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STRUCTURE OF POSITIVE BLOCK-MATRICES AND NONSTATIONARY PREDICTION

Gr.Arsene and T.Constantinescu

1. INTRODUCTION

What is called today "Schur analysis" originated in the paper [19]; there the structure of the Taylor coefficients of a *contractive* analytic function on the unit disc was done using as parameter the so called "Schur sequences". A lot of work was done to generalize this to the Hilbert space operators context; see [2] for a history of the subject and for a Schur analysis of contractive intertwining dilations using an operator generalization of Schur sequences called choice sequences. The Schur analysis of positive Toeplitz form done in [8] and [9] provides a geometric inside to factorizations, Naimark dilation, and Szegö limit theorems. Further, we applied it in [3] for showing the role of choice sequences in studying Gaussian stationary processes and for giving geometric interpretations to Szegö limit theorems and to the entropy.

Due to their algoritmic feature these phenomena have quite a few connections with questions in extrapolation theory, inverse problems, prediction and filtering, electrical engineering (transmission lines), geophisics (discretization of wave equations) and so on. It is then natural to expect the necessity of passing from stationary case to nonstationary case (i.e. from Toeplitz forms to general positive definite ones); see [16], [18] and the references therein. A Schur analysis of general positive block-matrices was done in [10].

The present paper uses [10] and some new geometric analysis for studying what may be called nonstationary prediction: the angles between parts of the past and the future of nonstationary processes versus the generalizations of Szegö limit theorems and their geometrical interpretations. These generalizations of Szegö-type phenomena include the first and the second limit theorems for nonstationary case, and a new scale of limit theorems which connects these.

Let us shortly describe the contents of this paper. Section 2 gives preliminary results concerning the structure of row (or column) contractions, elementary rotation associated to a contraction and some results (Lemmas 2.1, 2.2 and 2.3) about various properties of composition of elementary rotations, which are necessary in the next section. In Section 3 we recall first the Schur analysis for positive-definite block-kernels on $\mathbb{Z}$ and their Kolmogorov decomposition using generalized choice sequences. Then we consider the (Gaussian, nonstationary) process associated to the kernel and we define some angle operators in it. The main result of this section is Theorem 3.4; it gives a formula for angle operators which will be useful in computation of determinants. Theorem 3.4 uses Theorem 3.2 which has its own interest. Section 4 gives another variant for studying angle operators using Schur complements (Theorem 4.1). Section 5 contains applications. The first part describes thoroughly how to use previous sections in the stationary case; this presentation is generalized to the nonstationary case in the second part. Here we include a scale of Szegö-type limit theorems, their interpretation, and the relations with generalized choice sequences. The last part gives the interpretation in our setting of a result from [17] (where a particular nonstationary case was completely analysed using different methods).

Let us note that the general notations concerning Hilbert spaces operators are those of [21].

2. PRELIMINARIES

For two Hilbert spaces H and H', let $L(H,H')$ denote the space of all (linear, bounded) operators from H into H'; we write $L(H)$ for $L(H,H)$. If $T \in L(H,H')$ is a contraction (i.e., $\|T\| \leq 1$), then $D_T = (I - T^*T)^{\frac{1}{2}}$ and $\mathcal{D}_T = \overline{D_T(H)}$ are the defect operator, resp. the defect space of T.

If $H = \bigoplus_{k=1}^{\infty} H_k$, the structure of a row contraction $T \in L(H,H')$ (and of its defect spaces) was given in [9]. This structure goes as follows:

$$(2.1)_\infty \quad T = (T_1, D_{T_1^*}T_2, \ldots, D_{T_1^*} \ldots D_{T_{k-1}^*} T_k, \ldots),$$

where T_1 is a contraction in $L(H_1,H')$ and for each $k \geq 2$, T_k is a contraction in $L(H_k, \mathcal{D}_{T_{k-1}^*})$. For describing $\mathcal{D}_T$, consider (for each $k \geq 1$) the operators:

$$(2.2)_k \begin{cases} D_k(T) : \bigoplus_{j=1}^{k} H_j (= H^{[k]}) \to \bigoplus_{j=1}^{k} \mathcal{D}_{T_j} \subset \bigoplus_{j=1}^{\infty} \mathcal{D}_{T_j} \\[2ex] D_k(T) = \begin{bmatrix} D_{T_1} & -T_1^*T_2 & -T_1^*D_{T_2^*}T_3 \ldots & -T_1^*D_{T_2^*} \ldots D_{T_{k-1}^*} T_k \\ 0 & D_{T_2} & -T_2^*T_3 \quad \ldots & -T_2^*D_{T_3^*} \ldots D_{T_{k-1}^*} T_k \\ 0 & 0 & D_{T_3} \ldots & \\ \vdots & & \ldots & \\ 0 & 0 & 0 \quad \ldots & D_{T_k} \end{bmatrix}, \end{cases}$$

and the operator:

$$(2.2)_\infty \begin{cases} D_\infty(T) : H \to \bigoplus_{j=1}^{\infty} D_{T_j} \\ D_\infty(T) = \text{s-}\lim_{k\to\infty} D_k(T) P^{H}_{H^{[k]}}, \end{cases}$$

where for a (closed) subspace H' of H, $P^{H}_{H'}$ is the orthogonal projection of H onto H'. Then the operator:

$$(2.3)_\infty \begin{cases} \alpha(T) : D_T \to \bigoplus_{j=1}^{\infty} D_{T_j} = D(T) \\ \alpha(T) D_T = D_\infty(T) \end{cases}$$

is a unitary operator.

For the description of D_{T^*}, consider (for each $k \geq 1$) the operators:

$$(2.4)_k \begin{cases} H_k(T) : H' \to D_{T_k^*} \\ H_k(T) = D_{T_k^*} \dots D_{T_1^*} \quad , \end{cases}$$

and the operator

$$(2.4)_\infty \begin{cases} H_\infty(T) : H' \to H' \\ H^2_\infty(T) = \text{s-}\lim_{k\to\infty} H_k^*(T) H_k(T) . \end{cases}$$

Then the operator

$$(2.5)_\infty \begin{cases} \beta(T) : D_{T^*} \to \overline{\operatorname{Ran} H_\infty(T)} = D_*(T) \\ \beta(T) D_{T^*} = H_\infty(T) \end{cases}$$

is a unitary operator.

In the case where $H_k = 0$ for $k > n$, the relations $(2.1)_\infty$, $(2.3)_\infty$, and $(2.5)_\infty$ can be written as follows:

$$(2.1)_n \quad T = (T_1, D_{T_1^*} T_2, \dots, D_{T_1^*} \dots D_{T_{n-1}^*} T_n),$$

$$(2.3)_n \begin{cases} \alpha(T) : D_T \to \bigoplus_{j=1}^{n} D_{T_j} \\ \alpha(T) D_T = D_n(T) \mid D_T \end{cases}$$

(where $D_n(T)$ is defined in $(2.2)_n$), and

$$(2.5)_n \quad \begin{cases} \beta(T) : \mathcal{D}_{T^*} \to \mathcal{D}_{T_n^*} \\ \beta(T)D_{T^*} = H_n(T), \end{cases}$$

(where $H_n(T)$ is defined in $(2.4)_n$).

The useful feature of these formulas is that the knowledge of a decomposition of type $(2.1)_n$ for T ($n \in \mathbb{N} \cup \{\infty\}$) implies a upper triangularization for D_T having on the diagonal the sequence $\{D_{T_k}\}_{k=1}^n$.

For an arbitrary row contraction T as in $(2.1)_\infty$ the following operator will be useful in the sequel:

$$(2.6) \quad \begin{cases} W_+(T;H_1,H_2 \dots) = W_+(T) : H \to H' \oplus \mathcal{D}(T) \\ W_+(T) = \begin{bmatrix} T \\ D_\infty(T) \end{bmatrix} . \end{cases}$$

The operator $W_+(T)$ is clearly an isometry (see $(2.3)_\infty$); it is connected with the so-called adequate isometries in [7]. It is evident that $W_+(T)$ (and $\alpha(T)$, $\beta(T)$, $D_\infty(T)$ and so on) depends upon the decomposition of H as $\bigoplus_{j=1}^{\infty} H_j$; we will omit the decomposition in the notation only where the context is unambigous.

