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S. BOLINTINEANU

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FEEDBACK STABILIZATION OF PARABOLIC SYSTEMS BY A FINITE DIMENSIONAL COMPENSATOR

by

S. BOLINTINEANU *)

*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd. Pacii 220, 79622 Bucharest, Romania

Feedback stabilization of parabolic systems by a finite dimensional compensator

Ş. Bolintineanu*)

Abstract. A compensator, given by a linear system with constant coefficients of suitable finite dimension driven by the current output of a parabolic system, is proposed in order that the closed loop system obtained by using the output of the compensator as the input in the controlled system, be exponentially stable. We are assuming the output of the controlled system given by a finite number of point or boundary observations. We consider the case of boundary control, but the scheme can be also applied to the simpler case of distributed control or output.

1. Introduction. We will consider the following parabolic initial boundary value problem:

(1.1) $$\frac{\partial u}{\partial t} = \operatorname{div}(k(x)\nabla u) + q(x)u, \quad x\in\Omega,\ t>0$$

(1.2) $$\frac{\partial u}{\partial n} + \sigma(x)u = g(x)f(t), \quad x\in S,\ t>0$$

(1.3) $$u(o,x) = u_o(x), \quad x\in\Omega$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain, $S=\partial\Omega$ is a C^1 manifold, $f(t) = \mathrm{col}[f_1(t),\dots,f_r(t)]\in\mathbb{R}^r$ is the boundary input, $g(x) = \mathrm{row}[g_1(x),\dots,g_r(x)]\in(\mathbb{R}^r)^*$, $g\in L^2(S;(\mathbb{R}^r)^*)$; $\sigma\in C(S)$, $\sigma\geqslant 0$; $q\in C(\bar\Omega)$, $k\in C^1(\bar\Omega)$, $k(x)\geqslant k_o>0$ for $x\in\Omega$, are given functions.

Let us consider the functionals $\hat h_j : D(\hat h_j)\subset L^2(\Omega)\longrightarrow\mathbb{R}$ given by

*) Department of Mathematics I, Polytechnic Institute of Bucharest, Romania.

(1.4) $$\hat{h}_j(u) = (w_j, u)_S \ , \ D(\hat{h}_j) = H^1(\Omega), \quad j=1,\dots,P$$

and if $N \leq 3$ also by

(1.5) $$\hat{h}_j(u) = u(\xi_j), \ \xi_j \in \bar{\Omega} \ , \ D(\hat{h}_j) = C(\bar{\Omega}), \ j=1,\dots,P.$$

We will denote by $(\ , \ )_S$ and $(\ , \ )$ the usual inner product in $L^2(S)$ and $L^2(\Omega)$ respectively and by $\| \ \|_2$ the $L^2(\Omega)$ norm. In (1.4) $w_j \in L^2(S)$ are weighting functions.

Let $\hat{h} = \text{col}\,[\hat{h}_1,\dots,\hat{h}_P]$. The output for the system is given by

(1.6) $$y(t) = Y \cdot \hat{h}(u(t,\cdot)) \in \mathbb{R}^{P_1}, \quad t>0 ,$$

where Y is a $P_1 \times P$ matrix. We will denote $h = Y \cdot \hat{h}$. Hence:

(1.6') $$y(t) = h(u(t, \cdot)), \quad t>0.$$

Remark that if $\hat{h}$ is given by (1.4) we have boundary observations and in this case N is arbitrary.

The case (1.5) means point observations with sensors located in ξ_j , and in this situation consider only the case $N \leq 3$.

If $q(x) \geq 0$ the solution $u = 0$ of the problem (1.1), (1.2), (1.3) with $f(t) = 0$ is unstable. The problem is how to find the control $f(t)$ using only the output $y(t)$ to obtain exponential stability with a given rate $\mu^* > 0$ i.e. such that the corresponding solution satisfies $\|u(t)\| \leq Ke^{-\mu^* t}$, $t>0$, for a suitable norm, where K is a constant depending on u_0. To solve this problem we will construct matrices A, B, C of suitable dimension such that the auxiliary finite dimensional system:

(1.7) $$\frac{dV(t)}{dt} = AV(t) + By(t), \ V(0) = V_0 \ , \quad t>0$$

coupled with the original system by :

(1.8) $$f(t) = C'V(t), \quad t>0$$

more precisely, the closed loop system given by (1.1), (1.2), (1.3), (1.6), (1.7) and (1.8) will be exponentially stable.

Remark that the scheme which is proposed is applicable to the easier case of distributed control i.e. when (1.1) and (1.2) are replaced by

$$(1.1') \qquad \frac{\partial u}{\partial t} = \operatorname{div}(k(x)\nabla u)) + q(x)u + g(x)f(t), \quad x \in \Omega, \; t > 0$$

$$(1.2') \qquad \frac{\partial u}{\partial n} + \sigma(x) = 0, \quad x \in S$$

with $g \in L^2(\Omega;(\mathbb{R}^r)^*)$ and the output can be also distributed i.e. $\hat{h}_j$ will be given by :

$$(1.4') \qquad \hat{h}_j(u) = (w_j, u), \; w_j \in L^2(\Omega), \; D(\hat{h}_j) = L^2(\Omega), \; j=1,\ldots,P$$

A finite dimensional compensator for distributed system was proposed for the first time by Schumacher [10] but his approach is applicable only to the case of distributed control and observation. Later R.Curtain [3] proposed a general scheme to design finite dimensional compensators for some class of distributed systems with boundary or point control and observation.

It is difficult to see how to apply Curtain's scheme for the problem (1.1), (1.2), (1.3).

Though the construction of our compensator follows some ideas of Schumacher or Curtain we have used a different technique which allows to obtain existence and uniqueness for the closed loop system in the class of weak solutions.

Also we feel that the estimate for the order of the compensator is more explicit here and we get exponential stability with respect to several norms (possible, better than the L^2 one).

Feedback stabilization for parabolic systems was also studied by Y.Sakawa and T.Matsushita [9] for the system (1.1'), (1.2'), (1.3), (1.4'); T.Nambu [8] for the system (1.1), (1.2), (1.3), (1.4'), (1.6); N.Fujii [6] for the system (1.1), (1.2), (1.3), (1.4), (1.6); S.Bolintineanu [1] for the system (1.1), (1.2), (1.3), (1.5), (1.6) and the systems written above; but, in these papers only infinite dimensional compensators have been proposed.

2. Preliminary results and assumptions. Let us denote by $\{\lambda_n\}_n$ the eigenvalues of the eigenvalue problem:

$$\text{(2.1)} \qquad \begin{aligned} \lambda \emptyset &= \operatorname{div}(k(x)\nabla\emptyset) + q(x)\emptyset, & x \in \Omega \\ \frac{\partial\emptyset}{\partial n} &+ \sigma(x)\,\emptyset = 0, & x \in S \end{aligned}$$

and by $\{\emptyset_n\}_n$ the corresponding weak solutions i.e. $0 \neq \emptyset_n \in H^1(\Omega)$, where $H^1(\Omega)$ is the usual Sobolev space, and $\emptyset_n$ satisfies the relation :

$$\text{(2.2)} \quad \lambda_n \int_\Omega \emptyset_n \bar{v}\, dx = \int_\Omega (-k\nabla\emptyset_n \nabla\bar{v} + q\,\emptyset_n \bar{v})dx - \int_S k\sigma\emptyset_n \bar{v}\, dS$$

for all functions $v \in H^1(\Omega)$. We may define (Mihaĭlov [7, pp.174-179]) in $H^1(\Omega)$ an inner product equivalent to the usual one by:

$$\text{(2.3)} \qquad (\varphi,\psi)_1 = \int_\Omega (k\nabla\varphi\nabla\bar{\psi} + \tilde{q}\varphi\bar{\psi})dx + \int_S k\sigma\varphi\bar{\psi}\, dS$$

where

$$\text{(2.4)} \qquad \tilde{q}(x) = \hat{q} - q(x) + 1, \; x \in \bar{\Omega}\, ; \; \hat{q} = \max_{x\in\bar{\Omega}} q(x);$$

and so (2.2) may be written as:

$$\text{(2.2')} \qquad \lambda_n(\emptyset_n, v) = -(\emptyset_n, v)_1 + (1+\hat{q})(\emptyset_n, v).$$

We obtain that the eigenfunctions $\emptyset_1, \emptyset_2, \ldots$ are a complete orthonormal system in $L^2(\Omega)$ and the corresponding eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots$ have finite multiplicities $m_1, m_2, \ldots$, $\lim_{n\to\infty} \lambda_n = -\infty$ and $\lambda_1 \leq \hat{q}$. Also it is known that the functions $\frac{\emptyset_1}{\sqrt{1+\hat{q}-\lambda_1}}, \frac{\emptyset_2}{\sqrt{1+\hat{q}-\lambda_2}}, \ldots$ are a complete orthonormal system in

$H^1(\Omega)$ with respect to (2.3) and

(2.5) $$\lambda_n = O(n^{2/N}).$$

When we use point sensors, in the case $N \leq 3$ and (1.5) holds, we will make some additional assumptions and will distinguish two situations.

I) There are point sensors located on the boundary (SLB).

In this case we assume that:

(A1) S is a C^2 manifold and $\sigma \in C^1(S)$.

(A2) $g \in C^2(S; (\mathbb{R}^r)^*)$.

From (A1) we obtain that $\varnothing_n \in H^2(\Omega) \subset C(\bar{\Omega})$ (Mikhailov [7, p.229]). Let us denote

$$\|\varphi\|_c = \max_{x \in \bar{\Omega}} |\varphi(x)| \quad \text{for } \varphi \in C(\bar{\Omega}).$$

We will consider also that Ω, q,k and σ are such that the following assumption holds:

(A3) There exist a constant K_o and $\alpha > 0$ such that

(2.6) $$\|\varnothing_n\|_o \leq K_o \cdot n^{\frac{4-N}{2N} - \alpha} \quad \text{for all } n \geq 1.$$

II) All the point sensors are located in the domain (SLD). In this situation we will assume only (A1) and (A3); g may be a function of $L^2(S; (\mathbb{R}^r)^*)$.

Hereafter we will study simultaneously the cases of point sensors or boundary output and we will stress the differences whenever necessary.

Let $\mu^* > 0$ be a fixed number. Choose $\mu > \mu^*$ and an integer M such that

(2.7) $$\lambda_{M+1} < -\mu$$

Let us consider the matrices:

$$G = \text{col}\,[(\varnothing_1, kg)_S, \ldots; (\varnothing_M, kg)_S] \in L(\mathbb{R}^r; \mathbb{R}^M)$$

(2.8) $$\Lambda = \text{diag}\,(\lambda_1, \ldots, \lambda_M) \in L(\mathbb{R}^M; \mathbb{R}^M)$$

$$\chi = \text{row}\,[h(\varnothing_1), \ldots, h(\varnothing_M)] \in L(\mathbb{R}^M; \mathbb{R}^{P_1})$$

We have used above the following notation:
if $(\cdot,\cdot)$ is an inner product in a space X and $x=(x_1,\dots,x_n)\in X^n$ and $y\in X$, then the vector $[(x_1,y),\dots,(x_n,y)]$ will be denoted by (x,y) and it will be considered row or column like x. Also $\|x\| = \sqrt{\|x_1\|^2+\dots+\|x_n\|^2}$.

We will assume that $m_i \leq \min(r,P_1)$ for $i=1,\dots,M$, where m_i is the multiplicity of λ_i. If we denote by $\phi_{i1},\dots,\phi_{im_i}$ the eigenfunctions corresponding to λ_i then we will admit the following assumption:

(A4) The matrices:

$$G_i = \mathrm{col}\left[(\phi_{i1},kg)_S,\dots,(\phi_{im_i},kg)_S\right]$$

$$\hat{\chi}_i = \mathrm{row}\left[h(\phi_{i1}),\dots,h(\phi_{im_i})\right]$$

satisfy the conditions:

$$\text{(2.9)}\qquad \mathrm{rank}\, G_i = \mathrm{rank}\,\hat{\chi}_i = m_i, \qquad i=1,\dots,M.$$

According to Sakawa and Matsushita [9], the conditions (2.9) will ensure that the pair (Λ, G) is completely controlable and the pair (χ,Λ) is completely observable.

Let us choose distinct numbers $\mu_1,\mu_2,\dots,\mu_M$ with $\mathrm{Re}\,\mu_j<-\mu^*$; $\mu_j\neq\lambda_n$, $j=1,\dots,M$; $n\geq 1$. By the complete controlability of (Λ, G) we can find an $r\times M$ matrix R such that the spectrum of the matrix $\Lambda + GR$ is $\{\mu_1,\dots,\mu_M\}$ and also by the complete observability of (χ,Λ) we can choose an $M\times P_1$ matrix L such that the spectrum of $\Lambda - L\chi$ is in the half-plane $\mathrm{Re}\,\lambda \leq \lambda_{M+1}$.

We will consider next the functions:

$$\text{(2.10)}\qquad \begin{aligned} \rho(x) &= R\cdot \mathrm{col}\left[\phi_1(x),\dots,\phi_M(x)\right]\in \mathbb{R}^r \\ \ell(x) &= \left[\phi_1(x),\dots,\phi_M\right]\cdot L\in (\mathbb{R}^{P_1})^* \end{aligned}$$

We mention that all the notations and assumptions introduced are kept until the end of this paper.

We will state now some propositions and lemmas whose proof will be given in the appendix.

Proposition 2.1. If we denote by p_j an eigenvector of the matrix $\Lambda + GR$ corresponding to μ_j, then the problem:

$$\text{(2.11)} \quad \begin{aligned} \mu_j \psi &= \operatorname{div}(k(x)\nabla\psi) + q(x)\psi, \quad x \in \Omega \\ \frac{\partial\psi}{\partial n} + \sigma(x)\psi &= g(x)\, R\, p_j \quad, \quad x \in S \end{aligned}$$

has a unique weak solution $\psi_j \in H^1(\Omega)$, $j=1,\dots,M$, i.e. a unique ψ_j satisfying

$$\text{(2.11')} \quad \mu_j(\psi_j, v) = -(\psi_j, v)_1 + (1+\hat{q})(\psi_j, v) + (kgRp_j, v)_S$$
$$\text{for all } v \in H^1(\Omega).$$

When we deal with (SLD) we have in addition that

$$\text{(2.12)} \quad \psi_j \in H^2_{loc}(\Omega) \subset C(\Omega)$$

where $C(\Omega)$ is a Fréchet space endowed with the topology of uniform convergence on compact subsets.

If we have (SLB) then:

$$\text{(2.13)} \quad \psi_j \in H^2(\Omega) \subset C(\bar{\Omega}).$$

Proposition 2.2. The weak eigenfunctions of the problem:

$$\text{(2.14)} \quad \begin{aligned} \lambda u &= \operatorname{div}(k(x)\nabla u) + q(x)\, u, \quad x \in \Omega \\ \frac{\partial u}{\partial n} + \sigma(x) u &= g(x)(u, \rho) \quad, \quad x \in S \end{aligned}$$

i.e. the functions $0 \neq u \in H^1(\Omega)$ which satisfy the identity:

$$\text{(2.15)} \quad \lambda(u,v) = -(u,v)_1 + (\hat{q}+1)(u,v) - (kg, v)_S\,(u,\rho)$$
$$\text{for all } v \in H^1(\Omega),$$

are $\{\psi_1, \psi_2, \dots, \psi_M, \phi_{M+1}, \phi_{M+2}, \dots\}$ and the corresponding eigenvalues are $\{\mu_1, \mu_2, \dots, \mu_M, \lambda_{M+1}, \lambda_{M+2}, \dots\}$.

