INSTITUTUL
DE
MATEMATICĂ

INSTITUTUL NAȚIONAL
PENTRU CREAȚIE
ȘTIINȚIFICĂ ȘI TEHNICĂ

ISSN 0250 3638

ON THE STRONG SOLUTIONS OF THE DOLEANS-DADE AND PROTTER EQUATION

by

Constantin TUDOR

PREPRINT SERIES IN MATHEMATICS

No.6/1984

BUCUREȘTI

ON THE STRONG SOLUTIONS OF THE DOLEAS-DADE PROTTER EQUATION

by

Constantin TUDOR*)

February 1984

*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Pd.Pacii 220, 79622 Bucharest, Romania

On the strong solutions of the Doléans-Dade and Protter equation

C.Tudor
Faculty of Mathematics,University of Bucharest,14 Academy St., Bucharest,Romania

Existence,pathwise uniqueness and stability with respect to the compact convergence in probability of strong solutions for the Doléans-Dade and Protter equation are studied in a setting which covers the Lipschitz and monotone cases.

Let $(\Omega,\mathcal{F},P,(\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space which satisfies the usual assumptions and denote $D(R_+,R^d)$ the Skorokhod space equipped with the canonical filtration $(\mathcal{R}_t)$.

We consider the following stochastic differential equation

$$X(t)=V(t)+\int_0^t a(\cdot s,X)dZ(s) \qquad \text{(I)}$$

(Doléans-Dade and Protter's equation),where V is a d-dimensional càdlàg process, Z is a m-dimensional semimartingale that vanishes in o and $a=(a^{jk})_{j\leq d,k\leq q}$ is a $\mathcal{F}_t\otimes\mathcal{R}_t$-predictable functional with values in $R^d\otimes R^m$.

If Q is an increasing process then we introduce the following conditions:

Condition (α):There exists a positive predictable process γ_t such that

$$\int_0^t \gamma_s dQ_s<\infty \qquad \text{P-a.s. for each } t\geq 0 \qquad (1)$$

$$|a(t,\omega,f)|^2\leq \gamma_t(\omega)\left[1+\sup_{s<t}|f(s)|^2\right] \text{for every } t\geq 0,\omega\in\Omega,f\in D(R_+,R^d) \qquad (2)$$

Condition (β):For each positive integer n there exists a predictable process γ^n_t such that (1) holds and there exists a concave and increasing function $\varphi^n:R_+\longrightarrow R_+$ such that

$$\int_{0+}^1 \frac{dt}{\varphi^n(t)}=\infty \qquad (3)$$

$$|a(t,\omega,f)-a(t,\omega,g)|^2\leq \gamma^n_t(\omega)\varphi^n\left(\sup_{s<t}|f(s)-g(s)|^2\right) \qquad (4)$$

for all $t\geq 0,\omega\in\Omega,f,g\in D(R_+,R^d)$ with $\sup_{s\leq t}|f(s)|\leq n,\sup_{s\leq t}|g(s)|\leq n.$

Let Z=M+A be a decomposition of Z, where M is a locally square integrable martingale and A is càdlàg adapted process with finite variation.

Denote $Q=8(|A|+[M]+\langle M\rangle)$, where $|A|$ is the variation of A, $[M]$ is the quadratic variation of M and $\langle M\rangle$ is the Meyer process associated to M.

<u>Theorem 1</u>. If the conditions (α) and (β) hold (with Q defined as above) then there exists one and only one strong solution of (I) on the space $(\Omega,\mathcal{F},P)$.

<u>Proof</u>. Since (α), (β) imply the existence of a solution measure (see Jacod-Memin [4], Pellaumail [6]) we need to prove only the very good pathwise uniqueness (see [4], T.2.25, p.179).

Let X, Y be two solutions on a very good extension $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathcal{F}}_t,\tilde{P})$. We remark that Q is again a control process of Z considered as defined on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathcal{F}}_t,\tilde{P})$.