For a contraction $\Gamma \in L(H,H')$ the following unitary operator will be repeatedly used:

$$(2.7) \quad \begin{cases} J(\Gamma) : H \oplus \mathcal{D}_{\Gamma^*} \to H' \oplus \mathcal{D}_\Gamma \\ J(\Gamma) = \begin{bmatrix} \Gamma & D_{\Gamma^*} \\ D_\Gamma & -\Gamma^* \end{bmatrix} . \end{cases}$$

We will call $J(\Gamma)$ the *elementary rotation* given by Γ. Note that from the definitions it results that

$$(2.8) \quad J(\Gamma) = W_+(W_+(\Gamma^*;H')^*;H,\mathcal{D}_{\Gamma^*}).$$

On the other hand, having in mind $(2.3)_\infty$ and $(2.5)_\infty$ it is natural to consider the unitary operator

$$(2.9) \quad \begin{cases} W(T;H_1,H_2,\dots) = W(T) : \mathcal{D}_*(T) \oplus H \to H' \oplus \mathcal{D}(T) \\ W(T) = \begin{bmatrix} I & 0 \\ 0 & \alpha(T) \end{bmatrix} J(T) \begin{bmatrix} 0 & I \\ \beta(T) & 0 \end{bmatrix} . \end{cases}$$

Then we have that

(2.10) $W_+(T) = W(T)|\mathcal{H}$.

The elementary operations of deleting some subspaces and/or putting together some others in the decomposition of $\mathcal{H}$ as $\bigoplus_{j=1}^{\infty} \mathcal{H}_j$ have nice interpretations for the previous formulas; we will need only the following simple (but useful for induction) fact.

For a row contraction T as in $(2.1)_\infty$ and a positive integer $n \geq 2$, let us denote by $T_{(n)}$ the infinite row contraction

$$(2.11)_n \qquad \begin{aligned} &T_{(n)} : \bigoplus_{j=n}^{\infty} \mathcal{H}_j \to \mathcal{D}_{T^*_{n-1}} \\ &T_{(n)} = (T_n, D_{T^*_n} T_{n+1}, \ldots). \end{aligned}$$

Then we have

LEMMA 2.1. *With previous notation,*

(2.12) $W_+(T) = (J(T_1) \oplus I)(I \oplus W_+(T_{(2)}))$,

where the direct sums in the right hand side are written with respect to the decompositions $\mathcal{H}_1 \oplus \bigoplus_{j=2}^{\infty} \mathcal{H}_j$ *and* $(\mathcal{H}_1 \oplus \mathcal{D}_{T^*_1}) \oplus (\bigoplus_{j=2}^{\infty} \mathcal{D}_{T_j})$, *respectively.*

PROOF. The lemma follows by direct matrix computations, using formulas (2.7), (2.6), $(2.2)_\infty$ and $\{(2.2)_k\}_{k \in \mathbb{N}}$.

The iterative use of Lemma 2.1 gives an idea about the connections between $W_+(T)$ and the family $\{J(T_k)\}_{k=1}^{\infty}$ (see also [7]).

Our next aim is to establish a formula for the elementary rotation of a column contraction. We do not repeat the corresponding notation for columns; let us only note that in this case the operator α from (2.3) is the identification of the defect of the adjoint.

Let S be a contraction between $\mathcal{H}$ and $\mathcal{H}' = \mathcal{H}'_1 \oplus \mathcal{H}'_2$. Then

(2.13) $S = (S_1, S_2 D_{S_1})^t$,

where "t" stands for matrix transpose, and $S_1 \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$, $S_2 \in \mathcal{L}(\mathcal{D}_{S_1}, \mathcal{H}'_2)$ are contractions.

LEMMA 2.2. *For S as in (2.13) we have*

(2.14) $[I \oplus \beta(S)] J(S) [I \oplus \alpha(S)^*] = [I \oplus J(S_2)][J(S_1) \oplus I]$,

where the direct sums are written with respect to the decompositions, in the left hand *side* $(H)\oplus(D_{S_1^*}\oplus D_{S_2^*})$ *and* $(H')\oplus(D_S)$, *and in the right hand side* $(H\oplus D_{S_1^*})\oplus(D_{S_2^*})$ *and* $(H'_1)\oplus(D_{S_1}\oplus D_{S_2^*})$.

PROOF. Using (2.7) it follows that the operator in the right hand side of (2.14) is acting between $H\oplus D_{S_1^*}\oplus D_{S_2^*}$ and $H'_1\oplus H'_2\oplus D_{S_2^*}$, by the matrix

$$(2.15)\qquad \begin{bmatrix} S_1 & D_{S_1^*} & 0 \\ S_2 D_{S_1} & -S_2 S_1^* & D_{S_2^*} \\ D_{S_2} D_{S_1} & -D_{S_2} S_1^* & -S_2^* \end{bmatrix}.$$

The operator in the left hand side of (2.14) has, with respect to the decompositions $(H)\oplus(D_{S_1^*}\oplus D_{S_2^*})$ and $(H'_1\oplus H'_2)\oplus D_{S_2}$, the following matrix

$$(2.16)\qquad \begin{bmatrix} S & D_{S^*}\alpha(S)^* \\ \beta(S)D_S & -\beta(S)S^*\alpha(S)^* \end{bmatrix}.$$

Taking into account that

$$(2.17)\qquad \alpha(S)D_{S^*} = \begin{bmatrix} D_{S_1^*} & -S_1 S_2^* \\ 0 & D_{S_2^*} \end{bmatrix},$$

and that

$$(2.18)\qquad \beta(S)D_S = D_{S_2} D_{S_1},$$

the formulas (2.16) and (2.15) shows that it remains to prove that

$$(2.19)\qquad \beta(S)S^*\alpha(S)^* = (D_{S_2} S_1^* \quad S_2^*)\,|\,D_{S_1^*}\oplus D_{S_2^*}.$$

For proving (2.19), take $h\in H$ and $s_1 \in D_{S_1^*}$, $s_2 \in D_{S_2^*}$; put $a = D_{S_2} D_{S_1} h \in D_{S_2}$. Then:

$$(2.20)\qquad \langle \beta(S)S^*\alpha(S)^*(s_1\oplus s_2),\, a\rangle = \langle s_1\oplus s_2,\, \alpha(S)S\beta^*(S)D_{S_2}D_{S_1}h\rangle =$$

$$= \langle s_1\oplus s_2,\, \alpha(S)SD_S h\rangle = \langle s_1\oplus s_2,\, \alpha(S)D_{S^*}Sh\rangle =$$

$$= \langle s_1\oplus s_2,\, (D_{S_1^*}S_1 h - S_1S_2^*S_2D_{S_1}h)\oplus(D_{S_2^*}S_2D_{S_1}h\rangle =$$

$$= \langle s_1, S_1 D_{S_2}(D_{S_2} D_{S_1} h)\rangle + \langle s_2, S_2 D_{S_2} D_{S_1} h\rangle =$$

$$= \langle D_{S_2} S_1^* s_1, a\rangle + \langle S_2^* s_2, a\rangle.$$

As $\overline{\{D_{S_2} D_{S_1} h : h \in H\}} = D_{S_1}$, the formula (2.20) implies (2.19), and the lemma is completely proved.

We use now Lemmas 2.1 and 2.2 for obtaining a multiplicative formula for the operator W_+ of two "coupling" rows. More precisely let T be a row contraction as in $(2.1)_\infty$ and R another row contraction

$$(2.21)\quad \begin{cases} R : D(T) \to H'' \\ R = (R_1, D_{R_1^*} R_2, \ldots), \end{cases}$$

where $R_1 : D_{T_1} \to H''$ and $R_j : D_{T_j} \to D_{R_{j-1}^*}$ $(j \geq 2)$ are contractions. Then one can consider the column contractions:

$$(2.22)_1\quad \begin{cases} S_1 : H_1 \to H' \oplus H'' \\ S_1 = (T_1 \quad R_1 D_{T_1})^t, \end{cases}$$

and for $k \geq 2$

$$(2.22)_k\quad \begin{cases} S_k : H_k \to D_{S_{k-1}^*} \\ S_k = \alpha(S_1)^* \ldots \alpha(S_{k-1})^* (T_k \quad R_k D_{T_k})^t. \end{cases}$$

It is clear that

$$(2.22)_\infty\quad \begin{cases} S : \bigoplus_{k=1}^{\infty} H_k \to H' \oplus H'' \\ S = (S_1, D_{S_1^*} S_2, \ldots), \end{cases}$$

is an infinite row contraction written in the canonical form. We write $S = R \# T$, if S is constructed from T and R in the previous manner.