Proposition 2.3. The matrix

$$Q = \begin{bmatrix} (\psi_1, \phi_1) & \dots & (\psi_M, \phi_1) \\ \vdots & & \\ (\psi_1, \phi_M) & \dots & (\psi_M, \phi_M) \end{bmatrix} = \text{row}\,[p_1, \dots, p_M]$$

is invertible and the functions $\{\psi_1, \dots, \psi_M, \phi_{M+1}, \phi_{M+2}, \dots\}$ form a basis in $L^2(\Omega)$ and in $H^1(\Omega)$ i.e. each function $\varphi \in L^2(\Omega)$ (or $\varphi \in H^1(\Omega)$) can be developed in the $L^2(\Omega)$ topology (or in the $H^1(\Omega)$ topology) uniquely as :

$$(2.16) \qquad \varphi(\cdot) = \sum_{n=1}^{M} \varphi_n \psi_n(\cdot) + \sum_{n=M+1}^{\infty} \varphi_n \phi_n(\cdot)$$

The coefficients φ_n are determined from:

$$(2.17) \qquad \begin{aligned} &\text{col}\,[\varphi_1, \dots, \varphi_M] = Q^{-1}\,\text{col}\,[(\varphi, \phi_1), \dots, (\varphi, \phi_M)] \\ &\varphi_n = (\varphi, \phi_n) - [(\psi_1, \phi_n), \dots, (\psi_M, \phi_n)] \cdot \text{col}\,[\varphi_1, \dots, \varphi_M], \quad n > M. \end{aligned}$$

<u>Lemma 2.1</u>. The family $\{T_t,\ t \geq 0\}$ of linear operators defined by

$$(2.18) \qquad (T_t\varphi)(\cdot) = \sum_{n=1}^{M} e^{\mu_n t} \varphi_n \psi_n(\cdot) + \sum_{n=M+1}^{\infty} e^{\lambda_n t} \varphi_n \phi_n(\cdot)$$

(where φ and φ_n are as in (2.16) and (2.17)) is a strongly continuous semigroup on $L^2(\Omega)$ and on $H^1(\Omega)$, T_t satisfies the following relations:

$$(2.19) \qquad \|T_t\varphi\|_2 \leq K_1 e^{-\mu^* t} \|\varphi\|_2, \qquad t \geq 0, \quad \varphi \in L^2(\Omega),$$

$$(2.20) \qquad T_t(L^2(\Omega)) \subset H^1(\Omega), \qquad t > 0$$

and moreover there exists a function $\tilde{K}(t, \varphi)$ such that:

$$(2.21) \qquad \|T_t\varphi\|_1 \leq \tilde{K}(t, \varphi)\, e^{-\mu^* t}, \qquad t > 0, \quad \varphi \in L^2(\Omega)$$

$t \longmapsto \tilde{K}(t, \varphi)$ is nonincreasing and continuous on $]0, \infty[$, (2.22) $$\int_0^\infty \tilde{K}^2(t,\varphi)dt \leqslant K_2 \|\varphi\|_2^2, \quad \varphi \in L^2(\Omega).$$

If we deal with (SLB) then in addition

(2.23) $$T_t\ (L^2(\Omega)) \subset C(\bar{\Omega}) \quad \text{for } t > 0$$

and there exists a function $\hat{K}_0(t,\varphi)$ such that

(2.24) $$\|T_t\varphi\|_0 \leqslant \hat{K}_0(t,\varphi)\ e^{-\mu^* t}, \quad t > 0,\ \varphi \in L^2(\Omega),$$

$t \longmapsto K_0(t,\varphi)$ is nonincreasing and continuous on $]0,\infty[$,

(2.25) $$\int_0^\infty \hat{K}_0^2(t,\varphi)dt \leqslant \check{K}_0 \|\varphi\|_1^2, \quad \varphi \in H^1(\Omega).$$

For (SLD) we have instead of (2.23), (2.24), (2.25) that

(2.23') $$T_t(L^2(\Omega)) \subset C(\Omega), \quad t > 0,$$

and for any domain Ω_1 such that $\bar{\Omega}_1 \subset \Omega$ there exists a function $\hat{K}_{\Omega_1}(t,\varphi)$ nonincreasing, continuous for $t > 0$ and fixed $\varphi \in L^2(\Omega)$, such that

(2.24') $$\|T_t\varphi\|_{C(\bar{\Omega}_1)} \leqslant \hat{K}_{\Omega_1}(t,\varphi)e^{-\mu^* t}, \quad t > 0,\ \varphi \in L^2(\Omega)$$

(2.25') $$\int_0^\infty \hat{K}_{\Omega_1}^2(t,\varphi)dt \leqslant \check{K}_{\Omega_1}\|\varphi\|_1, \quad \varphi \in H^1(\Omega).$$

Here $K_1, K_2, \check{K}_0$, are constants which do not depend on t or φ.

<u>Lemma 2.2</u>. Let $u_0 \in L^2(\Omega)$ and $F(t,x)$ be a function such that $F(t,\cdot) \in H^1(\Omega)$, $t > 0$; $t \longmapsto F(t,\cdot)$ is continuous from $]0,\infty[$ to $H^1(\Omega)$, $\|F(t,\cdot)\|_1^2$ is integrable on $[0,t_0]$, $t_0 > 0$.

The problem

(2.26) $$\frac{\partial u}{\partial t} = \operatorname{div}(k(x)\nabla u) + q(x)u + F(t,x), \quad x \in \Omega,\ t > 0$$

(2.27) $$\frac{\partial u}{\partial n} + \sigma(x)u = g(x)\ (u(t,\cdot),\rho), \quad x \in S,\ t > 0$$

$$(2.28) \qquad u(0+,x) = u_o(x), \quad x \in \Omega$$

has a unique weak solution i.e. there exists a unique u such that:

(i) $t \longmapsto u(t,\cdot)$ is differentiable from $]0,\infty[$ to $L^2(\Omega)$

(ii) $t \longmapsto u(t,\cdot)$ is continuous from $[0,\infty[$ to $L^2(\Omega)$ and from $]0,\infty[$ to $H^1(\Omega)$; $u(0,\cdot) = u_o(\cdot)$.

(iii) u satisfies the identity:

$$(2.29) \qquad \left(\frac{\partial u}{\partial t}(t,\cdot), \eta(t,\cdot)\right) = -(u(t,\cdot), \eta(t,\cdot))_1 + ((1+\hat{q})u(t,\cdot) + F(t,\cdot), \eta(t,\cdot)) + (kg,\eta)_S (u(t,\cdot),\rho)$$

for all $t > 0$, $\eta(t,\cdot) \in H^1(\Omega)$.

(iv) In the case of (SLB) the function $t \longmapsto u(t,\cdot)$ is in addition continuous from $]0,\infty[$ to $C(\bar{\Omega})$ and for (SLD) $t \longmapsto u(t,\cdot)$ is continuous from $]0,\infty[$ to $C(\Omega)$.

The weak solution is given by:

$$(2.30) \qquad u(t,\cdot) = T_t u_o + \int_0^t T_{t-s}(F(s,\cdot))ds, \quad t \geq 0$$

hence it is also a mild solution.

<u>Lemma 2.3.</u> Let $n \geq M$ and $\tilde{L}_n$ be an $n \times P_1$ matrix of complex numbers. If one takes $\chi_n = \text{row}\,[h(\phi_1),\ldots,h(\phi_n)]$, $\Lambda_n = \text{diag}\,(\lambda_1,\ldots,\lambda_n)$ and:

$$(2.31) \qquad \tilde{Z}_t^n z_o = [\phi_1(\cdot),\ldots,\phi_n(\cdot)] \cdot \Big\{ \exp\,[(\Lambda_n - \tilde{L}_n\chi_n)t] \cdot \text{col}\,[(z_o,\phi_1),\ldots,(z_o,\phi_n)] - \int_0^t \exp\,[(\Lambda_n - \tilde{L}_n\chi_n)(t-s)] \cdot \tilde{L}_n \cdot h\Big(\sum_{p=n+1}^{\infty}(z_o,\phi_p)e^{\lambda_p s}\phi_p(\cdot)\Big)ds\Big\} + \sum_{p=n+1}^{\infty}(z_o,\phi_p)e^{\lambda_p t}\phi_p(\cdot),$$

$$t \geq 0, \; z_o \in L^2(\Omega)$$

then $\tilde{Z}_t^n$ is a strongly continuous semigroup on $L^2(\Omega)$ and also on $H^1(\Omega)$.

Moreover we have that

(2.32) $$\tilde{Z}_t^n (L^2(\Omega)) \subset H^1(\Omega), \qquad t > 0$$

and the function $t \longmapsto \|\tilde{Z}_t^n z_o\|_1^2$ is integrable on $[0,t_o]$, $t_o > 0$, $z_o \in L^2(\Omega)$.

When we deal with point sensors (SLB) or (SLD) we have in addition that

(2.33) $$\tilde{Z}_t^n (L^2(\Omega)) \subset C(\bar{\Omega}), \qquad t > 0$$

and the function $t \longmapsto \|\tilde{Z}_t^n z_o\|_o^2$ is integrable on $[0,t_o]$ for $t_o > 0$, $z_o \in H^1(\Omega)$.

<u>Lemma 2.4</u>. Let $z_o \in L^2(\Omega)$ and $\tilde{\ell}_n(x) = [\phi_1(x),\dots,\phi_n(x)]\cdot\tilde{L}_n$. If $F(t,x)$ fulfills the same conditions as in Lemma 2.2 then the problem:

(2.34) $$\frac{\partial z}{\partial t} = \operatorname{div}(k(x)\nabla z) + q(x)z - \tilde{\ell}_n(x)h(z(t,\cdot)) + F(t,x), \qquad x \in \Omega, \quad t > 0$$

(2.35) $$\frac{\partial z}{\partial n} + \sigma(x)\, z = 0, \qquad x \in S, \quad t > 0$$

(2.36) $$z(0+,x) = z_o(x), \qquad x \in \Omega$$

has a unique weak solution i.e. there exists only one function z such that:

(i) $t \longmapsto z(t,\cdot)$ is differentiable from $]0,\infty[$ to $L^2(\Omega)$;

(ii) $t \longmapsto z(t,\cdot)$ is continuous from $[0,\infty[$ to $L^2(\Omega)$ and from $]0,\infty[$ to $H^1(\Omega)$; $z(0,\cdot) = z_o(\cdot)$;

(iii) when we have point sensors (SLB) or (SLD) then in addition $t \longmapsto z(t,\cdot)$ is continuous from $]0,\infty[$ to $C(\bar{\Omega})$.

(iv) z satisfies the identity:

(2.37) $$\left(\frac{\partial z}{\partial t}(t,\cdot),\eta(t,\cdot)\right) = -(z(t,\cdot),\eta(t,\cdot))_1 + ((1+\hat{q})z(t,\cdot) + F(t,\cdot) - \tilde{\ell}_n(\cdot)\cdot h(z(t,\cdot)),\eta(t,\cdot))$$

for all $t > 0$, $\eta(t,\cdot) \in H^1(\Omega)$.

This weak solution is given by:

(2.38) $$z(t,\cdot) = \tilde{Z}_t^n z_o + \int_0^t \tilde{Z}_{t-s}^n (F(s,\cdot))ds$$

hence it is a mild solution.

Lemma 2.5. If in Lemma 2.3 we take $n = M$ and $\tilde{L}_n = L$, hence $\tilde{\ell}_n(x) = \ell(x)$ (see 2.10), and $\chi_n = \chi$, $\Lambda_n = \Lambda$ and if we denote by Z_t the corresponding semigroup i.e. $Z_t = \tilde{Z}_t^n$, then we have that:

(2.39) $$\| Z_t z_o\|_2 \leq C_1 e^{-\mu t} \| z_o\|_2, \quad t \geq 0, \; z_o \in L^2(\Omega),$$

and there exists a function $C_2(t,z_o)$ such that

(2.40) $$\| Z_t z_o\|_1 \leq C_2(t,z_o) e^{-\mu t}, \quad t > 0, \; z_o \in H^1(\Omega)$$

with $t \longmapsto C_2(t,z_o)$ nonincreasing, continuous on $]0,\infty[$, satisfying

(2.41) $$\int_0^\infty C_2^2(t,z_o)dt < \hat{C}_2^2 \| z_o\|_2^2, \quad z_o \in L^2(\Omega),$$

and $\lim_{\|z_o\|_2 \to 0} C_2(t,z_o) = 0$ for each $t > 0$.

When we have point sensors (SLD) or (SLB) then in addition we get that there exists a function $C_o(t,z_o)$ such that

(2.42) $$\| Z_t z_o\|_o \leq C_o(t,z_o) e^{-\mu t}, \quad t > 0, \; z_o \in L^2(\Omega),$$

$t \longmapsto C_o(t,z_o)$ is continuous, nonincreasing on $]0,\infty[$, satisfies

(2.43) $$\int_0^\infty C_o^2(t,z_o)dt \leq \hat{C}_o^2 \| z_o\|_1^2, \quad z_o \in H^1(\Omega)$$

and

(2.44) $$\lim_{\|z_o\|_2 \to 0} C_o(t,z_o) = 0 \quad \text{for each } t > 0.$$

Here C_1, $\hat{C}_2$ and $\hat{C}_o$ are constants with respect to t and z_o .

3. The stabilization by finite dimensional compensator. We will modify "slightly" the observer proposed in [1] in order to make it finite dimensional still ensuring exponential stability.

We shall consider separately the case of boundary output and the case of point sensors.

3.1. Stabilization using boundary output. Consider that $\hat{h}$ is given by (1.4). Let us remark that h is a bounded linear operator from $H^1(\Omega)$ to $\mathbb{R}^{p_1}$. Indeed, it is well known that there exists a unique bounded linear operator $T : H^1(\Omega) \longrightarrow L^2(S)$ such that $Tu = u|_S$ for any $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$ which defines the trace of the functions of $H^1(\Omega)$ (Mikhailov [7,p.137]) and hence

$$|\hat{h}(u)| = \sqrt{\hat{h}_1^2(u)+\dots+\hat{h}_p^2(u)} = \sqrt{(w_1,u)_S^2 +\dots+ (w_p,u)_S^2} =$$

$$= \sqrt{(w_1,Tu)_S^2 +\dots+ (w_p,Tu)_S^2} \leqslant \|T\|\cdot\|u\|_1\sqrt{\|w_1\|_S^2+\dots+\|w_p\|_S^2},$$

$$|h(u)| \leqslant \|Y\|\cdot\|T\|\sqrt{\|w_1\|_S^2 +\dots+ \|w_p\|_S^2}\ \|u\|_1$$

which means that h is bounded and

$$\|h\| \leqslant \|Y\|\cdot\|T\|\cdot\sqrt{\|w_1\|_S^2 +\dots+\|w_p\|_S^2} \tag{3.1}$$

(we denote by $\| \ \|$ the operator norm).

If we denote by ℓ^j the rows of L i.e. $L = \mathrm{col}[\ell^1,\dots,\ell^M]$ we can write (2.10) as:

$$\ell(x) = \ell^1\phi_1(x) +\dots+ \ell^M\phi_M(x) \tag{3.2}$$

and from Proposition 2.3. we obtain:

$$\ell(\cdot)= \ell_1\psi_1(\cdot) +\dots+ \ell_M\psi_M(\cdot)+\ell_{M+1}\phi_{M+1}(\cdot)+\ell_{M+2}\phi_{M+2}(\cdot)+\dots \tag{3.3}$$

where

$$\mathrm{col}\left[\ell_1,\dots,\ell_M\right] = Q^{-1} L$$

(3.4)

$$\ell_n = -\left[(\psi_1,\phi_n),\dots,(\psi_M,\phi_n)\right]\cdot Q^{-1} L, \quad n > M.$$

The series in (3.3) is convergent in $L^2(\Omega)$ and in $H^1(\Omega)$ because $\ell(\cdot) \in H^1(\Omega)$.

Let us consider the functions

(3.5) $$\hat{\ell}(\cdot) = \ell_1\psi_1(\cdot) + \dots + \ell_M\psi_M(\cdot) + \ell_{M+1}\phi_{M+1}(\cdot) + \dots + \ell_m\phi_m(\cdot)$$

and

(3.6) $$\delta(\cdot) = \ell(\cdot) - \hat{\ell}(\cdot) = \ell_{m+1}\phi_{m+1}(\cdot) + \ell_{m+2}\phi_{m+2}(\cdot) + \dots .$$

The number m will be chosen such that

(3.7) $$\mu - \|\delta\|_2^2\, \hat{C}_2^2\, \|h\| \geq \mu_1^*$$

which is possible because:

$$\|\delta\|_2^2 = |\ell_{m+1}|^2 + |\ell_{m+2}|^2 + \dots \longrightarrow 0 \text{ for } m \to \infty .$$

(the number $\hat{C}_2^2$ appears in (2.41) and is computed in (4.55)).

