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by

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Victor ŢIGOIU*)

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*) Department of Mathematics, National Institute for Scientific and Technical Creation, Bd. Pacii 220, 79622 Bucharest, Romania.

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0. INTRODUCTION

In this paper we deduce the complete thermodynamics restrictions which can be obtained from the Clausius-Duhem inequality for third order and third grade fluids without the supplementary hypothesis used by Fosdick and Rajagopal in [8] (which concerns to the existence of an absolute minimum on equilibrium states, for the free energy). Due to this hypothesis the stress response function of third grade fluid does not depend on the third Rivlin--Ericksen tensor A_3. On the other hand, as we have observed in Section 1, the dissipation function can attain his maximum on processes which are different from equilibrium (by equilibrium we understand as in [7,8] locally at rest) and for that we have considered that it is more suitable that we don't make use of the above mentioned hypothesis.

In the first section we obtain general thermodynamics restrictions (more complete than those obtained in [7,8]) on the class of non-Newtonian third order fluids and we annalyse the behaviour of response functions at equilibrium. In particular we observe that the heat flux must vanishes on each process if the temperature gradient vanishes. These restrictions are used in the second section when we deduce the heat propagation equation in the neighbourhood of the equilibrium states. Here we prove that the thermal conductivity tensor is symmetric and non-negative definite.

In the third section we obtain constitutive restrictions for third grade fluids, which preserve the fundamental character of this subclass of third order fluids. As a consequence of the residual dissipation inequality and of constitutive restrictions in the fourth section we can construct the class of functions which can be chosen as free energy for these fluids. Moreover we obtain an interesting interpretation for the normal stress coefficient α_1 and the complete description of the free energy which, on viscometric flows, is a linear function of trA_1^2.

1. THERMODYNAMICS OF THIRD ORDER FLUIDS

We shall keep, in this paper, the definitions given in [1,2] for a general thermodynamics theory. That is, we shall identify the material particle X of a body $\mathcal{B}$ with the position $X \in \mathcal{E}^3$ which is occupied by this particle in a fixed reference configuration $B \subseteq \mathcal{E}^3$. We shall denote by $\mathcal{V}$ the vector space attached to $\mathcal{E}^3$.

Let

$$x = \chi(X,t) \tag{1.1}$$

the motion of the body $\mathcal{B}$; then with the classical definitions we shall have for the deformation gradient and the velocity gradient associated with the motion χ :

$$(1.2)\qquad \begin{cases} F(X,t)=\text{Grad}\,\chi \\ L(x,t)=\text{grad}\,\dot{x}=\text{grad}\,v \end{cases}$$

and

$$(1.3)\qquad L=\dot{F}F^{-1}$$

whenever F is non-singular (and we shall preserve this assumption allways in this paper).

A thermokinetic process will be the pair $(\chi(X,t),\theta(X,t))$ where $\theta: \mathcal{B}\times R \longrightarrow R_+$ is the absolute temperature of the body $\mathcal{B}$. A thermodynamic process for the body $\mathcal{B}$ will be the following collection of eight functions defined on BxR:

$$(\chi(X,t),\theta(X,t),T(X,t),\varepsilon(X,t),\eta(X,t),q(X,t),b(X,t),r(X,t)),$$

(where T is the Cauchy stress tensor, ε -the specific internal energy per unit mass, η -the specific entropy per unit mass q -the heat flux; b-the specific body forces per unit mass, r-the radiant heating per unit mass), if the balance laws of momentum and energy and the classical Clausius-Duhem inequality are satisfied.

When sufficient smoothness is assumed and when we take into account the conservation of mass

$$(1.4)\qquad \rho_o(X)=\rho(X,t)\ \det F(X,t)\quad ,$$

where ρ_o is a positive function given once and for all allong with the body $\mathcal{B}$, the local equations of balance of linear momentum and energy are:

$$(1.5)\qquad \rho\dot{v}=\text{div}\,T+\rho b$$

$$(1.6)\qquad \rho\dot{\varepsilon}=T\cdot L-\text{div}\,q+\rho r.$$

In the same time the local form of the Clausius-Duhem inequality is:

(1.7) $\varrho(\dot{\Psi}+\eta\dot{\theta})-T\cdot L+(1/\theta)q\cdot \mathrm{grad}\,\theta \leq 0$

where we have introduced the Helmholtz specific free energy

(1.8) $\Psi=\Psi(X,t)=\varepsilon-\theta\eta$

Further on a particular class of fluids, namely the homogeneous and incompressible third order fluids, will be considered. For this kind of materials det F=1 (tr L=0).

We shall denote by $\mathrm{Lin}_o(\mathcal{V},\mathcal{V})$ the space of all linear and traceless transformations from $\mathcal{V}$ into $\mathcal{V}$ and by $\mathcal{D}=[\mathrm{Lin}_o(\mathcal{V},\mathcal{V})]^3 \times R_+ \times \mathcal{V}$. With these notations the above mentioned class of fluids is characterized by the existence of the following response fonctions

$$(1.9)\quad \begin{cases} \tilde{T}: \mathcal{D} \longrightarrow \mathrm{SLin}(\mathcal{V},\mathcal{V}) \\ \tilde{\varepsilon}: \mathcal{D} \longrightarrow R \\ \tilde{\eta}: \mathcal{D} \longrightarrow R \\ \tilde{q}: \mathcal{D} \longrightarrow \mathcal{V} \end{cases}$$

in terms of which:

$$(1.10)\quad \begin{aligned} T(x,t)&=\tilde{T}(L,\dot{L},\ddot{L},\theta,g) \\ \varepsilon(x,t)&=\tilde{\varepsilon}(L,\dot{L},\ddot{L},\theta,g) \\ \eta(x,t)&=\tilde{\eta}(L,\dot{L},\ddot{L},\theta,g) \\ q(x,t)&=\tilde{q}(L,L,L,\ ,g)\ , \end{aligned}$$

and where we have denoted by $\mathrm{SLin}(\mathcal{V},\mathcal{V})$ the space of all symmetric transformations from V into V and by $g=\mathrm{grad}\,\theta$. We shall suppose that all this functions are continuously differentiable.

According to the effective principle of determinisme , [2,3,4], we shall say that: the material response fonctions given by (1.10) are compatible with thermodynamics if for any thermo-kinetic process (χ,θ) the eight functions $(\chi,\theta,\varepsilon,\eta,T,q,r,b)$ are a thermodynamic process (because trL=0, p dose not appear in (1.7)). By consequence the response functions (1.10) will be compatible with thermodynamics if they will verify the inequality (1.7) for any thermokinetic process:

(1.11) $$\rho\,(\dot{\tilde{\psi}}+\tilde{\eta}\dot{\theta})-\tilde{T}\cdot L+\frac{\tilde{q}\cdot g}{\theta}\leq 0 \,,$$

where $\tilde{\psi}$ is given by

(1.12) $$\tilde{\psi}\,(M)\equiv\tilde{\varepsilon}\,(M)-\theta\,\tilde{\eta}\,(M)$$

for any $M\in\mathcal{D}$ and where for the calculus of $\dot{\tilde{\psi}}$ we apply a particular case of the Mizel-Wang chain rule [5]

(1.13) $$\dot{\tilde{\psi}}=\tilde{\psi}_{L}\cdot\dot{L}+\tilde{\psi}_{\dot{L}}\cdot\ddot{L}+\tilde{\psi}_{\ddot{L}}\cdot\dddot{L}+\tilde{\psi}_{\theta}\dot{\theta}+\tilde{\psi}_{g}\cdot\dot{g}$$