Denote $Q^n_t=\int_o^t \gamma^n_s dQ_s$ and define the stopping time

$$T(n,q)=\inf(t\geq o;|X(t)|+|Y(t)|\geq n)\wedge\inf(t\geq o;Q^n_t\geq q)$$

Then for every stopping time $S\leq T(n,q)$ we obtain

$$E(\sup_{t<S}|X(t)-Y(t)|^2)\leq nE(\int_{[o,S)}\varphi^n(\sup_{s<t}|X(s)-Y(s)|^2)dQ^n_t)$$

where from we get $\sup_{t<T(n,q)}|X(t)-Y(t)|=o$ P-a.s. (see [7]).

Taking $q\to\infty$ and $n\to\infty$ we deduce that X and Y are indistinguishable.

<u>Theorem 2</u>. Suppose we are given V, a, Z as in theorem 1 and let X be the strong solution of (I).

For every positive integer n let V^n, a^n that satisfy the assumptions of theorem 1 uniformly in n (with the same functions γ^k_t, φ^k).

Moreover assume that:

(i) V^n converges to V with respect to the compact convergence in probability,

(ii) For every n and R>o there exists a positive predictable process $\delta_t(n,R)$ such that

$$\int_o^t \delta_s(n,R)dQ_s \xrightarrow[n\to\infty]{p} o \quad \text{for each } t$$

$$|a^n(t,f)-a(t,f)|^2 \le \delta_t(n,R)$$

for every $t\ge o, f\in D(R_+,R^d)$ with $\sup_{s\le t}|f(s)|\le R$.

Let X^n be the solution of the equation

$$X^n(t)=V^n(t)+\int_o^t a^n(s,X^n)dZ(s)$$

Then X^n converges to X with respect to the compact convergence in probability.

Proof. Fix $o<\varepsilon<1, T, R>o$ and define the stopping times

$$S(R)=\inf(t\ge o; |X(t)|\ge R-1) \;;\; S(n,\varepsilon)=\inf(t\ge o; |X^n(t)-X(t)|\ge\varepsilon)$$

$$T_n=T(n,\varepsilon,T,R)=S(R)\wedge S(n,\varepsilon)\wedge T$$

Choose, for $i=1-4$, the stopping times $\theta^i_m\nearrow\infty$ such that (without loss of generality)

$$\sup_{t<\theta^1_m}|V^n(t)-V(t)| \xrightarrow[n\to\infty]{L^2} o \;; Q_{\theta^2_m-}\le m$$

$$Q^R_{\theta^3_m-}\le m \;; \int_{(o,\theta^4_m)}\delta_t(n,R)dQ_t \xrightarrow[n\to\infty]{L^1} o$$

If we denote by Y^u (resp. Y^{u-}) the stopping to u (resp. $u-$), then

$$(X^n-X)^{T_n}_t=(V^n-V)^{T_n}_t+\int_o^t\lambda_{[o,T_n]}\{a^n(s,(X^n)^{T_n-})-a^n(s,X^{T_n-})\}dZ(s)+$$

$$\int_o^t\lambda_{[o,T_n]}\{a^n(s,X^{T_n-})-a(s,X^{T_n-})\}dZ(s)$$

where from, for every stopping time $\tau\le\min_i\theta^i_m=\theta_m$, we obtain

$$E(\sup_{t<\tau}|(X^n-X)^{T_n}_t|^2)\le 3E(\sup_{t<\theta^1_m}|V^n(t)-V(t)|^2)+$$

$$3m\,E(\int_{(o,\tau)}\varphi^R(\sup_{s<t}|(X^n-X)^{T_n}_s|^2)dQ^R_t)+3mE(\int_{(o,\tau)}\delta_t(n,R)dQ_t)\le$$

$$C(m,n)+3mE(\int_{(o,\tau)}\varphi^R(\sup_{s<t}|(X^n-X)^{T_n}_s|^2)dQ^R_t) \qquad (5)$$

where $\lim_{n\to\infty}C(m,n)=o$.

