq f}} w(x_o)=f_{\mathcal{K}(W_o)}(x_o), \tag{3.74}$$

whence, by $X^{\#}+R\neq X^{\#}$ and (3.71), (3.72), it follows that, in general,

$$\beta_o^{G,h}=f_{\mathcal{K}(W_o)}(x_o)<f_{\mathcal{K}(W_o+R)}(x_o)=\beta_{pW_o}^{G,h}\leq\alpha; \tag{3.75}$$

thus, similarly to [6],[36], in order to obtain a smaller "duality gap" $\alpha-\beta$, one can replace the dual problem $(Q_o^{G,h})$ of (3.73) by the perturbational Lagrangian dual problem $(Q_{pW_o}^{G,h})$.

There arises naturally the problem, whether there exists a more general (non-anti-additive) "conjugation" $f\in\bar{R}^X\to f^c\in\bar{R}^{W_o}$, satisfying only (1.21), for which $(Q_o^{G,h})$ "behaves like" problem $(Q_{pW_o}^{G,h})$, i.e., such that $\lambda_o^{\{x_1\},f}$, $\beta_o^{G,h}$ satisfy (3.67) and (3.71) with c(n) replaced by c; note that, by (3.74), the second one of these equalities requires that $f_{\mathcal{K}(W_o)}(x_o)=f^{cc'}(x_o)$.

If we define

$$f^c(w)=\chi_{\{w'\in W_o|w'\leq f\}}(w)=0\quad \text{for } w\in W_o,\ w\leq f$$
$$=+\infty\quad \text{for } w\in W_o\setminus W^f, \tag{3.76}$$

then $f\to f^c$ satisfies (1.21) (but not (1.22)) and, by (3.73), (3.74) and (1.23), we have

$$\lambda_o^{\{x_1\},f}(w)=-f^c(w)\dot{+}w(x_o)\qquad (w\in W) \tag{3.77}$$

$$\beta_o^{G,h}=\sup_{w\in W}\{w(x_o)\dot{+}-f^c(w)\}=(f^c)^{c(n)'}(x_o); \tag{3.78}$$

however, the last term of (3.78) is only a "mixed" second conjugate of f at x_o.

Proposition 3.2. If $p=p_{G,h}:F\times X\to\bar{R}$ is h-separated, with x_o of (1.6) not depending on h, and if $W\subset\bar{R}^X$ satisfies (3.49)), then for $\pi_G:F\times X\to\bar{R}$ as in (3.20) and for $\gamma=\gamma_G:F\times W\to\bar{R}$ defined by

$$\gamma_G(y,w)=\inf_{x\in X}\{\pi_G(y,x)\dot{+}-w(x)\}\dot{+}w(x_o) \qquad (y\in F, w\in W), \qquad (3.79)$$

we have

$$L_{pW}^{G,h}(y,w)=L_{W\gamma}^{G,h}(y,w) \qquad (y\in F, w\in W), \qquad (3.80)$$

$$\lambda_{pW}^{G,h}(w)=\lambda_{W\gamma}^{G,h}(w) \qquad (w\in W), \qquad (3.81)$$

and hence the (pW)-dual to $(P_{G,h})$ coincides with the $(W\gamma)$-dual to $(P_{G,h})$.

Proof. For $\tilde{\gamma}_{(F,x_o)}$ defined by (3.61), γ_G of (3.22) becomes, by $\inf\emptyset=+\infty$, (3.49) and lemma 2.1,

$$\begin{aligned}\gamma_G(y,w)&=\inf_{\substack{x\in X\\(y,x)\in\mathrm{dom}\,\pi_G}}\{\pi_G(y,x)\dot{+}\{-w(x)\dot{+}w(x_o)\}\}=\\&=\inf_{\substack{x\in X\\(y,x)\in\mathrm{dom}\,\pi_G}}\{\pi_G(y,x)\dot{+}-w(x)\}\dot{+}w(x_o) \qquad (y\in F, w\in W),\end{aligned}$$

which, by $\inf\emptyset=+\infty$, coincides with (3.79).

Remark 3.10. a) By (3.21), we have

$$\pi_G(g,x_o)=0 \qquad (g\in G, h(g)\in R), \qquad (3.82)$$

whence, by (3.79), (3.49) and $+\infty\dot{+}-\infty=-\infty$, we obtain

$$\gamma_G(g,w)\leq -w(x_o)\dot{+}w(x_o)\leq 0 \qquad (g\in G, h(g)\in R, w\in W). \qquad (3.83)$$

b) It will be useful to note explicitly that, by (3.80), (2.4), (3.79), (3.49) and lemma 2.1 (or, alternatively, by (3.51), (3.20) and [16], formula (4.7)), for any h-separated p we have

$$L_{pW}^{G,h}(y,w)=\{h(y)\dot{+}\inf_{x\in X}\{\pi_G(y,x)\dot{+}-w(x)\}\}\dot{+}w(x_o) \quad (y\in F, w\in W). \qquad (3.84)$$

Now we shall consider the question, under what conditions (on an h-separated perturbation functional $p=p_{G,h}$, or, equivalently, on $\pi_G:F\times X\to\bar{R}$ as in (3.20)) is every (pW)-dual problem to $(P_{G,h})$ an unperturbational Lagrangian dual, i.e., a (V)-dual problem, to $(P_{G,h})$ with suitable $V\subset\bar{R}^F$.

Theorem 3.3. If $p=p_{G,h}:F\times X\to\bar{R}$ is h-separated, $W\subset\bar{R}^X$ and

$$w(x_o)=\inf_{g\in G}\sup_{x\in X}\{w(x)\dot{+}-\pi_G(g,x)\} \qquad (w\in W), \qquad (3.85)$$

(where π_G is as in (3.20)), then for

$$V=V_{W,\pi}=\{v_{w,\pi}\mid w\in W\}\subset\bar{R}^F, \qquad (3.86)$$

where

$$v_{w,\pi}(y)=\sup_{x\in X}\{w(x)\dot{+}-\pi_G(y,x)\} \qquad (y\in F, w\in W), \qquad (3.87)$$

we have

$$L_{pW}^{G,h}(y,w)=L_V^{G,h}(y,v_{w,\pi}) \qquad (y\in F, w\in W), \qquad (3.88)$$

$$\lambda_{pW}^{G,h}(w)=\lambda_V^{G,h}(v_{W,\pi}) \qquad (w\in W), \qquad (3.89)$$

$$(Q_{pW}^{G,h})\sim(Q_V^{G,h}). \qquad (3.90)$$

Proof. By (3.85) and (3.87), we have

$$w(x_o)=\inf v_{W,\pi}(G) \qquad (w\in W),$$

whence, by (3.49), (3.84), (3.87) and (2.29), we obtain (2.8) for V and

$$L_{pW}^{G,h}(y,w)=\{h(y)\dot{+}-v_{W,\pi}(y)\}\dot{+}\inf v_{W,\pi}(G)=$$
$$=L_V^{G,h}(y,v_{W,\pi}) \qquad (y\in F,\ w\in W),$$

i.e., (3.88). Hence, by (1.11), there follow (3.89), (3.90).

Remark 3.11. a) By (3.82) and (3.87), we have

$$w(x_o)\leq\sup_{x\in X}\{w(x)\dot{+}-\pi_G(g,x)\}=v_{W,\pi}(g) \qquad (g\in G,\ h(g)\in R,\ w\in W),$$

and hence, if $h(g)\in R$ for all $g\in G$, then

$$L_{pW}^{G,h}(y,w)\leq L_V^{G,h}(y,v_{W,\pi}) \qquad (y\in F,\ w\in W), \qquad (3.91)$$

$$\lambda_{pW}^{G,h}(w)\leq\lambda_V^{G,h}(v_{W,\pi}) \qquad (w\in W). \qquad (3.92)$$

b) One can also use proposition 3.1 and corollary 3.1 (or, alternatively, proposition 3.2 and theorem 2.2), to obtain (3.88)-(3.90) with $v_{W,\pi}$ replaced by $v_{W\gamma}$ of (2.39)(for γ_G of (3.79), provided that it satisfies (2.37) and x_o of (1.6) does not depend on h), i.e., by

$$v_{W\gamma}(y)=v_{W,\pi}(y)\dot{+}-w(x_o) \qquad (y\in F, w\in W). \qquad (3.93)$$

Moreover, if (2.37) holds for γ_G of (3.79), then, by [16], formula (4.8), we obtain (3.85), with both sides belonging to R; thus, (3.85) is slightly less restrictive than (2.37) (of course, if $w(x_o)\in R$ for all $w\in W$, then they are equivalent).

