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# THE COMMUTANT MODULO THE SET OF COMPACT OPERATORS OF A von NEUMANN ALGEBRA

by

Sorin POPA

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by

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## INTRODUCTION

Let $\mathcal{B}(\mathcal{H})$ be the algebra of all linear bounded operators acting on the Hilbert space $\mathcal{H}$ and $\mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ the ideal of compact operators on $\mathcal{H}$. Let $M \subset \mathcal{B}(\mathcal{H})$ be a norm closed $*$-subalgebra. Then the operators $T \in \mathcal{B}(\mathcal{H})$ that commute with M modulo the set of compact operators have an important significance in the study of the algebra M. A first problem to be settled about such operators is to decide whether or not they are compact perturbations of some operators in M′, the commutant of M in $\mathcal{B}(\mathcal{H})$. If T is the projection onto an infinite dimensional Hilbert subspace then a typical obstruction for it to be in $M' + \mathcal{K}(\mathcal{H})$ is to exist a unitary element in M whose restriction to the corresponding Hilbert subspace has nontrivial index. However, if we require M to be closed in $\mathcal{B}(\mathcal{H})$ in the weak operator topology, in other words if M is a von Neumann algebra, then such nontrivial index cannot appear. This fact was clarified by Johnson and Parrott who proved in [4] that actually any operator commuting modulo the compacts with an abelian von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$

has a compact perturbation that commutes with M. From this they derived the similar result for properly infinite von Neumann algebras and for finite von Neumann algebras with diffuse center, thus reducing the problem to the case when M is a type $II_1$ factor.

In this paper we settle this remaining case in the affirmative and so end up the proof of the following general result:

<u>THEOREM A</u>. If M is a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and T is an operator in $\mathcal{B}(\mathcal{H})$ that commutes modulo the set of compact operators with all the elements in M, then T is a compact perturbation of an operator that commutes with M.

In fact in their paper Johnson and Parrott study a more general problem: they consider derivations of the von Neumann algebra M into the compacts, i.e. linear applications $\delta: M \longrightarrow \mathcal{K}(\mathcal{H})$ satisfying $\delta(xy)=\delta(x)y+x\delta(y)$ for $x,y \in M$, and prove that if M has not type $II_1$ factors as direct summands then $\delta = \mathrm{ad} K$ for some $K \in \mathcal{K}(\mathcal{H})$. Thus, for these algebras the first cohomology group $H^1(M, \mathcal{K}(\mathcal{H}))$ vanishes. In particular, if $\delta = \mathrm{ad}\, T$ for an operator $T \in \mathcal{B}(\mathcal{H})$, this result implies that for some $K \in \mathcal{K}(\mathcal{H})$, $\mathrm{ad}\, T = \mathrm{ad}\, K$ on M and hence $T-K \in M'$. Thus theorem A is an immediate consequence of $H^1(M, \mathcal{K}(\mathcal{H}))=0$.

We shall actually study this derivation problem and the preceding theorem will be a consequence of the following:

<u>THEOREM B</u>. Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $\delta: M \longrightarrow \mathcal{K}(\mathcal{H})$ a derivation of M into the set of compact operators. Then there exists $K \in \mathcal{K}(\mathcal{H})$ such that $\delta = \mathrm{ad}\, K$.

The proof of the theorems will depnd heavily on our previous result [8], which shows that a derivation of a finite type $II_1$ factor M into the compacts is automatically continuous from the unit ball of M with the strong operator topology into $\mathcal{K}(\mathcal{H})$ with the norm topology. For the sake of competeness we therefore included an appendix with a proof of this result. The rest of the paper is devided into three sections. In the first one we prove the theorem up to a technical lemma. This lemma is proved in section II. In section III we make some comments and mention some consequences.

## 1. PROOF OF THE MAIN RESULTS

To prove the theorems we need some lemmas. In order to justify the lemmas we alternate them with a sketch of the proof of the theorems. Since we actually prove theorems A and B for type $II_1$ factors, we need to introduce first some notations regarding finite von Neumann algebras.

So, unless we made other specifications, we denote by M a generic finite von Neumann algebra with a fixed faithful normal trace $\tau$, $\tau(1)=1$. We let $\|x\|_2=\tau(x^*x)^{1/2}$ denote the Hilbert norm implemented on M by $\tau$ and $L^2(M,\tau)$ be the completion of M in this norm. When we regard the unit of M as a vector in $L^2(M,\tau)$ we denote it by $\xi_0$.

Note that M acts on $L^2(M,\tau)$ by left and right multiplication. We identify M with the left action on $L^2(M,\tau)$ so that $M'$, the commutant of M in $\mathcal{B}(L^2(M,\tau))$, is the set of operators of right multiplication with elements of M.

If $B\subset M$ is a von Neumann subalgebra (allways assumed to contain the unit of M) then $E_B$ denotes the unique normal trace preserving conditional expectation of M onto B ([10]). Then $E_B$ is just the restriction to M of the orthogonal projection of $L^2(M,\tau)$ onto the closure of $B\xi_0$ in $L^2(M,\tau)$.

Let now $\delta$ be a derivation of M into the ideal of compact operators on the Hilbert space $L^2(M,\tau)$, i.e. $\delta : M \longrightarrow \mathcal{K}(L^2(M,\tau))$ is a linear map with the property $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x,y \in M$. Let $A \subset M$ be a maximal abelian $*$-subalgebra of M and assume $\delta$ vanishes on A. In case M is a type $II_1$ factor it follows that A has no minimal projections. Thus if $K \in \mathcal{K}$ would be a compact operator such that $[K,x] = \delta(x)$ for $x \in M$, then in particular $[K,A]=0$. But a compact operator K cannot commute with a diffuse abelian von Neumann algebra unless $K=0$. Thus we have to prove that from $\delta_{|A}=0$ if actually follows that $\delta \equiv 0$.