Applying (5) to $\tau=\tau_u=\inf(t<\theta_m;\, Q^R_t>u)$ and using the time change theorem and the Jensen inequality we get

$$E(\sup_{t<\tau_u}|(X^n-X)_t^{T_n}|^2)\leq C(m,n)+3m\int_0^u \varphi^R(E(\sup_{s<\tau_t}|(X^n-X)_s^{T_n}|^2))dt$$

hence

$$\overline{\lim_{n\to\infty}}\ E(\sup_{t<\tau_u}|(X^n-X)_t^{T_n}|^2)\leq 3m\int_0^u \varphi^R(\overline{\lim_{n\to\infty}}\ E(\sup_{s<\tau_t}|(X^n-X)_s^{T_n}|^2))dt$$

It follows that

$$\overline{\lim_{n\to\infty}}\ E(\sup_{t<\tau_u}|(X^n-X)_t^{T_n}|^2)=0 \quad \text{for every } u\leq m$$

Taking u=m we obtain

$$\lim_{n\to\infty} E(\sup_{t<\theta_m}|(X^n-X)_t^{T_n}|^2)=0$$

which implies that $(X^n-X)^{T_n}$ converges to o with respect to the compact convergence in probability.In particular $X^n_{T_n}-X_{T_n}\xrightarrow{P}0$.Then

$$P(\sup_{t\leq S(R)\wedge T}|X^n_t-X_t|\geq\varepsilon)\leq P(S(n,\varepsilon)\leq S(R)\wedge T)\xrightarrow[n\to\infty]{}0$$

where from as $S(R)\wedge T\nearrow T$ when $R\to\infty$ we get

$$P(\sup_{t\leq T}|X^n_t-X_t|\geq\varepsilon)\longrightarrow 0$$

and this completes the proof.

Remark. Theorem 2 improves a result of Melnikov[5] and ,in some sense, is stronger than the results of Emery[1] and Jacod-Memin[4].

Let us make the following assumptions:

Condition (α_1): Z is a special semimartingale with the caninical decomposition $Z=\bar{M}+\bar{A}$.Let Q be an increasing predictable process with $Q_0=0$ such that $dQ_t\geq d\bar{A}_t$, $dQ_t\geq d\langle M\rangle_t$ and let $f=(f^j)$, $g=(g^{jk})$ be the predictable processes defined by

$$\langle \bar{M}^j,\bar{M}^k\rangle = g^{jk}.Q \quad ; \quad \bar{A}=f.Q$$

Assume, for each positive integer n, there exists positive predictable process γ^n such that (1) is satisfied(with the previous Q)and there exists a increasing concave function $\varphi^n:R_+\to R_+$ that satisfies (3), such that

$$2\langle x(t-)-y(t-),[a(t,\omega,x)-a(t,\omega,y)]f_t(\omega)\rangle +$$

$$+\Delta Q_t(\omega)\left|\left[a(t,\omega,x)-a(t,\omega,y)\right]f_t(\omega)\right|^2 +$$

$$\sum_{j\leq d;k,l\leq q}\left[a^{jk}(t,\omega,x)-a^{jk}(t,\omega,y)\right]g_t^{kl}\left[a^{jl}(t,\omega,x)-a^{jl}(t,\omega,y)\right]\leq$$

$$\gamma_t^n(\omega)\varphi^n(|x(t-)-y(t-)|^2)$$

for every $t\geq o, \omega\in\Omega, x,y\in D(R_+,R^d)$ with $\sup_{s\leq t}|x(s)|\leq n, \sup_{s\leq t}|y(s)|\leq n.$

Condition (β_1): The functional $a(t,\omega,.)$ is continuous on $D(R_+,R^d)$ endowed with the uniform topology, for $P(d\omega)xdQ_t(\omega)$-almost all (t,ω) (with Q defined in (α_1)).