Let us consider now the Lagrangians $L_{pW}^{G,h}$, for the h-separated perturbation functionals (3.29), in the general case and in the particular cases (3.34) and (3.37). For $p=p^n$ of (3.29) we have, by (3.84) and (3.30),

$$L_{p^nW}^{G,h}(y,w)=h(y)\dot{+}\inf_{x\in X}\{\chi_{\Gamma(x)}(y)\dot{+}-w(x)\}\dot{+}w(x_o)=$$
$$=\{h(y)\dot{+}\inf_{x\in\Gamma^{-1}(y)}(-w(x))\}\dot{+}w(x_o)=$$
$$=\{h(y)\dot{+}-\sup_{x\in\Gamma^{-1}(y)} w(x)\}\dot{+}w(x_o) \qquad (y\in F,\ w\in W), \qquad (3.94)$$

which, in the particular case when $W\subset R^X$, reduces to the "Lagrangian of Kurcyusz" (see [4] and the references therein). Furthermore, γ_G of (3.79) becomes now

$$\gamma_G(y,w)=\inf_{x\in\Gamma^{-1}(y)}(-w(x))\dot{+}w(x_o) \qquad (y\in F,\ w\in W);(3.95)$$

$w\in W$ (i.e., even when $h(g)\notin R$).

From (3.28) and theorem 3.3 we obtain

Corollary 3.6. If

$$w(x_0)=\inf_{y\in\Gamma(x_0)}\sup_{x\in\Gamma^{-1}(y)} w(x) \qquad (w\in W), \qquad (3.96)$$

then for $V=V_{W,\pi}$ of (3.86), where

$$v_{W,\pi}(y)=\sup_{x\in\Gamma^{-1}(y)} w(x) \qquad (y\in F,\ w\in W), \qquad (3.97)$$

and for $p=p^n$ of (3.29), we have (3.88)-(3.90).

Remark 3.12. By (3.31) and (3.97), there holds

$$w(x_0)\leq\inf_{y\in\Gamma(x_0)}\sup_{x\in\Gamma^{-1}(y)} w(x)=\inf v_{W,\pi}(G) \qquad (w\in W), \qquad (3.98)$$

whence (3.91), (3.92), for $p=p^n$ of (3.29) (even when $h(g)\notin R$ for some $g\in G$).

In the particular case (3.33), (3.34), where X is a linear space we have $x_0=0$ (by (3.34) and (3.28)), and hence, if $W\subset X^{\#}$, then

$$w(x_0)=0 \qquad (w\in W). \qquad (3.99)$$

Thus, assuming also that

$$\inf w(\Omega)<+\infty \qquad (w\in W), \qquad (3.100)$$

which is no restriction of the generality (this follows similarly to the argument of remark 2.3 f)), and taking into account (3.35), the Lagrangian (3.94) and γ_G of (3.95) become, respectively,

$$L^{G,h}_{p^n W}(y,w)=h(y)\dotplus\inf_{\omega\in\Omega}(-w(u(y)-\omega))=$$
$$=\{h(y)-w(u(y))\}\dotplus\inf w(\Omega) \qquad (y\in F,\ w\in W), \qquad (3.101)$$
$$\gamma_G(y,w)=-w(u(y))\dotplus\inf w(\Omega) \qquad (y\in F,\ w\in W). \qquad (3.102)$$

Remark 3.13. a) $L^{G,h}_{p^n W}$ of (3.101) encompasses, via (1.11), many usual dual problems. For example, considering a "semi-infinite" primal problem (3.40), i.e., with $F=R^n$, $X=R^I$ (with the product topology), where I is an infinite set, the dual problem to it, in the sense of [1], can be obtained by taking in (3.101), (1.11), $W=X^{*}=(R^I)^{*}$ and Ω of (3.41). For some other particular cases of (3.101), (1.11), see [21],[28],[30].

b) Let us show that some generalizations of $L^{G,h}_{p^n W}$ of (3.101) (and hence of the corresponding dual problems $(Q^{G,h}_{p^n W})$) can be also written as $(W\gamma)$-Lagrangians (respectively, $(W\gamma)$-dual problems), with suitable coupling functionals $\gamma=\gamma_G:F\times W\to\overline{R}$.

i) Firstly, let us consider the following generalization (see e.g. [11]) of (3.101) (when $F=R^n$, $X=R^m$, this also extends a generalization of (3.101), due to Gould [6]): Let $W=W^{G,h}$ be a set and let

$$L^{G,h}_1(y,w)=h(y)\dotplus\varphi_G(u(y),w) \qquad (y\in F,\ w\in W), \qquad (3.103)$$

where $u:F\to X$ is the (fixed) mapping occurring in (3.33) and where

$\varphi_G : u(F)\times W \to \overline{R}$ is an arbitrary coupling functional, with values $\varphi_G(u(y),w)\in\overline{R}$ not depending on h (they depend on G, at least via u). Define $\gamma_G^1 : F\times W\to\overline{R}$ by

$$\gamma_G^1(y,w)=\varphi_G(u(y),w) \qquad (y\in F,\ w\in W). \qquad (3.104)$$

ii) Let $W=W_G\subset\overline{R}^{F\times u(F)}$ be a family of coupling functionals $w: F\times u(F)\to\overline{R}$, with values $w(y,u(y))\in\overline{R}$ not depending on h, and let us consider the following generalization of (3.101) (which extends a generalization of (3.101), due to Klötzler[10]):

$$L_2^{G,h}(y,w)=h(y)\dot{+}w(y,u(y)) \qquad (y\in F,\ w\in W). \qquad (3.105)$$

Define $\gamma_G^2 : F\times W\to\overline{R}$ by

$$\gamma_G^2(y,w)=w(y,u(y)) \qquad (y\in F,\ w\in W). \qquad (3.106)$$

iii) Let $W=W_G\subset(\overline{R}^{u(F)})^F$, i.e., let W be a family of mappings $w: F\to\overline{R}^{u(F)}$, such that the numbers $w(y)(u(y))\in\overline{R}$ do not depend on h, and let us consider the following generalization of (3.101) (which extends a generalization of (3.101), due to Armin Hoffmann [7]):

$$L_3^{G,h}(y,w)=h(y)\dot{+}w(y)(u(y)) \qquad (y\in F,\ w\in W). \qquad (3.107)$$

Define $\gamma_G^3 : F\times W\to\overline{R}$ by

$$\gamma_G^3(y,w)=w(y)(u(y)) \qquad (y\in F,\ w\in W). \qquad (3.108)$$

Then, for i)-iii) we have, obviously,

$$L_i^{G,h}(y,w)=L_{W\gamma^i}^{G,h}(y,w) \qquad (y\in F,\ w\in W;\ i=1,2,3), \qquad (3.109)$$

$$\lambda_i^{G,h}(w)=\inf_{y\in F} L_i^{G,h}(y,w)=\lambda_{W\gamma^i}^{G,h}(w) \qquad (w\in W;\ i=1,2,3). \qquad (3.110)$$

Returning to (3.101), let us give

Corollary 3.7. If X *is a linear space,* $G=u^{-1}(\Omega)$, *where* $u: F\to X$, $\Omega\subset X$, $u(F)\supset\Omega$, *and if* $W\subset X^{\#}$ *satisfies*

$$\inf w(\Omega)\in R \qquad (w\in W), \qquad (3.111)$$

then for $V=V_{W,\pi}$ *of* (3.86), *where*

$$v_{w,\pi}(y)=v_{w,u,\Omega}(y)=w(u(y))\dot{+}-\inf w(\Omega) \qquad (y\in F,\ w\in W), \qquad (3.112)$$

and for $p=p^n$ *of* (3.29), *with* Γ *of* (3.34), *we have* (3.88)-(3.90).

Proof. By (3.34), $x_0=0$, (3.28), (3.35), $W\subset X^{\#}$, (3.111), $u(F)\supset\Omega$ and (3.99), we have

$$\inf_{y\in\Gamma(x_0)}\ \sup_{x\in\Gamma^{-1}(y)} w(x)=\inf_{\substack{y\in F\\ u(y)\in\Omega}}\ \sup_{\omega\in\Omega} w(u(y)-\omega)=$$

$$=\inf_{\substack{y\in F\\ u(y)\in\Omega}}\{w(u(y))\dot{+}-\inf w(\Omega)\}=0=w(x_0) \qquad (w\in W),$$

i.e., (3.96), so corollary 3.6 applies.