Let us consider the matrices:

$$\hat{Q} = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} \in L(\mathbb{R}^m,\mathbb{R}^m); \quad \hat{R} = (\mathbb{R},0) \in L(\mathbb{R}^m,\mathbb{R}^r)$$

(3.8) $$\hat{L} = \mathrm{col}\left[\ell_1,\dots,\ell_M,\ell_{M+1},\dots,\ell_m\right] \in L(\mathbb{R}^{P_1},\mathbb{R}^m);$$

$$H = \left[h(\psi_1),\dots,h(\psi_M),h(\phi_{M+1}),\dots,h(\phi_m)\right] \in L(\mathbb{R}^m,\mathbb{R}^{P_1});$$

$$\mathcal{M} = \mathrm{diag}\,(\mu_1,\dots,\mu_M,\lambda_{M+1},\dots,\lambda_m) \in L(\mathbb{R}^m,\mathbb{R}^m).$$

We can now state the

<u>Theorem 3.1</u>. Let $u_0 \in L^2(\Omega)$ and $V_0 \in \mathbb{R}^m$, then the closed loop system:

(3.9) $$\frac{\partial u}{\partial t} = \mathrm{div}(k(x)\nabla u) + q(x)u, \qquad x\in\Omega,\ t>0$$

(3.10) $$\frac{dV}{dt} = (\mathcal{M} - \hat{L}H)V + \hat{L}\, h(u(t,\cdot)), \qquad t>0$$

(3.11) $$\frac{\partial u}{\partial n} + \sigma(x)u = g(x)\,\hat{R}\,\hat{Q}\,V(t), \qquad x\in S,\ t>0$$

(3.12) $$u(0+, x) = u_0(x), \quad x \in \Omega$$

(3.13) $$V(0+) = V_0$$

has a unique weak solution (u,V) i.e. there exists a unique pair (u,V) such that

(i) $t \longmapsto u(t,\cdot)$ is differentiable from $]0,\infty[$ to $L^2(\Omega)$, continuous from $[0,\infty[$ to $L^2(\Omega)$ and from $]0,\infty[$ to $H^1(\Omega)$; $u(0,\cdot) = u_0(\cdot)$, $\|u(t)\|_1^2$ is integrable on $[0,t_0]$, $t_0 > 0$.

(ii) $t \longmapsto V(t)$ is differentiable from $]0,\infty[$ to $\mathbb{R}^m$, continuous from $[0,\infty[$ to $\mathbb{R}^m$ and satisfies (3.10),(3.13).

(iii) u satisfies the identity:

$$(\frac{\partial u}{\partial t}(t,\cdot),\eta(t,\cdot)) = -(u(t,\cdot),\eta(t,\cdot))_1 +$$

(3.14) $$+ (1+\hat{q})(u(t,\cdot),\eta(t,\cdot)) + (kg,\eta(t,\cdot))_S \cdot \hat{R}\hat{Q}V(t)$$

for all $\eta(t,\cdot) \in H^1(\Omega)$, $t > 0$.

Moreover the system is exponentially stable in the following sense: there exists constants K_4, K_5 and a nonincreasing function $K_6(t)$ $(t > 0)$ which depend on the initial state (u_0, V_0) such that:

(3.15) $$\|u(t,\cdot)\|_2 \leq K_4\, e^{-\mu^* t}, \qquad t \geq 0$$

(3.16) $$\|u(t,\cdot)\|_1 \leq K_6(t_0) e^{-\mu^* t}, \quad t \geq t_0 > 0, \ (t_0 \text{ arbitrary}).$$

(3.17) $$|V(t)| \leq K_5\, e^{-\mu^* t}, \qquad t \geq 0.$$

<u>Remark 3.1.</u> The theorem shows that if we put $A = \mathcal{M} - \hat{L}H$, $B = \hat{L}$, $C = \hat{R}\hat{Q}$ we obtain the stabilizing compensator (1.7) and the inputs for the original system (1.1), (1.2), (1.3) are given by (1.8).

To prove the theorem we need first the following:

<u>Lemma 3.1.</u> The system (3.9 –3.13) is equivalent to the system:

(3.18) $$\frac{\partial u}{\partial t} = \mathrm{div}(k(x)\nabla u) + q(x)u, \qquad x \in \Omega, \ t > 0$$

(3.19) $\frac{\partial v}{\partial t} = \mathrm{div}(k(x)\nabla v) + \hat{\ell}(x)\, h(u(t,\cdot) - v(t,\cdot)), \quad x\in\Omega, \; t>0$

(3.20) $\frac{\partial u}{\partial n} + \sigma(x)u = g(x)\; (v(t,\cdot),\rho), \qquad x\in S, \quad t>0$

(3.21) $\frac{\partial v}{\partial n} + \sigma(x)v = g(x)\; (v(t,\cdot),\rho), \qquad x\in S, \quad t>0$

(3.22) $u(o+\,,\,x) = u_o(x), \qquad x\in\Omega$

(3.23) $v(o+\,,\,x) =$

$= [\psi_1(x),\ldots,\psi_M(x),\phi_{M+1}(x),\ldots,\phi_m(x)]\cdot V_o = v_o(x), \; x\in\Omega\,,$

in the following sense:

- if (u,V) is a weak solution of (3.9-3.13) then (u,v) with

(3.24) $v(t,x) = [\psi_1(x),\ldots,\psi_M(x),\phi_{M+1}(x),\ldots,\phi_m(x)]\cdot V(t)$

is a weak solution of (3.18-3.23); conversely, if (u,v) is a weak solution of (3.18-3.23) then

$v(t,x)=v_1(t)\psi_1(x) +\ldots+ v_M(t)\psi_M(x)+v_{M+1}(t)\phi_{M+1}(x)+\ldots+v_m(t)\phi_m(x)$

and if we set $V(t) = \mathrm{col}\,[v_1(t),\ldots,v_m(t)]$ then (u,V) is a weak solution of (3.9-3.13).

Let us stress that by a weak solution for (3.18-3.23) we mean a pair (u,v) such that:

(i) the functions $t \longmapsto u(t,\cdot)$ and $t \longmapsto v(t,\cdot)$ are differentiable from $]0,\infty[$ to $L^2(\Omega)$, continuous from $[0,\infty[$ to $L^2(\Omega)$ and from $]0,\infty[$ to $H^1(\Omega)$; $u(o,\cdot) = u_o(\cdot)$; $v(o,\cdot)=v_o(\cdot)$.

(ii) u and v satisfy the identities:

(3.25) $(\frac{\partial u}{\partial t}(t,\cdot),\eta(t,\cdot)) = -(u(t,\cdot),\eta(t,\cdot))_1 +$

$+ (1+\hat{q})(u(t,\cdot),\eta(t,\cdot)) + (kg,\eta)_S \cdot (v(t,\cdot),\rho)$

(3.26) $(\frac{\partial v}{\partial t}(t,\cdot),\eta(t,\cdot)) = -(v(t,\cdot),\eta(t,\cdot))_1 +$

$+ ((1+\hat{q})v(t,\cdot)+\hat{\ell}h(u(t,\cdot)-v(t,\cdot)),\eta(t,\cdot)) +$

$+ (kg,\eta(t,\cdot))_S \cdot (v(t,\cdot),\rho)$

for all $t>0$ and $\eta(t,\cdot)\in H^1(\Omega)$.

(iii) $\|u(t)\|_1^2$ and $\|v(t)\|_1^2$ are integrable on $[0,t_o]$, $t_o > 0$.

The proof of this Lemma will also be given in the appendix.

Proof of Theorem 3.1. According to Lemma 3.1 it is sufficient to prove existence, uniqueness and asymptotic stability for the weak solution of (3.18-3.23).

We will replace the system (3.18-3.23) by an equivalent one obtained by the change of variables z = u-v

$$(3.27)\qquad \frac{\partial u}{\partial t} = \operatorname{div}(k(x)\nabla u) + q(x)u, \qquad x\in\Omega,\quad t>0$$

$$(3.28)\qquad \frac{\partial z}{\partial t} = \operatorname{div}(k(x)\nabla z) + q(x)z - \hat{\ell}(x)h(z(t,\cdot)),\quad x\in\Omega,\ t>0$$

$$(3.29)\qquad \frac{\partial u}{\partial n} + \sigma(x)u = g(x)(u(t,\cdot),\rho) - g(x)(z(t,\cdot),\rho),\quad x\in S,\ t>0$$

$$(3.30)\qquad \frac{\partial z}{\partial n} + \sigma(x)z = 0 \qquad x\in S,\quad t>0$$

$$(3.31)\qquad u(o+,x) = u_o(x),\quad x\in\Omega$$

$$(3.32)\qquad z(o+,x) = z_o(x),\quad x\in\Omega$$

where $z_o = u_o - [\psi_1,\dots,\psi_M,\varnothing_{M+1},\dots,\varnothing_m]\cdot V_o$ and it is obvious what a weak solution (u,z) does mean for (3.27-3.32).

The last system is decoupled and we will prove first the existence and uniqueness of a weak solution for the problem (3.28), (3.30), (3.32) for z. This problem differs of the problem (2.34), (2.35), (2.36) by the fact that $\hat{\ell}(\cdot)$ does not belong to a space spanned by a finite set of eigenfunctions $\{\varnothing_1,\dots,\varnothing_n\}$ as in Lemma 2.4.

Let us take $n > m$ and define:

$$(3.33)\qquad \hat{\ell}_n(\cdot) \equiv \ell(\cdot) - \ell_{m+1}\varnothing_{m+1}(\cdot) - \dots - \ell_n\varnothing_n(\cdot) =$$
$$= \ell^1\varnothing_1(\cdot) + \dots + \ell^M\varnothing_M(\cdot) - \ell_{m+1}\varnothing_{m+1}(\cdot) - \dots - \ell_n\varnothing_n(\cdot)$$

$$(3.34)\qquad \delta_m^n(\cdot) := \ell(\cdot) - \hat{\ell}_n(\cdot) = \ell_{m+1}\varnothing_{m+1}(\cdot) + \dots + \ell_n\varnothing_n(\cdot) \in H^1(\Omega).$$

It is obvious from (3.6) that:

(3.35) $$\|\delta_m^n\|_2^2 = |\ell_{m+1}|^2 + \dots + |\ell_n|^2 \leq \|\delta\|_2^2 \quad \text{for all } n > m,$$

(3.36) $$\lim_{n\to\infty} \delta_m^n = \delta \text{ in } H^1(\Omega) \text{ and in } L^2(\Omega),$$

$$\lim_{n\to\infty} \|\hat{\ell} - \hat{\ell}_n\|_2 = \lim_{n\to\infty} \|\ell_{n+1}\phi_{n+1} + \ell_{n+2}\phi_{n+2} + \dots\|_2 =$$

$$= \lim_{n\to\infty} \sqrt{|\ell_{n+1}|^2 + |\ell_{n+2}|^2 + \dots} = 0$$

(see (3.3), (3.5), (3.6)).

Now we will consider the problem:

(3.37) $$\frac{\partial z^n}{\partial t} = \operatorname{div}(k(x)\nabla z^n) + q(x)z^n - \hat{\ell}_n(x)h(z^n(t,\cdot)), \quad x\in\Omega, t>0$$

(3.38) $$\frac{\partial z^n}{\partial n} + \sigma(x)z^n = 0, \qquad x \in S, \ t > 0$$

(3.39) $$z^n(0+, x) = z_0(x), \quad x \in \Omega$$

which is obtained from (2.34), (2.35), (2.36) for $\tilde{\ell}_n(\cdot) = \hat{\ell}_n(\cdot)$ hence $\tilde{L}_n = \text{col}\,[\ell^1, \dots, \ell^M, 0, \dots, 0, \ell_{m+1}, \dots, \ell_n]$ and $F(t,x) = 0$. According to Lemma 2.4 this problem has a unique weak solution for each $n > m$, given by

$$z^n(t,\cdot) = \tilde{Z}_t^n z_0$$

Remark that $t \longmapsto \|z^n(t,\cdot)\|_1$ is continuous for $t > 0$ and square integrable on $[0, t_0]$, $t_0 > 0$.

If in (3.37) we put $\hat{\ell}_n = \ell + \delta_m^n$, using again Lemma 2.4 we obtain:

$$z^n(t,\cdot) = Z_t z_0 + \int_0^t Z_{t-s}(\delta_m^n(\cdot)\cdot h(z^n(s,\cdot)))ds$$

or

(3.40) $$z^n(t) = Z_t z_0 + \int_0^t Z_{t-s}(\delta_m^n)\cdot h(z^n(s))\, ds$$

(we will write in the sequel $z^n(t)$ instead of $z^n(t,\cdot)$ and $Z_{t-s}(\delta_m^n)$ means the vector function in $(R^{P_1})^*$ obtained by applying Z_{t-s} to each component of the vector function δ_m^n).

Because

$$\delta_m^{n+p} = \delta_m^n + \delta_n^{n+p}$$

and

$$(3.41) \qquad z^{n+p}(t) = Z_t z_o + \int_0^t Z_{t-s}(\delta_m^{n+p}) \cdot h(z^{n+p}(s))\, ds$$

we obtain:

$$(3.42) \qquad z^{n+p}(t)-z^n(t) = \int_0^t Z_{t-s}(\delta_m^n) \cdot h(z^{n+p}(s)-z^n(s))ds +$$

$$+ \int_0^t Z_{t-s}(\delta_n^{n+p}) \cdot h(z^{n+p}(s))\, ds \; .$$

From (3.41), (2.40) it follows that

$$\|z^{n+p}(t)\|_1 \leq C_2(t,z_o)e^{-\mu t} +$$

$$+ \|h\|\, e^{-\mu t} \int_0^t |C_2(t-s,\delta_m^{n+p})| e^{\mu s} \| z^{n+p}(s)\|_1\, ds$$

or, using Schwarz-Cauchy inequality:

$$e^{\mu t} \| z^{n+p}(t)\|_1 \leq C_2(t,z_o) +$$

$$+ \|h\| \left(\int_0^t |C_2(t-s,\delta_m^{n+p})|^2 ds\right)^{1/2} \cdot \left(\int_0^t e^{2\mu s} \| z^{n+p}(s)\|_1^2\, ds\right)^{1/2}$$

or

$$e^{2\mu t}\|z^{n+p}(t)\|_1^2 \leq 2\, C_2^2(t,z_o) +$$

$$+ 2\|h\|^2 \int_0^t |C_2\,(t-s,\delta_m^{n+p})|^2 ds \cdot \int_0^t e^{2\mu s} \| z^{n+p}(s)\|_1^2\, ds$$

and from (2.41) and (3.35) it follows that:

$$e^{2\mu t}\|z^{n+p}(t)\|_1^2 \leq 2\ C_2^2(t,z_o)+2\|h\|^2 \cdot \|\delta\|_2^2\ \hat{C}_2^2 \int_0^t e^{2\mu s}\|z^{n+p}(s)\|_1^2\ ds$$

for all $t > 0$.

Integrating on $[0,t]$ and using (2.41) we have that

$$\int_0^t e^{2\mu\tau}\|z^{n+p}(\tau)\|_1^2\ d\tau \leq 2\ \hat{C}_2^2\|z_o\|_2^2 + 2\|h\|^2\|\delta\|_2^2\ \hat{C}_2^2 \cdot$$

$$\cdot \int_0^t d\tau \int_0^\tau e^{2\mu s}\|z^{n+p}(s)\|_1^2\ ds$$

and from Gronwall's lemma we obtain

$$\int_0^t e^{2\mu\tau}\|z^{n+p}(\tau)\|_1^2\ d\tau \leq 2\ \hat{C}_2^2\|z_o\|_2^2\ \exp(2\|h\|^2\|\delta\|_2^2\ \hat{C}_2^2 t), \quad t \geq 0. \tag{3.43}$$

From (3.42) and (2.40) we have:

$$\|z^{n+p}(t)-z^n(t)\|_1 \leq \|h\|e^{-\mu t}\int_0^t |C_2(t-s,\delta_m^n)|e^{\mu s}\|z^{n+p}(s)-z^n(s)\|_1\ ds +$$

$$+\|h\|e^{-\mu t}\int_0^t |C_2(t-s,\delta_n^{n+p})|e^{\mu s}\|z^{n+p}(s)\|_1\ ds\ ;$$

multiplying by $e^{\mu t}$, using Schwarz-Cauchy inequality and (2.41) it follows that:

$$e^{\mu t}\|z^{n+p}(t)-z^n(t)\|_1 \leq \|h\|\ \hat{C}_2\|\delta_m^n\|_2\Big(\int_0^t e^{2\mu s}\|z^{n+p}(s)-z^n(s)\|_1^2\ ds\Big)^{1/2} +$$

$$+\|h\|\ \hat{C}_2\|\delta_n^{n+p}\|_2\ \Big(\int_0^t e^{2\mu s}\|z^{n+p}(s)\|_1^2\ ds\Big)^{1/2};$$

from (3.43), (3.35) we obtain:

$$e^{2\mu t}\|z^{n+p}(t)-z^n(t)\|_1^2 \leq 2\|h\|^2\ \hat{C}_2^2(\|\delta\|_2^2\int_0^t e^{2\mu s}\|z^{n+p}(s)-z^n(s)\|_1^2\ ds +$$

$$+ 2\|\delta_n^{n+p}\|_2^2 \cdot \hat{C}_2^2 \|z_o\|_2^2 \, e^{2\|h\|^2\|\delta\|_2^2 \hat{C}_2^2 t}).$$

Using again Gronwall's lemma we have that:

$$e^{2\mu t}\|z^{n+p}(t)-z^n(t)\|_1^2 \leq 4\|h\|^2 \hat{C}_2^4 \|\delta_n^{n+p}\|_2^2 \|z_o\|_2^2 \cdot e^{4\|h\|^2\|\delta\|_2^2 \hat{C}_2^2 t},$$

$$(3.44)\ \|z^{n+p}(t)-z^n(t)\|_1 \leq 2\|h\|\hat{C}_2^2\|\delta_n^{n+p}\|_2\|z_o\|_2 \, e^{(-\mu+2\|h\|^2\|\delta\|_2^2\hat{C}_2^2)t}$$

for all $t \geqslant 0$.