Concerning the restrictions for the respons functions (1.10) that we want to obtain, we are interested in finding a motion χ and a temperature field θ for which $L,\dot{L},\ddot{L},\dddot{L},\theta,\dot{\theta},g,\dot{g}$ may be arbitrary chosen, that is:

<u>LEMMA 1.1.</u> Let two arbitrary real numbers θ_o,θ'_o , $\theta_o>0$. Let $g_o,g'_o\in\mathcal{V}$ arbitrary and $L_o,L'_o,L''_o,L'''_o\in \mathrm{Lin}_o(\mathcal{V},\mathcal{V})$ arbitrary. Then, there exists a motion χ and a temperature field θ such that if X is a fixed particle and t a fixed time:

$$(1.14)\quad \begin{cases} L(x,t)=L_o, \dot{L}(x,t)=L'_o, \ddot{L}(x,t)=L''_o, \dddot{L}(x,t)=L'''_o \\ \theta(x,t)=\theta_o \ , \ \dot{\theta}(x,t)=\theta'_o \\ \text{grad}\,\theta(x,t)=g_o \ , \ \overset{\bullet}{\overline{\text{grad}\,\theta(x,t)}}=g'_o \ , \end{cases}$$

where $x=\chi(X,t)$.

PROOF. For the proof we first observe that it is sufficient to construct a velocity field v with the properties $(1.14)_{\overline{1,4}}$. Then, because

$$x=\chi(X,t)$$

and

$$y=\chi(X,\tau)$$

and on the other hand:

$$v(y,\tau)=\frac{\partial}{\partial\tau}\chi_t(x,\tau)\ ,$$

we shall obtain the following problem:

$$(1.15)\quad \begin{cases} \dfrac{dy}{d\tau}=v(y,\tau) \\ y(t)=x \end{cases}$$

If v is a Lipschitz function then this problem has a unique solution.

The construction of the field $v(y,\tau)$ is standard and for that we shall remember only that if we consider

$$(1.16)\qquad v(y,\tau)=v(x,t)+\frac{\partial}{\partial y}v(x,t)[y-x]+\frac{\partial}{\partial \tau}v(x,t)(\tau-t)+$$

$$+\frac{1}{2!}\frac{\partial^2}{\partial y^2}v(x,t)[y-x,y-x]+\frac{1}{2!}\frac{\partial^2}{\partial y\partial \tau}v(x,t)(\tau-t)[y-x]+$$

$$+\frac{1}{2!}\frac{\partial^2}{\partial \tau^2}v(x,t)(\tau-t)^2+\frac{1}{3!}\frac{\partial^3}{\partial \tau^2\partial y}v(x,t)(\tau-t)^2[y-x]+$$

$$+\frac{1}{3!}\frac{\partial^3}{\partial y^3}v(x,t)[y-x,y-x,y-x]+\frac{1}{3!}\frac{\partial^3}{\partial \tau\partial y^2}v(x,t)(\tau-t)[y-x,y-x]$$

$$+\frac{1}{4!}\frac{\partial^4}{\partial y^4}v(x,t)[y-x,y-x,y-x,y-x]+\frac{1}{4!}\frac{\partial^4}{\partial \tau^3\partial y}v(x,t)(\tau-t)^3[y-x]$$

then a simple functional calculus shows us that the gradient attached field is:

$$(1.17)\qquad L(y,\tau)[w]=L_o(x,t)[w]+\frac{1}{2!}\frac{\partial^2}{\partial y\partial \tau}v(x,t)(\tau-t)[w]+$$

$$+\frac{1}{2!}\frac{\partial^2}{\partial y^2}v(x,t)[w,y-x]+\frac{1}{2!}\frac{\partial^2}{\partial y^2}v(x,t)[y-x,w]+$$

$$+\frac{1}{3!}\frac{\partial^3}{\partial \tau^2\partial y}v(x,t)(\tau-t)^2[w]+\frac{1}{3!}\frac{\partial^3}{\partial y^3}v(x,t)\Big\{[w,y-x,y-x]+$$

$$+[y-x,w,y-x]+[y-x,y-x,w]\Big\}+\frac{1}{3!}(\tau-t)\frac{\partial^3}{\partial \tau\partial y^2}v(x,t)\Big\{[w,y-x]+[y-x,w]$$

$$+\frac{1}{4!}(\tau-t)^3\frac{\partial^4}{\partial \tau^3\partial y}v(x,t)[w]+\frac{1}{4!}\frac{\partial^4}{\partial y^4}v(x,t)\Big\{[w,y-x,y-x,y-x]+\ \ldots$$

$$\ldots+[y-x,y-x,y-x,w]\Big\}$$

and where we have denoted $L_o \equiv L_o(x,t) \equiv \frac{\partial}{\partial y}v(x,t)$.

The formulas (1.16), (1.17) give the response to the first part of the problem.

In the same way, the field

$$\theta(y,\tau)=\theta_o(x,t)+\frac{\partial}{\partial\tau}\theta(x,t)(\tau-t)+\frac{\partial}{\partial y}\theta(x,t)\cdot[y-x]+ \tag{1.18}$$

$$+\frac{\partial^2}{\partial y\partial\tau}\theta(x,t)(\tau-t)\cdot[y-x]+\frac{1}{2!}\frac{\partial^2}{\partial y^2}\theta(x,t)\cdot[y-x,y-x]$$

satisfy the properties $(1.14)_{\overline{5,8}}$ if we put

$$\theta_o'\equiv\frac{\partial}{\partial\tau}\theta(x,t)+g_o\cdot v(x,t)$$

(1.19)

$$g_o' \quad \frac{\partial^2}{\partial y\partial\tau}\theta(x,t)+\frac{\partial}{\partial y}g_o(x,t)[v(x,t)]$$

which complete the proof∎

It is a very simple fact to observe that the above construction permites an arbitrary choise, in the point (x,t), for the temporal derivatives and for the gradients of the motion and of the temperature field. With this we can give the proof of a Coleman theorem concerning the necessary and sufficient conditions in which the response fonctions (1.10) are compatible with thermodynamics.

<u>THEOREM 1.1</u>. The response functions (1.10) are compatbile with thermodynamics if and only if

$$\psi(x,t)=\hat{\psi}(L,\dot{L},\theta)\quad, \tag{1.20}$$

$$\eta(x,t)=\hat{\eta}(L,\dot{L},\theta)=-\hat{\psi}_\theta(L,\dot{L},\theta) \tag{1.21}$$

and the response functions $\hat{\psi}(.,.,.)$, $\tilde{T}(.,.,.,.,.,.)$ and $\tilde{q}(.,.,.,.,.,.)$ are such that the dissipation inequality

$$\rho\hat{\psi}_L(L,\dot{L},\theta)\cdot\dot{L}+\rho\hat{\psi}_{\dot{L}}(L,\dot{L},\theta)\cdot\ddot{L}-\tilde{T}(M)\cdot L+\frac{1}{\theta}\tilde{q}(M)\cdot g\leq 0 \tag{1.22}$$

holds for every thermokinetic process, where $M \in \mathcal{D}$.