Assume $\delta \neq 0$. Since M is spanned linearly by its unitary elements, there exists a unitary element $v \in M$ such that $\delta(v) \neq 0$.

Now we want to construct with the help of A and v some other abelian subalgebras of M on which $\delta$ behaves as bad as possible. The key technical part of this construction is contained in the next lemma. Its proof is quite elaborate but has little to do with the rest of the proof of the theorem. This is why we prove it separately, in the next section.

1.1. LEMMA. If $A \subset M$ is a diffuse abelian von Neumann subalgebra and $v \in M$ is a unitary element then there exists a sequence of unitary elements $(u_n)_n$ in A such that $((u_n v)^k)_n$ tends to zero, in the weak operator topology, for all $k \neq 0$.

The next lemma shows why Lemma 1.1 is important for us.

1.2. LEMMA. Let $\delta : M \longrightarrow \mathcal{K}(L^2(M,\tau))$ be a derivation, v and $(u_n)_n$ some unitary elements of M such that $\delta(u_n)=0$ for all n. If $((u_n v)^k)_n$ tends to zero in the weak operator topology for all $k \neq 0$ then $\langle \delta((u_n v)^k)\, \xi_o, (u_n v)^p\, \xi_o \rangle$ tends to $\langle \delta(v)\, \xi_o, v\, \xi_o \rangle$ for $k=p>0$ and to zero for all other k and p.

PROOF. Since $\delta(u_n)=0$, we have

$$\delta((u_n v)^k)=\sum_{s=0}^{k-1} (u_n v)^s u_n\, \delta(v)\, (u_n v)^{k-s-1}$$

and

$$\delta((v^{-1}u_n^{-1})^k)=\sum_{s=0}^{k-1} (v^{-1}u_n^{-1})^s\, \delta(v^{-1})\, u_n^{-1}\, (v^{-1}u_n^{-1})^{k-s-1}$$

for any positive integer k.

Since $\delta(v)$ and $\delta(v^{-1})$ are compact operators and $(u_n v)^{k-s-1}$, $u_n^{-1}(v^{-1}u_n^{-1})^{k-s-1}$ tends to zero in the weak operator topology it follows (see e.g. [2], chap. 5) that $\delta(v)(u_n v)^{k-s-1}\, \xi_o, k-s-1>0$, and $\delta(v^{-1})u_n^{-1}(v^{-1}u_n^{-1})^{k-s-1}\, \xi_o$, $k-s-1 \geq 0$, tend to zero in norm. Thus, since $u_n$, v are uniformly bounded in norm, we get:

(i) $\| \delta((u_n v)^k)\, \xi_o - (u_n v)^{k-1} u_n\, \delta(v)\, \xi_o \| \xrightarrow[n]{} 0$

(ii) $\| \delta((v^{-1}u_n^{-1})^k)\, \xi_o \| \xrightarrow[n]{} 0$

for any positive integer k. If $k<0$ then the statement follows now easily, by (ii).

If $k>0$ then by (i) we have for any p

$$|\langle \delta((u_n v)^k)\, \xi_o, (u_n v)^p\, \xi_o \rangle - \langle \delta(v)\, \xi_o, u_n^{-1}(u_n v)^{p-k+1}\, \xi_o \rangle| \xrightarrow[n]{} 0$$

But if $p-k+1 \neq 1$ then $u_n^{-1}(u_n v)^{p-k+1}$ tends to zero in the weak operator topology, so that $\delta(v)^* u_n^{-1}(u_n v)^{p-k+1} \xi_o$ tends to zero in norm.

The case $k=0$ is trivial, because then $(u_n v)^k = 1$ and $\delta(1)=0$. Q.E.D.

It turns out that the preceding lemma is a key observation in the proof of the theorem. To see this, let's suppose for simplicity that $\langle \delta(v)\, \xi_o, v\, \xi_o \rangle = 1$. If we assume further that $\tau((u_n v)^k)=0$, $n \geq 1$, $k \neq 0$, and denote $A_n = \{u_n v\}''$ then the derivations $\delta|_{A_n}$, restricted to the subspace $\overline{A_n \xi_o}$, are spatially isomorphic to a sequence of derivations $\delta_n : L^\infty(\mathbb{T}, \mu) \longrightarrow \mathcal{K}(L^2(\mathbb{T}, \mu))$, by sending $(u_n v)^k \xi_o$ to $z^k$ and the operator $u_n v$ to the multiplication by z operator. With these notations lemma 1.2 says that $\delta_n$ tend to behave as the derivation ad P implemented by the projection onto the Hardy space $H^2(\mathbb{T}, \mu)$. But by 4.1 it follows that $\delta_n$ are uniformly so-normic continuous, so that ad P would be so-normic continuous as well. This is easily seen to be false.

Of course the unitaries $u_n v$ may not generate abelian von Neumann algebras in the nice way we need. But we can slightly modify them to do so.

1.3. LEMMA. Suppose the finite von Neumann algebra M has no atoms and let $(w_n)_n$ be a sequence of unitary elements in M such that $\tau(w_n^k) \xrightarrow[n]{} 0$ for all $k \neq 0$. Then there exists a sequence of unitary elements $(v_n)_n$ in M such that $\tau(v_n^k)=0$, $k \neq 0$, and $\| w_n - v_n \| \longrightarrow 0$.