Since $\|\delta_n^{n+p}\|_2^2 = |\ell_{n+1}|^2+|\ell_{n+2}|^2+\ldots+|\ell_{n+p}|^2 \leq \|\ell-\ell_n\|_2^2 \longrightarrow 0$ for $n \longrightarrow \infty$, we conclude that the sequence $z^n(t)$ is Cauchy in $H^1(\Omega)$ for $t>0$ and converges uniformly in $H^1(\Omega)$ on any interval $[t_o,t_1] \subset]0,\infty[$ and also converges uniformly on any interval $[0,t_1]$ in the $L^2(\Omega)$ topology. Hence there exists:

$$(3.45) \qquad z(t) = \lim_{n\to\infty} z^n(t),$$

in $L^2(\Omega)$ for $t \geqslant 0$ and in $H^1(\Omega)$ for $t>0$ and the function $t \longmapsto z(t)$ is continuous from $[0,\infty[$ to $L^2(\Omega)$ and from $]0,\infty[$ to $H^1(\Omega)$.

On the other hand, from (2.39) we have that:

$$(3.46)\ \|Z_{t-s}(\delta_m^n)\cdot h(z^n(s))\|_2 \leq C_1\|\delta_m^n\|_2 \, e^{-\mu t} e^{\mu s}|h(z^n(s))| \leq$$

$$C_1\|\delta\|_2 \, e^{-\mu t}\|h\| e^{\mu s}\|z^n(s)\|_1, \quad 0<s\leq t .$$

Since

$$\int_0^t e^{\mu s}\|z^n(s)\|_1 \, ds \leq \sqrt{t}\ \left(\int_0^t e^{2\mu s}\|z^n(s)\|_1^2 \, ds\right)^{1/2} \leq$$

$$\leq \sqrt{t}\sqrt{2}\ \hat{C}_2 \|z_o\|_2 \exp(\|h\|^2\|\delta\|_2^2\hat{C}_2^2 t)$$

(the first inequality follows by Schwarz-Cauchy and the second from (3.43)), we can apply Lebesgue theorem to obtain:

$$(3.47)\quad \lim_{n\to\infty} \int_0^t Z_{t-s}(\delta_m^n)h(z^n(s))ds = \int_0^t Z_{t-s}(\delta)h(z(s))\, ds$$

(here we used also the fact that from (2.39) and (3.36) it follows that: $\lim_{n\to\infty} Z_{t-s}(\delta_m^n)\cdot h(z^n(s)) = Z_{t-s}(\delta)h(z(s))$ for each $s\in\,]0,\,t]$ in $L^2(\Omega)$).

Hence we can take the limit in (3.40) for $n \longrightarrow \infty$ and we will obtain:

$$(3.48)\qquad z(t) = Z_t\, z_o + \int_o^t Z_{t-s}(\delta)\, h\,(z(s))ds, \qquad t\geqslant 0.$$

Remark also that from (3.43) using again Lebesgue theorem we obtain that $t \longrightarrow \|z(t)\|_1^2$ is integrable on $[0,t_o]$, $t_o > 0$. Hence the function $F(t,x) = \delta(x)h(z(t))$ fulfills the conditions of Lemma 2.4 and we obtain that z is a weak solution of the problem (3.49), (3.30), (3.32) where:

$$(3.49)\qquad \frac{\partial z}{\partial t} = \operatorname{div}(k(x)\nabla z) + q(x)z - \ell(x)h(z(t,\cdot)) + \\ + \delta(x)h(t(t,\cdot)), \qquad x\in\Omega,\ \ t>0.$$

From (3.6) we obtain that z is a weak solution for (3.28), (3.30), (3.32).

To prove the uniqueness of the weak solution for (3.28), (3.30), (3.32) remark first that any weak solution of (3.28), (3.30), (3.32) is also for (3.49), (3.30), (3.32) hence it must fulfill (3.48). From (3.48), using the same computation as in the case of formula (3.43) we may obtain:

$$(3.50)\qquad \int_o^t e^{2\mu\tau}\|z(\tau)\|_1^2 d\tau \leqslant 2\, \hat{C}_2^2\|z_o\|_2^2\, e^{2\|h\|^2\|\delta\|_2^2\, \hat{C}_2^2\, t}, \qquad t>0$$

Let us consider $t_o > 0$ and $t\geqslant t_o$. From (3.48) and (2.40) we have that:

$$\|z(t)\|_1 \leqslant C_2(t,z_o)e^{-\mu t} + \int_o^{t_o} |C_2(t-s,\delta)|\,\|h\|e^{-\mu(t-s)}\|z(s)\|_1\, ds + \\ + \int_{t_o}^t |C_2(t-s),\delta|e^{-\mu(t-s)}\|h\|\,\|z(s)\|_1\, ds\,.$$

Using $(a+b+c)^2 \leq 4a^2 + 4b^2 + 2c^2$, Schwarz-Cauchy and (2.41), we obtain:

$$e^{2\mu t}\|z(t)\|_1^2 \leq 4C_2^2(t,z_o) + 4\|h\|^2 \int_0^{t_o} |C_2(t-s,\delta)|^2 ds \int_0^{t_o} e^{2\mu s}\|z(s)\|_1^2 ds +$$

$$+ 2\|h\|^2 \int_{t_o}^{t} |C_2(t-s,\delta)|^2 ds \int_{t_o}^{t} e^{2\mu s}\|z(s)\|_1^2\, ds \leq$$

$$\leq 4\, C_2^2(t_o,z_o) + 8\|h\|^2\|\delta\|_2^2\, \hat{C}_2^2\|z_o\|_2^2\, e^{2\|h\|^2\|\delta\|_2^2\, \hat{C}_2^2 t_o} +$$

$$+ 2\|h\|^2\, \hat{C}_2^2\|\delta\|_2^2 \int_{t_o}^{t} e^{2\mu s}\|z(s)\|_1^2\, ds \quad , \qquad t \geq t_o$$

By Gronwall's lemma we deduce:

$$e^{2\mu t}\|z(t)\|_1^2 \leq \left[4\, C_2^2(t_o,z_o) + 8\|h\|^2\|\delta\|_2^2\, \hat{C}_2^2\|z_o\|_2^2\, e^{2\|h\|^2\|\delta\|_2^2\hat{C}_2^2\, t_o}\right] \cdot$$

$$\cdot\, e^{2\|h\|^2\, \hat{C}_2^2\|\delta\|_2^2(t-t_o)} \quad ;$$

hence with

$$(3.51)\quad K^2(t_o,z_o) = 4\, C_2^2(t_o,z_o)\, e^{-2\|h\|^2\, \hat{C}_2^2\|\delta\|_2^2\, t_o} + 8\|h\|^2\|\delta\|_2^2\hat{C}_2^2\|z_o\|_2^2$$

we have

$$(3.52)\qquad \|z(t)\|_1 \leq K(t_o,z_o)\, e^{-\mu^* t}, \qquad t \geq t_o > 0.$$

Remark that $t \longmapsto K(t,z_o)$ is continuous and nonincreasing (see Lemma 2.5),

$$(3.53)\qquad \lim_{\|z_o\|_2 \to 0} K(t_o,z_o) = 0 \quad \text{for each } t_o > 0,$$

and

$$(3.54)\qquad K(t_o,0) = 0, \qquad t_o > 0.$$

We have thus proved that any weak solution z of the problem (3.28), (3.30), (3.32) has to fulfill (3.52). If z_1 and

z_2 are weak solutions for (3.28), (3.30), (3.32) then $z = z_1 - z_2$ is a weak solution for (3.28), (3.30) with the initial condition $z(o+, \cdot) = 0$ and from (3.52), (3.53), (3.54) we will obtain $z = 0$, hence $z_1 = z_2$.

Now we will prove the existence and the uniqueness of the weak solution u for the problem (3.27), (3.29), (3.31) where z is the unique weak solution of (3.28), (3.30), (3.32) which was considered above.

Let us denote

$$v(t,\cdot) = u(t,\cdot) - z(t,\cdot) \tag{3.55}$$

Obviously u is the weak solution for (3.27), (3.29), (3.31) iff v is a weak solution (3.56), (3.21), (3.23) where:

$$\frac{\partial v}{\partial t} = \mathrm{div}(k(x)\nabla v) + q(x)v + \hat{\ell}(x)h(z(t,\cdot)), \quad x \in \Omega, \ t > 0 \tag{3.56}$$

Remark that if we take $F(t,x) = \hat{\ell}(x)h(z(t,\cdot))$ then this function F will satisfy the conditions of Lemma 2.2 and we have that.

$$v(t,\cdot) = T_t v_o + \int_0^t T_{t-s}(\hat{\ell})\, h(z(s))\, ds \tag{3.59}$$

where $v_o(\cdot) = [\psi_1(\cdot), \dots, \psi_M(\cdot), \phi_{M+1}(\cdot), \dots, \phi_m(\cdot)]\, V_o$, is the unique weak solution for (3.56), (3.21), (3.23). The uniqueness of v is equivalent to the uniqueness of u and we obtain that :

$$u(t,\cdot) = T_t v_o + \int_0^t T_{t-s}(\hat{\ell})\cdot h(z(s,\cdot))ds + z(t,\cdot), \quad t \geq 0 \tag{3.60}$$

Remark that from (3.52) it follows that:

$$|h(z(t)| \leq \|h\| \cdot K(t,z_o)e^{-\mu^* t}, \quad t > 0 \tag{3.61}$$

Using (3.51), and Lemma 2.1 and (3.61), we obtain easily (3.15), (3.16), and from Lemma 3.1 and Lemma 2.1 and (3.59) we have also (3.17) which completes the proof of Theorem 3.1.

Remark 3.2. Using Green formula it is obvious that a classical solution for the closed loop system (3.9-3.13) is also a weak solution.

Remark 3.3. It is easy to see that $t \longmapsto h(u(t,\cdot))$ is continuous for $t > 0$ and integrable on $[0,t_0]$, $t_0 > 0$, and so the "observer" (3.10), (3.13) has a meaning.

3.2. Stabilization using point sensors output. Assume now that $\hat{h}$ is given by (1.5) and also that $u_0(\cdot) \in H^1(\Omega)$. In this case h is a bounded linear operator from $C(\bar{\Omega})$ to $\mathbb{R}^{P_1}$ because

$$|\hat{h}(u)| = \sqrt{\hat{h}_1^2(u) + \ldots + \hat{h}_p^2(u)} = \sqrt{u^2(\xi_1) + \ldots + u^2(\xi_p)} \leq \sqrt{P}\, \|u\|_0$$

and so

$$|h(u)| \leq \|Y\| \cdot \sqrt{P} \cdot \|u\|_0, \qquad u \in C(\bar{\Omega})$$

hence

$$(3.1') \qquad \|h\| \leq \|Y\| \cdot \sqrt{P}$$

(we can also see that in the last relation we may replace the sign " $\leq$ " by " $=$ ").

Also h is a continuous linear operator from the Fréchet space $C(\Omega)$ to $\mathbb{R}^{P_1}$.

The relations (3.2-3.6) are still true and we will choose the number m such that

$$(3.7') \qquad \mu - \|\delta\|_1^2\, \hat{C}_0^2 \|h\| \geq \mu^*$$

(see also (2.43) and (4.59)). (3.7') is possible since

$$\|\delta\|_1^2 = |\ell_{m+1}|^2 (1+\hat{q}-\lambda_{m+1}) + |\ell_{m+2}|^2 (1+\hat{q}-\lambda_{m+2}) + \ldots \longrightarrow 0 \quad \text{for } m \to \infty.$$

The matrices $\hat{Q}$, $\hat{R}$, $\hat{L}$, H and $\mathcal{M}$ are given by (3.8);

We have the following:

Theorem 3.1 . Let $u_0 \in H^1(\Omega)$ and $V_0 \in \mathbb{R}^m$, then the closed loop system (3.9-3.13) has a unique weak solution (u,V) i.e. there exists a unique pair (u,V) such that:

(i) $t \longmapsto u(t,\cdot)$ is differentiable from $]0,\infty[$ to $L^2(\Omega)$, continuous from $[0,\infty[$ to $L^2(\Omega)$ and from $]0,\infty[$ to $H^1(\Omega)$; for (SLB) $t \longmapsto u(t,\cdot)$ is continuous from $]0,\infty[$ to $C(\bar{\Omega})$ and for (SLD) $t \longmapsto u(t,\cdot)$ is continuous from $]0,\infty[$ to $C(\Omega)$; $u(0+,\cdot) = u_0(\cdot)$; $|h(u(t))|^2$ is integrable on $[0,t_0]$, $t_0 > 0$;

(ii) $t \longmapsto V(t)$ is differentiable from $]0,\infty[$ to $\mathbb{R}^m$, continuous from $[0,\infty[$ to $\mathbb{R}^m$ and satisfies (3.10), (3.13);

(iii) u satisfies (3.14) for all $\eta(t,\cdot) \in H^1(\Omega)$ and $t > 0$.

Moreover the system is exponentially stable in the sense that there exists the constants K_4 and K_5 and a nonincreasing function $K_6(t)$, $t > 0$ such that (3.15), (3.16), (3.17) are fulfilled; for (SLB) there exists also a nonincreasing function $K_7(t)$ such that

$$\|u(t,\cdot)\|_0 \leq K_7(t_0)e^{-\mu^* t}, \quad t \geq t_0 > 0, \tag{3.62}$$

while for (SLD) for any domain Ω_1 such that $\bar{\Omega}_1 \subset \Omega$, there exists a nonincreasing function $K_{\Omega_1}(t)$, $t > 0$ such that:

$$\|u(t,\cdot)\|_{C(\bar{\Omega}_1)} \leq K_{\Omega_1}(t_0)e^{-\mu^* t}, \quad t \geq t_0 > 0. \tag{3.61'}$$

<u>Proof</u>. Consider again (3.18-3.23) and define the weak solution as a pair (u,v) such that:

(i) the functions $t \longmapsto u(t,\cdot)$ and $t \longmapsto v(t,\cdot)$ are differentiable from $]0,\infty[$ to $L^2(\Omega)$, continuous from $[0,\infty[$ to $L^2(\Omega)$ and from $]0,\infty[$ to $H^1(\Omega)$; for (SLB) these functions are also continuous from $]0,\infty[$ to $C(\bar{\Omega})$ and for (SLD) these functions are continuous from $]0,\infty[$ to $C(\Omega)$.

(ii) u and v satisfy (3.25), (3.26) for all $t > 0$, $\eta(t,\cdot) \in H^1(\Omega)$.

(iii) $|h(u(t))|^2$, $|h(v(t))|^2$ are integrable on $[0,t_0]$, $t_0 > 0$.

It is easy to see that Lemma 3.1 still works and we will follow step by step the proof of theorem 3.1. With the same change of variables we obtain (3.27-3.32) and in this case the definition of a weak solution is also obvious. Remark that $z_o \in H^1(\Omega)$. We will define again $\hat{\ell}_n$ and δ_m^n by (3.33), (3.34) and will remark that:

$$(3.35') \qquad \|\delta_m^n\|_1^2 = |\ell_{m+1}|^2(1+\hat{q}-\lambda_{m+1})+\ldots+|\ell_n|^2(1+\hat{q}-\lambda_n) \leq \|\delta\|_1^2$$

for all $n > m$, (3.36) is true and

$$\lim_{n\to\infty} \|\hat{\ell} - \hat{\ell}_n\|_1 = 0.$$

Consider again the problem (3.37-3.39). We will remark using Lemma 2.4, that its weak solution

$$z^n(t,\cdot) = \tilde{Z}_t^n\, z_o \in C(\bar{\Omega}), \qquad t > 0$$

has also the property that $t \longmapsto \|z^n(t,\cdot)\|_o$ is continuous for $t > 0$ and square integrable on $[o,t_o]$, $t_o > 0$. We obtain in the same way formulas (3.40), (3.41) and (3.42).