PROOF. We shall choose a thermokinetic process which verify the conditions of Lemma 1.1. Then with (1.16) and (1.18) we can apply the chain rule and we obtain:

$$(1.23)\quad \rho\left\{\tilde{\psi}_L(M_o)\cdot L'_o+\tilde{\psi}_{\dot{L}}(M_o)\cdot L''_o+\tilde{\psi}_{\ddot{L}}(M_o)\cdot L'''_o+\tilde{\psi}_\theta(M_o)\cdot \theta'_o+\tilde{\psi}_g(M_o)\cdot g'_o\right\}+$$

$$+\rho\,\tilde{\eta}(M_o)\theta'_o-\tilde{T}(M_o)\cdot L_o+\frac{1}{\theta_o}\tilde{q}(M_o)\cdot g_o \leq 0\ ,$$

where $M_o \equiv (L_o, L'_o, L''_o, \theta_o, g_o)$, which must hold for all $(L_o, L'_o, L''_o, L'''_o, \theta_o, \theta'_o, g_o, g'_o)$. Moreover L'''_o, θ'_o, g'_o appear only linearly and hence (1.23) is equivalent to

$$(1.24)\quad \tilde{\psi}_{\ddot{L}}(M_o)\cdot L'''_o \equiv 0\ ;\ \tilde{\psi}_g(M_o)\cdot g'_o \equiv 0;\quad \rho\,(\tilde{\psi}_\theta(M_o)+\tilde{\eta}(M_o))\theta'_o \equiv 0$$

$$(1.25)\quad \rho\tilde{\psi}_L(M_o)\cdot L'_o+\rho\tilde{\psi}_{\dot{L}}(M_o)\cdot L''_o-\tilde{T}(M_o)\cdot L_o+\frac{1}{\theta_o}\tilde{q}(M_o)\cdot g_o \leq 0$$

for all $L_o, L'_o, L''_o, L'''_o \in \mathrm{Lin}_o(\mathcal{V},\mathcal{V})$, $\theta_o > 0$, $\theta'_o \in R$ and $g_o, g'_o \in \mathcal{V}$.

Thus (1.20), (1.21) and (1.22) necessarily follow from (1.24) and (1.25). The proof of the sufficiency is immediate ∎

It is not surprising that from $(1.24)_1$ it result that $\tilde{\psi}_{\ddot{L}}(M_o) \equiv 0$, because a simple calculus shows that for such functions the restriction of the derivative on $\mathrm{Lin}(\mathcal{V},\mathcal{V})$ to the subspace $\mathrm{Lin}_o(\mathcal{V},\mathcal{V})$ coincide with the derivative on $\mathrm{Lin}_o(\mathcal{V},\mathcal{V})$.

The results obtained in Lemma 1.1 and in Theorem 1.1 are similar with those given in [7,8] but with some specific interpretations that are necessary in the remainder of the paper.

REMARK. If we denote

$$(1.26)\qquad \Gamma(M) \equiv \rho\tilde{\Psi}_{L}(M)\cdot\dot{L}+\rho\tilde{\Psi}_{\dot{L}}(M)\cdot\ddot{L}-\tilde{T}(M)\cdot L+\frac{1}{\theta}\tilde{q}(M)\cdot g$$

we observe that the reduced dissipation inequality (1.22) implies that the function Γ be nonpositive over $\mathcal{D}$. Also we see that in equilibrium points (that is in $M_e \equiv (0,0,0,\theta_e,0)$; a more precise term would be: locally at rest) $\Gamma(M_e)=0$. However we observe that in the points $M_o' \equiv (0,0,\ddot{L}_o,\theta_o,0)$ the inequality (1.22) reduces to:

$$(1.27)\qquad \rho\hat{\Psi}_{\dot{L}}(0,0,\theta_o)\cdot\ddot{L}_o \leq 0$$

If we substitute now $\ddot{L}_o$ with $s\ddot{L}_o$ and employ the definition of the derivative of $\hat{\Psi}$ in relation with $\dot{L}$ we shall see that (1.27) is satisfied with equality and more, that the deviator of $\hat{\Psi}_{\dot{L}}(0,0,\theta_o)$ is nul for all θ_o. Then $\Gamma(M_o')=0$ and so Γ attains his maximum in two different processes ∎

In the paper of Dunn and Foskick [7] it is proved that for nonewtonian fluids of order two the points $(0,\dot{L}_o,\theta_o,0)$ are stationary points for the free energy (that means for this class of fluids that $\hat{\Psi}_{L}(0,\theta)=0$ and $\rho\hat{\Psi}_{LL}(0,\theta)[A,A]=\hat{T}(0,A,\theta,0)\cdot A$). We shall obtain any results of this type for fluids of third order.

COROLLARY 1.1. The points $M_o' \in \mathcal{D}$ are necessarily stationary points for the free energy.

PROOF. Supposing $\hat{\Psi}$ twice continuously differentiable we observe that the condition that $M_o' \in \mathcal{D}$ be a maximum (local) for Γ is equivalent with

$$(1.28)\qquad 0 = \left.\frac{d\Gamma}{ds}\right|_{s=0}$$

and

$$(1.29) \qquad 0 \geq \left.\frac{d^2\Gamma}{ds^2}\right|_{s=0}$$

If we perform the calculus in the right hand sides of (1.28) and (1.29) we shall obtain in a standard way the following necessary conditions

$$(1.30) \qquad \begin{cases} \hat{\Psi}_{L}(0,0,\theta_o)=0 \\ \hat{\Psi}_{\dot{L}}(0,0,\theta_o)=0 \\ \hat{\Psi}_{\dot{L}\dot{L}}(0,0,\theta_o)[\dot{L}_1]=0 \\ \rho\,\hat{\Psi}_{L\dot{L}}(0,0,\theta_o)[L_1,\ddot{L}_o]\equiv\tilde{T}(M_o')\cdot L_1 \\ \hat{\Psi}_{\dot{L}\theta}(0,0,\theta_o)=0 \\ \tilde{q}(M_o')=0 \end{cases}$$

which, in particular, gives the proof of the corollary ∎

<u>COROLLARY 1.2</u>. The points $M_e \in \mathcal{D}$ are necessarily stationary points for the free energy.

<u>PROOF</u>. Remainding that $\Gamma(M_e)=0$, the proof of the stationarity property is similar with those performed in the Corollary 1.1. The restrictions that we shall obtain by an appropriate use of conditions (1.28), (1.29) in the point M_e are:

$$(1.31) \qquad \begin{cases} \hat{\Psi}_{L}(0,0,\theta_o)=0 \\ \hat{\Psi}_{\dot{L}}(0,0,\theta_o)=0 \\ \tilde{T}(M_e)\cdot L_1=0 \quad , \quad (\forall)\ L_1\in \mathrm{Lin}(\mathcal{V},\mathcal{V}) \\ \tilde{q}(M_e)=0 \end{cases}$$

which in particular gives the proof of the corollary and moreover it results that on the equilibrium states the pressure tensor is hydrostatic.

By emploing the relation (1.29) we can obtain any useful supplementary restrictions on second derivatives of $\hat{\Psi}$ and first derivatives of $\tilde{T}$.

$$(1.32) \qquad \begin{cases} \varrho\, \hat{\Psi}_{LL}(0,0,\theta_o)[A,B]=0 \\ \varrho\, \hat{\Psi}_{LL}(0,0,\theta_o)[A,B]=T_L(M_e)[A,B] \end{cases}$$

for all $A,B \in Lin_o(\mathcal{V},\mathcal{V})$,

$$(1.33) \qquad \begin{cases} \varrho\, \hat{\Psi}_{\dot{L}L}(0,0,\theta_o)[A,A] \leq 0 \\ \tilde{T}_L(M_e)[A,A] \geq 0 \end{cases}$$

for all $A \in Lin_o(\mathcal{V},\mathcal{V})$.