<u>PROOF</u>. Since M has no atoms each $w_n$ is contained in some diffuse abelian von Neumann subalgebra $A_n \subset M$ with separable predual. Then $(A_n, \tau|_{A_n})$ can be identified by some measure preserving isomorphism $\varphi_n$ with $L^\infty(\mathbb{T}, \mu)$, where $\mu$ is the normalized Lebesgue measure on the thorus $\mathbb{T}$. Moreover $\varphi_n$ can be chosen so that $\varphi_n(w_n) = f_n$, where $f_n(e^{2\pi it}) = e^{2\pi i h_n(t)}$ for some nondicreasing function $h_n: [0,1] \to [0,1]$. By Helly's selection principle there exists a subsequence $(h_{k_n})_n$ tending everywhere to some nondicreasing function $h: [0,1] \to [0,1]$. Thus, if $f(e^{2\pi it}) = e^{2\pi i h(t)}$ then $f_{k_n}$ tends everywhere to f so that by Lebesgue's theorem $\int f_{k_n}^p d\mu \to \int f^p d\mu$ for all p, which by the hypothesis implies $\int f^p d\mu = 0$, $p \neq 0$. Thus $\int q(f) d\mu = \int q d\mu$ for Laureant polynomials q. Let $z = e^{2\pi it}$ and denote $g_z(e^{2\pi is}) = \begin{cases} 1 & \text{if } 0 \leq s < t \\ 0 & \text{if } t \leq s \leq 1 \end{cases}$. Then $g_z$ is a pointwise limit of continuous functions of norm one on the thorus. Thus by the Stone Weierstrass theorem $g_z$ is a pointwise limit of polynomials q with $\|q\|_\infty \leq 1$. It follows by Lebesgue's theorem and the preceding equality that $\int g_z \circ f d\mu = \int g_z d\mu$, which means that $\int_{h(s) \leq t} d\lambda(s) = t$, where $\lambda$ is the Lebesgue measure on $[0,1]$. This implies that $h(t) = t$ and hence $f(z) = z$ is the identity function on $\mathbb{T}$.

Now, since $h_{k_n}$ are monotone and converge everywhere to a continuous function it follows that $h_{k_n}$ converge uniformly to h, i.e. $\|h_{k_n} - h\|_\infty \to 0$, so that $\|f_{k_n} - f\|_\infty \to 0$. But f was taken to be any limit point of $f_{k_n}$ (in the everywhere convergence) and shown to be equal to the identity. Thus, f being the unique limit point, $\|f_n - f\|_\infty \to 0$.

We can take $v_n = \varphi_n^{-1}(f)$. Since $\int f^p = 0$, $\tau(v_n^p) = 0$ for all $p \neq 0$. Moreover $\|w_n - v_n\| = \|\varphi_n(w_n) - \varphi_n(v_n)\| = \|f - f_n\| \to 0$.

Q.E.D.

We are now ready to prove the theorems.

THEOREM B. Let $\mathcal{H}$ be a Hilbert space and $M\subset B(\mathcal{H})$ a von Neumann algebra. If $\delta : M \to \mathcal{K}(\mathcal{H})$ is a derivation of M into the ideal of compact operators then there exists a compact operator $K\in\mathcal{K}(\mathcal{H})$ such that $\delta = \mathrm{ad}\, K$.

PROOF. Note from the beginning that by 1.2 in [4], $\delta$ is automatically norm continuous. By [4] we only need to prove the statement in case M is a type $II_1$ factor. Then M has a unique normal faithful trace $\tau$ with $\tau(1)=1$, and we can use the notations introduced at the beginning of this section. We consider first the case when M acts standardly on $\mathcal{H}$, so that $\mathcal{H} = L^2(M,\tau)$ and M acts on it by left multiplication.

Let $A\subset M$ be a maximal abelian von Neumann subalgebra. Since M is of type $II_1$ it has no minimal projections, so that A is a diffuse abelian algebra. By [4] there exists a compact operator K such that $\delta(a)=[K,a]$ for all $a\in A$. Thus, by taking if necessary $\delta - \mathrm{ad}\, K$ instead of $\delta$, we may assume $\delta_{|A}=0$. We show that in fact $\delta=0$ on all M. To do this we proceed by contradiction.

If $\delta$ is not identically zero then there must be a unitary element $v\in M$ for which $\delta(v)\neq 0$. Since $\mathcal{H}=L^2(M,\tau)$ we have $\overline{M\xi_o} = \overline{\xi_o M} = \mathcal{H}$ so that there exists $x_1, x_2\in M$ such that $\langle \delta(v)(\xi_o x_1), v\xi_o x_2\rangle = 1$. Thus, if we denote by $x_1', x_2'$ the corresponding operators of right multiplication by $x_1$ and respectively $x_2$, then $\langle x_2'^{*}\delta(v)x_1'(\xi_o), v\xi_o\rangle = 1$. Note that $M\ni x \longmapsto x_2'^{*}\delta(x)x_1'$ is still a derivation of M into the compacts, vanishing on A. Thus, by replacing if necessary $\delta$ with $x_2'^{*}\delta(\ )x_1'$ we may assume, in addition to the preceding hypothesis, that $\langle \delta(v)\xi_o, v\xi_o\rangle = 1$.

By Lemma 1.1 there exist unitary elements $(u_n)_n$ such that $((u_n v)^k)_n$ tends to zero in the weak operator topology, for all $k \neq 0$. By 1.2, $\langle \delta((u_n v)^k)\xi_o, (u_n v)^p \xi_o \rangle$ tend to 1 for $k=p>0$ and to 0 for all other integers k and p. By 1.3 there exist unitary elements $v_n \in M$ such that $\| v_n - u_n v \| \longrightarrow 0$ and $\tau(v_n^k)=0$, for $k \neq 0$. Since $\delta$ is normic continuous this implies that $\langle \delta(v_n^k)\xi_o, v_n^p \xi_o \rangle$ tend to 1 for $k=p>0$ and otherwise to 0.

Let $A_n = \{v_n\}''$ and $p_n$ be the orthogonal projection of $\mathcal{H} = L^2(M,\tau)$ onto $\overline{A_n \xi_o}$. Then $p_n$ commutes with $A_n$. Therefore all the applications $A_n \ni a \longmapsto p_n \delta(a) p_n$ are derivations into the compacts and, since $\| p_n \delta(a) p_n \| \leq \| \delta(a) \|$, they are by 4.1 uniformly so-normic continuous on the unit balls of $A_n$.