From (3.41) and (2.42) we have:

$$\|z^{n+p}(t)\|_o \leq C_o(t,z_o)e^{-\mu t} + \|h\|e^{-\mu t}\int_0^t e^{\mu s}|C_o(t-s,\delta_m^{n+p})|\,\|z^{n+p}(s)\|_o\, ds$$

and using (2.43) and a similar computation as in theorem 3.1 we get

$$(3.43') \qquad \int_0^t e^{2\mu\tau}\|z^{n+p}(\tau)\|_o^2\, d\tau \leq 2\,\hat{C}_o^2\,\|z_o\|_1^2\, e^{2\|h\|^2\|\delta\|_1^2\hat{C}_o^2 t}, \quad t \geq 0$$

and

$$(3.44') \quad \|z^{n+p}(t)-z^n(t)\|_o \leq 2\|h\|\,\hat{C}_o^2\,\|\delta_n^{n+p}\|_1\|z_o\|_1\, e^{(-\mu+2\|h\|^2\|\delta\|_1^2\hat{C}_o^2)t}$$
$$t \geq 0.$$

Since $\|\delta_n^{n+p}\|_1 \longrightarrow 0$ for $n \longrightarrow \infty$, $p \geq 1$, we conclude that the sequence $z^n(t)$ is Cauchy in $C(\bar{\Omega})$ for $t > 0$ and converges uniformly in $C(\bar{\Omega})$ on any interval $[t_o, t_1] \subset]o,\infty[$ and in $L^2(\Omega)$ on any $[o, t_o]$, $t_o > 0$. Hence there exists

$$z(t) = \lim_{n\to\infty} z^n(t), \tag{3.45'}$$

in $L^2(\Omega)$ for $t \geq 0$ and in $C(\bar{\Omega})$ for $t > 0$ and $t \longmapsto z(t)$ is continuous from $[o,\infty[$ to $L^2(\Omega)$ and from $]o,\infty[$ to $C(\bar{\Omega})$.

We obtain also:

$$\|Z_{t-s}(\delta^n_m)h(z^n(s))\|_2 \leq C_1 \|\delta\|_2 \, e^{-\mu t} \|h\| e^{\mu s} \|z^n(s)\|_o \quad \text{for each } s \in]o,t]. \tag{3.46'}$$

Since from Schwarz-Cauchy inequality and (3.43') we have:

$$\int_o^t e^{\mu s} \|z^n(s)\|_o \, ds \leq \sqrt{t} \left(\int_o^t e^{2\mu s} \|z^n(s)\|_o^2 \, ds \right)^{1/2} \leq$$

$$\leq \sqrt{t}\sqrt{2}\, \hat{C}_o \|z_o\|_1 \, e^{\|h\|^2 \|\delta\|_1^2 \hat{C}_o^2 t}$$

we may apply Lebesgue theorem and we have (3.47) because $\lim Z_{t-s}(\delta^n_m)h(z^n(s)) = Z_{t-s}(\delta)h(z(s))$ for each $s \in]o,t]$ in $L^2(\Omega)$.

Hence, from (3.40) for $n \longrightarrow \infty$ we obtain (3.48).

From (3.43') and Lebesgue theorem we get that $t \longmapsto \|z(t)\|_o^2$ is integrable on $[o, t_o]$, $t_o > 0$ and so the function $F(t, x) = \delta(x)h(z(t))$ fulfills the conditions of Lemma 2.4. We conclude that z is a weak solution for (3.49), (3.30), (3.32) or equivalently for (3.28), (3.30), (3.32).

From (3.48) using (2.42), (2.43) and Gronwall's lemma we obtain in the same way as in theorem 3.1 that

$$\int_o^t e^{2\mu\tau} \|z(\tau)\|_o^2 \, d\tau \leq 2\, \hat{C}_o^2 \|z_o\|_1^2 \, e^{2\|h\|^2 \|\delta\|_1^2 \hat{C}_o^2 t}, \quad t \geq 0 \tag{3.50'}$$

and

$$\|z(t)\|_o \leq K'(t_o, z_o)\, e^{-\mu^* t}, \quad t \geq t_o > 0 \tag{3.52'}$$

where

$$(3.51') \quad K'^2(t_o,z_o) = 4C_o^2(t_o,z_o)\, e^{-2\|h\|^2 \hat{C}_o^2 \|\delta\|_1^2 t_o} +$$
$$+ 8\|h\|^2 \|\delta\|_1^2 \hat{C}_o^2 \|z_o\|_1^2 .$$

The uniqueness of z results in the same way as in theorem 3.1 and the existence and uniqueness of u with this z will result from (3.55), (3.56) and u will be given by (3.60).

From (3.52') it follows that

$$(3.61') \qquad |h(z(t))| \leq \|h\| \cdot K'(t,z_o)e^{-\mu^* t}, \quad t \geq 0$$

and from (3.51'), Lemma 2.1 and (3.61') we obtain easily (3.15), (3.16), (3.62) and (3.17), hence all the conclusions of the Theorem 3.1' are proved.

Remark 3.4. In the case of point sensors observation, it is very important that the state is a continuous function on $\bar{\Omega}$ or Ω because for the functions of $L^2(\Omega)$ or $H^1(\Omega)$ the point values have no meaning. Theorem 3.1' insures this condition, moreover $t \longmapsto h(u(t))$ is continuous for $t > 0$ and square integrable on $[0,t_o]$, $t_o > 0$ and so the "observer" (3.10), (3.13) is meaningful.

Let us apply the theory developed in the following

Example. Consider the problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + qu, \qquad t > 0,\ 0 < x < 1;$$
$$\frac{\partial u}{\partial x}(t,0) = f(t);\ \frac{\partial u}{\partial x}(t,1) = 0, \quad t > 0;$$
$$u(0,x) = u_o(x), \qquad 0 < x < 1;$$

and the output given by $y(t) = u(t,0)$. It is easy to see that $\lambda_n = q - (n-1)^2\pi^2$, $n=1,2,\ldots$ and $\phi_1(x) = 1$, $\phi_n(x) = \sqrt{2}\cos(n-1)\pi x$, $n=2,3,\ldots$. Also $G = \mathrm{col}\,[-\phi_1(0),\ldots,-\phi_M(0)] = -\mathrm{col}\,[1\ \sqrt{2}\ldots\sqrt{2}]$ and $\chi = \mathrm{row}\,[\phi_1(0),\ldots,\phi_M(0)] = \mathrm{row}\,[1\ \sqrt{2}\ldots\sqrt{2}]$. All the assumptions hold (in (2.6) we have $K_o = 1$; $\alpha = 3/2$). Hence we are able to construct a finite dimensional compensator and, to simplify, we will take $q = 1$. The functional $h(u) = u(0)$ is bounded on $C([0,1])$ (also on $H^1(0,1)$ but we consider here point sensor observation) and $\|h\| = 1$.

Because there exists only one unstable eigenvalue $\lambda_1 = 1$, we may choose $M = 1$ and thus $\Lambda = \lambda_1 = 1$, $G = -1$, $\chi = 1$ are scalars. Let us take $\mu = \pi^2 - 2$, $\mu^* = 1/2$, $R = 2$, $L = \pi^2$ ans so we obtain $\mu_1 = -1$, $\psi_1(x) = \frac{\sqrt{2}}{\sin\sqrt{2}} \cos\sqrt{2}(1-x)$, $Q = 1$, $p_1 = 1$. From (3.4) we obtain $\ell_1 = \pi^2$, $\ell_n = 2\sqrt{2}\pi^2/((n-1)^2\pi^2 - 2)$, $n=2,3,\ldots$. On the other hand, from (4.59) we obtain $\hat{C}_o^2 = 2$ (since $\check{C}_o^2 =$

$$= \sum_{n\geq 2} \frac{1}{(1+(n-1)^2\pi^2)(2(n-1)^2\pi^2-2-2\mu')} < 1 \text{ for } \mu' = \pi^2 - 3/2 \text{ and}$$

also $\tilde{C} = 1$). If we set $m = 3$ then by a simple computation we get that $\|\delta\|_1^2 = 8\pi^2 \sum_{n>3} ((n-1)^2\pi^2-1)/((n-1)^2\pi^2-2))^2 < 3.4$, hence (3.7') is fullfiled and we obtain the compensator:

$$\frac{dV}{dt} = (\mathcal{M} - \hat{L}H)\, V + L\, y(t), \quad t > 0;\ V(0) = V_0$$

with $\mathcal{M} = \mathrm{diag}\,(-1,\ 1-\pi^2,\ 1-4\pi^2)$, $\hat{L} = \pi^2 \,\mathrm{col}\,[1, 2\sqrt{2}/(\pi^2-2), 2\sqrt{2}/(4\pi^2-2)]$, $H = \mathrm{row}\,[\sqrt{2}\,\mathrm{cotan}\sqrt{2}, \sqrt{2}, \sqrt{2}]$ and the stabilizing input is $f(t) = \hat{R}\,\hat{Q}\,V(t) = [2\ \ 0\ \ 0]\,\mathrm{diag}\,(1\ \ 0\ \ 0)\,V(t) = [2\ \ 0\ \ 0]\,V(t)$.

The rate of convergence for (u,V) will be of the order $e^{-0.5\,t}$.

Remark that infinite dimensional compensators have been proposed for this problem in [1], [4] and [6].

4. Appendix

In this section we will give the proof of Propositions and Lemmas stated in the previous sections.

Proof of Proposition 2.1. It is known (Mikhaïlov [7,p.174]) that there exists a bounded linear one to one operator I from

$L^2(\Omega)$ into $H^1(\Omega)$ such that

(4.1) $$(u,v) = (I\,u,v)_1$$

for all $v \in H^1(\Omega)$. The restriction of I to $H^1(\Omega)$ is self-adjoint, compact and positive. Moreover λ_n is an eigenvalue for (2.1) iff $(1+\hat{q}-\lambda_n)$ is a characteristic number for the restriction of I to $H^1(\Omega)$.

So we obtain that ψ_j satisfies (2.11) iff

(4.2) $$(\psi_j-(1+\hat{q}-\mu_j)\,I\,\psi_j,v)_1 = (kgRp_j,v)_S$$

for all $v \in H^1(\Omega)$.

Because $v \longmapsto (kgRp_j, v)_S$ is a bounded linear functional on $H^1(\Omega)$ (see the explanations at the begining of subsection 3.1 concerning the operator $T : H^1(\Omega) \longrightarrow L^2(S)$), using F.Riesz theorem we obtain that there exists only one function $\varphi_j \in H^1(\Omega)$ such that

$$(kgRp_j,v)_S = (\varphi_j,v)_1 \quad \text{for all } v \in H^1(\Omega).$$

From (4.2) we obtain that ψ_j satisfies (2.11) iff

(4.3) $$\psi_j - (1+\hat{q}-\mu_j)\,I\,\psi_j = \varphi_j$$

Since $1+\hat{q}-\mu_j$ is not a characteristic number for I using Fredholm alternative we obtain the existence and uniqueness of ψ_j. The additional smoothness assumption in the case of point sensors will insure (2.12), (2.13) (Mikhaîlov [7, p.229]).

<u>Proof of Proposition 2.2.</u> Let (λ, u) be an eigenpair for (2.14). If in (2.15) we set $v = \emptyset_n$ we get, using also (2.2') that:

$$\lambda(u,\emptyset_n) = -(u,\emptyset_n)_1 + (\hat{q}+1)(u,\emptyset_n) + (kg,\emptyset_n)_S\,(u,\rho) =$$

$$= \lambda_n(u,\emptyset_n)_2 + (kg,\emptyset_n)_S\,(u,\rho)\,.$$

If we denote $p = \mathrm{col}\left[(u,\emptyset_1),\ldots,(u,\emptyset_M)\right]$ then from (2.8), (2.10) we obtain

(4.4) $$\lambda p = (\Lambda + GR)p.$$

If $p \neq 0$ then p is an eigenvector for the matrix $\Lambda + GR$ hence $p = p_j$ for some $j=1,\ldots,M$ and so $\lambda = \mu_j$. Since $(u,\rho) = R\,p_j$ it is obvious that u is the solution of (2.11) i.e. $u = \psi_j$.

If $p = 0$ then $(u,\rho) = 0$ hence u is an eigenfunction for (2.2) and λ the corresponding eigenvalue. Since $(u, \varnothing_n) = 0$, $n=1,\ldots,M$ we conclude that $u \in \{\varnothing_{M+1},\varnothing_{M+2},\ldots\}$ and $\lambda \in \{\lambda_{M+1},\lambda_{M+2},\ldots\}$. We see further that conversely $(\lambda_n, \varnothing_n)$ for $n > M$ is an eigenpair for (2.14) (because $(\varnothing_n,\rho) = 0$). Also let us prove that (μ_j,ψ_j) is an eigenpair for (2.14).

From (2.11') we have

$$\mu_j(\psi_j,v) = -(\psi_j,v)_1 + (\hat{q}+1)(\psi_j,v) + (kgRp_j,v)_S$$

for all $v \in H^1(\Omega)$. If we take $v = \varnothing_n$, using (2.2') we obtain

(4.5) $$\mu_j(\psi_j,\varnothing_n) = \lambda_n(\psi_j,\varnothing_n) + (kg,\varnothing_n)_S\, Rp_j, \quad n=1,2,\ldots$$

If we put $p_j' = \text{col}\left[(\psi_j,\varnothing_1),\ldots,(\psi_j,\varnothing_M)\right]$ we obtain

$$\mu_j\, p_j' = \Lambda\, p_j' + GR\, p_j$$

Since $GR\,p_j = +\mu_j p_j - \Lambda\, p_j$ we have that:

$$\mu_j(p_j'-p_j) = \Lambda(p_j'-p_j)$$

and $\mu_j \notin \{\lambda_1,\ldots,\lambda_n\}$ will imply that $p_j = p_j'$ and because $R\,p_j = R\,p_j' = (\psi_j,\rho)$ we will obtain that ψ_j will verify

$$\mu_j(\psi_j,v) = -(\psi_j,v)_1 + (\hat{q}+1)(\psi_j,v) + (kg,v)_S(\psi_j,\rho)$$

for all $v \in H^1(\Omega)$.

It can be proved also that the multiplicities of μ_j are equal to one (see [1]).

<u>Proof of Proposition 2.3.</u> The fact that $\{\psi_1,\ldots,\psi_M,\varnothing_{M+1},\varnothing_{M+2},\ldots\}$ is a basis in $L^2(\Omega)$ and (2.17) was proved in [1]. We will prove only the convergence of the series (2.16) in the

$H^1(\Omega)$ topology if $\varphi \in H^1(\Omega)$. Because $\sum_{n>M} \varphi_n \phi_n(\cdot) =$

$= \sum_{n>M} \varphi_n \sqrt{1+\hat{q}-\lambda_n} \frac{\phi_n(\cdot)}{\sqrt{1+\hat{q}-\lambda_n}}$, it is sufficient to prove the convergence of the series $\sum_{n>M} \varphi_n^2 (1+\hat{q}-\lambda_n)$. This statement follows from (2.17) if we remark that

$$\varphi_n^2 \leq (1+\varphi_1^2+\ldots+\varphi_M^2)((\varphi,\phi_n)^2 + (\psi_1,\phi_n)^2+\ldots+(\psi_M,\phi_n)^2)$$

and, because $\psi_j, \varphi \quad H^1(\Omega)$, the series:

$$\sum_{n>M} (\varphi,\phi_n)^2(1+\hat{q}-\lambda_n) \quad \text{and} \quad \sum_{n>M} (\psi_j,\phi_n)^2(1+\hat{q}-\lambda_n),\ j=1,\ldots,M$$

are convergent.

<u>Proof of Lemma 2.1.</u> The fact that T_t is a strongly continuous semigroup on $L^2(\Omega)$ or $H^1(\Omega)$ is obvious according to Proposition 2.3.