As $\hat{\Psi}$ has been supposed twice continuously differentiable we can conclude from $(1.33)_2$ and $(1.35)_1$ that

$$(1.34) \qquad \tilde{T}_{\ddot{L}}(M_e)[A,A] \leq 0$$

for all $A \in Lin_o(\mathcal{V},\mathcal{V})$ ■

REMARK 1.1. The conclusions of these two corollaries are in the sense of the Corollary 1 from §3 of [7]. We cannot say else about the mechanical power than it is inferior bounded by the projections of $\varrho\hat{\Psi}_L$ and $\varrho\hat{\Psi}_{\dot{L}}$ respectively on $\dot{L}$ and $\ddot{L}$.

REMARK 1.2. The complet set of necessary and sufficient conditions for the existence of a local maximum for Γ in $M_e \in \mathcal{D}$ is:

$$(1.35)\qquad \begin{cases} \Gamma(M_e)=0 \\ \Gamma_M(M_e)=0 \\ \Gamma_{M^2}(M_e)[M,M] \leq 0 \\ \Gamma_{M^2}(M_e)[M,M]^2 \leq \Gamma_{M^2}(M_e)[M,M]\cdot\Gamma_{M^2}(M_e)[N,N] \end{cases}$$

for all $M,N\in\mathfrak{D}$. If we performe the calculus (which is similar to the calculus performed in the Corollary 1.1) we can obtain any supplementary restriction on response functions:

$$(1.36)\qquad \begin{cases} \tilde{q}_{\theta}(M_e)=0. \\ \tilde{q}_{\ddot{L}}(M_e)=0 \\ \tilde{q}_g(M_e)[y,y] \leq 0 \\ \frac{1}{\theta_o}\tilde{q}_L(M_e)[A]\cdot y=\tilde{T}_g(M_e)[y]\cdot A \\ \hat{\psi}_{\theta L}(0,0,\theta_o)=\hat{\psi}_{\theta\dot{L}}(0,0,\theta_o)=0 \end{cases}$$

for all $y\in\mathcal{V}$, $A\in Lin_o(\mathcal{V},\mathcal{V})$, $\theta_o>0$.

Moreover, from (1.20), (1.21) and (1.8) it is easily to see that the specific internal energy is given by:

$$(1.37)\qquad \varepsilon(x,t)\equiv\tilde{\varepsilon}(M)=\hat{\psi}(L,\dot{L},\theta)-\theta\,\hat{\psi}_{\theta}(L,\dot{L},\theta)\equiv\hat{\varepsilon}(L,\dot{L},\theta)$$

If we use the principle of frame indifference [1,6,7,8], we can breafily conclude that the response functions can depend on the motion only by the first three Rivlin-Ericksen tensors A_1, A_2, A_3 which can be given by the recurrence formula:

$$(1.38)\qquad \begin{cases} A_1=L+L^T \\ A_n=\dot{A}_{n-1}+A_{n-1}L+L^TA_{n-1} \end{cases}.$$

So we can write:

$$(1.39)\quad \begin{cases} \Psi(x,t)=\bar{\Psi}(A_1,A_2,\theta) \\ \eta(x,t)=-\bar{\Psi}_\theta(A_1,A_2,\theta) \\ T(x,t)=-p\mathbb{I}+\bar{T}(A_1,A_2,A_3,\theta,g) \\ q(x,t)=\bar{q}(A_1,A_2,A_3,\theta,g) \end{cases}$$

and moreover the response functions $\bar{\Psi}$, $\bar{T}$ and $\bar{q}$ are isotropic functions. The isotropy of $\bar{q}$ leads to:

$$\bar{q}(A_1,A_2,A_3,\theta,0)=-\bar{q}(A_1,A_2,A_3,\theta,0)$$

and finally to:

$$(1.40)\quad \bar{q}(A_1,A_2,A_3,\theta,0)=0$$

Then we can conclude that the heat flux must vanishes, regardless of the motion and the temperature at a particle, if the temperature gradient vanishes.

2. HEAT PROPAGATION IN THIRD ORDER FLUIDS

Taking into account the relations (1.8), (1.20) and (1.21) and the chain rule the equation of balance of energy (1.6) can be written:

$$(2.1)\quad \rho(\hat{\Psi}_L(L,\dot{L},\theta)-\theta\hat{\Psi}_{L\theta}(L,\dot{L},\theta))\cdot\dot{L}+\rho(\hat{\Psi}_{\dot{L}}(L,\dot{L},\theta)-\theta\hat{\Psi}_{\dot{L}\theta}(L,\dot{L},\theta))\cdot\ddot{L}-$$
$$-\rho\theta\hat{\Psi}_{\theta\theta}(L,\dot{L},\theta)\dot{\theta}-\tilde{T}(M)\cdot L+\operatorname{div}\tilde{q}(M)-\rho r=0 .$$

Introducing the chain rule for q and separating the terms that vanishe if the motion vanishes, we can write:

$$(2.2)\qquad -\varrho\vartheta\hat{\psi}_{\theta\theta}(L,\dot{L},\vartheta)+\tilde{q}_g(M)\cdot(\nabla g)^T+\tilde{q}_\theta(M)\cdot g+\varrho(\hat{\psi}_L(L,\dot{L},\theta)-$$

$$-\theta\hat{\psi}_{L\theta}(L,\dot{L},\theta))\cdot\dot{L}+\varrho(\hat{\psi}_{\dot{L}}(L,\dot{L},\theta)-\theta\hat{\psi}_{\dot{L}\theta}(L,\dot{L},\theta))\cdot\ddot{L}-\tilde{T}(M)\cdot L+$$

$$+\tilde{q}_L(M)\cdot\nabla L+\tilde{q}_{\dot{L}}(M)\cdot\nabla\dot{L}+\tilde{q}_{\ddot{L}}(M)\cdot\nabla\ddot{L}-\varrho r,$$

where by ∇a we have noted the gradient of the function a. The equation (2.2) represent the heat propagation equation for a third order fluid.

On the other hand, we can write, in the neighbourhood of an equilibrium point M_e:

$$(2.3)\qquad \tilde{q}(M)=\tilde{q}(M_e)+\tilde{q}_\theta(M_e)\theta+\tilde{q}_g(M_e)[g]+\{\text{terms which vanishe if the motion vanishes}\}+0(\theta^2,|g|^2).$$

According to $(1.31)_4$ and $(1.36)_1$ the equation (2.3) can be written:

$$(2.4)\qquad \tilde{q}(M)=\tilde{q}_g(M_e)[g]+0(\theta^2,|g|^2)$$

Consequently, if additionally we suppose that the material is homogeneous in the equilibrium state, we have (in linear approximation):

$$(2.5)\qquad \operatorname{div}\tilde{q}(M)=\tilde{q}_g(M_e)\cdot(\nabla g)^T$$

If we denote by $\tilde{K}(M_e)\equiv-\tilde{q}_g(M_e)$, the thermal conductivity, then it results that $\tilde{K}(M_e)$ is a symmetric tensor and from $(1.36)_2$ we see that $\tilde{K}(M_e)$ is non-negative defined.

Then, if we rewrite the equation (2.2) in the neighbourhood of an equilibrium point in the absence of mechanical motion, we obtain the heat propagation equation for a third order fluid in the form:

(2.6) $$(\rho\theta \frac{\partial\tilde{\eta}}{\partial\theta}(M_e))\dot{\theta} - \tilde{K}_{ij}(M_e)\frac{\partial^2}{\partial x_i \partial x_j} - \rho r=0$$

If we suppose that $-\theta\hat{\Psi}_{\theta\theta}(0,0,\theta)>0$ and if we observe that $\theta\hat{\eta}_\theta(L,\dot{L},\theta)=c(L,\dot{L},\theta)$ is the specific heat, then (2.6) has the form of the classical heat equation.