LEMMA 2.3. *With the previous notation:*

$$(2.23)\quad (I \oplus (\beta(S_1) \oplus \beta(S_2) \oplus \ldots))W_+(R \# T) = (I \oplus W_+(R))W_+(T),$$

where the direct sums are written with respect to the decompositions, in the left hand side $(H' \oplus H'') \oplus (D_{S_1} \oplus D_{S_2} \oplus \ldots)$, *and in the right hand side* $(H') \oplus (D_{T_1} \oplus D_{T_2} \oplus \ldots)$.

PROOF. Using Lemma 2.1 it follows that:

$$
\begin{aligned}
&(I_{H'} \oplus W_+(R))W_+(T) = (I_{H'} \oplus J(R_1) \oplus I_{D(R_{(2)})})(I_{H' \oplus D_{T_1}} \oplus W_+(R_{(2)}))(J(T_1) \oplus \\
(2.24)\quad & \oplus I_{D(T_{(2)})})(I_{H_1} \oplus W_+(T_{(2)})) = (I_{H'} \oplus J(R_1) \oplus I_{D(R_{(2)})})(J(T_1) \oplus I_{D_{R_1^*}} \oplus I_{D(R_{(2)})}) \cdot \\
&\cdot (I_{H_1} \oplus I_{D_{T_1^*}} \oplus W_+(R_{(2)}))(I_{H_1} \oplus W_+(T_{(2)}))
\end{aligned}
$$

(here, and in what follows, the indices to the identity operators make clear the decompositions which appear in the direct sums). Using Lemma 2.2, the formula (2.24) becomes

$$
\begin{aligned}
&(I_{H'} \oplus W_+(R))W_+(T) = (((I_{H' \oplus H''} \oplus \beta(S_1))J(S_1)(I_{H_1} \oplus \alpha(S_1)^*)) \oplus I_{D(R_{(2)})}) \cdot \\
(2.25)\quad & \cdot (I_{H_1} \oplus ((I_{D_{T_1^*}} \oplus W_+(R_{(2)}))W_+(T_{(2)}))) = ((I_{H' \oplus H''} \oplus \beta(S_1))J(S_1) \oplus I_{D(R_{(2)})}) \cdot \\
&\cdot (I_{H_1} \oplus ((\alpha(S_1)^* \oplus I_{D(R_{(2)})})(I_{D_{T_1^*}} \oplus W_+(R_{(2)}))W_+(T_{(2)}))) .
\end{aligned}
$$

On the other hand, using again Lemma 2.1, we have

$$
\begin{aligned}
&(I_{H' \oplus H''} \oplus (\beta(S_1) \oplus \beta(S_2) \oplus \ldots))W_+(S) = \\
(2.26)\quad &= (I_{H' \oplus H''} \oplus (\beta(S_1) \oplus \beta(S_2) \oplus \ldots))(J(S_1) \oplus I_{D(S_{(2)})})(I_{H_1} \oplus W_+(S_{(2)})) = \\
&= ((I_{H' \oplus H''} \oplus \beta(S_1))J(S_1) \oplus I_{D(R_{(2)})})(I_{H_1} \oplus ((I_{D_{S_1^*}} \oplus \beta(S_2) \oplus \beta(S_3) \oplus \ldots)W_+(S_{(2)}))) = \\
&= ((I_{H' \oplus H''} \oplus \beta(S_1))J(S_1) \oplus I_{D(R_{(2)})})(I_{H_1} \oplus ((\alpha(S_1)^* \oplus \beta(S_2) \oplus \beta(S_3) \oplus \ldots)W_+(\bar{S}_{(2)}))),
\end{aligned}
$$

where the row contraction $\bar{S}_{(2)} = (\bar{S}_2, D_{\bar{S}_2}\bar{S}_3, \ldots)$ with $\bar{S}_k = \alpha(S_1)S_k$ $(k \geq 2)$ verifies $\bar{S}_{(2)} = R_{(2)} \# T_{(2)}$. From (2.25) and (2.26) it follows that (2.23) is reduced to

$$
(2.23)_2 \quad (I_{D_{T_1^*} \oplus D_{R_1^*}} \oplus \beta(\bar{S}_2) \oplus \beta(\bar{S}_3) \oplus \ldots)W_+(\bar{S}_{(2)}) = (I_{D_{T_1^*}} \oplus W_+(R_{(2)}))W_+(T_{(2)}),
$$

which is exacty (2.23) with T and R replaced by $T_{(2)}$, resp. $R_{(2)}$. Thus the procedure can be continued, and this provides, in fact, a proof of (2.23). This can be easily seen if one takes into account the upper triangular form of the operator $W_+(T)$.

Other multiplicative properties connected with the structure of contractions will be discussed in [1].

Let us note that the previous phenomena (and some which will be described later) have interpretations in terms of transmission lines, using ideas from [5], where the structure of a positive Toeplitz form (as given in [8]) was illustrated in circuits setting.

3. SOME ANGLES IN NONSTATIONARY PROCESSES

The main concern of this section is the study of positive-definite block-kernels on $\mathbf{Z}$. Given a family of Hilbert spaces $\{H_n\}_{n\in\mathbf{Z}}$, a positive-definite $(\{H_n\})$-kernel is an application T defined on $\mathbf{Z}\times\mathbf{Z}$ such that $T(i,j) = T_{i,j} \in L(H_j, H_i)$ for every $i,j \in \mathbf{Z}$, and the operators

$$(3.1)_{i,j}\quad \begin{cases} M_{i,j}(T) = M_{i,j} : \bigoplus_{k=i}^{j} H_k \to \bigoplus_{k=i}^{j} H_k \\ M_{i,j} = (T_{m,n})_{i\leq m,n\leq j} \end{cases}$$

for $i,j \in \mathbf{Z}$, $i\leq j$, are all positive. In what follows we will suppose that $T_{i,i} = I$, for each $i \in \mathbf{Z}$; this will simplify the formulas without being a serious restriction (see Remark 1.4 of [10]).

Before going into describing the connections with nonstationary Gaussian processes, let us recall the structure of positive-definite block-kernels on $\mathbf{Z}$, as presented in [10]. For the rest of this section, let us fix a positive-definite $(\{H_n\})$-kernel T; when this will cause no confusion, we will omit the indication of the dependence on T of the objects presented in what follows.

The structure of T can be described using generalized choice sequences (gcs) (see [10]). A generalized $(\{H_n\})$-choice sequence is a family $G = \{G_{i,j}\}_{i,j\in\mathbf{Z},\, i\leq j}$, where $G_{ii} = O_{H_i}$, for $i \in \mathbf{Z}$, and for each $i,j \in \mathbf{Z}$, $i<j$, $G_{i,j} : D_{G_{i+1,j}} \longrightarrow D_{G^*_{i,j-1}}$ is an arbitrary contraction (so $G_{i,i+1}$ acts between H_{i+1} and H_i).

As shown in [10], there exists a one-to-one correspondence between the set of all positive-definite $\{H_n\}$-kernels and the set of all generalized $\{H_n\}$-choice sequences. If the previous mentioned T and G correspond to each other under this correspondence, then one has:

$$(3.2)_{i,i+1}\quad T_{i,i+1} = G_{i,i+1}\ ,$$

for every $i \in \mathbf{Z}$, and for $i,j \in \mathbf{Z}$, $j > i+1$,

$$(3.2)_{i,j}\quad T_{i,j} = R_{i,j-1}U_{i+1,j-1}C_{i+1,j} + D_{G^*_{i,i+1}} \ldots D_{G^*_{i,j-1}} G_{i,j} D_{G_{i+1,j}} \ldots D_{G_{j-1,j}}\ ,$$

where the operators $R_{i,j}$, $U_{i,j}$ and $C_{i,j}$ will be defined immediately. (Note that $\{(3.2)_{i,j}\}_{i,j\in\mathbb{Z}, i<j}$ completely define T from G because $T_{i,i} = I$ and $T_{i,j} = T^*_{j,i}$ for $i,j \in \mathbb{Z}$ and $i > j$; this procedure can be also reversed.)