Let $L^\infty(\mathbb{T},\mu)$ act on $L^2(\mathbb{T},\mu)$ by left multiplication ($\mu$ is hte normalized Lebesgue measure on the thorus $\mathbb{T}$). Denote by $\psi_n$ the measure preserving isomorphism of $L^\infty(\mathbb{T},\mu)$ onto $(A_n, \tau|_{A_n})$ defined by $\psi_n(M_{z^k}) = v_n^k$, where $M_f$ is the operator of left multiplication by $f \in L^\infty(\mathbb{T},\mu)$, and by $U_n : L^2(\mathbb{T},\mu) \longrightarrow L^2(M,\tau)$ the corresponding isometry at the vector space level, i.e., $z^k$ being the usual orthonormal basis in $L^2(\mathbb{T},\mu)$, $U_n(z^k) = v_n^k \xi_o$. Thus $U_n U_n^* = p_n$. Then define $\delta_n : L^\infty(\mathbb{T},\mu) \to \mathcal{B}(L^2(\mathbb{T},\mu))$ by $\delta_n(M_f) = U_n^* \delta(\psi_n^{-1}(M_f)) U_n$. Since $U_n^* \psi_n^{-1}(M_f) U_n = M_f$ for all $f \in L^\infty(\mathbb{T},\mu)$, we have

$$\delta_n(M_{fg}) = U_n^* p_n \delta(\psi_n^{-1}(M_f)\psi_n^{-1}(M_g)) p_n U_n = U_n^* \delta(\psi_n^{-1}(M_f)) U_n U_n^* \cdot$$

$$\cdot \psi_n^{-1}(M_g) U_n + U_n^* \psi_n^{-1}(M_f) U_n U_n^* \delta(\psi_n^{-1}(M_g)) U_n = \delta_n(M_f) M_g + M_f \delta_n(M_g).$$

Thus $\delta_n$ are derivations and since $U_n^* \delta(x) U_n$ are compact operators, $\delta_n$ take values into the compact operators on $L^2(\mathbb{T},\mu)$. Moreover, since $\|\psi_n(M_f)\|_2 = \|M_f\|_2$ and $\| U_n^* T U_n \| \leq \| T \|$, for $f \in L^\infty(\mathbb{T},\mu)$ and $T \in \mathcal{B}(L^2(M,\tau))$, it follows that $\|\delta_n\| \leq \|\delta\|$ and that $\delta_n$ are uniformly continuous from the unit ball of $L^\infty(\mathbb{T},\mu)$ with the norm $\| \ \|_2$ into $\mathcal{K}(L^2(\mathbb{T},\mu))$ with the usual norm . By the definition, $\delta_n$ also satisfy: $\langle \delta_n(M_z k)1, z^p \rangle$ tend to 1 for $k=p>0$ and to 0 otherwise.

Let $\omega$ be a free ultrafilter on $\mathbb{N}$. For each $f \in L^\infty(\mathbb{T},\mu)$ denote $\Delta(M_f) = \text{w-}\lim_{n\to\omega} \delta_n(M_f)$. Then $\Delta$ is a linear map and, since $\Delta(M_{fg}) = \lim_{n\to\omega} \delta_n(M_{fg}) = \lim_{n\to\omega} (\delta_n(M_f)M_g + M_f\delta_n(M_g)) = \lim_{n\to\omega} \delta_n(M_f)M_g + \lim_{n\to\omega} M_f\delta_n(M_g) = \Delta(M_f)M_g + M_f\Delta(M_g)$, $\Delta$ is a derivation of $L^\infty(\mathbb{T},\mu)$ into $\mathcal{B}(L^2(\mathbb{T},\mu))$. Since $\Delta(M_f)$ is a weak limit point of $\delta_n(M_f)$, by the inferior semicontinuity of the norm in the weak operator topology, it follows that $\|\Delta(M_f)\| \leq \sup_n \|\delta_n(M_f)\|$. Thus $\|\Delta\| \leq \sup_n \|\delta_n\| \leq \|\delta\|$ and $\Delta$ is so-normic continuous on the unit ball of $L^\infty(\mathbb{T},\mu)$. Moreover $\langle \Delta(M_z k)1, z^p \rangle$ is equal to 1 for $k=p>0$ and to 0 otherwise.

Let now P be the orthogonal projection of $L^2(\mathbb{T},\mu)$ onto the Hardy subspace $H^2(\mathbb{T},\mu) = \overline{\text{span}}\{z^k \mid k>0\}$. Then an easy computation shows that

$$\langle (PM_z - M_zP)(z^k), z^p \rangle = \langle \Delta(M_z)z^k, z^p \rangle$$

and hence $\Delta$ coincides with ad P on Laureant polynomials. But, since $\Delta$ is so-normic continuous on bounded sets, this and the Kapalnsky density theorem imply that $\Delta$ = ad P on all $L^\infty(\mathbb{T},\mu)$. Thus ad P is so-normic continuous on the unit ball of $L^\infty(\mathbb{T},\mu)$. This fact is well known to be false (e.g., see [2], chap.7). But let's proceed with another argument, more in the spirit of this paper. Note that $\Delta(M_z) = [P, M_z]$ is a rank one operator so that (ad P)$(M_q)$ is a compact operator for any Laureant polynomial $q$. Hence, by the Kaplansky density theorem and the so-normic continuity of ad P on the unit ball of $L^\infty(\mathbb{T},\mu)$, it follows that ad P takes values into the compact operators. Thus by [4] there exists a compact operator K such that $[P-K, L^\infty(\mathbb{T},\mu)] = 0$. Since $L^\infty(\mathbb{T},\mu)$ is maximal abelian in $\mathcal{B}(L^2(\mathbb{T},\mu))$, it follows that $P-K=M_\varphi$ for some $\varphi \in L^\infty(\mathbb{T},\mu)$. Thus $\langle (P-K)(z^k),z^k\rangle = \int(\varphi(z)z^k)z^{-k}dz = \int\varphi(z)dz$, for all k. But $\lim_{|k|\to\infty} \langle K(z^k),z^k\rangle = 0$, while $\langle P(z^k),z^k\rangle$ is 1 for $k>0$ and 0 for $k \leq 0$. This gives the desired contradiction.