Remark 3.14. a) Condition (3.111) is rather restrictive (see remark 2.3 e)). If $w_0\in W$, $\inf w_0(\Omega)=-\infty$, then, by (3.112), we have

V), we obtain

$$L^{G,h}_{p^n_W}(y,w_o)=-\infty=L^{G,h}_V(y,v_{w_o,u,\Omega}) \qquad (y\in \mathrm{dom}\ h),$$

and hence (see (2.30)) $\lambda^{G,h}_{p^n_W}(w_o)=-\infty=\lambda^{G,h}_V(v_{w_o,u,\Omega})$. Thus, (3.89), (3.90) hold also under the weaker assumption (3.100).

b) By (3.93) and (3.99), we have now $v_{w_T}=v_{w,\pi}$ $(w\in W)$. Also, if $u(F)\supset\Omega$ and (3.111) holds, then τ_G of (3.102) satisfies (2.37).

In view of remark 3.14 a), let us also give

Theorem 3.4. *If* X *is a linear space*, $G=u^{-1}(\Omega)$, *where* $u:F\to X$, $\Omega\subset X$, $u(F)\supset\Omega$, *and if* $W\subset X^{\#}$ *satisfies* (3.100), *then for*

$$V=V_{W,u}=\{v_{w,u}\mid w\in W\}\subset \overline{R}^F, \qquad (3.113)$$

where

$$v_{w,u}(y)=w(u(y)) \qquad (y\in F, w\in W), \qquad (3.114)$$

and for $p=p^n$ *of* (3.29), *with* Γ *of* (3.34), *we have* (3.88)-(3.90).

Proof. In [32], remark 1.2, we have observed that

$$v_{w,u}(G)=(w\circ u)(u^{-1}(\Omega))=w(u(F)\cap\Omega) \qquad (w\in W), \qquad (3.115)$$

and thus, if $u(F)\supset\Omega$, then

$$v_{w,u}(G)=w(\Omega) \qquad (w\in W). \qquad (3.116)$$

Hence, by (3.101), (3.114) and (2.29), we obtain

$$L^{G,h}_{p^n_W}(y,w)=\{h(y)-v_{w,u}(y)\}+\inf v_{w,u}(G)=$$
$$=L^{G,h}_V(y,v_{w,u}) \qquad (y\in F,\ w\in W), \qquad (3.117)$$

which proves (3.88). Hence, by (1.11), there follow (3.89), (3.90).

Remark 3.15. If $u(0)=0$, then $v_{w,u}(0)=0$ $(w\in W)$. If u is linear, then $V_{W,u}\subset F^{\#}$.

In the particular case when $X=F$ and $u=I_F$, condition $u(F)\supset\Omega$ is satisfied and $V_{W,u}=W$, $v_{w,u}=w$, and hence, from theorem 3.4 we obtain

Corollary 3.8. *If* F *is a linear space, then for* $W\subset F^{\#}$ *satisfying* (2.8), *and* $p=p^s:F\times F\to\overline{R}$ *of* (3.38), *we have*

$$L^{G,h}_{p^s_W}(y,w)=L^{G,h}_W(y,w) \qquad (y\in F,\ w\in W), \qquad (3.118)$$

$$\lambda^{G,h}_{p^s_W}(w)=\lambda^{G,h}_W(w) \qquad (w\in W), \qquad (3.119)$$

and hence the (p^sW)-*dual to* $(P_{G,h})$ *coincides with the* (W)-*dual to* $(P_{G,h})$.

Remark 3.16. a) While the perturbational Lagrangian dual to $(P_{G,h})$ is obtained from the unperturbational Lagrangian dual to $(\tilde{P}_{(F,x_o),p})$, using $\tilde{W}$ of (3.6) (see remark 3.7 a)), corollary 3.8 shows that, when F is a linear space and $W\subset F^{\#}$, the unperturbational Lagrangian dual to $(P_{G,h})$ is the particular case $X=F$, $p=p^s$ of the perturbational Lagrangian dual to $(P_{G,h})$, with the same W.

b) There are some perturbation functionals $p:F\times X\to\overline{R}$ which are not h-separated, but can be replaced by an h-separated perturbation functional, so as to obtain the same primal functional (3.64), and hence, by (3.67), the same Lagrangian dual problem. For example, if $X=F$ is a linear space and $p^o=p^o_{G,h}:F\times F\to\overline{R}$ is defined by

$$p^o(y,x)=h(y+x)\dot{+}\chi_G(y) \qquad (y,x\in F), \qquad (3.120)$$

which satisfies (1.6) with $x_o=0$ (such perturbation functionals are used in the theory of best approximation, e.g. in the problem of infimizing, for an element x_1 of a normed linear space F and for y ranging in a subset G of F, the functional $h(y)=\|x_1-y\|$, $y\in F$), then p^o is not h-separated, but can be replaced by the "standard perturbation functional" $p^s=p^s_{G,h}:F\times F\to\overline{R}$ of (3.38). Indeed, if f^o, $f^s:F\to\overline{R}$ are the primal functionals (3.64) corresponding to p^o and p^s respectively, then

$$f^o(x)=\inf_{g\in G} h(g+x)=f^s(x) \qquad (x\in F), \qquad (3.121)$$

and hence, by (3.67), it follows that

$$\lambda^{G,h}_{p^oW}(w)=\lambda^{G,h}_{p^sW}(w) \qquad (w\in W). \qquad (3.122)$$

Let us also note that

$$p^o(y,x)=p^s(y+x,x) \qquad (y,x\in F). \qquad (3.123)$$

From corollary 3.3 and theorem 3.2, we obtain

Corollary 3.9. For an arbitrary perturbation functional $p:F\times X\to\overline{R}$, if X is a linear space and $W\subset X^{\#}$, then

$$\lambda^{G,h}_{pW}(w)=\lambda^{\{x_o\},f}_{p^sW}(w) \qquad (w\in W), \qquad (3.124)$$

where $p^s=p^s_{\{x_o\},f}:X\times X\to\overline{R}$ is the standard perturbation functional

$$p^s(x,z)=f(x)\dot{+}\chi_{z+\{x_o\}}(x) \qquad (x,z\in X). \qquad (3.125)$$

Hence,

$$\{(P_{G,h}),(Q^{G,h}_{pW})\}\sim\{(P_{\{x_o\},f}),(Q^{\{x_o\},f}_{p^sW})\}. \qquad (3.126)$$

Proof. By corollary 3.8, we have

$$\lambda^{\{x_o\},f}_{W}(w)=\lambda^{\{x_o\},f}_{p^sW}(w) \qquad (w\in W), \qquad (3.127)$$

whence, by (3.67), there follows (3.124); hence, by (3.66), we obtain (3.126).

Remark 3.17. Corollary 3.9 is a generalization of [30], formula (2.6).

Corollary 3.10. If F is a linear space, $W\subset F^{\#}$, $p^s=p^s_{G,h}:F\times F\to\overline{R}$ is the standard perturbation functional (3.38), and $f^s=f^s_{G,h}:F\to\overline{R}$ the "standard primal functional", corresponding to p^s, i.e.,

$$f^s(x)=\inf p^s(y,x)=\inf h(g+x) \qquad (x\in F), \qquad (3.128)$$

then

$$\lambda_W^{G,h}(w)=\lambda_W^{\{0\},f^s}(w)=-(f^s)^{c(n)}(w) \qquad (w\in W), \qquad (3.129)$$

$$\{(P_{G,h}),(Q_W^{G,h})\}\sim\{(P_{\{0\},f^s}),(Q_W^{\{0\},f^s})\}. \qquad (3.130)$$

Proof. By corollary 3.8 and theorem 3.2 for $p=p^s$ (which satisfies (1.6) with $x_0=0$), and since $w(0)=0$ ($w\in W$), we have

$$\lambda_W^{G,h}(w)=\lambda_{p^sW}^{G,h}(w)=\lambda_W^{\{0\},f^s}(w)=-(f^s)^{c(n)}(w) \qquad (w\in W);$$

alternatively, (3.129) follows also directly from (2.9), $W\subset F^{\#}$ and (3.128).