Let us denote $T_t'\varphi = \sum_{n=1}^{M} e^{\mu_n t} \varphi_n \psi_n(\cdot)$; $T_t''\varphi = \sum_{n>M} e^{\lambda_n t} \varphi_n \phi_n(\cdot)$

Hence $T_t = T_t' + T_t''$ and since T_t' verifies obviously the relations (2.19-2.25), ~~(2.24'),~~ (2.25') it is sufficient to prove these relations for T_t''. We have from Parseval relation and (2.17) that

$$(4.6) \quad \|T_t''\varphi\|_2^2 = \sum_{n>M} e^{2\lambda_n t} \varphi_n^2 = e^{-2\mu t} \sum_{n>M} e^{2(\lambda_n+\mu)t} \varphi_n^2 \leq$$

$$\leq e^{-2\mu t} \sum_{n>M} \varphi_n^2 \leq e^{-2\mu t} \|\varphi\|_2^2 \cdot K_1''$$

because:

$$(4.7)\ \varphi_n^2 \leq 2(\varphi,\phi_n)^2 + 2((\psi_1,\phi_n)^2+\ldots+(\psi_M,\phi_n)^2)(\varphi_1^2+\ldots+\varphi_M^2) \leq$$

$$\leq 2(\varphi,\phi_n)^2 + 2((\psi_1,\phi_n)^2+\ldots+(\psi_M,\phi_n)^2)\|Q^{-1}\|^2((\varphi,\phi_1)^2+\ldots+(\varphi,\phi_M)^2) \leq$$

$$\leq 2(\varphi,\phi_n)^2 + 2((\psi_1,\phi_n)^2+\ldots+(\psi_M,\phi_n)^2)\|Q^{-1}\|^2\|\varphi\|_2^2$$

and

$$(4.8)\quad \sum_{n>M} \varphi_n^2 \leq 2\|\varphi\|_2^2 + 2(\|\psi_1\|_2^2 + \ldots + \|\psi_M\|_2^2)\|Q^{-1}\|^2 \|\varphi\|_2^2 = K_1''\|\varphi\|_2^2 .$$

Also, because for $t > 0$ the series $\sum_{n>M} e^{2\lambda_n t}\varphi_n^2(1+\hat{q}-\lambda_n)$ is convergent ($e^{2\lambda_n t}(1+\hat{q}-\lambda_n) \leq \max(e^{2(1+\hat{q})t-1}/2t\ ,\ e^{-2\mu t}(1+\hat{q}-\mu))$) we obtain

$$(4.9)\qquad T_t''(L^2(\Omega)) \subset H^1(\Omega), \quad t > 0$$

Also

$$\|T_t''\|_1^2 = \sum_{n>M} e^{2\lambda_n t}\varphi_n^2(1+\hat{q}-\lambda_n) = e^{-2\mu t}\sum_{n>M} e^{2(\lambda_n+\mu)t}\varphi_n^2(1+\hat{q}-\lambda_n) .$$

Take $\tilde{K}''^2(t,\varphi) = \sum_{n>M} e^{2(\lambda_n+\mu)t}\varphi_n^2(1+\hat{q}-\lambda_n)$. It is obvious that $t \longmapsto K''(t,\varphi)$ is nonincreasing and continuous for $t > 0$ and

$$(4.10)\qquad \int_0^\infty \tilde{K}''^2(t,\varphi)\,dt = \sum_{n>M} \frac{1}{2}\,\frac{1+\hat{q}-\lambda_n}{-\lambda_n-\mu}\varphi_n^2 < \frac{1+\hat{q}-\lambda_{M+1}}{-2(\lambda_{M+1}+\mu)}\sum_{n>M}\varphi_n^2 \leq$$

$$\leq \frac{1+\hat{q}-\lambda_{M+1}}{-2(\lambda_{M+1}+\mu)}\|\varphi\|_2^2\, K_1''$$

In the case of point sensors the differences between (SLB) and (SLD) affect only T_t' (because $\psi_j \in C(\bar{\Omega})$ in the case of (SLB) and $\psi_j \in C(\Omega)$ for (SLD) but $\emptyset_n \in C(\bar{\Omega})$ in both cases).

Let us take

$$(4.11)\qquad \beta = \frac{4-N}{2N} - \alpha .$$

Using (2.6) we have:

$$\|T_t''\varphi\|_0 \leq K_0 \sum_{n>M} e^{\lambda_n t} n^\beta \varphi_n = K_0\, e^{-\mu t}\sum_{n>M} e^{(\lambda_n+\mu)t} n^\beta\varphi_n =$$

$$= e^{-\mu t}\,\hat{K}_0''(t,\varphi), \qquad t > 0 .$$

The series $\sum_{n>M} e^{\lambda_n t} n^\beta \varphi_n$ is convergent for $t > 0$ from (2.5) and so

$$(4.12)\qquad T_t''(L^2(\Omega)) \subset C(\bar{\Omega}), \qquad t > 0$$

On the other hand, if we take $\varphi \in H^1(\Omega)$ then from Schwarz-Cauchy inequality and (4.7) we obtain:

$$\hat{K}''^2_o(t,\varphi) \leq K_o^2 \sum_{n>M} e^{2(\lambda_n+\mu)t} \frac{n^{2\beta}}{1+\hat{q}-\lambda_n} \sum_{n>M} \varphi_n^2 \, (\hat{q}+1-\lambda_n) \leq$$

$$\leq \tilde{K}''_o \|\varphi\|_1^2 \sum_{n>M} e^{2(\lambda_n+\mu)t} \cdot \frac{n^{2\beta}}{1+\hat{q}-\lambda_n}$$

Also

$$\int_0^\infty \hat{K}''^2_o(t,\varphi)dt \leq \tilde{K}''_o \|\varphi\|_1^2 \sum_{n>M} \frac{n^{2\beta}}{-2(\lambda_n+\mu)(1+\hat{q}-\lambda_n)}$$

and the last series is convergent according to (2.5) which completes the proof of the Lemma 2.1.

Proof of Lemma 2.2. From (4.5) we obtain

$$\lambda_n\left[(\psi_1,\varnothing_n),\ldots,(\psi_M,\varnothing_n)\right] + (kg,\varnothing_n)_S RQ = \tag{4.13}$$
$$= \left[(\psi_1,\varnothing_n),\ldots,(\psi_M,\varnothing_n)\right] \operatorname{diag}(\mu_1,\ldots,\mu_M).$$

Since $F(t,\cdot) \in H^1(\Omega)$, $t>0$ we have from Proposition 2.3 :

$$F(t,\cdot) = \sum_{n=1}^{\infty} (F(t,\cdot),\varnothing_n)\, \varnothing_n(\cdot) = \tag{4.14}$$
$$= \sum_{n=1}^{M} F_n(t)\psi_n(\cdot) + \sum_{n>M} F_n(t)\varnothing_n(\cdot)$$

where

$$\operatorname{col}\left[F_1(t),\ldots,F_M(t)\right] = Q^{-1} \operatorname{col}\left[(F(t),\varnothing_1),\ldots,(F(t),\varnothing_M)\right]$$

(4.15)

$$F_n(t) = (F(t),\varnothing_n) -$$
$$- \left[(\psi_1,\varnothing_n),\ldots,(\psi_M,\varnothing_n)\right] \operatorname{col}\left[F_1(t),\ldots,F_M(t)\right], \quad n>M .$$

We will prove first the uniqueness of the weak solution for (2.26), (2.27), (2.28).

Let u be a weak solution for this problem.

We have

$$(4.16)\quad u(t,\cdot) = \sum_{n=1}^{\infty} (u(t,\cdot),\phi_n(\cdot))\ \phi_n(\cdot) = \sum_{n=1}^{M} u_n(t)\psi_n(\cdot) + \sum_{n>M} u_n(t)\phi_n(\cdot)$$

where, if we denote $U(t) = \mathrm{col}\left[u_1(t),\ldots,u_M(t)\right]$ and $\hat{U}(t) = \mathrm{col}\ \left[(u(t,\cdot),\phi_1),\ldots,(u(t,\cdot),\phi_M)\right]$, then from (2.17) :

$$(4.17)\quad \begin{aligned} U(t) &= Q^{-1}\ \hat{U}(t) \\ u_n(t) &= (u(t,\cdot),\phi_n) - \left[(\psi_1,\phi_n),\ldots,(\psi_M,\phi_n)\right] U(t), \quad n > M. \end{aligned}$$

On the other hand, if in (2.29) we take $\eta(t,\cdot) = \phi_n(\cdot)$ then from (2.2') we have:

$$(4.18)\quad \left(\frac{\partial u}{\partial t}(t,\cdot),\phi_n\right) = \lambda_n(u(t,\cdot) + F(t,\cdot),\phi_n) + (kg,\ \phi_n)_S\ (u(t,\cdot),\rho) \qquad n \geq 1$$

From (4.17) we obtain:

$$\frac{d}{dt}\ (u(t,\cdot),\phi_n) = \frac{du_n(t)}{dt} + \left[(\psi_1,\phi_n),\ldots,(\psi_M,\phi_n)\right]\ \frac{dU(t)}{dt}\ , \quad n > M$$

hence from (4.18) and (2.10) we have that

$$(4.19)\quad \begin{aligned} \frac{du_n}{dt}(t) &= \lambda_n(u(t),\phi_n) + (F(t),\phi_n) + (kg,\phi_n)_S\ R\hat{U}(t) - \\ &- \left[(\psi_1,\phi_n),\ldots,(\psi_M,\phi_n)\right]\ \frac{dU}{dt}(t), \qquad n > M \end{aligned}$$

and

$$(4.20)\quad \frac{d\hat{U}(t)}{dt} = \Lambda\hat{U} + \mathrm{col}\left[(F(t),\phi_1),\ldots,(F(t),\phi_M)\right] + GR\ \hat{U}(t)$$

But $\Lambda + GR = Q\ \mathrm{diag}\ (\mu_1,\ldots,\mu_M)\ Q^{-1}$ and $\frac{d\hat{U}}{dt} = Q\ \frac{dU}{dt}$ hence, using also (4.15), we have:

$$(4.21)\quad \frac{dU(t)}{dt} = \mathrm{diag}(\mu_1,\ldots,\mu_M)\cdot U(t) + \mathrm{col}\left[F_1(t),\ldots,F_M(t)\right]$$

Also from (4.15), (4.19), (4.13) and (4.21) we obtain:

$$\begin{aligned} \frac{du_n(t)}{dt} &= \lambda_n(u(t),\phi_n) + (F(t),\phi_n) + (kg,\phi_n)_S\ RQU(t) - \\ &- \left[(\psi_1,\phi_n),\ldots,(\psi_M,\phi_n)\right]\ \mathrm{diag}(\mu_1,\ldots,\mu_M)U(t) + \mathrm{col}\left[F_1(t),\ldots,F_M(t)\right] \end{aligned}$$

$$= \lambda_n\Big\{(u(t),\phi_n) - \big[(\psi_1,\phi_n),\dots,(\psi_M,\phi_n)\big]\,U(t)\Big\} + F_n(t), \quad n > M$$

or by (4.17)

$$\frac{du_n(t)}{dt} = \lambda_n u_n(t) + F_n(t), \quad n > M \tag{4.22}$$

If we write

$$u_o(\cdot) = \sum_{n=1}^{M} u_n^o \psi_n(\cdot) + \sum_{n>M} u_n^o \phi_n(\cdot) \tag{4.23}$$

then from (2.28) we have that:

$$u_n(0+) = u_n^o, \quad n \geqslant 1 \tag{4.24}$$

From (4.21), (4.22), (4.24) we get

$$u_n(t) = \begin{cases} e^{\mu_n t} u_n^o + \int_0^t e^{\mu_n(t-s)} F_n(s)ds, & 1 \leq n \leq M \\ e^{\lambda_n t} u_n^o + \int_0^t e^{\lambda_n(t-s)} F_n(s)ds, & n > M. \end{cases} \tag{4.25}$$

Using the hypothesis about F and (4.15) (4.8) one obtain easily

that $\sum_{n>M} F_n^2(s) \leqslant K_1'' \| F(s,\cdot)\|_2^2, \quad s \in]0,t]$

and so, applying Lebesgue theorem we have:

$$\sum_{n>M} \Big(\int_0^t e^{\lambda_n(t-s)} F_n(s)ds\Big)\phi_n(\cdot) = \int_0^t \Big(\sum_{n>M} e^{\lambda_n(t-s)} F_n(s)\phi_n(\cdot)\Big)ds \tag{4.26}$$

in $L^2(\Omega)$. Since from (4.15) one obtains easily, using (4.7),

that:

$$\sum_{n>M} (1+\hat{q}-\lambda_n)F_n^2(s) \leqslant \tilde{K}_1'' \| F(s,\cdot)\|_1^2 \qquad s \in]0,t] \tag{4.26'}$$

we have that (4.26) is fulfilled also in $H^1(\Omega)$. The relation (4.26) subsists in $C(\bar{\Omega})$ too because

$$(4.26'')\ \Big\| \sum_{n=p+1}^{p+\hat{k}} \int_0^t e^{\lambda_n(t-s)} F_n(s)ds\, \emptyset_n(\cdot)\Big\|_o^2 \leqslant$$

$$\leqslant K_o^2 \Big(\int_0^t \sum_{n=p+1}^{p+\hat{k}} n^{\beta} e^{\lambda_n(t-s)} |F_n(s)|\, ds\Big)^2 \leqslant$$

$$\leqslant K_o^2 \Big(\int_0^t \Big(\sum_{n=p+1}^{p+\hat{k}} \frac{n^{2\beta} e^{2\lambda_n(t-s)}}{1+\hat{q}-\lambda_n}\Big)^{1/2} \Big(\sum_{n=p+1}^{p+\hat{k}} (1+\hat{q}-\lambda_n) F_n^2(s)\Big)^{1/2} ds\Big)^2 \leqslant$$

$$\leqslant K_o^2 \sum_{n=p+1}^{p+\hat{k}} \frac{(1-e^{2\lambda_n t})n^{2\beta}}{(1+\hat{q}-\lambda_n)(-2\lambda_n)} \cdot \int_0^t \sum_{n=p+1}^{p+\hat{k}} (1+\hat{q}-\lambda_n) F_n^2(s)ds \leqslant$$

$$\leqslant K^* \int_0^t \sum_{n=p+1}^{p+\hat{k}} (1+\hat{q}-\lambda_n) F_n^2(s)ds \longrightarrow 0 \text{ for } p \longrightarrow \infty,\ \hat{k} \geqslant 1$$

from (2.6), (2.5), (4.11), (4.26') and the integrability of $\|F(s,\cdot)\|_1^2$ on $[0, t_o]$, $t_o > 0$.

In this way from (4.16), (4.25), (4.26) we obtained that if there exists a weak solution u for (2.26), (2.27), (2.28) then u must be given by (2.30) hence the uniqueness was proved.

Let us prove that the function u given by (2.30) is a weak solution for (2.26), (2.27), (2.28). We will consider again formulas (4.14), (4.15), (4.23), (4.26) and so, u will be:

$$(4.27)\quad u(t,\cdot) = \sum_{n=1}^{M} \Big[e^{\mu_n t} u_n^o + \int_0^t e^{\mu_n(t-s)} F_n(s)ds\Big] \psi_n(\cdot) + \sum_{n>M} \Big[e^{\lambda_n t} u_n^o + \int_0^t e^{\lambda_n(t-s)} F_n(s)\Big] \emptyset_n(\cdot)$$

in $L^2(\Omega)$ for $t \geqslant 0$ and in $H^1(\Omega)$ (and $C(\bar{\Omega})$ or $C(\Omega)$ when (SLB) or (SLD)) for $t > 0$.

Take:

$$(4.28)\quad u^p(t,\cdot) = \sum_{n=1}^{M} \Big[e^{\mu_n t} u_n^o + \int_0^t e^{\mu_n(t-s)} F_n(s)ds\Big] \psi_n(\cdot) +$$

$$+ \sum_{n=M+1}^{p} \left[e^{\lambda_n t} u_n^o + \int_0^t e^{\lambda_n (t-s)} F_n(s)ds \right] \phi_n(\cdot), \qquad p > M.$$

It is obvious that u^p verifies the condisions (i), (ii), (iv) of Lemma 2.2.