For a fixed $i \in \mathbb{Z}$, the family $\{G_{i,k}\}_{i<k<\infty}$ defines a row contraction

$$(3.3)_i \quad \begin{cases} R_i : \bigoplus_{k=i+1}^{\infty} \mathcal{D}_{G_{i+1,k}} \to \mathcal{H}_i \\ R_i : (G_{i,i+1}, D_{G^*_{i,i+1}} G_{i,i+2}, \ldots); \end{cases}$$

if $j > i$, the operator $R_{i,j}$ which appears in the formula $(3.2)_{i,j+1}$ is the restriction of R_i to $\bigoplus_{k=i+1}^{j} \mathcal{D}_{G_{i+1,k}}$. Analogously, for a fixed $j \in \mathbb{Z}$, the family $\{G_{-k,j}\}_{-j<k<\infty}$ defines a column contraction

$$(3.4)_j \quad \begin{cases} C_j : \mathcal{H}_j \to \bigoplus_{k=-(j-1)}^{\infty} \mathcal{D}_{G^*_{-k,j-1}} \\ C_j = (G_{j-1,j}, G_{j-2,j} D_{G_{j-1,j}}, \ldots)^t; \end{cases}$$

if $i < j$, the operator $C_{i,j}$ which appears in the formula $(3.2)_{i-1,j}$ is the compression of C_j to $\bigoplus_{k=-(j-1)}^{-i} \mathcal{D}_{G^*_{-k,j-1}}$.

The operators $U_{i,j}$ are "generalized rotations" associated to G. So for every $i \in \mathbb{Z}$

$$(3.5)_{i,i} \quad U_{i,i} = I_{\mathcal{H}_i} : \mathcal{D}_{G^*_{i,i}} \to \mathcal{D}_{G_{i,i}},$$

and for $j > i$

$$(3.5)_{i,j} \quad \begin{cases} U_{i,j} : \bigoplus_{k=-j}^{-i} \mathcal{D}_{G^*_{-k,j}} \to \bigoplus_{k=i}^{j} \mathcal{D}_{G_{i,k}} \\ U_{i,j} = J_j(G_{i,i+1}) J_j(G_{i,i+2}) \ldots J_j(G_{i,j})(U_{i+1,j} \oplus I_{\mathcal{D}_{G^*_{i,j}}}), \end{cases}$$

where the subscript j at $J(G_{i,i+k})$ means that for $1 \le k \le j-i$

$$(3.6)^k_{i,j} \quad \begin{cases} J_j(G_{i,i+k}) : (\bigoplus_{m=1}^{k-1} \mathcal{D}_{G_{i+1,i+m}}) \oplus (\mathcal{D}_{G_{i+1,i+k}} \oplus \mathcal{D}_{G^*_{i,i+k}}) \oplus (\bigoplus_{m=k+1}^{j} \mathcal{D}_{G_{i,i+m}}) \to \\ \to (\bigoplus_{m=1}^{k-1} \mathcal{D}_{G_{i+1,i+m}}) \oplus (\mathcal{D}_{G^*_{i,i+k-1}} \oplus \mathcal{D}_{G_{i,i+k}}) \oplus (\bigoplus_{m=k+1}^{j} \mathcal{D}_{G_{i,i+m}}) \\ J_j(G_{i,i+k}) = I \oplus J(G_{i,i+k}) \oplus I. \end{cases}$$

A first useful byproduct of this analysis is the possibility of obtaining triangular factorizations for each $M_{i,j}(T)$, $i,j \in \mathbb{Z}$, $i \leq j$ (see [10]). For this, consider for $i \in \mathbb{Z}$

$$(3.7)_{i,i} \quad \begin{cases} F_{i,i} : H_i \to H_i \\ F_{i,i} = I \end{cases}$$

and for $j > i$

$$(3.7)_{i,j} \quad \begin{cases} F_{i,j} : \bigoplus_{k=i}^{j} H_k \to \bigoplus_{k=i}^{j} D_{G_{i,k}} \\ F_{i,j} = \begin{bmatrix} F_{i,j-1} & U_{i,j-1}C_{i,j} \\ 0 & D_{G_{i,j}} \cdots D_{G_{j-1,j}} \end{bmatrix} . \end{cases}$$

Then we have for $i,j \in \mathbb{Z}$, $i \leq j$, that

$$(3.8)_{i,j} \quad M_{i,j} = F^*_{i,j} F_{i,j} .$$

It is worth mentioning that $\{F_{i,j}\}$ verify also the following relations (see [10], relation (1.13)):

$$(3.7)'_{i,j} \quad F_{i,j} = \begin{bmatrix} I_{H_i} & R_{i,j}F_{i+1,j} \\ 0 & D_j(R_{i,j})F_{i+1,j} \end{bmatrix} , \qquad (i,j \in \mathbb{Z},\ j > i)$$

where $D_j(R_{i,j})$ is defined as in $(2.2)_j$.

The previous analysis is also useful for describing the so-called Kolmogorov decomposition of T, which means the indication of a discrete process which has F as its covariance matrix. This construction goes as follows ([10]).

For each $i \in \mathbb{Z}$, we apply the analysis of Section 2 to the row contraction R_i defined in $(3.3)_i$. Let us denote by D_i the space $D(R_i)$ considered in $(2.3)_\infty$, and by $D_{i,*}$ the space $D_*(R_i)$ which appears in $(2.5)_\infty$. Consider also the Hilbert spaces:

$$(3.9)_i \quad K_i = \bigoplus_{j=-\infty}^{i-1} D_{j,*} \oplus H_i \oplus D_i .$$

(Note that R_i is defined on $H_{i+1} \oplus D_{i+1}$). Define the unitary operators:

$$(3.10)_i \quad \begin{cases} W_i : K_{i+1} \to K_i \\ W_i = I \oplus W(R_i) , \end{cases}$$

where the direct sum is written with respect to the decompositions $K_{i+1} = (\bigoplus_{j=-\infty}^{i-1} D_{j,*}) \oplus (D_{i,*} \oplus H_{i+1} \oplus D_{i+1})$ and $K_i = (\bigoplus_{j=-\infty}^{i-1} D_{j,*}) \oplus (H_i \oplus D_i)$, and $W(R_i)$ was defined in (2.9).

Putting $K_i^+ = H_i \oplus D_i$, we have that $W_+(R_i)$ (denoted in what follows by W_i^+) is (see (2.6))

$$(3.11)_i \quad \begin{cases} W_i^+ : K_{i+1}^+ \to K_i^+ \subset K_i \\ W_i^+ = W_i | K_{i+1}^+ . \end{cases}$$

The (minimal) Kolmogorov decomposition of T is then the sequence $V(T) = V = \{V(n)\}_{n \in \mathbb{Z}}$, defined by

$$(3.12)_n \quad \begin{cases} V(n) : H_n \to K_o \\ V(n) = \begin{cases} W_{-1}^* W_{-2}^* \ldots W_n^* | H_n, & n < 0 \\ (P_{H_o}^{K_o})^*, & n = 0 \\ W_o W_1 \ldots W_{n-1} | H_n, & n > 0 ; \end{cases} \end{cases}$$

this means that $T_{i,j} = V(i)^* V(j)$ for each $i,j \in \mathbb{Z}$ and that $K_o = \bigvee_{n \in \mathbb{Z}} V(n)H_n$. The sequence V is an identification of the Gaussian discrete (nonstationary) process which has T as its covariance matrix.

The minimality condition and the triangular structure of each $D_\infty(R_i)$, $i \in \mathbb{N}$, (see $(2.2)_\infty$) imply that $K_o^+ = \bigvee_{n=0}^{\infty} V(n)H_n$ is exactly $H_o \oplus D_o$. The same argument shows that for any $n \geq 1$, $\bigoplus_{k=0}^{n-1} D_{G_{o,k}}$ is equal to $\bigvee_{k=0}^{n-1} V(k)H_k$; we denote this space with $K_{o,n}$. More generally, for two integers $p < q$, $K_{p,q}$ is, by definition, the space $\bigvee_{k=p}^{q-1} V(k)H_k$. We will also need the space $K_o^- = \bigvee_{n=-\infty}^{-1} V(n)H_n$.