Remark 3.12. By theorem 2.4, if (2.46) holds, then (unperturbational surrogate) $(W\Delta)$-duality is a "particular case of" V-duality, namely, for $V=V_{W\Delta}\subset\overline{R}^F$ of (2.51), so it is natural to ask whether one can apply corollary 3.10 to this V, in order to conclude that $\lambda_{W\Delta}^{G,h}(w)=-(f^s)^{c(n)}(-\chi_{\Delta_{G,w}})$ ($w\in W$). The answer is negative, since if F is a linear space and W is a set, then the family $V=V_{W\Delta}\subset\overline{R}^F$ defined by (2.51) does not satisfy $V\subset F^{\#}$ (when $\Delta_{G,w}\neq F$ for some $w\in W$). Moreover, in [32], theorem 3.1, we have shown that for $F=R^2$ there exists no coupling functional $\varphi:F\times(F^{\#}\times R)\to\overline{R}$ (independent on G,h), such that

$$\beta_{F^{\#}\Delta\delta s}^{G,h}=\sup\{-(f^s)^{c(-\varphi)}(F^{\#},0)\} \qquad (G\in 2^F, h\in\overline{R}^F),$$

where $\beta_{F^{\#}\Delta\delta s}^{G,h}$ is as in remark 2.9 d) above and 2^F denotes the family of all subsets of F; hence, by (1.20), there exists no family $V=\{v_{W\varphi}|w\in(F^{\#},0)\}\subset\overline{R}^F$ (independent on G,h), such that

$$\beta_{F^{\#}\Delta\delta s}^{G,h}=\sup_{w\in(F^{\#},0)}\{-(f^s)^{c(n)}(v_{W\varphi})\} \qquad (G\in 2^F,\ h\in\overline{R}^F), \qquad (3.131)$$

However, let us mention that for $\beta_{F^{\#}\Delta\gamma s}^{G,h}$ of remark 2.9 d) there exists (by [32], formula (3.24) and [34]) $\varphi:F\times(F^{\#}\times R)\to\overline{R}$ such that

$$\lambda_{F^{\#}\Delta\gamma s}^{G,h}(\Psi,0)=-(f^s)^{c(-\varphi)}(\Psi,0) \qquad (\Psi\in F^{\#},\ G\in 2^F,\ h\in\overline{R}^F).$$

3.4. Perturbational surrogate dual problems

We recall that, assuming (1.6), if $\tilde{\Delta}_{(F,x_0),w}\subset F\times X$ ($w\in W$), then the $(X,p,x_0,W,\tilde{\Delta})$-dual, or, briefly, the $(pW\tilde{\Delta})$-dual, to $(P)=(P_{G,h})$, is defined [32] as the supremization problem

$$(Q)=(Q_{pW\tilde{\Delta}}^{G,h}) \qquad \beta=\beta_{pW\tilde{\Delta}}^{G,h}=\sup\lambda(W);\lambda(w)=\lambda_{pW\tilde{\Delta}}^{G,h}(w)=$$

$$=\inf p(\tilde{\Delta}_{(F,x_0),w}) \qquad (w\in W), \quad (3.132)$$

and any such $(pW\tilde{\Delta})$-dual is called [32] a PS-dual (perturbational surrogate dual) problem to (P). We shall also assume (3.2) (as in [32]), in order to work with $\tilde{W}$ of (3.6), and that, for each fixed

$w\in W^{G,h}$, the set $\widetilde{\Delta}_{(F,x_o),w}\subset F\times X$ does not depend on p.

Remark 3.19. In [32] we have used the term "PES-dual" ("perturbational extended surrogate dual"), where the word "extended" stands to indicate that $(Q^{G,h}_{pW\widetilde{\Delta}})$ "comes from" the surrogate dual to the "extended problem" $(\widetilde{P})$ of (3.3); however, in the unified framework of the present paper, the word "extended" will be omitted.

Definition 3.6. We define the $(pW\widetilde{\Delta})$-Lagrangian $L=L^{G,h}_{pW\widetilde{\Delta}}:F\times W\to\overline{R}$ for $\{(P_{G,h}),(Q^{G,h}_{pW\widetilde{\Delta}})\}$, by

$$L^{G,h}_{pW\widetilde{\Delta}}(y,w)=\inf_{\substack{x\in X\\(y,x)\in\widetilde{\Delta}_{(F,x_o),w}}} p(y,x)=\inf p(\widetilde{\Delta}_{(F,x_o),w}\cap(y,X)) \qquad (y\in F,\ w\in W). \qquad (3.133)$$

Remark 3.20. a) $(Q^{G,h}_{pW\widetilde{\Delta}})$ and $L^{G,h}_{pW\widetilde{\Delta}}$ are the perturbational dual problem and Lagrangian corresponding to $(Q^{G,h}_{W\Delta})$ and $L^{G,h}_{W\Delta}$ of (2.44) and (2.45), by the scheme of definition 3.3 and remark 3.3. Indeed, if we replace $w\in\overline{R}^F$, $y\in F, G, h, W\subset\overline{R}^F$ and $\Delta_{G,w}$ by $(0,w)\in(0,\overline{R}^X)$, $(y,x)\in F\times X$, (F,x_o), p, $\widetilde{W}=(0,W)\subset(0,\overline{R}^X)$ and $\widetilde{\Delta}_{(F,x_o),(0,w)}=\widetilde{\Delta}_{(F,x_o),w}$ respectively, then $\lambda^{G,h}_{W\Delta}$ and $L^{G,h}_{W\Delta}$ of (2.44),(2.45) will be replaced, respectively, by

$$\widetilde{\lambda}^{(F,x_o),p}_{\widetilde{W}\widetilde{\Delta}}(0,w)=\inf p(\widetilde{\Delta}_{(F,x_o),w}) \qquad (w\in W), \qquad (3.134)$$

$$\widetilde{L}^{(F,x_o),p}_{\widetilde{W}\widetilde{\Delta}}((y,x),(0,w))=p(y,x)\dot{+}\chi_{\widetilde{\Delta}_{(F,x_o),w}}(y,x)\ (y\in F,x\in X,w\in W), \qquad (3.135)$$

whence, by (3.132), (3.133), (3.7) and (3.9), we obtain

$$\lambda^{G,h}_{pW\widetilde{\Delta}}(w)=\lambda^{G,h}_{pW\widetilde{\lambda}}(w) \qquad (w\ W), \qquad (3.136)$$

$$L^{G,h}_{pW\widetilde{\Delta}}(y,w)=L^{G,h}_{pW\widetilde{\lambda}}(y,w) \qquad (y\in F,\ w\in W); \qquad (3.137)$$

formula (3.7) for $\lambda=\lambda^{G,h}_{pW\widetilde{\Delta}}$, $\widetilde{\lambda}=\widetilde{\lambda}^{(F,x_o),p}_{\widetilde{W}\widetilde{\Delta}}$ has been also observed in [32], remark 4.1 b).

b) By a), we have (1.11), (1.12) for $L^{G,h}_{pW\widetilde{\Delta}}$, $\lambda^{G,h}_{pW\widetilde{\Delta}}$ and $\beta^{G,h}_{pW\widetilde{\Delta}}$, so $L^{G,h}_{pW\widetilde{\Delta}}$ is indeed a Lagrangian. Furthermore, if

$$(F,x_o)\subset\widetilde{\Delta}_{(F,x_o),w} \qquad (w\in W), \qquad (3.138)$$

then, by (1.6), (3.138) and (3.133), we obtain

$$h(y)\dot{+}\chi_G(y)=p(y,x_o)\geqslant\inf_{\substack{x\in X\\(y,x)\in\widetilde{\Delta}_{(F,x_o),w}}} p(y,x)=L^{G,h}_{pW}(y,w)\ (y\in F,w\in W), \qquad (3.139)$$

whence (1.14).

Proposition 3.3. Under the assumptions of definition (3.132), for $\widetilde{\varphi}_{(F,x_o)}:(F\times X)\times\widetilde{W}\to\overline{R}$ defined by

$$\widetilde{\varphi}_{(F,x_o)}((y,x),(0,w))=\chi_{\widetilde{\Delta}_{(F,x_o),w}}(y,x) \qquad (y\in F,x\in X,w\in W), \qquad (3.140)$$

<u>we have</u>

$$L^{G,h}_{pW\tilde{\Delta}}(y,w)=L^{G,h}_{pW\tilde{\gamma}}(y,w) \qquad (y\in F, w\in W), \quad (3.141)$$

$$\lambda^{G,h}_{pW\tilde{\Delta}}(w)=\lambda^{G,h}_{pW\tilde{\gamma}}(w) \qquad (w\in W), \quad (3.142)$$

<u>so the</u> $(pW\tilde{\Delta})$<u>-dual to</u> $(P_{G,h})$ <u>coincides with the</u> $(pW\tilde{\gamma})$<u>-dual to</u> $(P_{G,h})$.