Since

$$\frac{\partial u^p}{\partial t}(t,\cdot) = \sum_{n=1}^{M} \left\{ \mu_n \left[e^{\mu_n t} u_n^o + \int_0^t e^{\mu_n (t-s)} F_n(s)ds \right] + F_n(t) \right\} \psi_n(\cdot) +$$

$$+ \sum_{n=M+1}^{P} \left\{ \lambda_n \left[e^{\lambda_n t} u_n^o + \int_0^t e^{\lambda_n (t-s)} F_n(s)ds \right] + F_n(t) \right\} \phi_n(\cdot)$$

it is easy to verify using (2.2') and (2.15), that

(4.29) $$\left(\frac{\partial u^p}{\partial t}(t,\cdot), \eta(t,\cdot)\right) = -(u^p(t,\cdot), \eta(t,\cdot))_1 +$$

$$+ ((1+\hat{q})u^p(t,\cdot) + F^p(,\cdot), \eta(t,\cdot)) + (kg, \eta(t,\cdot))_S \, (u^p(t,\cdot), \rho)$$

for all $t > 0$, $\eta(t,\cdot) \in H^1(\Omega)$

where

(4.30) $$F^p(t,x) = \sum_{n=1}^{M} F_n(t)\psi_n(x) + \sum_{n=M+1}^{P} F_n(t)\phi_n(x)$$

(4.31) $$u_o^p(x) = \sum_{n=1}^{M} u_n^o \psi_n(s) + \sum_{n=M+1}^{P} u_n^o \phi_n(x)$$

hence u^p is a weak solution for (2.26-2.28) with F^p and u_o^p instead of F and u_o.

We have that

$$\| u^{p+\hat{k}}(t,\cdot) - u^p(t,\cdot) \|_1^2 = \sum_{n=p+1}^{p+\hat{k}} \left[e^{\lambda_n t} u_n^o + \right.$$

$$\left. + \int_0^t e^{\lambda_n (t-s)} F_n(s)ds \right]^2 (1+\hat{q}-\lambda_n) \leq 2 \sum_{n=p+1}^{p+\hat{k}} \left[e^{2\lambda_n t} (u_n^o)^2 (1+\hat{q}-\lambda_n) + \right.$$

$$\left. + \frac{1+\hat{q}-\lambda_n}{-2\lambda_n} (1 - e^{2\lambda_n t}) \int_0^t F_n^2(s)ds \right]$$

and because $\sum_{n>1} (u_n^o)^2 < \infty$; $e^{2\lambda_n t}(1+\hat{q}-\lambda_n) \leq \max\{1+\hat{q}, \frac{1}{2t} e^{2t(1+\hat{q})-1}\}$,

$t>0$; $\frac{1+\hat{q}-\lambda_n}{-2\lambda_n} < \frac{1+\hat{q}+\mu}{2\mu}$, $n \geq M$; $\sum_{n\geq M} \int_0^t F_n^2(s)ds \leq K_1'' \int_0^t \|F(s,\cdot)\|_2^2\, ds$

we conclude that $u^p(t,\cdot)$ is a Cauchy sequence in $H^1(\Omega)$ for each $t>0$ hence

(4.32) $\lim_{p\to\infty} u^p(t,\cdot) = u(t,\cdot)$, in $H^1(\Omega)$ for each $t>0$.

Moreover the limit (4.32) is uniform on any interval $[t_o,t_1] \subset]o,\infty[$ hence $t \longmapsto u(t,\cdot)$ is continuous from $]o,\infty[$ to $H^1(\Omega)$.

On the other hand:

$$\|\frac{\partial u^{p+\hat{k}}}{\partial t}(t,\cdot) - \frac{\partial u^p}{\partial t}(t,\cdot)\|_2^2 = \sum_{n=p+1}^{p+\hat{k}} \Big\{\lambda_n\Big[e^{\lambda_n t}u_n^o + \int_0^t e^{\lambda_n(t-s)} F_n(s)ds\Big] +$$

$$+ F_n(t)\Big\}^2 \leq 2 \sum_{n=p+1}^{p+\hat{k}} \Big[\lambda_n^2(2e^{2\lambda_n t} u_n^o + \frac{1-e^{2\lambda_n t}}{-2\lambda_n} \cdot \int_0^t F_n^2(s)ds) + F_n^2(t)\Big]$$

and since

$$\lambda_n^2 e^{2\lambda_n t} \leq \frac{1}{t^2} e^{-2},\ t>0;\ \sum_{n>M} F_n^2(t) \leq K_1'' \|F(t,\cdot)\|_2^2 ,$$

$$\sum_{n>M} -\lambda_n(1-e^{2\lambda_n t}) \int_0^t F_n^2(s)ds \leq \sum_{n>M} \frac{-\lambda_n}{1+\hat{q}-\lambda_n} \int_0^t (1+\hat{q}-\lambda_n)F_n^2(s)ds \leq$$

$$\leq \frac{\mu}{1+\hat{q}+\mu} \sum_{n>M} \int_0^t (1+\hat{q}-\lambda_n) F_n^2(s)ds \leq \frac{K_1''\mu}{1+\hat{q}+\mu} \int_0^t \|F(s,\cdot)\|_1^2\, ds$$

We obtain that the sequence $\frac{\partial u^p}{\partial t}(t,\cdot)$ is uniformly convergent in $L^2(\Omega)$ on any interval $[t_o,t_1] \subset]o,\infty[$ hence $t \longmapsto u(t,\cdot)$ is differentiable for $t>0$ and

(4.33) $\frac{\partial u}{\partial t}(t,\cdot) = \lim_{p\to\infty} \frac{\partial u^p}{\partial t}(t,\cdot)$ in $L^2(\Omega)$, $t>0$.

From (4.32), (4.33) and since $\lim_{p\to\infty} F^p(t,\cdot) = F(t,\cdot)$, $t>0$ in

$H^1(\Omega)$ (hence in $L^2(\Omega)$) we may pass to limit in (4.29) and will obtain that u verifies (2.29) for all $t>0$, $\eta(t,\cdot)\subset H^1(\Omega)$ and since it is obvious that $\lim_{t\to+0}\|u(t,\cdot)-u_o(\cdot)\|_2 = 0$ we obtain that u verifies the conditions (i), (ii), (iii) of the Lemma 2.2. The condition (iv) follows imediately since

$$\| u^{p+\hat{k}}(t,\cdot) - u^p(t,\cdot)\|_o \leq$$

$$\leq K_o\left[\sum_{n=p+1}^{p+\hat{k}} e^{\lambda_n t}|u_n^o| + \int_0^t \sum_{n=p+1}^{p+\hat{k}} e^{\lambda_n(t-s)} F_n(s)ds\right] \longrightarrow 0$$

for $p\longrightarrow\infty$, $\hat{k}\geq 1$, $t>0$ using (4.26") and (2.5) and so $u^p(t,\cdot)$ converges uniformly in $C(\bar{\Omega})$ if (SLB), or in $C(\Omega)$ if (SLD), on any $[t_o,t_1]\subset\,]0,\infty[$.

<u>Proof of Lemma 2.3.</u> Let us denote

$$\check{Z}_t^n z_o = \sum_{p=n+1}^{\infty} e^{\lambda_p t}(z_o,\phi_p)\,\phi_p(\cdot) \tag{4.34}$$

$$\hat{Z}_t^n z_o = \left[\phi_1(\cdot),\ldots,\phi_n(\cdot)\right]\Big\{\exp\,(\Lambda_n - \tilde{L}_n\chi_n)t\,\cdot \tag{4.35}$$

$$\cdot\,\mathrm{col}\left[(z_o,\phi_1),\ldots,(z_o,\phi_n)\right] - \int_0^t \exp\,(\Lambda_n-\tilde{L}_n\chi_n)(t-s)\,\tilde{L}_n h(\check{Z}_s^n z_o)ds\Big\}$$

Hence $\tilde{Z}_t^n z_o = \hat{Z}_t^n z_o + \check{Z}_t^n z_o$.

Since for $t>0$ the series $\sum_{p=n+1}^{\infty} e^{2\lambda_p t}(z_o,\phi_p)^2(1+\hat{q}-\lambda_p)$ is convergent we have that

$$\check{Z}_t^n z_o\in H^1(\Omega),\qquad t>0,\; z_o\in L^2(\Omega). \tag{4.36}$$

Also $t\longmapsto\|\check{Z}_t^n z_o\|_1^2$ is integrable on $[0,t_o]$, $t_o>0$ because

$$\int_0^{t_o}\|\check{Z}_t^n z_o\|_1^2\,dt = \int_0^{t_o}\sum_{p\geq n+1} e^{2\lambda_p t}(z_o,\phi_p)^2(1+\hat{q}-\lambda_p)dt =$$

$$= \sum_{p>n+1} \frac{1+\hat{q}-\lambda_p}{-2\lambda_p} (1-e^{2\lambda_p t_o})(z_o,\varnothing_p)^2 \leq \frac{1-\hat{q}-\lambda_{n+1}}{-2\lambda_{n+1}} \sum_{p>n+1} (z_o,\varnothing_p)^2 < \infty$$

Let us consider now the case when h is given by (1.4). We have that

$$|h(\check{Z}_t^n z_o)| \leq \|h\| \cdot \|\check{Z}_t^n z_o\|_1 , \quad t>0$$

and so $t \longmapsto |h(Z_t^n z_o)|^2$ is integrable on $[0,t_o]$, hence $\hat{Z}_t^n z_o$ is well defined and it is easy to see that $\tilde{Z}_t^n$ is a strongly continuous semigroup on $L^2(\Omega)$ and $H^1(\Omega)$. From (4.36) we get also (2.32).

If we have point sensors then $\hat{h}$ is given by (1.5) and from (2.6), (4.11) we obtain:

(4.37) $$\check{Z}_t^n z_o \in C(\bar{\Omega}), \quad t>0, \quad z_o \in L^2(\Omega)$$

because the series $\sum_{p>n+1} e^{\lambda_p t} (z_o,\varnothing_p) n^\beta$ is convergent for $t>0$ from (2.5) and also

(4.38) $$\|\check{Z}_t^n z_o\|_o \leq K_o \sum_{p>n+1} e^{\lambda_p t} |(z_o,\varnothing_p)| n^\beta, \quad t>0, \ z_o \in L^2(\Omega)$$

Since

$$|h(\check{Z}_t^n z_o)| \leq \|h\| \cdot \|\check{Z}_t^n z_o\|_o , \quad t>0$$

and

$$\int_0^{t_o} \|\check{Z}_t^n z_o\|_o \, dt \leq K_o \sum_{p>n+1} \frac{1-e^{\lambda_p t_o}}{-\lambda_p} |(z_o,\varnothing_p)| n^\beta \leq$$

$$K_o \Big(\sum_{p>n+1} \frac{n^{2\beta}}{\lambda_p^2}\Big)^{1/2} \Big(\sum_{p>n+1} (z_o,\varnothing_p)^2\Big)^{1/2} < \infty \quad , \ t_o > 0, \ z_o \in L^2(\Omega)$$

we obtain that $\hat{Z}_t^n z_o$ is well defined and so $\tilde{Z}_t^n$ is again a strongly continuous semigroup on $L^2(\Omega)$ and $H^1(\Omega)$ and in this case we have (2.32) and (2.33).

<u>Proof of Lemma 2.4.</u> The proof of this Lemma is adapted from [1]. To prove the uniqueness let z be a weak solution for

(2.34), (2.35), (2.36). If in (2.37) we take $\eta = \phi_p$, from (2.2') we obtain

$$\left(\frac{\partial z}{\partial t}(t,\cdot),\phi_p\right) = \lambda_p(z(t,\cdot),\phi_p)+(F(t,\cdot),\phi_p) - (\tilde{\ell}_n(\cdot),\phi_p)h(z(t,\cdot))$$

and if we denote $\tilde{L}_n = \mathrm{col}\left[L_n^1,\ldots,L_n^n\right]$ then $\tilde{\ell}_n(\cdot) = L_n^1\phi_1(\cdot)+\ldots+L_n^n\phi_n(\cdot)$ and

$$(4.39)\quad \frac{d}{dt}(z(t,\cdot),\phi_p) = \lambda_p(z(t,\cdot),\phi_p) + (F(t,\cdot),\phi_p) - L_n^p h(z(t,\cdot)), \quad 1\leq p\leq n, \quad t>0,$$

$$(4.40)\quad \frac{d}{dt}(z(t,\cdot),\phi_p) = \lambda_p(z(t,\cdot),\phi_p)+(F(t,\cdot),\phi_p), \; p>n, \; t>0.$$

Since $z_o(\cdot) = \sum_{p\geq 1}(z_o,\phi_p)\phi_p(\cdot)$ and $z(t,\cdot) = \sum_{p\geq 1}(z(t,\cdot),\phi_p)\phi_p(\cdot)$, $t\geq 0$, we obtain that

$$(4.41)\quad (z(o+,\cdot),\phi_p) = (z_o,\phi_p) \qquad p\geq 1 .$$

From (4.40), (4.41) we have that

$$(z(t,\cdot),\phi_p) = e^{\lambda_p t}(z_o,\phi_p) + \int_o^t e^{\lambda_p(t-s)}(F(s,\cdot),\phi_p)ds, \; p\geq n$$

and so by (4.34) we get

$$(4.42)\quad z(t,\cdot) = \sum_{p=1}^{n}(z(t,\cdot),\phi_p)+\check{Z}_t^n z_o + \int_o^t \check{Z}_{t-s}^n(F(s,\cdot))ds,$$

if we remark as in the proof of the Lemma 2.2, relation (4.26) that:

$$(4.43)\quad \sum_{p>n}\left[\int_o^t e^{\lambda_p(t-s)}(F(s),\phi_p)ds\right]\phi_p(\cdot) = \int_o^t\left[\sum_{p>n} e^{\lambda_p(t-s)}(F(s),\phi_p)\phi_p(\cdot)\right]ds.$$

in $L^2(\Omega)$, $H^1(\Omega)$ (and in $C(\bar{\Omega})$ when we deal with point sensors).

We obtain from (4.42), (4.39) that:

$$(4.44)\quad \frac{d}{dt}\mathrm{col}\left[(z(t,\cdot),\phi_1),\ldots,(z(t,\cdot),\phi_n)\right] =$$

$$= \left[\Lambda_n - \tilde{L}_n \chi_n\right] \operatorname{col}\left[(z(t,\cdot),\phi_1),\ldots,(z(t,\cdot),\phi_n)\right] +$$

$$+ \operatorname{col}\left[(F(t,\cdot),\phi_1),\ldots,(F(t,\cdot)\phi_n)\right] - \tilde{L}_n h(\check{Z}^n_t z_o + \int_0^t \check{Z}^n_{t-s}(F(s,\cdot))ds)$$

and from the last linear system of ordinary differential equation using the variation of parameters fromula we obtain that z is given by (2.38) hence if it exists a weak solution this is unique.

To prove the existence of the weak solution for (2.34), (2.35), (2.36) it is sufficient to remark first that a weak solution for the problem:

$$\frac{\partial \check{z}}{\partial t} = \operatorname{div}(k(x)\nabla \check{z}) + q(x)\check{z} + \check{F}(t,x), \quad x \in \Omega, \ t > 0$$

$$\frac{\partial \check{z}}{\partial n} + \sigma(s)\check{z} = 0, \qquad x \in S, \ t > 0$$

$$\check{z}(0+, x) = \check{z}_o(x), \qquad x \in \Omega$$

with $\check{F}(t,\cdot) = \sum_{p>n} (F(t,\cdot),\phi_p)\,\phi_p(\cdot)$ and $\check{z}_o(\cdot) = \sum_{p>n} (z_o,\phi_p)\phi_p(\cdot)$

is given by

$$\check{z}(t,\cdot) = \check{Z}^n_t \check{z}_o + \int_0^t \check{Z}^n_{t-s}(\check{F}(s,\cdot))ds = \check{Z}^n_t z_o + \int_0^t \check{Z}^n_{t-s}(F(s,\cdot))ds$$

which is well known.