Theorem 3. Under $(\alpha),(\alpha_1),(\beta_1)$, there exists one and only one strong solution of (I) on the space $(\Omega,\mathcal{F},P)$.

Proof. Let X,Y be two solutions on a good extension $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathcal{F}}_t,\tilde{P})$. It suffices to prove that X and Y are indistinguishable.

Let $\tau_n=\inf(t\geq o; |X(t)|+|Y(t)|\geq n)$, $R_t^n=\int_0^t\gamma_s^n dQ_s+t.$

The Ito formula implies

$$|X(t\wedge\tau_n)-Y(t\wedge\tau_n)|^2=\int_0^{\tau_n\wedge t}\Big\{2\langle X(s-)-Y(s-),[a(s,X)-a(s,Y)]f_s\rangle+$$

$$\Delta Q_s|[a(s,X)-a(s,Y)]f_s|^2+\sum_{j\leq d;k,l\leq q}[a^{jk}(s,X)-a^{jk}(s,Y)]$$

$$g_s^{kl}[a^{jl}(s,X)-a^{jl}(s,Y)]\Big\}dQ_s+\bar{m}(t\wedge\tau_n)\leq$$

$$\int_0^{\tau_n\wedge t}\varphi^n(|X(s-)-Y(s-)|^2)dR_s^n+\bar{m}(t\wedge\tau_n) \qquad (6)$$

where $\bar{m}$ is a local martingale with $\bar{m}(o)=o$.

For each $t\geq o$ and stopping times $(\sigma_i)_{i=1,2,3}$ we put $S=t\wedge\tau_n\wedge\sigma_1, T=\sigma_2\wedge\sigma_3$.

From (6) we deduce

$$|X^S(T)-Y^S(T)|^2\leq\int_0^T\varphi^n(|X^S(p-)-Y^S(p-)|^2)dR_p^n+\bar{m}^S(T) \qquad (7)$$

Let σ^i be a sequence of stopping times which reduces the local martingale $\bar{m}$. Applying (7) to $\sigma_2=\sigma^i$ and taking the expectation we get

$$E(|X^S(\sigma_3\wedge\sigma^i)-Y^S(\sigma_3\wedge\sigma^i)|^2)\leq E\left(\int_0^{\sigma_3\wedge\sigma^i}\varphi^n(|X^S(p-)-Y^S(p-)|^2)dR_p^n\right)$$

where from letting $i\to\infty$ we obtain

$$E(|X^S(\sigma_3)-Y^S(\sigma_3)|^2) \leq E(\int_0^{\sigma_3} \varphi^n(|X^S(p-)-Y^S(p-)|^2)dR_p^n)$$

where from

$$E(|X^S(\sigma-)-Y^S(\sigma-)|^2) \leq E(\int_{(0,\sigma)} \varphi^n(|X^S(p-)-Y^S(p-)|^2)dR_p^n) \quad (8)$$

for every predictable stopping time σ.

Applying (8) to $\sigma^n(p)=\inf(s\geq 0; R_s^n \geq p)$ and utilising the inequality

$$\int_{(0,\sigma^n(p))} f(s)dR_s^n \leq \int_0^u f(\sigma^n(s))ds$$

and the Jensen inequality we obtain

$$E(|X^S(\sigma^n(p)-)-Y^S(\sigma^n(p)-)|^2) \leq \int_0^p \varphi^n(E(|X^S(\sigma^n(u)-)-Y^S(\sigma^n(u)-)|^2))du$$

Therefore for all p we have

$$E(|X^S(\sigma^n(p)-)-Y^S(\sigma^n(p)-)|^2)=0 \quad (9)$$

Let, for all n, a sequence of stopping times $\alpha^n(p)$ which increase to ∞ as $p\to\infty$ such that $\alpha^n(p)<\sigma^n(p)$ (see Jacod[3],L.1.37,p.17).