The evolution (and the prediction) of the process V is connected with some angles between these subspaces of K_o. In this respect consider, for each $n \geq 1$, the projections P_n^+ and P_n^- of K_o onto $K_{o,n}$, resp. $K_{-n,0}$; and $P^+ = P_{K_o^+}^{K_o}$, $P^- = P_{K_o^-}^{K_o}$.

The key operator to be studied is

$$(3.13)\quad \begin{cases} B : K_o \to K_o \\ B = P^- P^+ P^- , \end{cases}$$

which is a measure of the angle between the past (K_o^-) and the future (K_o^+). (Compare with [13], [15], where this operator is denoted by B_1). B is approximated by the operators

$$(3.14)_n \quad \begin{cases} B_n : K_o \to K_o \\ B_n = P_n^- P^+ P_n^- , \quad (n \geq 1), \end{cases}$$

which measure the angle between parts of the past ($K_{-n,-1}$) and the entire future.

This analysis can go further by considering, for each $n \geq 1$ and $k \geq 1$, the operator

$$(3.15)_{n,k} \quad \begin{cases} B_{n,k} : K_o \to K_o \\ B_{n,k} = P_n^- P_k^+ P_n^- . \end{cases}$$

These operators, which measure the angle between $K_{-n,-1}$ and $K_{o,k}$, will be studied in the next section.

Our next aim will be the study of the structure of the operators $I - B_n$, $n \geq 1$. For this we consider the spaces $K_{0,n}^{(m)} \subset K_m$, defined as $K_{0,n}^{(m)} = \bigoplus_{k=0}^{n-1} \mathcal{D}_{G_{m,m+k}}$, $(n \geq 1$, $m \in \mathbb{Z})$; clearly $K_{0,n}^{(0)} = K_{0,n}$. Denote by $P_n^{+(m)}$ the projection of K_m onto $K_{0,n}^{(m)}$; clearly $P_n^{+(0)} = P_n^+$. With these notations, we have:

LEMMA 3.1. *For every* $n \geq 1$,

$$(3.15)_n \quad I - B_n = W_{-1}^* \cdots W_{-n}^* (I_{-n-1} \oplus D_{A_n^*}^2) W_{-n} \cdots W_{-1},$$

$$\bigoplus_{j=-\infty} \mathcal{D}_{j,*}$$

where

$$(3.17)_n \quad \begin{cases} A_n : K_0^+ \to K_{-n}^+ \\ A_n = P_n^{+(-n)} W_{-n}^+ \cdots W_{-1}^+ . \end{cases}$$

PROOF. First, note that

$$(3.18)_n \quad P_n^- = W_{-1}^* \cdots W_{-n}^* P_n^{+(-n)} W_{-n} \cdots W_{-1} .$$

This relation follows from the equality

$$(3.19)_n \quad W_{-n} \cdots W_{-1} (K_{-n,0}) = K_{0,n}^{(-n)};$$

this is equivalent with the fact that the closed linear span of H_{-n}, $W_{-n}H_{-n+1}, \ldots,$ $\ldots, W_{-n}\ldots W_{-2}H_{-1}$ is $H_{-n} \oplus D_{G_{-n,-n+1}} \oplus \ldots \oplus D_{G_{-n,-1}}$, which can be easily proved by induction using the upper triangular form of each W_i^+, $i \in \mathbb{Z}$, and the structure of the diagonals (see relations $(2.2)_k$, $k \geq 1$).

Using $(3.18)_n$, we have

$$I_{K_o} - B_n = I_{K_o} - P_n^- P^+ P_n^- =$$

$$= I_{K_o} - W^*_{-1}\ldots W^*_{-n} P_n^{+(-n)} W_{-n}\ldots W_{-1} P^+ W^*_{-1}\ldots W^*_{-n} P_n^{+(-n)} W_{-n}\ldots W_{-1} =$$

$$= W^*_{-1}\ldots W^*_{-n}(I_{K_{-n}} - P_n^{+(-n)} W_{-n}\ldots W_{-1} P^+ W^*_{-1}\ldots W^*_{-n} P_n^{+(-n)}) W_{-n}\ldots W_{-1} =$$

$$= W^*_{-1}\ldots W^*_{-n}\Big(I_{\bigoplus_{j=-\infty}^{-n-1} D_{j,*}} \oplus \big(I_{K^+_{-n}} - P_n^{+(-n)} W^+_{-n}\ldots W^+_{-1} W^{+*}_{-1}\ldots W^{+*}_{-n} \cdot$$

$$\cdot P_n^{+(-n)}\big)\Big) W_{-n}\ldots W_{-1},$$

and the lemma is proved.

The main step is the use of Lemma 2.3 to obtain a nice structure for the product $W^+_{-n}\ldots W^+_{-1}$; this will imply immediately the structure of A_n, and so that of $I - B_n$. To this end, for a given $n \geq 1$ let us define the following row contractions:

$$(3.20)^n_{-n} \qquad S^{(n)}_{-n} = R_{-n}$$

(see $(3.3)_{-n}$), and for every $1 \leq k \leq n-1$, put

$$(3.20)^n_{-n+k} \qquad S^{(n)}_{-n+k} = (S^{(n)}_{-n+k-1})_{(2)} \# R_{-n+k}$$

(for the index (2) see $(2.11)_2$, and for the operation $\#$ see (2.21) and $(2.22)_{k=1}^{\infty}$). These definitions make sense. Indeed the domain of the result of the operation $\#$ is the domain of the *last* term; and the domain of $(R_i)_{(2)}$ is $D(R_{i+1})$ for any $i \in \mathbb{Z}$ (see $(3.3)_i$), exactly as asked in (2.21). Now we can state:

THEOREM 3.2. (i) *For every* $n \geq 1$

$$(3.21)_n \qquad W^+_{-n} W^+_{-n+1}\ldots W^+_{-1} = U'_n W_+(S^{(n)}_{-1}),$$

where U'_n *is a unitary operator, and* $S^{(n)}_{-1}$ *was defined in* $(3.20)^n_{-1}$.

(ii) *The unitary operator* U'_n *from (i) has the property that it is diagonal with the exception of its* $n \times n$ *corner.*

PROOF. (i) The case $n = 3$ contains all the ingredients for the general situation, so we will consider only this case.

Using Lemma 2.1, we have

$$(3.22)\quad W^+_{-3} = W_+(R_{-3}) = U'_{3,1}(I_{H_{-2}} \oplus W_+((R_{-3})_{(2)})),$$

where $U'_{3,1}$ is a unitary operator. Now, Lemma 2.3 implies

$$(3.23)\quad [I_{H_{-2}} \oplus W_+((R_{-3})_{(2)})]W_+(R_{-2}) = U'_{3,2}W_+((R_{-3})_{(2)} \# R_{-2}) = U'_{3,2}W_+(S^{(3)}_{-2}),$$

where $U'_{3,2}$ is a unitary operator, and $S^{(3)}_{-2}$ was defined in $(3.20)^3_{-2}$. Applying again Lemma 2.1 to $S^{(3)}_{-2}$ and Lemma 2.3 for $(S^{(3)}_{-2})_{(2)}$ and R_{-1}, and taking into account (3.22) and (3.23), we obtain:

$$(3.24)\quad W^+_{-3}W^+_{-2}W^+_{-1} = U'_3W_+((S^{(3)}_{-2})_{(2)} \# R_{-1}) = U'_3W_+(S^{(3)}_{-1}),$$

where U'_3 is a unitary operator. But (3.24) is $(3.21)_3$, and the case $n = 3$ is proved.