With this we finished the proof of the case when the type $II_1$ factor M acts standardly on $\mathcal{H}$.

In the general case M′ is anyway semifinite and by [6] there exists a unique semifinite trace $\tau'$ on M′ such that for any projection $e' \in M'$, $\tau'(e')$ is the coupling constant of M in $\mathcal{B}(e'\mathcal{H})$.

Let $A \subset M$ be a maximal abelian $*$-subalgebra and assume $\delta|_A = 0$.

If $v'_1, v'_2 \in M'$ are partial isometries with $v'^*_1 v'_1 = v'_2 v'^*_2 = e'$ and $\tau'(e') \leq 1$ then $M \ni x \longmapsto v'_2\delta(x)v'_1$ can be viewed as a derivation of M acting in its standard form. By the first part of the proof it follows that $v'_2\delta(x)v'_1 = 0$ for any $x \in M$. Since $Mv'_2$ and $v'_1M$, where $v'_1, v'_2$ satisfy the above conditions, form a total subset of M, it follows that $\delta(x)=0$, $x \in M$.

Q.E.D.

<u>THEOREM A</u>. Let M be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$. If T is an operator on $\mathcal{H}$ such that $Tx-xT \in \mathcal{K}(\mathcal{H})$ for all $x \in M$ then there exists $T' \in M'$ such that $T-T' \in \mathcal{K}(\mathcal{H})$.

<u>PROOF</u>. Let $\delta(x) = [T,x]$, $x \in M$. Then $\delta$ is a derivation and $\delta(M) \subset \mathcal{K}(\mathcal{H})$ so that by the preceding theorem there exists $K \in \mathcal{K}(\mathcal{H})$ such that $\delta(x) = [K,x]$, $x \in M$. Hence $[T-K,x]=0$ for all $x \in M$ so that $T' = T-K \in M'$.

Q.E.D.

## 2. PROOF OF THE TECHNICAL RESULT

We prove a slightly more general result than 1.1.

2.1. LEMMA. Let M be a finite von Neumann algebra and $A \subset M$ a diffuse abelian von Neumann subalgebra. Let $(v_n)_n$ be a sequence of unitary elements in M. There exists a sequence $(u_n)_n$ of unitary elements in A such that $(\prod_{i=1}^{k}(u_n v_i))_n$ tends $\sigma$-weakly to zero, for any $k \geq 1$.

PROOF. Since there exists a partition of the unity $(p_i)_i$ with projections in the center of M such that $Mp_i$ is countably decomposable for each i, it follows that it suffices to prove the statement for M with a normal faithful trace $\tau$, $\tau(1)=1$. We may further assume M has separable predual. Indeed, instead of A we can take a diffuse countably generated $*$-subalgebra of it. If M is countable decomposable this subalgebra will generate together with the sequence $(v_n)_n$ an algebra with separable predual.

We use from now on the notations introduced at the begining of section 1.

Let $\varepsilon>0$, $p \in \mathbb{N}$ and $\mathcal{F}$ be a finite set of elements in M. We denote by $\mathcal{W}$ the set of partial isometries w in A such that $|\tau(\prod_{i=1}^{k}(wy_i))| < \varepsilon\tau(w^*w)$ for any $1 \leq k \leq p$ and any k-tuple of elements $y_1,\ldots,y_k \in \mathcal{F}$. Endow $\mathcal{W}$ with the usual order: $w_o \leq w_1$ if $w_o$ is a restriction of $w_1$, i.e. $w_o=w_1 w_o^* w_o$. It is easily seen that $(\mathcal{W}, \leq)$ is inductively ordered. Let u be a maximal element of it. We want to show that u is a unitary element.

Suppose $e=1-u^*u\neq 0$. Since A is abelian and $u \in A$, $e \in A$ and eAe is a diffuse subalgebra of eMe. We denote the corresponding reduced algebras $A_e \subset M_e$ by $A_o \subset M_o$. Let $\mathfrak{F}_o = \{ey_1 \prod_{i=2}^{k} (uy_i)e \mid 1 \le k \le p, y_i \in \mathfrak{F}\}$. $\mathfrak{F}_o$ is clearly a finite set and we denote by N its cardinality.

Since $A_o$ has separable predual, there exists an increasing sequence of finite dimensional $*$-subalgebras $(A_n)_{n\ge 1}$ of $A_o$, $1 \in A_n$ for all n, such that $\overline{\bigcup_{n\ge 1} A_n}^w = A_o$. Then

$\| E_{A_n' \cap M_o}(y) - E_{A_o' \cap M_o}(y) \|_2 \longrightarrow 0$ for any $y \in M_o$ (see for instance 1.2 in [7]). (Here and in the sequel $E_B$, for $B \subset M_o$ with the same unit as $M_o$, denotes the conditional expectation of $M_o$ onto B that preserves the induced trace $\tau_o$ on $M_o$, $\tau_o(x) = \tau(e)^{-1}\tau(x)$, $x \in eMe = M_o$. However the elements $E_B(x)$ are allways regarded as elements in $M \supset eMe = M_o$ and the norm $\| \ \|_2$ refferes as usual to the $\tau$-norm in M). Thus if $\delta > 0$ then there exists some n for which $\sum_{y \in \mathfrak{F}_o} \| E_{A_n' \cap M_o}(y) - E_{A_o' \cap M_o}(y) \|_2^2 < \delta^2 \|e\|_2^2$. If $e_1, \ldots, e_m$ are the minimal projections of $A_n$ then $\sum e_i = e$ and $E_{A_n' \cap M_o}(y) = \sum e_i y e_i$. Moreover, since $e_i$ commute with $E_{A_o' \cap M_o}(y)$, we have $E_{A_o' \cap M_o}(y) = \sum_i e_i E_{A_o' \cap M_o}(y) e_i$. Thus, using that $e_i(y - E_{A_o' \cap M_o}(y))e_i$ are mutually orthogonal, $1 \le i \le m$, we get

