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STRONGLY PLURISUBHARMONIC EXHAUSTION FUNCTION ON 1-CONVEX SPACES

by

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STRONGLY PLURISUBHARMONIC EXHAUSTION

FUNCTIONS ON 1-CONVEX SPACES

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Strongly plurisubharmonic exhaustion functions on 1 - convex spaces

§ . Introduction. In [3] the first author has proved the following :

Theorem 1.1. Let X be a complex space which carries a strongly plurisubharmonic exhaustion function $\varphi : X \to [-\infty, \infty)$. Then X is holomorphically convex and is obtained from a Stein space by blowing up finitely many points .

The conclusion " obtained from a Stein space by blowing up finitely many points " means precisely that there are :

- a compact analytic set $S \subset X$ with $\dim_x S > 0$ for any $x \in S$,

- a Stein space Y , a finite set $A \subset Y$ and a proper holomorphic map $p : X \to Y$ inducing a biholomorphism $X \setminus S \cong Y \setminus A$ and which satisfies $p_* \mathcal{O}_X \cong \mathcal{O}_Y$.

Following a customary terminology X is called an 1 - convex space , S its exceptional set and Y the Remmert reduction of X .

Theorem 1.1. was conjectured by Fornaess-Narasimhan in [5] .

In this paper we prove the converse :

Theorem 1.2. Let X be an 1 - convex space. Then X carries a strongly plurisubharmonic exhaustion function $\varphi : X \to [-\infty, \infty)$. Moreover φ can be chosen $-\infty$ exactly on the exceptional set S of X and real analytic outside S .

This theorem together with Theorem 1.1 and the results of Narasimhan [9] and Andreotti-Grauert [1] gives the following characterization of 1 - convex spaces :

Theorem Let X be an 1 - convex space . Then the following statements are equivalent :

i) X is an 1 - convex space .

ii) $\dim_{\mathbb{C}} H^q(X,\mathcal{F}) < \infty$ for any $q > 0$ and any coherent analytic sheaf $\mathcal{F}$ on X .

iii) X carries a continuous exhaustion function which is strongly pseudoconvex outside a compact set .

iv) X carries a strongly plurisubharmonic exhaustion function $\varphi : X \to [-\infty, \infty)$

i) $\Leftrightarrow$ iv) could be called the Levi problem with discontinuous functions .

The key ingredients of the proof of Theorem 1.2. are the identity between plurisubharmonic functions and weakly plurisubharmonic functions ([5] , Theorem 5.3.1.) and the analytic version of Chow's Lemma .

§2. Preliminaries All complex spaces are assumed to be reduced and countable at infinity .

An upper semicontinuous function on a complex space $\varphi : X \to [-\infty,\infty)$ is said to be plurisubharmonic if for every holomorphic map $f : D \to X$ (D = the unit disc in $\mathbb{C}$) it follows that $\varphi \circ f$ is subharmonic on D (possibly $\equiv -\infty$) . φ is called strongly plurisubharmonic if for any C^∞ real-valued function θ with compact support there is an $\varepsilon_0 > 0$ such that $\varphi + \varepsilon\theta$ is plurisubharmonic for $|\varepsilon| \leq \varepsilon_0$.

It is known (cf. Fornaess-Narasimhan [5] , Theorem 5.3.1.) that a (strongly) plurisubharmonic function is locally the restriction of a (strongly) plurisubharmonic function in an open set in some $\mathbb{C}^N$ in which X is locally embedded .

Proposition 2.1. Let X , Y be complex spaces and $p: X \longrightarrow Y$ be a proper , surjective holomorphic map. Let $\varphi: Y \longrightarrow [-\infty, \infty)$ be an upper semicontinuous function such that $\varphi \circ p$ is plurisubharmonic on X . Then φ is plurisubharmonic on Y .

Proof (sketch) In case is real-valued and continuous but p is supposed only holomorphic and surjective this is exactly Proposition 1.3. in Borel-Narasimhan [2]. In the general case the proof is a slight modification of theirs . For the sake of completeness we indicate the necessary modifications to be done .