We conclude this section by observing that from (2.4) we can write in neighbouring states to equilibirum

(2.7) $$\tilde{q}(M)\cdot\nabla\theta \simeq -\tilde{K}(M_e)\ |\nabla\theta|^2 \leq 0$$

and according with this relation it results that in neighbouring states with equilibrium the Fourier inequality holds and the Fourier law is a good approximation for the constitutive law of the heat flux.

3. THERMODYNAMICS OF THIRD ORDER, AND THIRD GRADE FLUIDS

In this section we shall concern with the restrictions that can be derived from the second law of thermodynamics (in the classical Clausius-Duhem form) for the stresse response function. We don't make use of the supplementary hypotesis concerning the existence of an absolute minimum for the free energy on equilibrium states as in the paper of Fosdick and Rajagopal [8].

The incompressible third order and third grade fluid are described by the following Cauchy stress response function (see for example [8] and the References of [8])

(3.1) $$T(x,t)=-p(\theta)I+\mu(\theta)A_1+\alpha_1(\theta)A_2+\alpha_2(\theta)A_1^2+\beta_1(\theta)A_3+$$
$$+\beta_2(\theta)(A_1A_2+A_2A_1)+\beta_3(\theta)(trA_1^2)A_1$$

where $-p(\theta)I$ is the stress due to the constraint of incompressibility, $\mu(\theta)$ is the kinematic viscosity, $\alpha_1(\theta)$ and $\alpha_2(\theta)$ are the coefficients of normal stresses and $\beta_1(\theta)$, $\beta_2(\theta)$, $\beta_3(\theta)$ are material functions connected with the notion of shear viscosity. For simplicity we shall call this class of fluids - third grade fluids.

The conclusions derived in the first two sections remain valid and, moreover, due to the particular form of the stress response function we shall obtain supplementary information about the free energy and any important restrictions on constitutive coefficients from (3.1). In this section we shall deal only with constitutive restrictions on the stress response function. In this sense we give the following theorem (which, with a different proof can be found in the paper of Fosdick and Rajagopal [8]).

THEOREM 3.1. The free energy of a third grad fluid is lineary dependent on $\dot{A}_1$.

PROOF. If we substitute the stress T given by (3.1) into (1.7) and if we have into account the restrictions impose by the Theorem 1.1 we obtain:

$$\rho\hat{\psi}_{\dot{L}}(L_o,\dot{L}_o,\theta_o)\cdot\ddot{L}_o-\beta_1(\theta_o)L_o\cdot(\ddot{L}_o+\ddot{L}_o^T)=0 \tag{3.2}$$

for all $\theta_o>0$ and $L_o,\dot{L}_o,\ddot{L}_o\in Lin_o(\mathcal{V},\mathcal{V})$.

Now, choosing $\ddot{L}_o\in Skew\ Lin(\mathcal{V},\mathcal{V})$ it results that $\rho\hat{\psi}_{\dot{L}}(L_o,\dot{L}_o,\theta_o)\in SimLin_o(\mathcal{V},\mathcal{V})$. By Skew $Lin(\mathcal{V},\mathcal{V})$ we have denoted the space of all skew linear transformation from $\mathcal{V}$ into $\mathcal{V}$. Then, if $\ddot{L}_o\in SLin_o(\mathcal{V},\mathcal{V})$ we shall have that $\rho\hat{\psi}_{\dot{L}}(L_o,\dot{L}_o,\theta_o)-2\beta_1(\theta_o)L_o\equiv A\in$ $\in Skew\ Lin(\mathcal{V},\mathcal{V})$. These properties finally leads to the following relation:

$$\rho\hat{\Psi}_{\dot{L}}(L_o,\dot{L}_o,\theta_o)=\beta_1(\theta_o)(L_o+L_o^T)=\beta_1(\theta_o)A_1$$

Then

$$(3.3)\quad \rho\hat{\Psi}(L_o,\dot{L}_o,\theta_o)\equiv\rho\overline{\Psi}(A_1,A_2,\theta_o)=2\beta_1(\theta_o)L_o^s\cdot\dot{L}_o^s+\rho\overline{\overline{\Psi}}(L_o,\theta_o)$$

where $L^s\equiv\frac{1}{2}(L+L^T)$ is the symmetrical part of L.

REMARK 3.1. With these, the Clausius-Duhem inequality written on processes with $\nabla\theta=0$ has the following form:

$$(3.4)\quad \left\{2\beta_1(\theta_o)(\dot{L}_o)^s+\rho\overline{\overline{\Psi}}_L(L_o,\theta_o)\right\}\cdot\dot{L}_o-\tilde{T}(L_o,\dot{L}_o,\ddot{L}_o,\theta_o,0)\cdot L_o\leq 0$$

for all $\theta_o>0$ and $L_o,\dot{L}_o,\ddot{L}_o\in \mathrm{Lin}_o(\mathcal{V},\mathcal{V})$.

Using (3.1) and groupping in a suitable manner the terms, the inequality (3.4) becomes:

$$(3.5)\quad 2\beta_1(\theta)\dot{L}^s\cdot\dot{L}+\left\{\rho\overline{\overline{\Psi}}_L(L,\theta)-\beta_1(\theta)A_1-\beta_1(\theta)\left[3A_1^2+2(WA_1-A_1W)\right]-\right.$$
$$\left.-2\beta_2(\theta)A_1^2\right\}\cdot\dot{L}-\left\{\frac{1}{4}\left[\beta_1(\theta)+2\beta_2(\theta)+2\beta_3(\theta)\right](\mathrm{tr}A_1^2)^2+\right.$$
$$\left.+\frac{1}{2}\left[\alpha_1(\theta)+\alpha_2(\ )\right]\mathrm{tr}A_1^3+\frac{1}{2}\mu(\theta)\mathrm{tr}A_1^2+\beta_1(\theta)\mathrm{tr}\left[A_1^2W^2-(A_1W)^2\right]\right\}\leq 0$$

for all $\theta>0$, $A_1,A_2\in \mathrm{SLin}_o(\mathcal{V},\mathcal{V})$ and $W\in \mathrm{SkewLin}(\mathcal{V},\mathcal{V})$, where $W\equiv L^a\equiv\frac{1}{2}(L-L^T)$ is the skew part of L.

For simplicity, we shall denote:

$$(3.6)\quad \begin{cases} a\equiv 2\beta_1(\theta) \\ B\equiv\rho\overline{\overline{\Psi}}_L(L,\theta)-\alpha_1(\theta)A_1-\beta_1(\theta)\left[3A_1^2+2(WA_1-A_1W)\right]-2\beta_2(\theta)A_1^2 \\ c\equiv -\frac{1}{2}\left\{\frac{1}{2}\left[\beta_1(\theta)+2\beta_2(\theta)+2\beta_3(\theta)\right](\mathrm{tr}A_1^2)^2+\left[\alpha_1(\theta)+\alpha_2(\theta)\right]\mathrm{tr}A_1^3+\right. \\ \quad\left.+\mu(\theta)\mathrm{tr}A_1^2+2\beta_1(\theta)\mathrm{tr}\left[A_1^2W^2-(A_1W)^2\right]\right\} \end{cases}$$

Now we are ready to give the following theorem:

THEOREM 3.2. The residual inequality (3.5) holds, for all thermokinetic process if and only if

$$(3.7) \qquad \begin{cases} a \leq 0, \quad c \leq 0 \\ B' \cdot B' \leq 4ac \end{cases}$$

where $B' \equiv B - \frac{1}{3}(\mathrm{tr}B)I$ is the deviator of B.