$$\sum_{y \in \mathfrak{F}_o} \sum_i \| e_i y e_i - e_i E_{A_o' \cap M_o}(y) e_i \|_2^2 = \sum_{y \in \mathfrak{F}_o} \| \sum_i e_i (y - E_{A_o' \cap M_o}(y)) e_i \|_2^2 <$$

$$< \delta^2 \sum_i \| e_i \|_2^2 .$$

It follows that for some $1 \le i \le m$

$$\sum_{y \in \mathcal{F}_0} \| e_i y e_i - e_i E_{A_0' \cap M_0}(y) e_i \|_2^2 < \delta^2 \| e_i \|_2^2$$

so that by the discrete version of the Cauchy-Schwartz inequality if we denote this $e_i$ by $e_0$ we have:

$$(1) \qquad \sum_{y \in \mathcal{F}_0} \| e_0 y e_0 - e_0 E_{A_0' \cap M_0}(y) e_0 \|_2 < N^{1/2} \delta \| e_0 \|_2 \quad .$$

Note that, in particular, $e_0 \neq 0$.

Denote $\mathcal{F}_1 = \left\{ \prod_{i=1}^{k} e_0 E_{A_0' \cap M_0}(y_i) e_0 \mid 1 \leq k \leq p,\ y_1, \dots, y_k \in \mathcal{F}_0 \right\}$.

Let now $w$ be a unitary element in $A_0 e_0$ such that $\tau(w^n) = 0$, for $n \neq 0$. The choice of $w$ is possible because $A_0 e_0$ is diffuse. It follows that $(w^n)_{n \geq 1}$ is $\sigma$-weakly convergent to zero so that if $n$ is larger than some $n_0$, we have $|\tau(w^n z)| < \delta \tau(e_0)$ for any $z$ in the finite set $\mathcal{F}_1$.

Finally, we denote by $w_0 = w^{n_0+1} \in A_0 e_0$ and regard it as a partial isometry in $A$. Since $w_0 \in A_0$ has the support orthogonal to $u^* u$, it follows that $u_0 = u + w_0$ is a partial isometry in $A$ that extends $u$. Since $w_0 \neq 0$, $u_0 \neq u$. We show that actually $u_0 \in \mathcal{W}$. So let $y_1, \dots, y_k$ be an arbitrary k-tuple in $\mathcal{F}$, $1 \leq k \leq p$. Then

$$(2) \qquad \left| \tau\left(\prod_{i=1}^{k} (u_0 y_i)\right) \right| = \left| \tau\left(\prod_{i=1}^{k} ((u + w_0) y_i\right) \right| \leq$$

$$\leq \left| \tau\left(\prod_{i=1}^{k} (u y_i)\right) \right| + \sum \left| \tau\left(\prod_{j=1}^{s} (w_0 z_j)\right) \right|$$

where the last sum is taken over all s-tuples $z_1, \dots, z_s \in \mathcal{F}_0$ and $1 \leq s \leq p$. The last inequality follows by the definition of $\mathcal{F}_0$ and by the fact that any product of the form $\left(\prod_{i=1}^{t-1} (u y_i)\right)(w_0 y_t) x$ has the same trace as $w_0 e y_t x \left(\prod_{i=1}^{t-1} (u y_i)\right) e$.

For a fixed s-tuple $z_1,\dots,z_s \in \tilde{\mathcal{F}}_o$ we have the formula

$$(3)\qquad \prod_{j=1}^{s}(w_o z_j)e_o=\sum_{t=1}^{s}\Big(\prod_{j=1}^{t-1}(w_o e_o E_{A_o'\cap M_o}(z_j)e_o)\Big)w_o(e_o z_t e_o-$$

$$-e_o E_{A_o'\cap M_o}(z_t)e_o)\Big(\prod_{j=t+1}^{s}(w_o z_j)\Big)e_o+\prod_{j=1}^{s}(w_o e_o E_{A_o'\cap M_o}(z_j)e_o)$$

Applying the Cauchy-Schwartz inequality we get the estimate

$$(4)\qquad \Big|\tau\Big(\Big(\prod_{j=1}^{t-1}(w_o z_j)\Big)(e_o z_t e_o-e_o E_{A_o'\cap M_o}(z_t)e_o)\prod_{j=t+1}^{s}(w_o e_o\,\cdot$$

$$\cdot E_{A_o'\cap M_o}(z_j)e_o)\Big)\Big|=\Big|\tau\Big((e_o z_t e_o-e_o E_{A_o'\cap M_o}(z_t)e_o)\,\cdot$$

$$\cdot\Big(\prod_{j=t+1}^{s}(w_o e_o E_{A_o'\cap M_o}(z_j)e_o)\Big)\prod_{j=1}^{t-1}(w_o z_j)\Big)\Big|\le$$

$$\le\|e_o z_t e_o-e_o E_{A_o'\cap M_o}(z_t)e_o\|_2\,\Big\|\Big(\prod_{j=t+1}^{s}(w_o e_o E_{A_o'\cap M_o}(z_t)e_o)\Big)\Big(\prod_{j=1}^{t-1}(w_o z_j)\Big)\Big\|_2 .$$

Since $w_o$ is supported on $e_o$ and $\|E_{A_o'\cap M_o}(z_j)\|\le\|z_j\|$, it follows that

$$(5)\qquad \Big\|\Big(\prod_{j=t+1}^{s}(w_o e_o E_{A_o'\cap M_o}(z_j)e_o)\Big)\prod_{j=1}^{t-1}(w_o z_j)\Big\|_2\le(\max_j\|z_j\|)^s\|e_o\|_2$$