Exactly as in [2] we may assume that Y is the unit disc in $\mathbb{C}$. Since p is proper for any irreducible component X_o of X it follows that $p|_{X_o}$ is constant or $p(X_o) = Y$. Therefore we may assume X irreducible . This hypothesis together with the maximum principle yields the following equality :

$$\overline{\lim_{\substack{y \to y_o \\ y \neq y_o}}} \varphi(y) = \varphi(y_o) \quad \text{for any } y_o \in Y .$$

Now the proof can easily be concluded using the following two remarks :

a) Let $x_o \in X$ be such that $p(x_o) = y_o$. Then there exists an analytic curve C in a neighbourhood of x_o in X such that $p|_C$ is a ramified covering of a neighbourhood N of y_o .(see for exemple Fischer [4] , 3.3)

b) Let $N \subset \mathbb{C}$ be a domain , $y_o \in N$ and $\varphi: N \to [-\infty, \infty)$ an upper semicontinuous function such that :

i) $\varphi|_{N \setminus \{y_o\}}$ is subharmonic

ii) $\overline{\lim}_{\substack{y \to y_o \\ y \neq y_o}} \varphi(y) = \varphi(y_o)$

Then φ is subharmonic on N . (see for instance Grauert-Remmert [6], Satz 5)

Remark Simple examples show that the assumption that p is proper cannot be dropped in Proposition 2.1.

Corollary 2.2. Let X , Y be complex spaces and $p:X \to Y$ be a proper , surjective holomorphic map . Let $\varphi:Y \to [-\infty,\infty)$ be an upper semicontinuous function such that $\varphi \circ p$ is strongly plurisubharmonic . Then φ is strongly plurisubharmonic on Y .

Applying Proposition 2.1. to $\pi:\hat{X} \to X$ (the normalisation of X) one easily verifies :

Corollary 2.3. Let X be a complex space and $\varphi: X \to [-\infty,\infty)$ be an upper semicontinuous function .

Then φ is (strongly) plurisubharmonic on X iff restricted to any irreducible component of X is (strongly) plurisubharmonic .

Another fact which will be used in the proof of Theorem 1.2. is the following :

Lemma 2.4. Let Y be a Stein space and $\mu: Y \to [-\infty,\infty)$ a function which is continuous outside a compact set containing $\{\mu = -\infty\}$. Then one can find a real analytic strongly plurisubharmonic function $\lambda: Y \to \mathbb{R}$ such that $\tau = \lambda + \mu$ is an exhaustion function on Y, i.e. $Y_c = \{\tau < c\} \subset\subset Y$ for any $c \in \mathbb{R}$.

The proof is obtained slightly modifying the proof of the well known fact that any Stein space carries a real analytic strongly plurisubharmonic exhaustion function (see for instance Narasimhan [8]) . λ can be chosen as a convergent series

$\sum_{j \in \mathbb{N}} |f_j|^2$, $f_j \in \Gamma(Y, \mathcal{O}_Y)$.

Finally , to construct strongly plurisubharmonic functions on a given 1 - convex space we shall use the following analytic version of Chow's lemma :

Lemma of Chow (Hironaka [7]) Let X be an 1 - convex space , $S \subset X$ its exceptional set and $p: X \longrightarrow Y$ the Remmert reduction of X . Suppose that S is rare .

Then there exist a coherent ideal $\mathcal{J} \subset \mathcal{O}_Y$ such that $\mathrm{supp}(\mathcal{O}_Y/\mathcal{J}) = p(S)$ and a commutative diagram :

$$\begin{array}{ccc} Y^* & \xrightarrow{f} & X \\ & \pi \searrow \quad \swarrow p & \\ & Y & \end{array}$$

where $\pi : Y^* \longrightarrow Y$ is the blowing-up of Y with the center $(p(S), (\mathcal{O}_Y/\mathcal{J})|_{p(S)})$ and f is holomorphic, proper and surjective .

As a general reference for the construction and basic properties of analytic blowing-up we refer to Fischer ([4]).

§3. Proof of Theorem 1.2. From now on X will be an 1 - convex space , S its exceptional set , $p: X \longrightarrow Y$ the Remmert reduction and $A = p(S)$. Recall that Y is Stein , p is proper, holomorphic , surjective and A is finite .

The proof will be divided into several steps :

i) S is rare (i.e. S does not contain any irreducible component of X) and $\dim X < \infty$.

ii) S is rare and no assumption on $\dim X$.

iii) the general case .

Step i) Consider $\mathcal{J} \subset \mathcal{O}_Y$ the ideal such that $A = \operatorname{supp}(\mathcal{O}_Y/\mathcal{J})$ given by Chow's lemma and $\pi : Y^* \longrightarrow Y$ the blowing-up of Y with center $(A, (\mathcal{O}_Y/\mathcal{J})|_A)$.

The idea is to construct on Y a strongly plurisubharmonic exhaustion function $\tau : Y \to [-\infty, \infty)$ such that $\tau = -\infty$ on A and τ is real analytic outside A, in such a way that $\tau \circ \pi$ is strongly plurisubharmonic on Y^*. Then, using Corollary 2.2., $\varphi = \tau \circ \rho$ is strongly plurisubharmonic on X. The other desired properties of φ are easily verified.