Then by using (9) with $\sigma_1=\alpha^n(p)$ we obtain

$$E(|X(t\wedge\tau_n\wedge\alpha^n(p))-Y(t\wedge\tau_n\wedge\alpha^n(p))|^2)=0$$

where from letting $p\to\infty$ and $n\to\infty$ we deduce that $X(t)=Y(t)$ a.s.

Remark. The result of theorem 3 covers the results of Gyongy-Krylov[2] and Jacod-Memin[4] in the monotone case.

Theorem 4. Let a, Z be as in theorem 2 and let X be the strong solution of (I) with $V(t)=V_0$ for every t, where V_0 a $\mathcal{F}_0$-measurable random variable with values in R^d.

For every positive integer n let V_0^n a R^d-valued $\mathcal{F}_0$-measurable random variable and a functional a^n which satisfies the assumptions of theorem 3 uniformly in n.

Let X_n be the strong solution of

$$X_n(t)=V_0^n+\int_0^t a_n(s,X_n)dZ(s) \quad (10)$$

Moreover assume that $V_0^n \xrightarrow{p} V_0$ and that the hypothesis (ii) of theorem 2 is satisfied (with Q given as in (α_1)).

Then X_n converges to X with respect to the compact convergence in probability.

We need the following result (which may there exists somewhere in the literature).

Lemma. Let $X,(X_n)_{n\geq 1}$ be predictable processes such that $X_n(o)=o$ for all n. Then X_n converge to X with respect to the compact convergence in probability if and only if every sub-sequence contains another sub-sub-sequence which converges locally in L^p to X (here $1\leq p<\infty$ is fixed).

Proof. It suffices to prove that if X_n converges to o with respect to the compact convergence in probability then it contains a sub-sequence which converges locally in L^p to o.

Without loss of generality we may assume that X_n converges to o with respect to the compact convergence almost surely.

Let S_n be the predictable stopping time defined by $S_n=\inf(t\geq o;|X_n(t)|\geq 1)$. Since $X_n(o)=o$ it follows that $S_n>o$ and therefore there exists a stopping time $T_n<S_n$ such that $P(T_n\leq S_n-1/n)\leq 1/2^n$.

Define $U_n=\inf_{p\geq n} T_p$. We claim that $U_n\nearrow\infty$

Lemma Borel-Cantelli implies that a.s. $S_n-1/n\leq T_n$ for n sufficiently large.

It follows that, for k fixed and for almost all ω, there exists a integer $N(\omega)\geq k$ such that $\sup_{t\leq k}|X_n(t,\omega)|<1$ and $S_n(\omega)-1/n\leq T_n(\omega)$.

Then $S_n(\omega)\geq k$ and thus $T_n(\omega)\geq k-1/k$ for $n\geq N(\omega)$, so that $U_{N(\omega)}(\omega)\geq k-1/k$ and this shows that $U_n(\omega)\nearrow\infty$.

Now, for fixed n, let Y_m be the process X_m stopped at $U_n\wedge n$. Since

$$Y_m^*=\sup_t|Y_m|\leq\sup_{t\leq n}|X_m(t)|\xrightarrow{a.s.}o \text{ as } m\to\infty$$ and $|Y_m^*|\leq 1$ it follows that $Y_m\to o$ in L^p.

Proof of theorem 4. By multiplying (I) and (1o) with $\exp(-\sup_n|V_o^n|)$ which is $\mathcal{F}_o$-measurable and positive a.s., we see that we do not loss of generality if we assume that $V_o^n\xrightarrow{L^2}V_o$.