Thus, if we put together (3), (4), (5) and denote $K=(\max_{y\in\tilde{\mathcal{F}}_o}\|y\|)^p$, then we have

$$(6)\qquad \Big|\tau\Big(\prod_{j=1}^{s}(w_o z_j)\Big)\Big|\le K\sum_{t=1}^{s}\|e_o z_t e_o-e_o E_{A_o'\cap M_o}(z_t)e_o\|_2\|e_o\|_2+$$

$$+\Big|\tau\Big(\prod_{j=1}^{s}(w_o e_o E_{A_o'\cap M_o}(z_j)e_o)\Big)\Big| .$$

Since $w_o \in A_o$ commutes with $E_{A_o' \cap M_o}(z_j)$, $1 \le j \le s$, and since $w_o^s = w_o^{s(n_o+1)}$ and $s(n_o+1) > n_o$, it follows by (1), (6) that

$$(7) \qquad |\tau(\prod_{j=1}^{s}(w_o z_j))| \le KN^{1/2}\delta\tau(e_o) + \delta\tau(e_o) .$$

This last inequality gives the desired estimate for the terms of the sum in the right hand side of (2). Noting that there are at most $pN^p$ terms in that sum, it follows by (2) and the definition of u that

$$|\tau(\prod_{i=1}^{k}(u_o y_i))| \le \varepsilon\tau(u^* u) + pN^p(KN^{1/2}+1)\delta\tau(w_o^* w_o) .$$

Thus if we chose $\delta$ so that $pN^p(KN^{1/2}+1)\delta \le \varepsilon$ then $u_o = u + w_o$ satisfies

$$|\tau(\prod_{i=1}^{k}(u_o y_i))| \le \varepsilon\tau(u_o^* u_o)$$

for any k-tuple $y_1, \dots, y_k \in \mathcal{F}$, $1 \le k \le p$. Thus $u_o \in \mathcal{W}$ and, as we have seen, $u_o \ge u$, $u_o \ne u$. This contradicts the maximality of u. Hence u must be a unitary element in A and it satisfies $|\tau(\prod_{i=1}^{k}(u y_i))| \le \varepsilon$ for any k-tuple of elements in $\mathcal{F}$, $1 \le k \le p$.

It is now straighforward to construct the sequence $(u_n)_n \subset A$: We fix a sequence $(x_n)_{n \ge 0} \subset M$, with $x_o = 1$, dense in M in the Hilbert norm $\| \ \|_2$ and for each n denote by $\mathcal{F}_n = \{v_i x_j \mid 1 \le i \le n, 0 \le j \le n\}$. By the first part of the proof there exists a unitary element $u_n \in A$ such that

$$|\tau(\prod_{i=1}^{k}(u_n y_i))| < 2^{-n}$$

for any $1\leq k\leq 2n$ and any k-tuple $y_1,\ldots,y_k \in \mathcal{F}_n$. In particular for $y_1=v_1$ , $y_2=v_2$ , $y_{k-1}=v_{k-1}$ , $y_{k+j}=v_k x_j$ , $0\leq j\leq n$, we get

$$\left| \tau\left(\left(\prod_{i=1}^{k} (u_n v_i)\right)x_j\right)\right| \leq 2^{-n}, \quad 1\leq k\leq n, \quad 0\leq j\leq n.$$

Since $\left(\prod_{i=1}^{k} (u_n v_i)\right)_n$ are uniformly bounded in norm (being unitary elements), this implies that they converge to zero $\sigma$-weakly, for any $k\geq 1$.

Q.E.D.

PROOF of 1.1. Taking $v_i=v$ in the preceding lemma, we obtain a sequence of unitary elements $(u_n)_n \subset A$ such that $((u_n v)^k)_n$ converge $\sigma$-weaky to zero for any $k\geq 1$. But then $(u_n v)^{-k}=$ $=((u_n v)^k)^*$ also converge $\sigma$-weakly to zero.

Q.E.D.

3. FINAL REMARKS

3.1. It is quite clear that the study of (maximal) abelian $*$-subalgebras of a von Neumann algebra M and of their interrelations should be important in order to understand the algebra M itself. This fact seems even more evident in problems of cohomological nature on the algebra M, because cohomological problems can in general be solved for the abelian subalgebras of M (cf. [3], [4], [5]). So the whole problem remains to get the compatibility of the respective solutions, by relating in some way the abelian subalgebras. However this idea has not yet been proved to be so useful. The proof of theorem B seems to us the first succesful approach close to this line.

3.2. A diffuse abelian subalgebra A of M can be regarded in two ways: as generated by its projections or by its unitary elements. The first point of view means that we can refine inductively finite partitions of the unity in A so that the corresponding algebras generate A as a von Neumann algebra. This is like regarding the Borel $\sigma$-algebra of the interval $[0,1]$ as being generated by the diadic intervals $[\frac{k}{2^n}, \frac{k+1}{2^n})$, or equivalently taking a Haar system in $L^\infty([0,1],\lambda)$, where $\lambda$ is the Lebesgue measure. The second point of view means that we regard A as being isomorphic with $L^\infty(T,\mu)$. If $u \in A$ is the unitary element corresponding to the identity function on $\mathbb{T}$, then u generates A as a von Neumann algebra in a very nice way, as Fourier series in u. The first point of view proved to be useful in many situations: see for example [7] or even the proof of 2.1 in this paper. The proof of theorem B shows that the second point of view may also be of interest.

3.3. During the proof of theorem B we avoided to use the general result in [3], that a derivation $\delta : M \to \mathcal{B}(\mathcal{H})$, where $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra, is automatically $\sigma$-weakly continuous. However this result is fully used by Johnson and Prarrott when solving the non $II_1$ cases and also by us in proving the continuity result 4.1 ([8]).

3.4. After E.Christensen's solution to the cohomology problem $H^1(M,\mathcal{K}(\mathcal{H}))=0$ for $II_1$ factors in standard form ([1]), it was probably noted by many people that the statements of theorems A and B (or equivalently properties $(P_1)$ and $(P_2)$ in [4]) are actually equivalent in general. The argument is the same as the one we give at the end of the proof of theorem B, when we get the general case from the case when M acts standardly on $\mathcal{H}$.