PROOF. Denoting $X \equiv \dot{L} \in Lin_o(\mathcal{V},\mathcal{V})$ and using (3.6) the inequality (3.5) can be written as follows

$$(3.8) \qquad aX^s \cdot X + B \cdot X + c \leq 0$$

for all $X \in Lin_o(\mathcal{V},\mathcal{V})$. Then if we chose $X \equiv \lambda B^a$ with $\lambda \in R$ arbitrary chosen (3.8) becomes:

$$(3.9) \qquad \lambda B^a \cdot B^a + c \leq 0$$

for all $\lambda \in R$, which imply that $B^a \equiv 0$. Then from the relation $(3.6)_2$ it results that $\bar{\bar{\mathcal{T}}}_L(L,\theta) \in SLin(\mathcal{V},\mathcal{V})$. On the other hand the inequality (3.8) becomes:

$$(3.10) \qquad aX \cdot X + B' \cdot X + c \leq 0$$

for all $X \in SLin_o(\mathcal{V},\mathcal{V})$.

Then if we chose $X \equiv \lambda B'$ for arbitrary $\lambda \in R$ we immediately see that (3.10) is valids if and only if the relations (3.7) are valid, which concludes the proof of the necessity. For the sufficiency we apply the Schwartz inequality and the proof is immediate ∎

REMARK 3.2. The three conditions (3.7) will be separately analysed. For this first we shall observe that $(3.7)_1$ implies $\beta_1(\theta)\leq 0$. The equality leads to the degeneracy of the constitutive law (3.1) by the disappearance of A_3 (it is the case which has been obtained as necessary by Fosdick and Rajagopal considering the supplemntary hypoteses mentioned to the beginning of this section). We shall not consider this hypotesis and by consequence $\beta_1(\theta)<0$ ∎

The condition $(3.7)_2$ together with $(3.6)_3$ lead to:

$$\beta_1(\theta)\,\mathrm{tr}\left[A_1^2W^2-(A_1W)^2\right]+\mu(\theta)\,\mathrm{tr}A_1^2+\left[\alpha_1(\theta)+\alpha_2(\theta)\right]\mathrm{tr}A_1^3+ \tag{3.11}$$
$$+\frac{1}{2}\left[\beta_1(\theta)+2\beta_2(\theta)+2\beta_3(\theta)\right](\mathrm{tr}A_1^2)^2\geq 0$$

for all $\theta>0$ and $A_1\in \mathrm{SLin}_o(\mathcal{V},\mathcal{V})$, $W\in \mathrm{SkewLin}(\mathcal{V},\mathcal{V})$. Then taking into account the condition $(3.7)_1$ we can prove the following proposition:

PROPOSITION 3.1. The condition (3.11) with $\beta_1(\theta)<0$ is equivalent to:

$$\mu(\theta)\,\mathrm{tr}A_1^2+\left[\alpha_1(\theta)+\alpha_2(\theta)\right]\mathrm{tr}A_1^3+\frac{1}{2}\left[\beta_1(\theta)+2\beta_2(\theta)+2\beta_3(\theta)\right](\mathrm{tr}A_1^2)^2\geq 0 \tag{3.12}$$

for all $\theta>0$ and $A_1\in \mathrm{SLin}_o(\mathcal{V},\mathcal{V})$.

PROOF. The necessity is immediate if we choose $W\equiv 0$. For the sufficiency we denote by $\mathcal{A}(\theta,A_1)$ the left hand side of (3.12) and we choose $W\neq 0$, $W\in \mathrm{SkewLin}(\mathcal{V},\mathcal{V})$. Let λ_i, $i=1,2,3$, the proper values of A_1 (not necessary equal). Let $w_{12}=\alpha$, $w_{13}=\beta$, $w_{23}=\gamma$ the representation of W in the proper vector base of A_1. A simple calculus leads to the following inequality

$$(3.13)\qquad \mathcal{A}(\theta,A_1)-\beta_1(\theta)\left[\alpha^2(\lambda_1-\lambda_2)^2+\beta^2(\lambda_3-\lambda_1)^2+\gamma^2(\lambda_2-\lambda_3)^2\right]\geq 0$$

for all $\theta>0$, $\lambda_i\in R$ ($i=1,2,3$, $\lambda_1+\lambda_2+\lambda_3=0$) and $\alpha,\beta,\gamma\in R$, which gives the proof of the proposition.∎

With these we can state the main theorem of this section which gives the necessary and sufficient conditions for the validity of the inequality (3.12):

THEOREM 3.3. The condition (3.12) is equivalent to the following restrictions on the constitutive coefficients:

$$(3.14)\qquad \begin{cases}\mu(\theta)\geq 0\\ \beta_1(\theta)+2\beta_2(\theta)+2\beta_3(\theta)\geq 0\\ \alpha_1(\theta)+\alpha_2(\theta)=0\end{cases}$$

for all $\theta>0$.

PROOF. The sufficiency is immediate. For the necessity we denote, as in the proof of the Proposition 3.1, with λ_i, $i=1,2,3$ the proper values of A_1, $\lambda_1+\lambda_2+\lambda_3=0$. Then we can write (3.12) as follows:

$$(3.15)\qquad 2\mu(\theta)\left[(\lambda_1+\lambda_2)^2-\lambda_1\lambda_2\right]-3\left[\alpha_1(\theta)+\alpha_2(\theta)\right]\lambda_1\lambda_2(\lambda_1+\lambda_2)+$$
$$+2\left[\beta_1(\theta)+2\beta_2(\theta)+2\beta_3(\theta)\right]\left[(\lambda_1+\lambda_2)^2-\lambda_1\lambda_2\right]^2\geq 0$$

for all $\lambda_1,\lambda_2\in R$.

Denoting $\lambda_1+\lambda_2=s$, $\lambda_1\lambda_2=p$ and $s^2-p=Z$ it results that $Z\geq 0$ and (3.15) becomes:

$$(3.16)\qquad 2\left[\beta_1(\theta)+2\beta_2(\theta)+2\beta_3(\theta)\right]Z^2+\left[2\mu(\theta)+3(\alpha_1(\theta)+\alpha_2(\theta))s\right]Z-$$
$$-3s^3\left[\alpha_1(\theta)+\alpha_2(\theta)\right]\geq 0\ ,$$

for all $\theta > 0$, $Z \geq 0$ and $s \in R$. Then with necessity we have:

$$(3.17)\qquad \begin{cases} \mu(\theta) \geq 0 \\ \beta_1(\theta) + 2\beta_2(\theta) + 2\beta_3(\theta) \geq 0 \end{cases}$$

for all $\theta > 0$.

Moreover, a necessary condition for the achievement of (3.16) is:

$$(3.18)\qquad -s^3[\alpha_1(\theta) + \alpha_2(\theta)] \geq 0$$

for all $\theta > 0$ and $s \in R$, which gives the relation $(3.14)_3$ and concludes the proof.∎

In conclusion, taking into account the Remark of the end of the Section 2 (the relation (2.7)) according to which the Fourier inequality is valid in the neighbourhood of the equilibrium states, it results that the Clausius-Duhem inequality is valid in the neighbourhood of the equilibrium states if and only if the constitutive coefficients of the material verify the restrictions:

$$(3.19)\qquad \begin{cases} \beta_1(\theta) < 0, \quad \mu(\theta) \geq 0 \\ \beta_1(\theta) + 2\beta_2(\theta) + 2\beta_3(\theta) \geq 0 \\ \alpha_1(\theta) + \alpha_2(\theta) = 0 \end{cases}$$

for all $\theta > 0$ and the residual inequality $(3.7)_3$.

On far from equilibrium states the conditions (3.19) are only necessary conditions (and are also sufficient for processes in which $\nabla\theta = 0$).