(ii) The operator U'_n is a product of unitary operators resulting by succesive applications of Lemma 2.1 and Lemma 2.3. The statement follows from the observation that the unitary operators arising from Lemma 2.3 (as $U'_{3,2}$ in (3.23)) are diagonal, and the ones arising from Lemma 2.1 (as $U'_{3,1}$ in (3.22)) do not affect the entries off the $n \times n$ corner. ●

COROLLARY 3.3. *For every* $n \geq 1$, *the operator* A_n *from* $(3.17)_n$ *has the form:*

$$(3.25)_n\quad A_n = U''_n S^{(n)}_{-1},$$

where U''_n *has the property* $U''_n U''^*_n = P^{+(-n)}_n$.

PROOF. The corollary follows (taking $U''_n = P^{+(-n)}_n U'_n$) from $(3.17)_n$, Theorem 3.2 (i), Theorem 3.2 (ii), and (2.6). ●

Having in mind the usefulness of $(3.25)_n$ for the structure of $I - B_n$ (see $(3.16)_n$), it is necessary to clarify the structure of $S^{(n)}_{-1}$ as a row contraction. We put all these facts together in the following:

THEOREM 3.4. *For every* $n \geq 1$,

$$(3.26)_n\quad I - B_n = W'^*_n\Big(I_{\bigoplus_{j=-\infty}^{-n-1} D_{j,*}} \oplus U'''_n D^2_{K^{(n)}} U'''^*_n \oplus I_{K^+_{-n} \ominus K_{0,n}^{(-n)}}\Big)W'_n,$$

where W'_n *and* U'''_n *are unitary operators, and the components* $\{K^{(n)}_j\}^{\infty}_{j=1}$ *of* $K^{(n)}$ *as a row contraction (see* $(2.1)_\infty$*) are, up to some unitary operators, the following* $(1 \leq k \leq \infty)$:

$(3.27)_n^j \quad K_j^{(n)} = (G_{-1,j-1}, G_{-2,j-1}D_{G_{-1,j-1}}, \ldots, G_{-n,j-1}D_{G_{-n+1,j-1}} \ldots D_{G_{-1,j-1}})^t.$

PROOF. The theorem results from Lemma 3.1 and Corollary 3.3, putting $W'_n = W^+_{-n} \ldots W^+_{-1}$, $U'''_n = U''_n | K_{0,n}^{(-n)}$, $T^{(n)} = S_{-1}^{(n)}$; the structure of $K^{(n)}$ follows from $\{(3.20)^n_{-n+k}\}_{k=0}^{n-1}$ and $\{(2.22)_k\}_{k=1}^{\infty}$.

The formula $(3.26)_n$ gives the connection between B_n and the gcs associated to the process; this will allow us to compute the determinant of $I - B_n$ (when this has a meaning) in terms of the gcs.

4. QUALITATIVE NONSTATIONARY SZEGÖ-TYPE PHENOMENA

Consider again the positive-definite $(\{H_n\})$-kernel T with its associated gcs G and the generated process V.

This section is devoted to showing that the operators $\{B_{n,k}\}_{n,k\geq 1}$ (see (3.15)) are connected with a sort of "Schur complements" in the matrices $\{M_{-n,k-1}\}_{n,k\geq 1}$. This will give a simpler alternate way (aside Theorem 3.4) of computing the determinants of $\{I - B_{n,k}\}_{n,k\geq 1}$. Also it will point out that behind the numbers which converge in Szegö limit theorems there are *convergences of some angle operators.*

Consider (for $n \geq 1$ and $k \geq 1$) the operators

$$(4.1)_{n,k} \quad \hat{G}_{n,k} = P_n^{+(-n)} W_{-n} W_{-n+1} \ldots W_{-1} | K_{0,k}.$$

From $(3.15)_{n,k}$ and $(3.18)_n$ it follows that.

$$(4.2)_{n,k} \quad I_{K_0} - B_{n,k} = W^*_{-1} \ldots W^*_{-n} D^2_{\hat{G}_{n,k}} W_{-n} \ldots W_{-1}.$$

On the other hand, for $n \geq 1$ and $k \geq 0$, the matrices $M_{-n,k}$ (see $(3.1)_{-n,k}$) have the form

$$(4.3)_{n,k} \quad M_{-n,k} = \begin{bmatrix} M_{-n,1} & Q_{-n,k} \\ Q^*_{-n,k} & M_{0,k} \end{bmatrix},$$

where $Q_{-n,k} = (T_{i,j})_{-n\leq i\leq -1,\, 0\leq j\leq k}$; the matrix in $(4.2)_{n,k}$ is written with respect to the decomposition of $\bigoplus_{i=-n}^{k} H_i$ as $(\bigoplus_{i=-n}^{-1} H_i) \oplus (\bigoplus_{i=0}^{k} H_i)$.

We can state now:

THEOREM 4.1. *For every* $n \geq 1$ *and* $k \geq 0$, *we have:*

(i) $$Q_{-n,k} = F^*_{-n,-1}\hat{G}_{n,k+1}F_{0,k},$$

the operators $\{F_{i,j}\}$ *being defined in* $(3.7)_{i,j}$;

(ii) $$M_{-n,k} = \begin{bmatrix} F^*_{-n,-1} & 0 \\ 0 & F^*_{0,k} \end{bmatrix} \begin{bmatrix} I & \hat{G}_{n,k+1} \\ \hat{G}^*_{n,k+1} & I \end{bmatrix} \begin{bmatrix} F_{-n,-1} & 0 \\ 0 & F_{0,k} \end{bmatrix};$$

(iii) *If the operators* $\{F_{i,j}\}_{j \geq i}$ *are all invertible, then*

$$I_{K_0} - B_{n,k+1} = W^*_{-1}\cdots W^*_{-n}(I_{K_n} - F^{*-1}_{-n,-1}Q_{-n,k}M^{-1}_{0,k}Q^*_{-n,k}F^{-1}_{-n,-1})W_{-n}\cdots W_{-1}.$$

PROOF. (ii) follows immediately from (i), using $(3.8)_{-n,-1}$ and $(3.8)_{0,k}$; (iii) results from (i) and $(4.2)_{n,k+1}$. So, it remains to prove (i). A proof of (i) is virtually contained in the proof of Lemma 1.2 and Proposition 2.1 of [10]. For completeness we indicate here how the formula (i) follows from the recursive relations verified by the operators $\{F_{i,j}\}_{i \leq j}$. To this end, note first that $\{(3.7)'_{i,j}\}$ imply that for each $i,j \in \mathbb{Z}$, $i \leq j$,

$$(4.4)_{i,j} \quad F_{i,j} = (P_{i,j}|H_i, P_{i,j}W_i|H_{i+1}, \ldots, P_{i,j}W_iW_{i+1}\cdots W_{j-1}|H_j),$$

where we denote in this proof by $P_{i,j}$ the projection of K_i onto $K^{(i)}_{0,j-i+1}$. This can be proved (for a fixed $j \in \mathbb{Z}$) by induction on $-\infty < i \leq j$. Indeed, $(4.4)_{j,j}$ is clear because $F_{j,j} = I_{H_j}$; the induction step uses $(3.7)'_{i,j}$ and the formulas

$$(4.5)_{i,j} \quad P_{i,j}W_i|K^{(i+1)}_{0,j-i} = \begin{bmatrix} R_{i,j} \\ D_j(R_{i,j}) \end{bmatrix},$$

and

$$(4.6)_{i,j} \quad P_{i,j}W_iP_{i+1,j}W_{i+1}\cdots W_j|H_j = P_{i,j}W_i\cdots W_j|H_j,$$

which follows by definitions and from the triangular form of the operators $W_i|H_i \oplus D_i$.

The formula (i) follows now by simple matrix computations using (4.4), (4.5) and the fact that

$$(4.7)_{i,j} \quad T_{i,j} = P^{K_i}_{H_i}W_i\cdots W_{j-1}|H_j, \qquad i,j \in \mathbb{Z},\ i<j.$$

Theorem 4.1 (and the triangular form of $(F_{i,j})_{i \leq j}$) shows the connections between the gcs of the process and various angles associated to the process. These facts will be analysed in the next section.

5. CONSEQUENCES

In this section each term which appears in the sequence $\{H_n\}_{n\in\mathbb{Z}}$ from the definition of the positive-definite kernel T is a *finite dimensional* Hilbert space.