The construction of τ goes as follows.

Since Y is a Stein space of bounded dimension a well known argument which uses theorem B of H.Cartan shows that there are $h_1, \dots, h_l \in \Gamma(Y, \mathcal{J})$ such that :

$$A = \{ h_1 = \dots = h_l = 0 \}$$

Choose $h_{l+1}, \dots, h_s \in \Gamma(Y, \mathcal{J})$ which generate the fiber $\mathcal{J}_y$ for any $y \in A$ (A is finite!).

Then the germs $h_{1,y}, \dots, h_{s,y}$ generate $\mathcal{J}_y$ for any $y \in Y$. Hence $h = (h_1, \dots, h_s) : Y \longrightarrow \mathbb{C}^s$ is a holomorphic map such that :

$$h^{-1}(0) = (A, (\mathcal{O}_Y/\mathcal{J})|_A)$$

According to Lemma 2.4. we can choose a real analytic strongly plurisubharmonic function $\lambda : Y \longrightarrow \mathbb{R}$ such that $\tau = \lambda + \log(\sum_{j=1}^{s} |h_j|^2)$ is an exhaustion function on Y.

It is clear that τ is strongly plurisubharmonic on Y, $\{\tau = -\infty\} = A$ and τ is real analytic outside A.

Let $\varphi = \tau \circ \rho : X \longrightarrow [-\infty, \infty)$. We claim that φ has the required properties. In fact we only have to check that φ

is strongly plurisubharmonic because the other properties are obviously verified .

According to Corollary 2.2. , it is enough to show that $\tau\circ\pi$ is strongly plurisubharmonic on Y . To do this we need the explicit description of analytic blowing-up .

Let $\mathfrak{m}\subset\mathcal{O}_{\mathbb{C}^s}$ be the sheaf of ideals of the origin . There is an exact sequence on $\mathbb{C}^s$:

$$(*)\qquad \mathcal{O}_{\mathbb{C}^s}^{\binom{s}{2}} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{C}^s}^{s} \longrightarrow \mathfrak{m} \longrightarrow 0$$

where α is given by the holomorphic $s\times\binom{s}{2}$ - matrix

$$\begin{bmatrix} x_2 & & \vdots & & \vdots \\ -x_1 & & \vdots & & \vdots \\ & \cdots & x_j & \cdots & \vdots \\ & & \vdots & & \vdots \\ & & -x_i & & x_s \\ & & \vdots & & -x_{s-1} \end{bmatrix}$$

and $x_1,\dots,x_s$ denote the coordinate functions on $\mathbb{C}^s$.

Since h^* (the analytic inverse image) is right exact we get an exact sequence on Y :

$$(**)\qquad \mathcal{O}_{Y}^{\binom{s}{2}} \xrightarrow{h^*\alpha} \mathcal{O}_{Y}^{s} \longrightarrow h^*\mathfrak{m} \longrightarrow 0$$

Let $\xi_1:\mathbb{P}(\mathcal{J})\longrightarrow Y$ and $\xi_2:\mathbb{P}(h^*\mathfrak{m})\longrightarrow Y$ be the projective varieties over Y associated to $\mathcal{J}$, respectively to $h^*\mathfrak{m}$ (in general they are not reduced) . The canonical epimorphism $h^*\mathfrak{m}\longrightarrow\mathcal{J}$ yields an embedding $\mathbb{P}(\mathcal{J})\hookrightarrow\mathbb{P}(h^*\mathfrak{m})$.

Since by the construction of the analytic blowing-up

Y^* is a closed subspace of $\mathbb{P}(\mathcal{J})$ in order to verify that $\tau\circ\pi$ is strongly plurisubharmonic it will be enough to test that $\tau\circ\xi_2 : \mathbb{P}(h^*\mathfrak{m}) \longrightarrow [-\infty,\infty)$ is strongly plurisubharmonic (note that $\pi = \xi_2|_Y$) .

From (**) $\mathbb{P}(h^*\mathfrak{m}) \hookrightarrow Y \times \mathbb{P}_{s-1}$ is given by the equations :

$$h_j(y)z_i - h_i(y)z_j = 0 \quad , \quad 1 \leq i < j \leq s$$

where $(z_1 : \ldots : z_s)$ are the homogeneous coordinates on $\mathbb{P}_{s-1}$.