<u>Proof.</u> By (3.133), (1.7) and (3.140), (3.14), we have

$$L^{G,h}_{pW\tilde{\Delta}}(y,w)=\inf_{x\in X}\{p(y,x)\dot{+}\chi_{\tilde{\Delta}_{(F,x_0),w}}(y,x)\}=$$

$$=L^{G,h}_{pW\tilde{\gamma}}(y,w) \qquad (y\in F, w\in W),$$

which proves (3.141). Hence, by (1.11) (see remark 3.20 b)), we obtain (3.142) (alternatively, (3.142) follows also from (3.132), (1.7), (3.140) and (3.13).

<u>Remark 3.21</u> . a) One can give an alternative proof of proposition 3.3, combining remark 3.20 a), theorem 2.3 (applied to $(\tilde{Q}^{(F,x_0)}_{\tilde{W}\tilde{\Delta}};P)$ and $\tilde{\gamma}$) and remark 3.4 a).

b) By proposition 3.3, the perturbational surrogate dual problems (3.132) constitute a particular class of $(pW\tilde{\gamma})$-dual problems.

c) If we define $\Delta^o_{\{x_0\},w}\subset X$ $(w\in W)$ by

$$\Delta^o_{\{x_0\},w}=Pr_X\tilde{\Delta}_{(F,x_0),w}=\{x\in X\mid \exists y'\in F, (y',x)\in\tilde{\Delta}_{(F,x_0),w}\} \quad (w\in W), (3.143)$$

and if (3.138) holds, then, for $f:X\to\overline{R}$ of (3.64), we have only

$$\lambda^{G,h}_{pW\tilde{\Delta}}(w)=\inf p(\tilde{\Delta}_{(F,x_0),w})\geq\inf_{y\in F, x\in\Delta^o_{\{x_0\},w}} p(y,x)=$$

$$=\inf f(\Delta^o_{\{x_0\},w})=\lambda^{\{x_0\},f}_{W\Delta^o}(w) \qquad (w\in W), (3.144)$$

so a result corresponding to theorem 3.2 does not hold for Δ^o of (3.143); formula (3.144) has been observed in [32], remark 4.1 d) and formula (5.10).

Combining proposition 3.3 and theorem 3.1, we obtain

<u>Proposition 3.4. If</u> $p=p_{G,h}:F\times X\to\overline{R}$ <u>(satisfying</u> (1.6)) <u>is h-separated and</u> $W=W^{G,h}\subset\overline{R}^X$, <u>and if the sets</u> $\tilde{\Delta}_{(F,x_0),w}\subset F\times X$ $(w\in W)$ <u>do not depend on h, then for</u> $\gamma_G:F\times W\to\overline{R}$ <u>defined by</u>

$$\gamma_G(y,w)=\inf_{\substack{x\in X\\(y,x)\in\tilde{\Delta}_{(F,x_0),w}}}\pi_G(y,x)=\inf \pi_G(\tilde{\Delta}_{(F,x_0),w}\cap(y,X)) \qquad (y\in F, w\in W), \quad (3.145)$$

<u>where</u> π_G <u>is as in</u> (3.20), <u>we have</u>

$$L^{G,h}_{pW\tilde{\Delta}}(y,w)=L^{G,h}_{W\gamma}(y,w) \qquad (y\in F, w\in W), \quad (3.146)$$

$$\lambda^{G,h}_{pW\tilde{\Delta}}(w)=\lambda^{G,h}_{W\gamma}(w) \qquad (w\in W), \quad (3.147)$$

<u>and hence the</u> $(pW\tilde{\Delta})$<u>-dual to</u> $(P_{G,h})$ <u>coincides with the</u> $(W\gamma)$<u>-dual to</u> $(P_{G,h})$.

Proof. For $\tilde{\gamma}_{(F,x_0)}$ defined by (3.140), γ_G of (3.22) becomes

$$\gamma_G(y,w)=\inf_{x\in X}\{\pi_G(y,x)\dot{+}\chi_{\tilde{\Delta}_{(F,x_0),w}}(y,x)\} \qquad (y\in F,\ w\in W),$$

i.e., (3.145).

Remark 3.22. a) If (3.138) holds, then, by (3.145) and (3.82), we have

$$\gamma_G(g,w)\leq\pi_G(g,x_0)=0 \qquad (g\in G, h(g)\in R, w\in W). \qquad (3.148)$$

b) It will be useful to note explicitly that, by (3.146), (3.147) and (2.4) (or, alternatively, by (3.133), (3.20), and [16], formula (4.7)), for any h-separated p we have

$$L^{G,h}_{pW\tilde{\Delta}}(y,w)=h(y)\dot{+}\inf_{\substack{x\in X\\(y,x)\in\tilde{\Delta}_{(F,x_0),w}}}\pi_G(y,x) \qquad (y\in F, w\in W). \qquad (3.149)$$

Let us consider now the question, under what conditions (on an h-separated perturbation functional $p=p_{G,h}$, or, equivalently, on $\pi_G:F\times X\to\overline{R}$ as in (3.20)) is every $(pW\tilde{\Delta})$-dual problem to $(P_{G,h})$ an unperturbational Lagrangian dual, i.e., a (V)-dual problem, to $(P_{G,h})$ with suitable $V\subset\overline{R}^F$. Corresponding to remark 3.11 b), we have

Proposition 3.5. If $p=p_{G,h}:F\times X\to\overline{R}$ is h-separated and $W=W^{G,h}\subset\overline{R}^X$ and if the sets $\tilde{\Delta}_{(F,x_0),w}\subset F\times X$ $(w\in W)$ do not depend on h and satisfy

$$\sup_{g\in G}\ \inf_{\substack{x\in X\\(g,x)\in\tilde{\Delta}_{(F,x_0),w}}}\pi_G(g,x)=0 \qquad (w\in W) \qquad (3.150)$$

(where π_G is as in (3.20)), then for $V=V_{W\gamma}\subset\overline{R}^F$ of (2.38), (2.39), with $\gamma_G:F\times W\to\overline{R}$ of (3.145), we have

$$L^{G,h}_{pW\tilde{\Delta}}(y,w)=L^{G,h}_V(y,v_{w\gamma}) \qquad (y\in F, w\in W), \qquad (3.151)$$

$$\lambda^{G,h}_{pW\tilde{\Delta}}(w)=\lambda^{G,h}_V(v_{w\gamma}) \qquad (w\in W), \qquad (3.152)$$

$$(Q^{G,h}_{pW\tilde{\Delta}})\sim(Q^{G,h}_V). \qquad (3.153)$$

Proof. For γ_G of (3.145), condition (2.37) becomes (3.150), and hence, by proposition 3.3 and corollary 3.1 (or, alternatively, by proposition 3.4 and theorem 2.2), we obtain (3.151)-(3.153).

Now we shall consider the question, under what conditions (on an h-separated perturbation functional $p=p_{G,h}$, or, equivalently, on $\pi_G:F\times X\to\overline{R}$ as in (3.20)), is every $(pW\tilde{\Delta})$-dual problem to $(P_{G,h})$ an unperturbational surrogate, more precisely, a $(W\Delta)$-dual problem, to $(P_{G,h})$, with the same $W\subset\overline{R}^X$ (this is slightly different from the corresponding question for unperturbational Lagrangian dual problems, where we have needed another $V\subset\overline{R}^F$, because of definition 2.3).

Proposition 3.6. If $p_{G,h}:F\times X\to\overline{R}$ is h-separated and $W=W^{G,h}\subset\overline{R}^X$, and if the sets $\tilde{\Delta}_{(F,x_0),w}\subset F\times X$ $(w\in W)$ do not depend on h and satisfy

$$\inf_{\substack{x\in X\\ (y,x)\in\tilde{\Delta}_{(F,x_0),w}}} \ldots \Delta_{G,w} \ldots \qquad (y\in F,\ w\in W), \quad (3.154)$$

then

$$L^{G,h}_{pW\tilde{\Delta}}(y,w)=L^{G,h}_{W\Delta}(y,w) \qquad (y\in F,\ w\in W), \quad (3.155)$$

$$\lambda^{G,h}_{pW\tilde{\Delta}}(w)=\lambda^{G,h}_{W\Delta}(w) \qquad (w\in W). \quad (3.156)$$

Proof. This follows from (3.149), (2.45) and (1.11).

Remark 3.23. a) For $\gamma_G: F\times W \to \bar{R}$ of (3.145), condition (3.154) becomes (2.48). Hence, by theorem 2.3 and remark 2.5 (applied to theorem 2.3), formulae (3.155), (3.156) are equivalent to (3.146), (3.147), with γ_G of (3.145) satisfying (3.154). In the sequel we shall be concerned with (3.155), (3.156).

b) By a) above and remark 2.9 b), for γ_G of (3.145), condition (3.150) (i.e., (2.37)) is equivalent to (2.46).