Fix $o<\varepsilon<1$, $T,R>o$ and a stopping time τ. Define $Q_t^R=\int_o^t\gamma_t^R dQ_t$. We may assume that there exists a sequence of stopping times $\sigma(m,R)$ ~~such that~~

$$Q_{\sigma(m,R)}+Q^R_{\sigma(m,R)}\leq m\;;\int_o^{\sigma(m,R)}\delta_t(n,R)dQ_t\xrightarrow{L^1}o \text{ as } n\to\infty$$

(eventually we pass to a sub-sequence by using the previous lemma).
Now if we denote $U=U(n,\varepsilon,T,R,m,\tau)=T_n\wedge\sigma(m,R)\wedge\tau$ (we keep the notations of theorem 2) then a simple calculation (by utilising the Ito formula) yields

$$E(|X_n^U(\sigma-)-X^U(\sigma-)|^2)\leq E(|V_o^n-V_o|^2)+\sum_{j=1}^{5}E\Big[\int_o^U I_t(j,n)dQ_t\Big]+$$

$$E\Big[\int_{(o,\sigma)}\varphi^R(|X_n^U(t-)-X^U(t-)|^2)dQ_t^R\Big]$$

for every predictable stopping time σ, where

$$I_t(1,n)=2|X(t-)-X_n(t-)|\,\big|[a(t,X_n)-a_n(t,X_n)]f_t\big|$$

$$I_t(2,n)=\Delta Q_t\big|[a(t,X_n)-a_n(t,X_n)]f_t\big|^2$$

$$I_t(3,n)=2\Delta Q_t\big|[a(t,X_n)-a(t,X)]f_t\big|\,\big|[a(t,X_n)-a_n(t,X_n)]f_t\big|$$

$$I_t(4,n)=\sum_{j\leq d;k,l\leq q}\big|[a^{jk}(t,X_n)-a_n^{jk}(t,X_n)]g_t^{kl}[a^{jl}(t,X_n)-a_n^{jl}(t,X_n)]\big|$$

$$I_t(5,n)=2\sum_{j\leq d;k,l\leq q}\big|[a^{jk}(t,X)-a^{jk}(t,X_n)]g_t^{kl}[a^{jl}(t,X_n)-a_n^{jl}(t,X_n)]\big|$$

On the interval $[o,U]$ we have the following estimations

$$I_t(1,n)\leq 2\varepsilon\delta_t^{1/2}(n,R)\ ;\ I_t(2,n)\leq m\,\delta_t(n,R)$$

$$I_t(3,n)\leq C(R)\Delta Q_t[\gamma_t^R\delta_t(n,R)]^{1/2};\ I_t(4,n)\leq\delta_t(n,R)$$

$$I_t(5,n)\leq C(R)[\gamma_t^R\delta_t(n,R)]^{1/2}$$

It follows that

$$E\Big[\int_o^U I_t(1,n)dQ_t\Big]\leq 2m^{1/2}\varepsilon\Big\{E\Big[\int_o^{\sigma(m,R)}\delta_t(n,R)dQ_t\Big]\Big\}^{1/2}\xrightarrow[n\to\infty]{}o$$

$$E\Big\{\int_o^U[I_t(2,n)+I_t(4,n)]dQ_t\leq(m+1)E\Big[\int_o^{\sigma(m,R)}\delta_t(n,R)dQ_t\Big]\xrightarrow[n\to\infty]{}o$$

$$E\Big[\int_o^U I_t(3,n)dQ_t\Big]\leq C(R)E\Big[\int_o^{\sigma(m,R)}(\gamma_t^R)^{1/2}dQ_t\Big]E\Big[\int_o^{\sigma(m,R)}\delta_t^{1/2}(n,R)dQ_t\Big]\leq$$

$$C(m,R)\Big\{E\Big[\int_o^{\sigma(m,R)}\delta_t(n,R)dQ_t\Big]\Big\}^{1/2}\xrightarrow[n\to\infty]{}o$$

$$E\Big[\int_0^U I_t(5,n)\,dQ_t\Big] \le C(R)\,E\Big[\Big|\int_0^{\sigma(m,R)} \gamma_t^R\,dQ_t\Big|^{1/2}\Big]E\Big[\Big|\int_0^{\sigma(m,R)} \delta_t(n,R)\,dQ_t\Big|^{1/2}\Big]$$

$$\le C(m,R)\Big\{E\Big[\int_0^{\sigma(m,R)} \delta_t(n,R)\,dQ_t\Big]\Big\}^{1/2} \xrightarrow[n\to\infty]{} 0$$