A. Stationary case. We start with the analysis of the situation in which T is a Toeplitz form. Aside classical analysis of asymptotics of Toeplitz determinants and generalizations of these phenomena (see for example [14], [22], [4]), the study of various angles in stationary processes is done in [13], [15], [11], [23], and so on; the connection between these two is presented (in the context of this paper) in [8], [9], [3]. We remind this connection because it gives a paradigm for the general case.

T is a Toeplitz form if $T_{i,j} = T_{i+n,j+n}$ for every $i,j,n \in \mathbb{Z}$, $i \leq j$; the form is determined by the sequence $T_i = T_{1,i+1}$, $i \in \mathbb{N}$. This implies that $H_n = H_m$ for every $n,m \in \mathbb{N}$; denote this space by H. The parameter (gcs) G becomes a usual choice sequence: $G_{i,j} = G_{i+n,j+n}$ for $i,j,n \in \mathbb{Z}$, $i \leq j$, and denoting $G_i = G_{1,i+1}$ for $i \in \mathbb{N}\cup\{0\}$, we have that the sequence $\{G_i\}$ verifies that $G_o = O_H$ and for $i \geq 1$ G_i is a contraction from $D_{G_{i-1}}$ into $D_{G^*_{i-1}}$. The process associated to this form is stationary; the Kolmogorov decomposition becomes the Naimark dilation, i.e. $W_i = W_j$ for each $i,j \in \mathbb{Z}$, and $V(n) = W^n$, $(n \in \mathbb{N})$, where $W = W_i$. We have that $M_{i,j} = M_{i+p,j+p}$ and $F_{i,j} = F_{i+p,j+p}$ for each $i,j,p \in \mathbb{Z}$; we denote $M_i = M_{1,i+1}$ and $F_i = F_{1,i+1}$, $i \in \mathbb{N}$.

It is clear that the form is completely determined by its part on $\mathbb{N}$ (this is not at all the case in general). Moreover, the position of the origin is not important; this implies that the operators $\{B_{n,k}\}_{n,k\in\mathbb{N}}$ do not depend on the choice of the origin in the process.

The computations depend upon the following:

COROLLARY 5.1. *For each* $n \geq 1$ *and* $k \geq 1$

(i) $$\det(I - B_{n,k}) = \Big(\prod_{i=1}^{(n\wedge k)-1} (\det D_{G_i})^{2i}\Big)\Big(\prod_{i=n\wedge k}^{n\vee k} (\det D_{G_i})^{2(n\wedge k)}\Big)\Big(\prod_{i=(n\vee k)+1}^{n+k-1} (\det D_{G_i})^{2(n+k-i)}\Big),$$

where $n\wedge k = \min\{n,k\}$ *and* $n\vee k = \max\{n,k\}$;

(ii) $$\det(I - B_n) = \Big(\prod_{i=1}^{n-1} (\det D_{G_i})^{2i}\Big)\Big(\prod_{i=n}^{\infty} (\det D_{G_i})^{2n}\Big).$$

PROOF. Relation (ii) follows from (i) noting that

$$(5.1)_n \quad \operatorname*{s-lim}_{k\to\infty} (I - B_{n,k}) = (I - B_n).$$

The proof of (i) results from Theorem 4.1 (ii). Indeed, from the quoted formula we have that

$$(5.2)_{n,k} \quad \det(I - B_{n,k}) = (\det M_{n+k-1})/((\det M_{n-1})(\det M_{k-1})).$$

(We consider throughout this section only the nondegenerate case where all choice operators have the defects with nonzero determinant, i.e. they are completely nonunitary contractions. The degenerate case can be easily worked out.) From (3.8) it follows that for every $i \in \mathbf{N}$

$$(5.3)_i \quad \det M_i = (\det F_i)^2 .$$

From (3.7), we have for every $i \in \mathbf{N}$

$$(5.4)_i \quad \det F_i = (\det D_{G_1})^i (\det D_{G_2})^{i-1} \dots (\det D_{G_i}).$$

Relations $(5.2)_{n,k}$, (5.3) and (5.4) imply (i).

REMARK 5.2. Formula (ii) (with appeared in [3], Proposition 5.1 with a different proof)) of Corollary 5.1 can be also proved using Theorem 3.4. Indeed, from $(3.26)_n$ we have

$$(5.5)_n \quad \det(I - B_n) = (\det D_{K^{(n)}})^2 .$$

From $(2.3)_\infty$ and (2.2), we infer that:

$$(5.6)_n \quad \det D_{K^{(n)}} = \det D_\infty(K^{(n)}) = \prod_{j=1}^{\infty} \det D_{K_j^{(n)}} .$$

From $(3.27)_n^j$ and $(2.5)_n$ (for colums contractions), it follows that:

$$(5.7)_j^n \quad \det D_{K_j^{(n)}} = \prod_{k=j}^{j+n-1} \det G_k .$$

From $(5.5)_n$, $(5.6)_n$, and $(5.7)_j^n$ it results the formula (ii) of Corollary 5.1.

Formula (i) can also be obtained in this way, using truncations of $K^{(n)}$.

Let us note that Theorem 3.2 (i), besides being the key ingredient in obtaining Theorem 3.4, has an independent interest. It gives the possibility of understanding the evolution of the process, via the relations (3.12).

We give now the geometric interpretation of the two Szego limit theorems (see [9], [3]); we show also that these two theorems are the first and the last term from a whole scale of Szegö-type limit theorems.

COROLLARY 5.3.

(i) $\lim_{n\to\infty} (\det M_n)/(\det M_{n-1}) = \det (I - B_1) \left(= \prod_{n=1}^{\infty} \det D_{G_n}{}^2 \right)$

(this limit is usually called the geometrical mean of the process and is denoted by $g(T) = g$*).*

(ii) *For every integer* $p \geq 0$

$$\lim_{n\to\infty} [(\det M_n)/(\det M_{n-1})]^{p+1}/(\det M_p) = \lim_{n\to\infty} [(\det M_{p+n})/(\det M_p)]/(\det M_n) =$$
$$= \det (I - B_{p+1}) \left(= g^{p+1}/(\det M_p) \right).$$

(iii) $\lim_{n\to\infty} (\det M_n)/(g^{n+1}) = 1/\det (I - B) \left(= 1/ \prod_{n=1}^{\infty} (\det D_{G_n})^{2n} \right).$

The proof follows from the formulas (5.2) and Corollary 5.1.

REMARK 5.4. Let us make some comments on Corollary 5.3.

(i) The first part of Corollary 5.3 shows that the limit in the first Szegö limit theorem is in fact the approximation of the angle operator B_1 with the sequence $\{B_{1,n}\}$ (or $\{B_{n,1}\}$). It gives also the formula of the geometrical mean of the process in terms of its associated choice sequence. As pointed out in [3], using the notion of the entropy of a process (defined as $h(T) = -\frac{1}{2}\ln g(T)$), one obtains a nice connection with the maximum entropy spectral analysis of Burg [6]. Indeed, because

$$(5.8) \qquad h(T) = - \sum_{n=1}^{\infty} \ln \det D_{G_n} ,$$

if the $n\times n$ corner of T is fixed, the extremal entropy continuation of it is obtained taking $G_k = 0$ for $k \geq n$.

(ii) The second part of Corollary 5.3 is the announced "scale" of Szegö-type theorems. It gives the interpretations for all angle operators B_n, $n \geq 1$, in terms of determinants from T. Case $p = 0$ in (ii) is exactly (i); the second Szegö limit theorem contained in (iii) is the "limit case" in (ii).

(iii) The third part of Corollary 5.3 shows that the limit in the second Szegö limit theorem is in fact the approximation of the angle operator B with the sequence $\{B_n\}$. See[3], Theorems 5.2 and 6.1 for a discussion about the (suggested by (iii)) connection between B being trace-class and the convergence of the product $\prod_{j=1}^{\infty} (\det D_{G_n})^{2n}$.

B. Nonstationary case. We come back now to the general situation of a positive-definite ($\{H_n\}$)-kernel T with its associated gcs G and the generated process V; recall that all H_n are - in this section - finite dimensional. The analysis in Part A clearly indicates how to generalize the Szegö phenomena and their geometrical interpretations to this case. (Szegö limit theorems for positive-definite kernels were given in [10].) This general context will give the possibility to understand some "hidden" features of the formulas from the stationary case.