Let us consider now the h-separated perturbation functionals (3.29), in the general case and in the particular cases (3.34), (3.37).

Corollary 3.11. Given $\Gamma: X\to 2^F$ satisfying (3.28) and $\tilde{\Delta}_{(F,x_0),w}\subset F\times X$ $(w\in W)$, for $p=p^n$ of (3.29) and for $\Delta_{G,w}\subset F$ $(w\in W)$ defined by

$$\Delta_{G,w}=\{y\in F \mid \tilde{\Delta}_{(F,x_0),w}\cap(y,\Gamma^{-1}(y))\neq\emptyset\} \qquad (w\in W) \quad (3.157)$$

we have (3.155), (3.156).

Proof. By proposition 3.6, it will be sufficient to show that

$$\inf_{\substack{x\in X\\ (y,x)\in\tilde{\Delta}_{(F,x_0),w}}} \chi_{\Gamma(x)}(y)=\chi_{\{y'\in F \mid \tilde{\Delta}_{(F,x_0),w}\cap(y',\Gamma^{-1}(y'))\neq\emptyset\}}(y)$$

$$(y\in F,\ w\in W)\ ; \quad (3.158)$$

indeed, then the sets $\Delta_{G,w}$ of (3.157) will satisfy condition (3.154), with π_G of (3.30). But, the left hand side of (3.158) is $=0$ if and only if there exists $x\in X$ with $(y,x)\in\tilde{\Delta}_{(F,x_0),w}$ such that $y\in\Gamma(x)$, i.e., such that $x\in\Gamma^{-1}(y)$, which happens if and only if the right hand side of (3.158) is $=0$.

Remark 3.24. a) Formula (3.157) has been obtained in [32], theorem 4.3 (where these sets have been denoted by $\Delta^n(\tilde{\Delta})_{\Gamma(x_0),w}$). However, formula (3.155), with $\Delta_{G,w}$ of (3.157), has been taken in [32] as the definition of $L^{G,h}_{p^nW\tilde{\Delta}}$ (see also remark 1.1 b) above).

b) Since the sets $\Delta_{G,w}$ of (3.157) satisfy (3.154) (with π_G of (3.30)), for γ_G of (3.145) we have (3.148) for all $g\in G$, $w\in W$ (i.e., even when $h(g)\notin R$) if and only if

$$\chi_{\Delta_{G,w}}(g)\leq\chi_{\Gamma(x_0)}(g)=\chi_G(g)=0 \qquad (g\in G,\ w\in W),$$

i.e., if and only if (2.46) holds. Note also that, for $\Delta_{G,w}$ of (3.157), we have (2.46) (or, equivalently, (3.150)) if and only if

$$\tilde{\Delta}_{(F,x_o),w} \cap (g,\Gamma^{-1}(g)) \neq \emptyset \qquad (g\in G,\ w\in W); \qquad (3.159)$$

in particular, this is satisfied if (3.138) holds, or, even if

$$(G,x_o) \subset \tilde{\Delta}_{(F,x_o),w} \qquad (w\in W) \qquad (3.160)$$

(since then, by (3.160), (3.28) and (3.31), we have $(g,x_o)\in\tilde{\Delta}_{(F,x_o),w} \cap (g,\Gamma^{-1}(g))$ for all $g\in G$, $w\in W$) and hence, in this case, (3.151)-(3.15[illegible]) hold.

In the converse direction, let us give

Corollary 3.12. Given $\Delta_{G,w}\subset F$ $(w\in W)$ and $\Gamma: X\to 2^F$ satisfying (3.28) and

$$\Gamma^{-1}(y)\neq\emptyset \qquad (y\in F) \qquad (3.161)$$

(or, equivalently, $\Gamma(X)=\bigcup_{x\in X}\Gamma(x)=F$), for the sets $\tilde{\Delta}_{(F,x_o),w}\subset F\times X$ $(w\in W)$ defined by

$$\tilde{\Delta}_{(F,x_o),w}=(\Delta_{G,w},\Gamma^{-1}(\Delta_{G,w})) \qquad (w\in W) \qquad (3.162)$$

(where $\Gamma^{-1}(A)=\bigcup_{y\in A}\Gamma^{-1}(y)$) there holds (3.157), and hence, for $p=p^n$ of (3.29), we have (3.155), (3.156).

Proof. If $y\in\Delta_{G,w}$, then, by (3.161), (3.162), for any $x\in\Gamma^{-1}(y)\subset\Gamma^{-1}(\Delta_{G,w})$ we have $(y,x)\in\tilde{\Delta}_{(F,x_o),w}\cap(y,\Gamma^{-1}(y))$. Conversely, if $(y,x)\in\tilde{\Delta}_{(F,x_o),w}\cap(y,\Gamma^{-1}(y))$, then, by (3.162), we have $y\in\Delta_{G,w}$, which proves (3.157). Hence, by corollary 3.11, we obtain (3.155), (3.156).

In the particular case (3.33), (3.34), where X is a linear space we have $x_o=0$ and (3.35), and hence, from corollary 3.11 we obtain

Corollary 3.13. If X is a linear space and $G=u^{-1}(\Omega)$, where $u: F\to X$, $\Omega\subset X$, $u(F)\cap\Omega\neq\emptyset$, and if $W\subset\overline{R}^X$, $\tilde{\Delta}_{(F,0),w}\subset F\times X$ $(w\in W)$, then for $p=p^n$ of (3.29), with Γ of (3.34), and for $\Delta_{G,w}\subset F$ $(w\in W)$ defined by

$$\Delta_{G,w}=\{y\in F\mid \tilde{\Delta}_{(F,0),w}\cap(y,u(y)-\Omega)\neq\emptyset\} \qquad (w\in W), \qquad (3.163)$$

we have (3.155), (3.156).

Similarly, observing that, for Γ of (3.34), condition (3.161) is satisfied (by (3.35)), from corollary 3.12 we obtain

Corollary 3.14. If X is a linear space and $G=u^{-1}(\Omega)$, where $u:F\to X$, $\Omega\subset X$, $u(F)\cap\Omega\neq\emptyset$, and if $W\subset\overline{R}^X$, $\Delta_{G,w}\subset F$ $(w\in W)$, then for $\tilde{\Delta}_{(F,0),w}\subset F\times X$ $(w\in W)$ defined by

$$\tilde{\Delta}_{(F,0),w}=(\Delta_{G,w},\ u(\Delta_{G,w})-\Omega) \qquad (w\in W), \qquad (3.164)$$

there holds (3.163), and hence, for $p=p^n$ of (3.29), with Γ of (3.34) we have (3.155), (3.156).

In the particular case when $X=F$ and $u=I_F$, from corollary 3.13 there follows

Corollary 3.15. If F is a linear space, $W\subset\overline{R}^F$ and $\tilde{\Delta}_{(F,0),w}\subset F\times X$ $(w\in W)$, then for $p=p^c$ of (3.38) and for $\Delta_{[illegible]}\subset F$ $(w\in W)$ defined by

$\Delta_{G,w}=\{y\in F|\tilde{\Delta}_{(F,0),w}\cap(y,y-G)\neq\emptyset\}$ $\qquad (w\in W), \qquad (3.165)$

we have (3.155), (3.156).

Remark 3.25. a) The parts (3.156) of corollaries 3.13 and 3.15, have been obtained in [32], remark 4.2.

b) For these particular cases, condition (3.159) reduces, respectively, to

$$\tilde{\Delta}_{(F,0),w}\cap(y,u(y)-\Omega)\neq\emptyset \qquad (y\in u^{-1}(\Omega), w\in W), \quad (3.166)$$

$$\tilde{\Delta}_{(F,0),w}\cap(g,g-G)\neq\emptyset \qquad (g\in G, \ w\in W). \quad (3.167)$$

Similarly, from corollary 3.14, there follows

Corollary 3.16. If F is a linear space, $W\subset\bar{R}^F$ and $\Delta_{G,w}\subset F$ $(w\in W)$, then for $\tilde{\Delta}_{(F,0),w}\subset F\times F$ $(w\in W)$ defined by

$$\tilde{\Delta}_{(F,0),w}=(\Delta_{G,w}, \Delta_{G,w}-G) \qquad (w\in W) \quad (3.168)$$

there holds (3.165), and hence, for $p=p^s$ of (3.38), we have (3.155), (3.156).