In [4] Johnson and Parrott also obtained some other interesting results related to theorems A and B. They proved these results for von Neumann algebras satisfying the statements of these theorems. Since we now know that all von Neumann algebras satisfy them, it worth mentioning these corollaries. The proof of the first one is a trivial consequence of theorem A. The proof of the next two can be found by the interested reader in [4].

3.5. <u>COROLLARY</u>. Let $\pi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ be the quotient map into the Calkin algebra. If $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra then the commutant of $\pi(M)$ in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the image by $\pi$ of the commutant of M in $\mathcal{B}(\mathcal{H})$. In particular the bicommutant of $\pi(M)$ in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ equals $\pi(M)$.

3.6. <u>COROLLARY</u>. Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $\delta : M+\mathcal{K}(\mathcal{H}) \longrightarrow M+\mathcal{K}(\mathcal{H})$ a derivation. Then $\delta = \mathrm{ad}\, T$ for some $T \in M+M'+\mathcal{K}(\mathcal{H})$.

3.7. <u>COROLLARY</u>. Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra in standard form and $U \in \mathcal{B}(\mathcal{H})$ a unitary operator with $U(M+\mathcal{K}(\mathcal{H}))U^* = M+\mathcal{K}(\mathcal{H})$. Then U can be splitted as $U=U_1U_2$ where $U_1, U_2 \in \mathcal{B}(\mathcal{H})$ are unitary operators such that $U_1MU_1^*=M$, $U_2 - I \in \mathcal{K}(\mathcal{H})$.

## 4. APPENDIX

In this section we prove a continuity result for derivations into the compacts ([8]).

4.1. THEOREM. Let M be a type $II_1$ factor acting on a Hilbert space $\mathcal{H}$. Let $\delta : M \to \mathcal{K}(\mathcal{H})$ be a derivation. Then $\delta$ is continuous from the unit ball of M with the strong operator topology to $\mathcal{K}(\mathcal{H})$ with the norm topology.

PROOF. Note first that by [4], $\delta$ is norm continuous and by [3] it is $\sigma$-weakly continuous. Since M is a factor it has a unique trace $\tau$, $\tau(1)=1$ and we denote as in Section 1 by $\| \ \|_2$ the norm implemented by $\tau$. Since $\| \ \|_2$ induces the strong operator topology on the unit ball of M, we have to show that if $(x_n)_n$ is a bounded sequence in M with $\| x_n \|_2 \to 0$ then $\| \delta(x_n) \| \to 0$. It is clear that we only need to prove this implication in the case $x_n$ are selfadjoint elements. Moreover, since $\| \, |x_n| \, \|_2 = \| x_n \|_2$, it follows that if $\| x_n \|_2 \to 0$ then $\| (x_n)_+ \|_2 \to 0$ and $\| (x_n)_- \|_2 \to 0$, so that it is sufficient to prove that if $x_n \in M$ are positive elements and $\| x_n \|_2 \to 0$ (equivalently $\tau(x_n) \to 0$) then $\| \delta(x_n) \| \to 0$. Let's first prove this in case $x_n$ are projections in M. Suppose this is false, so that there exists a sequence of projections $(f_n)_n$ in M with $\tau(f_n) \to 0$ and $\| \delta(f_n) \| \geq c > 0$ for all n. By taking a subsequence if necessary, we may assume $\sum \tau(f_n) < \infty$. Let $g_n$ be the supremum of the projections $(f_k)_{k \geq n}$. Then $\tau(g_n) \leq \sum_{k \geq n} \tau(f_k)$ tends to zero with n. Let $s_{nm}$ be the support of $f_m g_n f_m$. Then $s_{nm} \leq f_m$ and $s_{nm}$ is majorised by $g_n$ so that $\tau(s_{nm}) \leq \tau(g_n) \to 0$, for each fixed m. Moreover since $g_n$ is dicreasing, $f_m g_n f_m$ is dicreasing in n, so that $s_{nm}$ is dicreasing in n. Thus $(f_m - s_{nm})_n$ increases to $f_m$ so that $\delta(f_m - s_{nm})$ is weakly convergent to $\delta(f_m)$, and so, by the inferior semicontinuity of the norm, for n big enough we have

$$\| \delta(f_m - s_{nm}) \| \geq c/2 .$$

We may thus get by induction an increasing sequence of integers $n_1, n_2, \ldots$ such that the projections $h_k = f_{n_k} - s_{n_{k+1}, n_k}$ satisfy $\|\delta(h_k)\| \geq c/2$. These projections also satisfy $\tau(h_k) \leq \leq \tau(f_{n_k}) \longrightarrow 0$. Moreover a simple inspection of the definitions show that $(h_k)_k$ are mutually orthogonal projections. By [4] this is a contradiction.

Let now $(x_n)_n$ be positive elements in M with $\| x_n \| \leq 1$ and $\tau(x_n) \longrightarrow 0$. Let $x_n = \sum_{n \geq 1} 2^{-m} e_m^n$ be the diadic decomposition of $x_n$. It follows that $\tau(e_m^n) \xrightarrow[n]{} 0$ for each $m \geq 1$. Let $\varepsilon > 0$ and $m_o \geq 1$ so that $2^{-m_o} < \varepsilon / 2 \|\delta\|$. Then by the first part of the proof there exists $n_o$ such that for $n \geq n_o$, $\| \delta(e_m^n) \| < \varepsilon/2$ for any $m \leq m_o$. Thus, for $n \geq n_o$ we get $\| \delta(x_n) \| \leq \sum_{m=1}^{m_o} 2^{-m} \| \delta(e_m^n) \| + \|\delta\| \sum_{m > m_o} 2^{-m} \leq$
$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Q.E.D.

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