Let us note that there are two simple operations on the parameter G which generate new processes. First, for each $n \in \mathbb{Z}$, consider the gcs $G^{(n)}$ defined by the family $G^{(n)}_{i,j} = G_{i+n,j+n}$, $(i,j \in \mathbb{Z},\ i \leq j)$. (This corresponds to the changing of the origin in the process.) Then define $G^{(-)}$ by the family $G^{(-)}_{i,j} = G^{*}_{-j,-i}$, $(i,j \in \mathbb{Z},\ i \leq j)$. (This corresponds to the interchange between past and future.) The stationary processes are invariant to the first operation; the second operation shows that in the stationary case there is no difference in the behavior near $+\infty$ and $-\infty$. In the general case these operations produce a whole bunch of Szegö-type phenomena. Because the reader can easily work out the details for $G^{(n)}$ and $G^{(-)}$, we will consider in what follows limit phenomena to $+\infty$, starting with a fixed origin.

COROLLARY 5.5. *For each* $n \geq 1$ *and* $k \geq 1$

(i) $$\det(I - B_{n,k}) = \prod_{i=-n}^{-1} \prod_{j=0}^{k-1} \det D^2_{G_{i,j}} ;$$

(ii) $$\det(I - B_n) = \prod_{i=-n}^{-1} \prod_{j=0}^{\infty} \det D^2_{G_{i,j}} .$$

The proof follows from Theorem 4.1 (ii) (or Theorem 3.4) as in Corollary 5.1 (or in Remark 5.2). Note that the formulas are even clearer in this general case. The key equalities are:

$$(5.9)_{n,k} \quad \det(I - B_{n,k}) = (\det M_{-n,k-1})/[(\det M_{-n,-1})(\det M_{0,k-1})], \quad (n \geq 1,\ k \geq 1).$$

COROLLARY 5.6. *For each* $m \in \mathbb{Z}$

$$(5.10)_m \quad \lim_{k \to \infty} [(\det M_{m,k})/(\det M_{m+1,k})] = \prod_{j=m+1}^{\infty} \det D^2_{G_{m,j}} .$$

Denote this limit by $g_m(T) = g_m$. *Then*

$$\det(I - B_1) = g_{-1}. \tag{5.11}$$

These follows from the fact that for $i,j \in \mathbb{Z}$, $i \leq j$,

$$\det M_{i,j} = \prod_{p=i}^{j-1} \prod_{q=p+1}^{j} D^2_{G_{p,q}}, \tag{5.12$_{i,j}$}$$

(see (3.7) and (3.8)), and from Corollary 5.5 (ii).

This is the analogue of the first Szegö limit theorem and of its geometrical interpretation. As for each $m \in \mathbb{Z}$

$$g_m(T) = g_{-1}(T^{(m+1)}), \tag{5.13$_m$}$$

where $T^{(m+1)}$ is the kernel associated to $G^{(m+1)}$, it is natural to call g_m the *geometrical mean of order* m for *T*; its geometrical interpretation follows from $(5.13)_m$ and (5.11).

Note that $\{(5.10)_m\}$ show the right procedure in forming the ratio for the first Szegö limit theorem.

Owing (5.8), it is natural to define for each $m \in \mathbb{Z}$

$$h_m(T) = -\tfrac{1}{2} \ln g_m(T) = -\sum_{j=m+1}^{\infty} \ln \det D_{G_{m,j}}, \tag{5.14$_m$}$$

to be the *entropy of order* m for *T*. These lead to an extremal entropy spectral analysis for nonstationary processes (see [12] for related ideas).

Note that due to the analysis in Sections 3 and 4, it is clear that behind the numbers which represent entropies, there are (even in infinite dimensional case) some angle operators (as $I - B_1$).

As in the stationary case, Corollary 5.6 is a first step in a scale of Szegö-type limit theorems, the last one being the analogue of the second Szegö limit theorem.

COROLLARY 5.7. (i) *For each* $p \geq 1$

$$\lim_{k\to\infty} \Big[\prod_{j=-p}^{-1} (\det M_{j,k})/(\det M_{j+1,k}) \Big] / (\det M_{-p,-1}) =$$

$$= \lim_{k\to\infty} \Big[\prod_{j=0}^{k} (\det M_{-p,j})/(\det M_{-p,j-1}) \Big] / (\det M_{0,k}) =$$

$$= \det(I - B_p) \Big(= \Big(\prod_{j=-p}^{-1} g_j\Big) / (\det M_{-p,-1})\Big).$$

(ii) $\lim_{n\to\infty} (\det M_{-n,-1})/(\prod_{j=-n}^{-1} g_j) = 1/\det(I - B_1) \left(= 1/\prod_{i=-\infty}^{-1} \prod_{j=0}^{\infty} \det D^2_{G_{i,j}}\right).$

In (i) above we use the products in the first two terms in order to point out the idea that these are generalizations of the fact that the ratio of two consecutive determinants from T is a measure of some angle (see [20] for a construction of gcs as "angles" in T).

C. Krein-Spitkovskii case. One of the problems discussed in [17] is the following. Consider a positive-definite Toeplitz H-kernel T (where H is finite dimensional). We use the notation from subsection A; in particular W is the Naimark dilation of T. For a fixed integer $k \geq 1$, define

$$(5.15)_{n,k} \qquad d_{n,k} = \det \left[\begin{array}{c|ccc} I & T_k & T_{k+1} & \cdots\ T_{k+n} \\ \hline T_k^* & & & \\ \vdots & & M_n & \\ T_{k+n}^* & & & \end{array}\right], \quad n \in \mathbb{N}.$$

Then in [17] it is proved that $\lim_{n\to\infty} (\det d_{n,k})/(\det M_n) = d_k$ exists and a geometrical interpretation of these numbers as "angles" is described. (In [17] it is proved also that $\lim_{n\to\infty} d_k$ exists.)

This problem is an example of a nonstastionary situation as studied in Subsection 3. Let us explicitate this.

Fix an integer $k \geq 1$ and consider the process $V^{[k]} = \{V^{[k]}_{(n)}\}_{n\in\mathbb{Z}}$, where

$$(5.16)_k \qquad V^{[n]}_n = \begin{cases} W^{(n-k+1)}|H & n < 0 \\ (P^K_H)^* & n = 0 \\ W^n|H & n > 0\ . \end{cases}$$

It is easy to verify that the correlation matrix of $V^{[k]}$ is the following

$$(5.17)\ T^{[k]} = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots\ I & T_1 & T_2 & T_{k+2} & \cdots & \cdots & \cdots & \cdots \\ \cdots\ T_1^* & I & T_1 & T_{k+1} & T_{k+2} & \cdots & \cdots & \cdots \\ \cdots & T_1^* & I & T_k & T_{k+1} & T_{k+2} & \cdots & \cdots \\ \cdots & \cdots & T_k^* & I & T_1 & T_2 & T_3 & \cdots \\ \cdots & \cdots & \cdots & T_1^* & I & T_1 & T_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & T_1^* & I & T_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

(the marked position is the (0,0) one). We use the exponent [k] to indicate the objects associated to the process $V^{[k]}$. It is clear that for each $n \geq 1$ we have (see $(5.15)_{n,k}$):

$$(5,18)_{n,k} \qquad d_{n,k} = \det M^{[k]}_{-1,n} .$$

From (5.9) it follows that

$$(5.18)_{n,k} \qquad d_{n,k}/(\det M_n) = (\det M^{[k]}_{-1,n})/(\det M^{[k]}_{0,n}) = \det (I - B^{[k]}_{1,n+1}) .$$

From Corollary 5.6 we have that

$$(5.19)_k \quad \lim_{n\to\infty} d_{n,k}/(\det M_n) = \det (I - B^{[k]}_1) = g^{[k]}_{-1} = \prod_{j=0}^{\infty} \det D^2_{G^{[k]}_{-1,j}} ,$$

which shows the existence of the limit d_k and its geometrical interpretation in the process $V^{[k]}$.

Note that in the classical case ([14]) or in the generalizations of Szegö limit theorems (e.g. [22], [4], [17]) there were obtained nice integral formulas for the limits in terms of the spectral function of the process; the parameter gcs and the formulas using it are intendent as a "discrete" replacement for them.

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