Remark 3.26. While the $(pW\tilde{\Delta})$-dual to $(P_{G,h})$ is obtained from the $(\tilde{W}\tilde{\Delta})$-dual to $(\tilde{P}_{(F,x_0),p})$, with $\tilde{W}$ of (3.6) (see remark 3.20 a)), corollary 3.16 shows that, when F is a linear space, the (unperturbational surrogate) $(W\Delta)$-dual is the particular case $p=p^c$ and $\tilde{\Delta}=(3.168)$, of the perturbational surrogate dual to $(P_{G,h})$, with the same W.

Finally, let us consider the particular case of DPS-dual (decomposed perturbational surrogate dual) problems to $(P_{G,h})$, in the sense of [32]. Namely, if the sets $\tilde{\Delta}_{(F,x_0),w}\subset F\times X$ $(w\in W)$ are of the form

$$\tilde{\Delta}_{(F,x_0),w}=F\times\Delta^{o}_{\{x_0\},w} \qquad (w\in W), \quad (3.169)$$

where $\Delta^{o}_{\{x_0\},w}\subset X$ $(w\in W)$, then problem $(Q^{G,h}_{pW\tilde{\Delta}})$ of (3.132) becomes the $(pW,F\times\Delta^{o})$-dual problem

$$(Q)=(Q^{G,h}_{pW,F\times\Delta^{o}}) \quad \beta=\beta^{G,h}_{pW,F\times\Delta^{o}}=\sup\lambda(W); \ \lambda(w)=\lambda^{G,h}_{pW,F\times\Delta^{o}}(w)=$$

$$=\inf_{y\in F, x\in\Delta^{o}_{\{x_0\},w}} p(y,x) \qquad (w\in W), \qquad (3.170)$$

and we shall call it a DPS-dual problem, to $(P_{G,h})$ (actually, in [32] we have called it a DPES-dual problem to $(P_{G,h})$, where the E stands for "extended", which we shall omit here, according to remark 3.19). Furthermore, for $\tilde{\Delta}_{(F,x_0),w}$ of (3.169), the $(pW\tilde{\Delta})$-Lagrangian (3.133) becomes the $(pW,F\times\Delta^{o})$-Lagrangian

$$L^{G,h}_{pW,F\times\Delta^{o}}(y,w)=\inf_{x\in\Delta^{o}_{\{x_0\},w}} p(y,x) \qquad (y\in F, \ w\in W), \qquad (3.171)$$

which coincides with the Lagrangian for $\{(P_{G,h}),(Q^{G,h}_{pW,F\times\Delta^{o}})\}$ defined (directly) in [32].

Remark 3.27. If (3.169) holds, then, for any $y\in F$, $x\in X$ and $w\in W$,

$$(y,x)\in\widetilde{\Delta}_{(F,x_0),w} \Longleftrightarrow x\in\Delta^{o}_{\{x_0\},w}\,. \qquad (3.172)$$

Consequently, the above results on general $(pW\widetilde{\Delta})$-dual problems and $(pW\widetilde{\Delta})$-Lagrangians remain also valid for this particular case, with the additional features that formulae (3.137), (3.140) and condition (3.138) become now, respectively,

$$\widetilde{L}^{(F,x_0),p}_{\widetilde{W}\widetilde{\Delta}}((y,x),(0,w))=p(y,x)\dot{+}\chi_{\Delta^{o}_{\{x_0\},w}}(x) \quad (y\in F, x\in X, w\in W), \quad (3.173)$$

$$\widetilde{\Psi}_{(F,x_0)}((y,x),(0,w))=\chi_{\Delta^{o}_{\{x_0\},w}}(x) \qquad (y\in F, x\in X, w\in W), \quad (3.174)$$

$$x_0\in\Delta^{o}_{\{x_0\},w} \qquad (w\in W). \qquad (3.175)$$

Also, if $p=p_{G,h}$ is h-separated, with π_G as in (3.20), then (3.145), (3.149) and condition (3.154) become now, respectively,

$$\gamma_G(y,w)=\inf_{x\in\Delta^{o}_{\{x_0\},w}}\pi_G(y,x) \qquad (y\in F, w\in W), \qquad (3.176)$$

$$L^{G,h}_{pW,F\times\Delta^{o}}(y,w)=h(y)\dot{+}\inf_{x\in\Delta^{o}_{\{x_0\},w}}\pi_G(y,x) \qquad (y\in F, w\in W), \qquad (3.177)$$

$$\inf_{x\in\Delta^{o}_{\{x_0\},w}}\pi_G(y,x)=\chi_{\Delta_{G,w}}(y) \qquad (y\in F, w\in W). \qquad (3.178)$$

Furthermore, formula (3.157) becomes now

$$\Delta_{G,w}=\{y\in F\mid \Delta^{o}_{\{x_0\},w}\cap\Gamma^{-1}(y)\neq\emptyset\}=\Gamma(\Delta^{o}_{\{x_0\},w}) \qquad (w\in W); \qquad (3.179)$$

for $\widetilde{\Delta}_{(F,x_0),w}$ of (3.169) and $\Delta_{G,w}$ of (3.179), corollary 3.11 has been given in [52], theorem 5.3 and remark 5.4. Similar results are also valid for (3.179) with Γ of (3.34), (3.37) and for corollaries 3.13, 3.15 (see [32], remark 5.5), of which we mention the last one (corresponding to corollary 3.8 above):

Corollary 3.17. If F is a linear space, $W\subset\overline{R}^{F}$ and $\Delta^{o}_{\{0\},w}\subset F$ $(w\in W)$, then for $p=p^{S}$ of (3.38) and for

$$\Delta_{G,w}=\{y\in F\mid \Delta^{o}_{\{0\},w}\cap(y-G)\neq\emptyset\}=G+\Delta^{o}_{\{0\},w} \qquad (w\in W), \qquad (3.180)$$

we have (3.155), (3.156). Hence, in particular, if $\Delta^{o}:2^{F}\times W\rightarrow 2^{F}$ is "translative" in the sense of [32], i.e.,

$$\Delta^{o}_{G,w}=G+\Delta^{o}_{\{0\},w} \qquad (G\in 2^{F},\ w\in W), \qquad (3.181)$$

then

$$L^{G,h}_{p^{S}W,F\times\Delta^{o}}(y,w)=L^{G,h}_{W\Delta^{o}}(y,w) \qquad (y\in F,\ w\in W), \qquad (3.182)$$

$$\lambda^{G,h}_{p^sW,F\times\Delta^o}(w)=\lambda^{G,h}_{W\Delta^o}(w) \qquad (w\in W). \qquad (3.183)$$

For $\tilde{\Delta}_{(F,x_o),w}$ of (3.169), there holds (3.143) and we have the following improvement of remark 3.21 c)(corresponding to theorem 3.2):

Theorem 3.5. For any $p:F\times X\to\bar{R}$ (satisfying (1.6)), $W\subset R^X$ and $\Delta^o_{\{x_o\},w}\subset X$ $(w\in W)$, we have

$$\lambda^{G,h}_{pW,F\times\Delta^o}(w)=\lambda^{\{x_o\},f}_{W\Delta^o}(w) \qquad (w\in W) \qquad (3.184)$$

(with $f=f_{G,h}:X\to\bar{R}$ of (3.64)), and hence

$$\{(P_{G,h}),(Q^{G,h}_{pW,F\times\Delta^o}\}\sim\{(P_{\{x_o\},f}),(Q^{\{x_o\},f}_{W\Delta^o})\}. \qquad (3.185)$$

Proof. By (3.170), (3.64) and (2.44), we have

$$\lambda^{G,h}_{pW,F\times\Delta^o}(w)=\inf f(\Delta^o_{\{x_o\},w})=\lambda^{\{x_o\},f}_{W\Delta^o}(w) \qquad (w\in W),$$

i.e., (3.184), which, together with (3.66), yields (3.185).

From corollary 3.17 and theorem 3.5, we obtain the following result (corresponding to corollary 3.9):

Corollary 3.18. Let X be a linear space and $W\subset\bar{R}^X$, and let $\Delta^o:2^X\times W\to 2^X$ be "locally translative" in the sense of [32], i.e.,

$$\Delta^o_{\{x\},w}=x+\Delta^o_{\{0\},w} \qquad (x\in X,\ w\in W). \qquad (3.186)$$

Then for any perturbation functional $p:F\times X\to\bar{R}$, satisfying (1.6) for some $x_o\in X$, and for the standard perturbation functional $p^s=p^s_{\{x_o\},f}:X\times X\to\bar{R}$ of (3.125), we have

$$\lambda^{G,h}_{pW,F\times\Delta^o}(w)=\lambda^{\{x_o\},f}_{p^sW,X\times\Delta^o}(w) \qquad (w\in W), \qquad (3.187)$$

$$\{(P_{G,h}),(Q^{G,h}_{pW,F\times\Delta^o}\}\sim\{(P_{\{x_o\},f}),(Q^{\{x_o\},f}_{p^sW,X\times\Delta^o})\}. \qquad (3.188)$$

Proof. By corollary 3.17, there holds

$$\lambda^{\{x_o\},f}_{p^sW,X\times\Delta^o}(w)=\lambda^{\{x_o\},f}_{W\Delta^o}(w) \qquad (w\in W), \qquad (3.189)$$

whence, by (3.184), we obtain (3.187), (3.188).