Set $U_i = \{ (y,z) \in Y \times \mathbb{P}_{s-1} \ / \ z_i \neq 0 \}$, $\alpha_i : U_i \longrightarrow Y \times \mathbb{C}^{s-1}$

$\alpha_i(y,z) = (y, z_1/z_i , \ldots, z_{i-1}/z_i , z_{i+1}/z_i , \ldots, z_s/z_i)$

and define :

$$\psi_i = \lambda + \log(1 + \sum_{j=1}^{s-1} |t_j|^2) + \log(|h_i|^2)$$

where $(t_1, \ldots, t_{s-1})$ are the affine coordinates on $\mathbb{C}^{s-1}$.

Then ψ_i is strongly plurisubharmonic on $Y \times \mathbb{C}^{s-1}$ and $\tau\circ\xi_2 = \psi_i \circ \alpha_i$ on $U_i \cap \mathbb{P}(h^*\mathfrak{m})$.

This proves that $\tau\circ\xi_2$ is strongly plurisubharmonic on $\mathbb{P}(h^*\mathfrak{m})$ thus ending the proof of Step i) .

Step ii) We drop the assumption that X has finite dimension .

This is done by carefully analysing the arguments given above .

Let $U \subset\subset Y$ be a relatively compact open neighbourhood of A and $V = p^{-1}(U) \subset\subset X$. Let X_o be an analytic subset of X having finite dimension and containing V . Then $Y_o = p(X_o)$ is an analytic subset of Y and $U \subset Y_o$. Finally , let $\mathcal{J}$ be the ideal on Y given by Chow's lemma .

Using again Cartan's theorem B we find (since Y_o is of bounded dimension) $h_1, \ldots, h_l \in \Gamma(Y, \mathcal{J})$ such that

$$A = \{ y \in Y_o \ / \ h_1(y) = \ldots = h_l(y) = 0 \}$$

Choose $h_{l+1}, \dots, h_s \in \Gamma(Y, \dot{\mathcal{J}})$ which generate $\dot{\mathcal{J}}_y$ for any $y \in A$. Then the germs $h_{1,y}, \dots, h_{s,y}$ generate $\dot{\mathcal{J}}_y$ for any $y \in Y_o$. Next choose a countable set $\{g_k\}_{k\in\mathbb{N}} \subset \Gamma(Y, \mathcal{O}_Y)$ such that $Y_o = \bigcap_{k\in\mathbb{N}} \{g_k = 0\}$.

We may suppose that the series $\sum_{k\in\mathbb{N}} |g_k|^2$ converges uniformly on compact sets of Y .

By Lemma 2.4. we find a real analytic strongly plurisubharmonic function $\lambda : Y \longrightarrow \mathbb{R}$ such that

$$\tau = \lambda + \log\left(\sum_{j=1}^{s} |h_j|^2 + \sum_{k\in\mathbb{N}} |g_k|^2 \right)$$

is an exhaustion function on Y .

Clearly $\tau = -\infty$ exactly on A and is real analytic on $Y \setminus A$. Moreover $\tau|_U = \lambda + \log(\sum_{j=1}^{s} |h_j|^2)$ since $g_k = 0$ on $Y_o \supset U$.

For that reason if we set $\varphi = \tau \circ p$ the same arguments as in Step i) prove that φ is a strongly plurisubharmonic exhaustion function on X ., $S = \{\varphi = -\infty\}$ and φ is real analytic outside S .

<u>Remark</u> Under the assumptions of Step ii) one can prove that given a finite set $B \subset X \setminus S$ there is a strongly plurisubharmonic exhaustion function $\varphi : X \to [-\infty, \infty)$ such that $\{\varphi = -\infty\}$ exactly on $S \cup B$ and φ is real analytic outside $S \cup B$.

The proof is straightforward and so will be omitted .

Step iii) S is not necessarily rare . Let $\tilde{X}$ be the union of those irreducible components of X not contained in S . Being a closed subspace of an 1 - convex space

X is itself 1 - convex and is clear that its exceptional set is rare . However $\tilde{S} = \tilde{X} \cap S$ contains the exceptional set of X and (eventually) a finite set . Anyhow by the Remark ending Step ii) there is a strongly plurisubharmonic exhaustion function $\tilde{\varphi} : \tilde{X} \to [-\infty, \infty)$ such that $\tilde{S} = \{ \tilde{\varphi} = -\infty \}$ and $\tilde{\varphi}$ is real analytic outside $\tilde{S}$.

Define $\varphi : X \to [-\infty, \infty)$ by $\varphi = \tilde{\varphi}$ on $\tilde{X}$ and $\varphi = -\infty$ on S . Then φ is an upper semicontinuous exhaustion function on X , real analytic outside S and $S = \{\varphi = -\infty\}$. Because of Corollary 2.3. φ is strongly plurisubharmonic and we are done .

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