The interpretation of the residual inequality $(3.7)_3$, which is closely connected with the free energy is the object of the following section.

4. THE FREE ENERGY OF A THIRD GRADE FLUID

In this section we are interested in the construction of the class of functions which can be chosen as free energy for a third grade fluid. For this it is essential to interpret the inequality $(3.7)_3$.

LEMMA 4.1. If $(3.7)_3$ holds then

$$(4.1) \qquad \bar{\bar{\Psi}}_L(L,\theta)A_1 = A_1\bar{\bar{\Psi}}_L(L,\theta)$$

for all $L \in Lin_o(\mathcal{V},\mathcal{V})$.

PROOF. The inequality $(3.7)_3$ is linear in W and must be satisfied for all $W \in SkewLin(\mathcal{V},\mathcal{V})$, then:

$$(4.2) \qquad 4\dot{\rho}\beta_1(\theta)\left[\bar{\bar{\Psi}}'_L(L,\theta)A_1 - A_1\bar{\bar{\Psi}}'_L(L,\theta)\right]\cdot W \geq 0 ,$$

for all $L \in Lin_o(\mathcal{V},\mathcal{V})$ and $W \in SkewLin(\mathcal{V},\mathcal{V})$.

REMARK 4.1. The inequality $(3.7)_3$ (as we see from the Lemma 4.1) is equivalent with the relation (4.1) and the following inequality:

$$(4.3) \qquad |\rho\bar{\bar{\Psi}}'_L(L,\theta) - \alpha_1(\theta)A_1 - [3\beta_1(\theta)+2\beta_2(\theta)]A_1^2|^2 \leq$$

$$\leq -4\beta_1(\theta)\mu(\theta)\,trA_1^2 + \frac{1}{3}\left[3\beta_1^2(\theta)+4\beta_2^2(\theta)-12\beta_1(\theta)\beta_3(\theta)\right]\cdot(trA_1^2)^2 ,$$

where we have denoted $|f|^2 \equiv f\cdot f$ for all $f \in Lin(\mathcal{V},\mathcal{V})$.

From (4.1) we easily deduce the isotropy of $\bar{\bar{\Psi}}_L$ and so we have the following chain rule

(4.4) $\bar{\bar{\Psi}}_L(L,\theta) \equiv 2\tilde{\tilde{\Psi}}_{A_1}(A_1,\theta)=4A_1\partial_x\tilde{\tilde{\Psi}}(x,y,\theta)+6A_1^2\partial_y\tilde{\tilde{\Psi}}(x,y,\theta)$

where we have denoted $x=trA_1^2$, $y=trA_1^3$ and:

(4.5) $\bar{\bar{\Psi}}(L,\theta) \equiv \tilde{\tilde{\Psi}}(A_1,\theta) \equiv \tilde{\tilde{\Psi}}(x,y,\theta)$.

With these we are ready to prove the following lemma:

LEMMA 4.2. If the Clausius-Duhem inequality is valid then with necessity:

(4.6) $3\rho x^2\partial_y\mathring{\Psi}+4\rho y\partial_x\mathring{\Psi}+\mu(\theta)x \geq 0$

for all $x\in R_+$, $\theta>0$, $y\in R$ and where:

(4.7) $\rho\mathring{\Psi}(x,y,\theta) \equiv \rho\tilde{\tilde{\Psi}}(x,y,\theta)-\frac{\alpha_1(\theta)}{4}x-\frac{3\beta_1(\theta)-2\beta_3(\theta)}{6}y$

PROOF. Having in view the relations (4.5) and (3.5) we shall observe that (3.5) becomes (with $W\equiv 0$, $A_2\equiv 0$):

(4.8) $\rho\tilde{\tilde{\Psi}}_{A_1}(A_1,\theta)\cdot A_1^2 \geq -\frac{1}{2}\mu(\theta)trA_1^2+\frac{1}{2}\alpha_1(\theta)trA_1^3+\frac{3\beta_1(\theta)-2\beta_3(\theta)}{4}\cdot (trA_1^2)^2$

which, toghether with (4.4), leads to:

(4.9) $4\rho y\partial_x\tilde{\tilde{\Psi}}(x,y,\theta)+3\rho x^2\partial_y\tilde{\tilde{\Psi}}(x,y,\theta)-\alpha_1(\theta)y-\frac{3\beta_1(\theta)-2\beta_3(\theta)}{2}(trA_1^2)^2+\mu(\theta)x \geq 0$

for all $(x,y)\in R_+xR$ and $\theta>0$. With this the proof is finished.■

REMARK 4.2. We can remark that the free energy on viscometric motion is given by:

$$\tilde{\tilde{\Psi}}(x,0,\theta) \equiv \tilde{\tilde{\Psi}}_0(x,\theta) \tag{4.10}$$

The problem of finding the solutions of the inequation (4.6) with the condition

$$\mathring{\Psi}(x,0,\theta) = \mathring{\Psi}_0(x,\theta)$$

where $\mathring{\Psi}_0(x,\theta) \equiv \tilde{\tilde{\Psi}}_0(x,\theta) - \frac{\alpha_1(\theta)}{4\rho}x$ is thus reduced to the following problem:

$$\begin{cases} 3\rho x^2 \partial_y \mathring{\Psi} + 4\rho y \partial_x \mathring{\Psi} + \mu(\theta)x - f(x,y,\theta) = 0 \\ \mathring{\Psi}(x,0,\theta) = \mathring{\Psi}_0(x,\theta) \end{cases} \tag{4.11}$$

in the domain $R_+ x R$, for all $\theta > 0$ and where

$f \in \mathcal{M} \equiv \{ g : R_+ x R x R_+ \longrightarrow R_+ \mid g \in C^1(R_+ x R) \}$ ■

The Lemmas 4.1, 4.2 and the Remark 4.2 lead us to the following main result:

Theorem 4.1. The problem (4.11) has a unique solution given by:

$$\mathring{\Psi}_f(x,y;\theta) = \mathring{\Psi}_0((x^3-2y^2)^{1/3};\theta) + \frac{1}{3\rho}\int_0^y \left\{ f((x^3-2y^2+2Y^2)^{1/3}, Y;\theta) - \mu(\theta)(x^3-2y^2+2Y^2)^{1/3} \right\} \cdot \frac{dY}{(x^3-2y^2+2Y^2)^{2/3}} \tag{4.12}$$

PROOF. The uniqueness is obvious. For the existence we apply the classic method for solving the linear hyperbolic equations, observing that the problem (4.11) is equivalent to:

$$(4.13)\quad \begin{cases} \dfrac{dx}{dt} = 4\rho y(t) \\[2ex] \dfrac{dy}{dt} = 3\rho x(t) \\[2ex] \dfrac{dz}{dt} = f(x(t),y(t),\theta)-\mu(\theta)x(t) \end{cases}$$

and

$$(4.14)\quad \begin{cases} x(0) = u \\ y(0) = 0 \\ z(0) = \mathring{\varphi}_o(u;\theta) \end{cases}$$

The solution of this new problem lead to

$$(4.15)\quad \begin{cases} u=(x^3(t)-2y^2(t))^{1/3} \\[1ex] 3\rho t= \displaystyle\int_o^{y(t)} \frac{dy}{(u^3+2y^2)^{2/3}} \\[2ex] z(t,u;\theta)=\mathring{\varphi}_o(u;\theta)+\displaystyle\int_o^t [f(x(s,u),\ y(s,u);\theta)-\mu(\theta)x(s,u)]\,ds \end{cases}$$