Remark 3.28. a) Corollary 3.18 is a generalization of [30], formula (2.3), concerning $W\subset X^\#$ and $\Delta^{\delta s}_{\{x\},w}$ of (2.55), which is locally translative for $W\subset X^\#$(see also remark 3.29 below).

b) The result corresponding to corollary 3.10, according to which if F is a linear space, $W\subset\bar{R}^F$ and $\Delta^o_{\{0\},w}\subset F$ $(w\in W)$, then for $f=f^s$ of (3.128) and $\Delta_{G,w}\subset F$ $(w\in W)$ of (3.180), we have $\lambda^{G,h}_{W\Delta}=\lambda^{\{0\},f^s}_{W\Delta^o}$ (and hence, if $\Delta^o:2^F\times W\to 2^F$ is translative, then $\lambda^{G,h}_{W\Delta^o}=\lambda^{\{0\},f^s}_{W\Delta^o}$), is now an immediate consequence of (2.44) and (3.128) (of course, it follows also from corollary 3.17 and theorem 3.5 for $p=p^s$ of (3.38)).

Finally, let us show some relations between decomposed perturbational surrogate dual and perturbational Lagrangian dual problems, with the same p, but a different $V=V_W\subset \overline{R}^X$. Corresponding to theorem 2.4, we have

Theorem 3.6. If (3.175) holds, then for

$$V=V_{W\triangle^o}=\{-\chi_{\triangle^o_{\{x_o\},w}} \mid w\in W\}\subset \overline{R}^X \tag{3.190}$$

we have

$$L^{G,h}_{pW,F\times\triangle^o}(y,w)=L^{G,h}_{pV}(y,-\chi_{\triangle^o_{\{x_o\},w}}) \qquad (y\in F,\ w\in W), \tag{3.191}$$

$$\lambda^{G,h}_{pW,F\times\triangle^o}(w)=\lambda^{G,h}_{pV}(-\chi_{\triangle^o_{\{x_o\},w}}) \qquad (w\in W), \tag{3.192}$$

$$(Q^{G,h}_{pW,F\times\triangle^o})\sim(Q^{G,h}_{pV}). \tag{3.193}$$

Proof. By (3.175), we have $-\chi_{\triangle^o_{\{x_o\},w}}(x_o)=0$ $(w\in W)$, whence, by (3.171) and (3.51),

$$L^{G,h}_{pW,F\times\triangle^o}(y,w)=\inf_{x\in X}\{p(y,x)\dot{+}\chi_{\triangle^o_{\{x_o\},w}}(x)\}=$$

$$=L^{G,h}_{pV}(y,-\chi_{\triangle^o_{\{x_o\},w}}) \qquad (y\in F,\ w\in W),$$

and hence, by (1.11), we obtain (3.192) and (3.193).

Remark 3.29. a) Theorem 3.6 shows that if (3.175) holds, then $(pW,F\times\triangle^o)$-duality is a "particular case" of (pV)-duality, for a suitable modification V of W. If (3.175) does not hold for some $w\in W$, then $-\chi_{\triangle^o_{\{x_o\},w}}(x_o)=-\infty$, whence $L^{G,h}_{pV}(y,-\chi_{\triangle^o_{\{x_o\},w}})\equiv-\infty$.

b) By (3.192) and by (3.71) applied to W replaced by V of (3.190), we have

$$\beta^{G,h}_{pW,F\times\triangle^o}=\sup_{v\in V}\{v(x_o)\dot{+}-f^{c(n)}(v)\}=f^{c(n)c(n)'}(x_o), \tag{3.194}$$

where $f^{c(n)}$ and $f^{c(n)c(n)'}$ are taken with respect to V; hence, by (3.72),

$$\beta^{G,h}_{pW,F\times\triangle^o}=f_{\mathcal{H}(V+R)}(x_o)=\sup_{\substack{v\in V,\ d\in R\\ -\chi_{\triangle^o_{\{x_o\},w}}+d\leq f}} f(x_o), \tag{3.195}$$

the "(V+R)-convex hull of f at x_o", with V of (3.190). Also, by [33], theorem 3.5, or theorem 4.2 (or formula (5.10)),

$$\beta^{G,h}_{pW,F\times\triangle^o}=f_{Q(\mathcal{M})}(x_o), \tag{3.196}$$

the "$\mathcal{M}$-quasi-convex hull of f at x_o" (in the sense of [33]), where

$$\mathcal{M}=\{X\setminus\Delta^{o}_{\{x_o\},w}|w\in W\}\subset 2^{X}; \qquad (3.197)$$

consequently, $\alpha=\beta^{G,h}_{pW,F\times\Delta^o}$ if and only if $f(x_o)=f_{Q(\mathcal{M})}(x_o)$ (i.e., if and only if f is "$\mathcal{M}$-quasi-convex at x_o" [35]). Therefore, the DPS-dual problem $(Q^{G,h}_{pW,F\times\Delta^o})$ may be also called the "$\mathcal{M}$-quasi-convex dual" problem to $(P_{G,h})$, with $\mathcal{M}$ of (3.197); in the particular case when $F=R^n$, $X=R^m$ and $\Delta^o=\Delta^{\delta s}$ of (2.56), $(Q^{G,h}_{pW,F\times\Delta^o})$ and $L^{G,h}_{pW,F\times\Delta^o}$ become, respectively, the "quasi-convex dual" problem of [2] (encompassing [5],[12], [8]) and the corresponding Lagrangian of [13],[14],[28]). For further particular cases (e.g, the "pseudo-dual" [25] and the "semi-dual" [29], [30] problems to $(P_{G,h})$) and some related results, see [31]-[33].

c) In the opposite direction, we have only the following remark: If in (2.55) or (2.56) we replace F,G and w(g) ($g\in G$) by X, $\{x_o\}$ and $w(x_o)$ respectively (see remark 3.3), we obtain

$$\Delta^{\gamma s}_{\{x_o\},w}=\Delta^{\delta s}_{\{x_o\},w}=\{x\in X|\, w(x)\geqslant w(x_o)\}=\{x\in X|\, 0\geqslant -w(x)\dot{+}w(x_o)\} \qquad (w\in W).\ (3.198)$$

Then $x_o\in\Delta^{\delta s}_{\{x_o\},w}$ $(w\in W)$, whence, by (1.6),

$$h(y)\dot{+}\chi_G(y)=p(y,x_o)\geqslant \inf_{x\in\Delta^{\delta s}_{\{x_o\},w}} p(y,x)=L^{G,h}_{pW,F\times\Delta^{\delta s}}(y,w)\geqslant$$

$$\geqslant \inf_{x\in\Delta^{\delta s}_{\{x_o\},w}} \{p(y,x)\dot{+}\{-w(x)\dot{+}w(x_o)\}\}\geqslant L^{G,h}_{pW}(y,w) \quad (y\in F, w\in W);\ (3.199)$$

hence, by $\alpha=\inf_{y\in F}\{h(y)\dot{+}\chi_G(y)\}$ and (1.11),

$$\alpha\geqslant\lambda^{G,h}_{pW,F\times\Delta^{\delta s}}(w)\geqslant\lambda^{G,h}_{pW}(w) \qquad (w\in W),\ (3.200)$$

and thus

$$\alpha\geqslant\beta^{G,h}_{pW,F\times\Delta^{\delta s}}\geqslant\beta^{G,h}_{pW}. \qquad (3.201)$$

Consequently, <u>if</u> $\alpha=\beta^{G,h}_{pW}$, <u>then</u> $\alpha=\beta^{G,h}_{pW,F\times\Delta^{\delta s}}$.

d) By c), problem $(Q^{G,h}_{pW,F\times\Delta^o})$ is the perturbational dual problem corresponding, via the scheme of definition 3.3 and remark 3.3, to both $(Q^{G,h}_{W\Delta^{\gamma s}})$ and $(Q^{G,h}_{W\Delta^{\delta s}})$ of remark 2.9 d), which are of rather different character (see remark 3.18 above).

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