Therefore for every predictable stopping time σ we have

$$E(|X_n^U(\sigma-)-X^U(\sigma-)|^2) \le C(n,m,R)+E\Big[\int_{(0,\sigma)} \varphi^R(|X_n^U(t-)-X^U(t-)|^2\,d(Q_t^R+t)\Big]$$

Applying the above inequality with $\sigma=\sigma^R(t)=\inf(p\ge 0;\, Q_p^R+p\ge t)$ and then taking $\overline{\lim}_n$ we get

$$\overline{\lim_n}\ E(|X_n^U(\sigma^R(t)-)-X^U(\sigma^R(t)-)|^2) \le$$

$$\int_0^t \varphi^R\Big[\overline{\lim_n}\ E(|X_n^U(\sigma^R(u)-)-X^U(\sigma^R(u)-)|^2)\Big]du$$

where from

$$\overline{\lim_n}\ E(|X_n^U(\sigma^R(t)-)-X^U(\sigma^R(t)-)|^2)=0 \qquad (11)$$

Choose a sequence of stopping times $\alpha^R(p)\nearrow\infty$ such that $\alpha^R(p)<\sigma^R(p)$. Applying (11) with $\tau=\tau^R(p)$ we deduce

$$\lim_n E(|X_n(T_n\wedge\sigma(m,R))-X(T_n\wedge\sigma(m,R))|^2)=0$$

hence $P(|X_n(T_n\wedge\sigma(m,R))-X(T_n\wedge\sigma(m,R))|\ge\varepsilon)\longrightarrow 0$ as $n\to\infty$

and in particular

$$P(S(n,\varepsilon)\le S(R)\wedge T\wedge\sigma(m,R))\longrightarrow 0 \text{ as } n\to\infty$$

Consequantly

$$P\Big(\sup_{t\le S(R)\wedge T\wedge\sigma(m,R)} |X_n(t)-X(t)|\ge\varepsilon\Big)\longrightarrow 0 \text{ as } n\to\infty$$

where from letting $m\to\infty$, $R\to\infty$ we get

$$P\Big(\sup_{t\le T}|X_n(t)-X(t)|\ge\varepsilon\Big)\longrightarrow 0$$

and this finishes the proof of the theorem.

Remark. Theorem 4 improves a result of Gyongy-Krylov[2].

References

1. Emery, M.: Equations différentielles stochastiques lipschitziennes: étude de la stabilité. Séminaire de Probabilités XIII. Lecture Notes in Math. 721. Berlin-Heidelberg-New York: Springer-Verlag 1979

2. Gyongy, I., Krylov, N.V.: On stochastic equations with respect to semimartingales I. Stochastics 4, 1-21 (198o)

3. Jacod, J.: Calcul stochastique et problèmes de martingales. Lecture Notes in Math. 714. Berlin-Heidelberg-New York: Springer-Verlag 1979

4. Jacod, J., Mémin, J.: Weak and strong solutions of stochastic differential equations: existence and stability, "Stochastic Integrals". Lecture Notes in Math. 851. Berlin-Heidelberg-New York: Springer-Verlag 1981

5. Melnikov, A.V.: On properties of strong solutions of stochastic equations with respect to semimartingales. Stochastics 8, 1o3-119 (1982)

6. Pellaumail, J.: Solutions faibles et semimartingales. Séminaire de Probabilités XV. Lecture Notes in Math. 85o. Berlin-Heidelberg-New York: Springer-Verlag 1981.

7. Tudor, C.: On the Doleans-Dade-Protter equation. A paraître in Séminaire de Probabilités de Strasbourg Vol XVIII. Lecture Notes in Math. Berlin-Heidelberg-New York: Springer-Verlag.