Then denoting by $\hat{u}$ the inverse of u and $t=\hat{t}(x,y)$ we have:

$$(4.16)\quad \mathring{\varphi}_f(x,y;\theta)=\mathring{\varphi}_o(\hat{u}(x,y);\theta)+\int_o^{\hat{t}(x,y)} [f(x(s,\hat{u}(x,y)),y(s,\hat{u}(x,y));\theta) - \mu(\theta)x(s,\hat{u}(x,y))]\,ds$$

We complete the proof if we observe that the function:

$$Y \longmapsto \frac{1}{3\rho}\int_o^Y \frac{dy}{(u^3+2y^2)^{2/3}}$$

is invertible. Then proceeding in (4.16) to the following change of variables

$$s=\frac{1}{3\rho}\int_0^{\hat{Y}}\frac{d\eta}{(x^3-2y^2+2\eta^2)^{2/3}}\quad,$$

where:

$$\hat{Y}=Y\left(\frac{1}{3\rho}\int_0^{\hat{Y}}\frac{d\eta}{(x^3-2y^2+2\eta^2)^{2/3}}\;,\;(x^3-2y^2)^{1/3}\right)\;,$$

we easily obtain the solution on the form given in the relation (4.12). ∎

<u>REMARK 4.3</u>. From (4.12), (4.10) and (4.7) it results the representation for the free energy $\tilde{\tilde{\Psi}}$ and finally we shall have:

$$(4.17)\qquad \rho\bar{\Psi}_f(A_1,A_2,\theta)=\frac{1}{2}\beta_1(\theta)A_1\cdot A_2+\frac{\alpha_1(\theta)}{4}trA_1^2+$$

$$+\frac{3\beta_1(\theta)-2\beta_3(\theta)}{6}trA_1^3+\rho\mathring{\Psi}_f(trA_1^2,trA_1^3,\theta)$$

Denoting by $z=trA_1A_2$ an elementary calculus leads us to:

$$(4.18)\qquad \rho\bar{\Psi}_f(A_1,A_2,\theta)\equiv\rho\check{\Psi}_f(x,y,z,\theta)=\frac{1}{2}\beta_1(\theta)z-\frac{1}{3}\beta_3(\theta)y+$$

$$+\frac{1}{4}\alpha_1(\theta)\left[x-(x^3-2y^2)^{1/3}\right]+\rho\tilde{\tilde{\Psi}}_o((x^3-2y^2)^{1/3};\theta)+$$

$$+\frac{1}{3}\int_0^y\left[f((x^3-2y^2+2\hat{Y}^2)^{1/3},\overset{\theta}{\underset{\vee}{Y}})-\mu(x^3-2y^2+2\hat{Y}^2)^{1/3}\right]\frac{d\hat{Y}}{(x^3-2y^2+2\hat{Y}^2)^{2/3}}\qquad ∎$$

<u>REMARK 4.4</u>. When (4.18) and restrictions (1.31)-(1.33) are taken into account we observe that the partial derivatives of $\check{\Psi}$ must be bounded on all paths in the neighbourghood of $(0,0,0,\theta)$. Then from the boundedness of $\partial_y\check{\Psi}$ we can easily deduce that

$$(4.19)\qquad 4\rho\,\partial_x\tilde{\tilde{\Psi}}_o(0,\theta)=\alpha_1(\theta)$$

That means that $\alpha_1(\theta)$ is the measure of the variation of free energy on viscometric motion in the neighbourhood of equilibrium state. ■

REMARK 4.5. From the same condition concerning the boundedness of partial derivatives of $\check{\psi}$ we deduce that

$$f(x,y;\theta)=g(x,y;\theta)+\mu(\theta)x \tag{4.20}$$

with

$$\lim_{x,y\to 0}\frac{g(x,y;\theta)}{x^2}=0 \tag{4.21}$$

REMARK 4.6. As $\mathring{\psi}_f(x,y,\theta)$ must verify the equation $(4.11)_1$ for y=0, then the boundedness of the derivative $\partial_y\mathring{\psi}_f\big|_{y=0}$ and the Remark (4.4) lead to

$$\partial_x\tilde{\tilde{\psi}}_o(x,\theta)=\frac{\alpha_1(\theta)}{4\rho} \tag{4.22}$$

and therefore the free energy of a third grade fluid on viscometric motion is linear in $x=\mathrm{tr}A_1^2$:

$$\tilde{\tilde{\psi}}_o(x,\theta)=\frac{1}{4\rho}\alpha_1(\theta)x+\tilde{\tilde{\psi}}_o(0,\theta) \tag{4.23}$$

In addition if we compare the viscometric stress relation (see for example [6]) :

$$T(x,t)=-pI+\frac{\tau(|k|)}{|k|}(M+M^T)+\frac{\sigma_1(|k|)}{|k|^2}M^TM+\frac{\sigma_2(|k|)}{|k|^2}MM^T \tag{4.24}$$

with (3.1) written on viscometric flows, that is:

$$T(x,t)=-pI+\mu(\theta)(M+M^T)+\alpha_1(\theta)(M^TM-MM^T)+ +2\beta_2(\theta)|k|^2(M+M^T)+2\beta_3(\theta)|k|^2(M+M^T) \tag{4.25}$$

where we have emploied the well known relations $A_1\equiv M+MT$,

$A_2 \equiv 2M^TM$ and $A_3 \equiv 0$, we are led to:

$$(4.26)\quad \begin{cases} \dfrac{\tau(|k|)}{|k|} = \mu(\theta) + 2\beta_3(\theta)k^2 + 2\beta_2(\theta)k^2 \\ \dfrac{\sigma_1(|k|)}{k^2} = \alpha_1(\theta) \\ \dfrac{\sigma_2(|k|)}{k^2} = -\alpha_1(\theta) \end{cases}$$

Moreover from the constitutive restrictions $(3.14)_2$ we deduce that $\tau(|k|) \geq 0$, for little $|k|$ ($|k| \to 0$) $\frac{\tau(|k|)}{|k|} = 0(1)$ and for great $|k|$ ($|k| \to \infty$) $\frac{\tau(|k|)}{|k|} = 0(k^2)$. If we employ the relations $(4.26)_{2,3}$ we easily see that $\sigma_2(|k|) - \sigma_1(|k|) = -2\alpha_1 k^2 \neq 0$ and then according to [6], for example, this fluid can experience a Weissemberg effect.

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REFERENCES

[1] W.Noll, Materially uniform simple bodies with inhomogeneities, Arch.Rat.Mech.Anal., 27, 1-32, 1967.

[2] C.Truesdell, Rational thermodynamics. A course of lectures on selected topics, N.Y. Mc.Graw Hill, 1969.

[3] B.D.Coleman; W.Noll, The thermodynamics of elastic materials with heat conduction and viscosity, Arch. Rat. Mech. Anal., 13, 167-178, 1963.

[4] B.D.Coleman, Thermodynamics of materials with memory, Arch. Rat. Mech. Anal., 17, 1-46, 1964.

[5] V.J.Mizel; C.C.Wang, A fading memory hypothesis which suffices for chain rules, Arch. Rat. Mech. Anal., 23, 124-134, 1966.

[6] C.Truesdell; W.Noll, The nonlinear field theories of mechanics, Hand. der Phys. III/3, Springer-Verlag 1965.

[7] J.E.Dunn; R.L.Fosdick, Thermodynamics, stability and boundedness of fluids of complexity 2 and fluids of second grade, Arch. Rat. Mech. Anal. 56, 191-252, 1974.

[8] R.L.Fosdick; K.R.Rajagopal, Thermodynamics and stability of fluids of third grade, Proc. Roy. Soc. London, A339, 351-377, 1980.