Second, if we put

$$\hat{z}(t,\cdot) = \hat{Z}^n_t z_o + \int_0^t \hat{Z}^n_{t-s}(F(s,\cdot))ds = \left[\phi_1(\cdot),\ldots,\phi_n(\cdot)\right] \cdot$$

$$\cdot \Big\{ \exp\left[(\Lambda_n - \tilde{L}_n\chi_n)t\right] \operatorname{col}\left[(\hat{z}_o,\phi_1),\ldots,(\hat{z}_o,\phi_n)\right] +$$

$$+ \int_0^t \exp\left[(\Lambda_n - \tilde{L}_n\chi_n)(t-s)\right] \Big[\operatorname{col}\left[(\hat{F}(s,\cdot),\phi_1),\ldots,(\hat{F}(s,\cdot),\phi_n)\right] -$$

$$- \tilde{L}_n h(\check{z}(s,\cdot))\Big] ds \Big\}$$

(the second equality may be obtained by a simple computation).

with

$$\hat{F}(t,\cdot) = \sum_{p=1}^{n} (F(t,\cdot),\emptyset_p)\emptyset_p(\cdot); \quad \hat{z}_o(\cdot) = \sum_{p=1}^{n} (z_o,\emptyset_p)\emptyset_p(\cdot),$$

then we obtain that $\hat{z}(t,\cdot)$ verifies (4.44) and from here and (2.2') we obtain that $\hat{z}(t,\cdot)$ is a weak solution for the problem:

$$\frac{\partial \hat{z}}{\partial t} = \operatorname{div}(k(x)\nabla \hat{z}) + q(x)\hat{z} + \hat{F}(t,x) - \tilde{\ell}_n(x)h(\hat{z}(t,\cdot)+\check{z}(t,\cdot)), \quad x\in\Omega, t>0$$

$$\frac{\partial \hat{z}}{\partial n} + \sigma(x)\hat{z} = 0, \qquad x\in S, \quad t>0$$

$$\hat{z}(0+, x) = \hat{z}_o(x), \qquad x\in\Omega$$

It is obvious that $z(t,\cdot) = \hat{z}(t,\cdot) + \check{z}(t,\cdot)$ is a weak so- solution for (2.34), (2.35), (2.36).

<u>Proof of Lemma 2.5</u>. According to (4.34), (4.35) since n=M we will omit the letter "n" hence

$$\text{(4.34')} \qquad \check{Z}_t\, z_o = \sum_{p>M} e^{\lambda_p t}(z_o,\emptyset_p)\,\emptyset_p(\cdot)$$

$$\text{(4.35')} \quad \hat{Z}_t\, z_o = [\emptyset_1,\ldots,\emptyset_M]\Big\{ e^{(\Lambda-L\chi)t}\operatorname{col}[(z_o,\emptyset_1),\ldots,(z_o,\emptyset_M)] - \int_0^t e^{(\Lambda-L\chi)(t-s)}\, L\, h\,(\check{Z}_s\, z_o)\, ds$$

and

$$\text{(4.45)} \quad Z_t\, z_o = \hat{Z}_t\, z_o + \check{Z}_t\, z_o, \qquad t\geq 0, \quad z_o\in L^2(\Omega).$$

We have

$$\text{(4.46)} \quad \|Z_t\, z_o\|_i^2 = \|\hat{Z}_t\, z_o\|_i^2 + \|\check{Z}_t\, z_o\|_i^2, \ i=1,2; \ t>0; \ z_o\in L^2(\Omega).$$

Let μ' be a number such that:

$$\text{(4.47)} \qquad \lambda_{M+1} < -\mu' < -\mu .$$

We have

$$\text{(4.48)} \quad \|\check{Z}_t z_o\|_2^2 = \sum_{p>M} e^{2\lambda_p t}(z_o,\emptyset_p)^2 = e^{-2\mu t}\sum_{p>M} e^{2(\lambda_p+\mu)t}(z_o,\emptyset_p)^2 \leq$$

$$\leq e^{-2\mu t}\,\|\check{z}_o\|_2^2, \qquad t\geq 0$$

if we denote

(4.49) $z_o = \hat{z}_o + \check{z}_o$; $\hat{z}_o = \sum_{p=1}^{M} (z_o, \phi_p)\phi_p(\cdot)$; $\check{z}_o = \sum_{p>M} (z_o, \phi_p)\phi_p(\cdot)$.

and remarke that

(4.49') $\|z_o\|_i^2 = \|\check{z}_o\|_i^2 + \|\hat{z}_o\|_i^2$, $i=1,2$.

Also we have:

(4.50) $$\|\check{Z}_t\, z_o\|_1^2 = \sum_{p>M} (z_o, \phi_p)^2 (1+\hat{q}-\lambda_p)\, e^{2\lambda_p t} =$$

$$= e^{-2\mu' t} \sum_{p>M} (z_o, \phi_p)^2 (1+\hat{q}-\lambda_p)\, e^{2(\lambda_p+\mu')t} = e^{-2\mu' t}\, \bar{C}_2^2(t, z_o),$$

$t > 0$, $z_o \in L^2(\Omega)$.

Obviously $t \longmapsto \bar{C}_2^2(t, z_o)$ is continuous and nonincreasing for $t > 0$ and

(4.51) $$\int_0^\infty \bar{C}_2^2(t, z_o)dt = \sum_{p>M} (z_o, \phi_p)^2 \frac{1+\hat{q}-\lambda_p}{-2(\lambda_p+\mu')} < \frac{1+\hat{q}-\lambda_{M+1}}{-2(\lambda_{M+1}+\mu')} \cdot$$

$$\cdot \sum_{p>M} (z_o, \phi_p)^2 = \frac{1+\hat{q}-\lambda_{M+1}}{-2(\lambda_{M+1}+\mu')} \|\check{z}_o\|_2^2 = \check{C}_2^2 \|\check{z}_o\|_2^2$$

Let $\tilde{C} > 0$ be such that

(4.52) $$|e^{(\Lambda - L\chi)t}| \leq \tilde{C}\, e^{-\mu' t}, \quad t \geq 0.$$

Consider first the case when $\hat{h}$ is given by (1.4). Then:

$$|h(\check{Z}_s\, z_o)| \leq \|h\| \cdot \|\check{Z}_s\, z_o\|_1 \leq \|h\|\, e^{-\mu' s}\, \bar{C}_2(s, z_o), \qquad s > 0$$

and so by Schwarz-Cauchy inequality we get:

(4.53) $$\left|\int_0^t e^{(\Lambda-L\chi)(t-s)} h(\check{Z}_s z_o)ds\right| \leq \|h\|\tilde{C}\, e^{-\mu' t}\sqrt{t}\; \check{C}_2 \|\check{z}_o\|_2 \leq$$

$$\leq \|h\|\, \tilde{C}\, \frac{1}{\sqrt{2e(\mu'-\mu)}}\, \check{C}_2\, e^{-\mu t} \|\check{z}_o\|_2, \qquad t \geq 0$$

Also

$$\|\hat{Z}_t z_o\|_2^2 \leq 2|e^{(\Lambda-L\chi)t}|^2 \|\hat{z}_o\|_2^2 + 2\left|\int_0^t e^{(\Lambda-L\chi)(t-s)} h(\check{Z}_s z_o)ds\right|^2 \leq$$

$$\leqslant 2\,\tilde{C}^2 e^{-2\mu t}(\|\hat{z}_o\|_2^2 + \|h\|^2 \frac{1}{2e(\mu'-\mu)} \check{C}_2^2 \|\check{z}_o\|_2^2), \qquad t \geqslant 0$$

Using (4.48), (4.49') and (4.46) we obtain (2.39).

On the other hand

$$\|\hat{Z}_t z_o\|_1^2 \leqslant 2(1+\hat{q}-\lambda_M)\cdot 2\{|e^{(\Lambda-L\chi)t}|^2 \|\hat{z}_o\|_2^2 + |\int_0^t e^{(\Lambda-L\chi)(t-s)} h(\check{Z}_s z_o)ds|^2$$

$$\leqslant 2(1+\hat{q}-\lambda_M)\tilde{C}^2 e^{-2\mu t}(\|\hat{z}_o\|_2^2 + \|h\|^2 \frac{1}{2e(\mu'-\mu)} \check{C}_2^2 \|\check{z}_o\|_2^2), \quad t > 0$$

Hence, if we take

$$(4.54)\quad C_2^2(t,z_o) = \bar{C}_2^2(t,z_o) + 2(1+\hat{q}-\lambda_M)e^{-2\mu t}\tilde{C}^2 \cdot \cdot (\|\hat{z}_o\|_2^2 + \frac{\|h\|^2}{2e(\mu'-\mu)} \check{C}_2^2 \|\check{z}_o\|_2^2)$$

and

$$(4.55)\quad \hat{C}_2^2 = \max\{(\check{C}_2^2 + (1+\hat{q}-\lambda_M)\tilde{C}^2 \frac{\|h\|^2 \check{C}_2^2}{e(\mu'-\mu)}), 2(1+\hat{q}-\lambda_M)\tilde{C}^2\} =$$

$$= \max\{\frac{1+\hat{q}-\lambda_{M+1}}{-2(\lambda_{M+1}+\mu')}(1 + \frac{1+\hat{q}-\lambda_M}{e(\mu'-\mu)}\tilde{C}^2\|h\|^2), 2(1+\hat{q}-\lambda_M)\tilde{C}^2\}$$

then we obtain immediately (2.40), (2.41).

Let us consider now the case of point sensors hence $\hat{h}$ is given by (1.5). From (2.6), (4.11) we have:

$$(4.56)\quad \|\check{Z}_t z_o\|_o \leqslant K_o \sum_{p>M} |(z_o,\phi_p)| e^{\lambda_p t} p^\beta = K_o e^{-\mu' t} \sum_{p>M} |(z_o,\phi_p)| \cdot$$

$$\cdot e^{(\lambda_p+\mu')t} p^\beta = e^{-\mu' t}\bar{C}_o(t,z_o);$$ it is obvious that $t \longmapsto \bar{C}_o(t,z_o)$ is continuous and nonincreasing for $t > 0$, and for $z_o \in H^1(\Omega)$:

$$(4.57)\quad \bar{C}_o(t,z_o) = K_o \sum_{p>M} |(z_o,\phi_p)| \sqrt{1+\hat{q}-\lambda_p}\, p^\beta \frac{e^{(\lambda_p+\mu')t}}{\sqrt{1+\hat{q}-\lambda_p}} \leqslant$$

$$K_o(\sum_{p>M}(z_o,\phi_p)^2(1+\hat{q}-\lambda_p))^{1/2}(\sum_{p>M}\frac{p^{2\beta}e^{2(\lambda_p+\mu')t}}{1+\hat{q}-\lambda_p})^{1/2} =$$

$$= K_o\|\check{z}_o\|_1 \sum_{p>M} \frac{p^{2\beta}e^{2(\lambda_p+\mu')t}}{1+\hat{q}-\lambda_p}.$$

Hence $\int_0^\infty \bar{C}_o^2(t,z_o)dt \leq K_o^2 \|\check{z}_o\|_1^2 \sum_{p>M} \frac{p^{2\beta}}{(1+\hat{q}-\lambda_p)(-2\lambda_p-2\mu')} =$

$$= \|\check{z}_o\|_1^2 \cdot \check{C}_o^2 < \infty$$

since from (2.5), (4.11) follows the convergence of the last series.

Now we have:

$$|h(\check{Z}_s z_o)| \leq \|h\| \, \|\check{Z}_s \, z_o\|_o \leq \|h\| \, e^{-\mu' s} \, \bar{C}_o(s,z_o), \qquad s>0$$

and will obtain like (4.53):

$$(4.53') \quad \left| \int_0^t e^{(\Lambda-L\chi)(t-s)} h(\check{Z}_s z_o) ds \right| \leq \|h\| \tilde{C} \frac{1}{\sqrt{2e(\mu'-\mu)}} \check{C}_o \, e^{-\mu t} \|\check{z}_o\|_1 .$$

Also

$$\|Z_t z_o\|_o \quad K_o M^{\beta+\frac{1}{2}} (\tilde{C} e^{-\mu' t} \|\hat{z}_o\|_2 + \|h\| \frac{\tilde{C}}{\sqrt{2e(\mu'-\mu)}} \check{C}_o \, e^{-\mu t} \|\check{z}_o\|_1) \leq$$

$$\leq K_o \; M^{\beta+\frac{1}{2}} \tilde{C} \, e^{-\mu t} \left(\frac{\|\hat{z}_o\|_1}{\sqrt{1+\hat{q}-\lambda_M}} + \frac{\|h\| \, \check{C}_o}{\sqrt{2e(\mu'-\mu)}} \|\check{z}_o\|_1 \right)$$

and so, if we set

$$(4.58) \quad C_o^2(t,z_o) = \bar{C}_o^2(t,z_o) + 2 \, K_o^2 \; M^{2\beta+1} \, \tilde{C}^2 \, e^{-2\mu t} \left(\frac{\|\hat{z}_o\|_1^2}{1+\hat{q}-\lambda_M} + \frac{\|h\|^2 \, \check{C}_o^2}{2e(\mu'-\mu)} \|\check{z}_o\|_1^2 \right)$$

and

$$(4.59) \quad \hat{C}_o^2 = \max \left\{ \check{C}_o^2 \, (1+K_o^2 \, M^{2\beta+1} \, \tilde{C}^2 \frac{\|h\|^2}{e(\mu'-\mu)}), \; 2 \, K_o^2 \, M^{2\beta+1} \cdot \tilde{C}^2 \frac{1}{1+\hat{q}-\lambda_M} \right\}$$

$$\check{C}_o^2 = \sum_{p>M} \frac{p^{2\beta}}{(1+\hat{q}-\lambda_p)(-2\lambda_p-2\mu')} < \infty$$

then we get (2.42), (2.43), (2.44) which completes the proof of Lemma 2.5.

<u>Proof of Lemma 3.1.</u> Let (u,V) be a weak solution for

(3.9 - 3.13) and let v be defined by (3.24). If in (2.10) we will denote $R = \text{row}\left[R^1,\dots,R^M\right]$ were $R^1,\dots,R^M$ are column vectors in $\mathbb{R}^r$ then

$$\rho(x) = R^1 \phi_1(x)+\dots+R^M\phi_M(x)$$

and so we have:

$$(4.60) \qquad (v(t,\cdot),\rho) = \sum_{i,j=1}^{M} v_i(t)(\psi_i,\phi_j)\, R^j = $$
$$= RQ\,\text{col}\left[v_1(t),\dots,v_M(t)\right] = \hat{R}\,\hat{Q}\,V(t), \quad t \geq 0.$$

On the other hand

$$(4.61) \qquad \frac{\partial v}{\partial t}(t,\cdot) = \left[\psi_1,\dots,\psi_M,\phi_{M+1},\dots,\phi_m\right]\frac{d\,V(t)}{dt} =$$
$$= \left[\psi_1,\dots,\psi_M,\phi_{M+1},\dots,\phi_m\right]\left\{(\mathcal{M} - \hat{L}H)V(t) + \hat{L}\,h(u(t,\cdot))\right\}$$

and

$$(4.62) \qquad h(v(t,\cdot)=v_1(t)h(\psi_1)+\dots+v_M(t)h(\psi_M)+v_{M+1}(t)h(\phi_{M+1})+\dots$$
$$\dots v_m(t)h(\phi_m) = HV(t).$$

Hence

$$(4.63) \qquad \frac{\partial v}{\partial t}(t,\cdot) = \sum_{i=1}^{M} \mu_i v_i(t)\psi_i(\cdot) + \sum_{i=M+1}^{m} \lambda_i v_i(t)\phi_i(\cdot) +$$
$$+ \hat{\ell}(\cdot)h(u(t,\cdot) - v(t,\cdot))$$

and from here, using Proposition 2.2 we have:

$$\left(\frac{\partial v}{\partial t}(t,\cdot),\eta(t,\cdot)\right) = \sum_{i=1}^{M} v_i(t)(\mu_i\psi_i,\eta(t,\cdot))+$$

$$+\sum_{i=M+1}^{m} v_i(t)(\lambda_i\phi_i,\eta(t,\cdot)) + (\hat{\ell}(\cdot),\eta(t,\cdot))h(u(t,\cdot) - v(t,\cdot)) =$$

$$= \sum_{i=1}^{M} v_i(t)\left[-(\psi_i,\eta(t,\cdot))_1 + (1+\hat{q})(\psi_i,\eta(t,\cdot)) - (kg,\eta(t,\cdot))_S(\psi_i,\rho)\right]+$$

$$+\sum_{i=M+1}^{m} v_i(t)\left[-(\phi_i,\eta(t,\cdot))_1 + (1+\hat{q})(\phi_i,\eta(t,\cdot)) -\right.$$
$$\left.- (kg,\eta(t,\cdot))_S(\phi_i,\rho)\right] + (\hat{\ell}(\cdot)h(u(t,\cdot) - v(t,\cdot)),\eta(t,\cdot)) =$$

$$= -(v(t,\cdot),\eta(t,\cdot))_1 + (1+\hat{q})(v(t,\cdot),\eta(t,\cdot)) -$$

$$- (kg,\eta(t,\cdot))_S(v(t,\cdot),\rho) , \qquad t>0;\ \eta(t,\cdot)\in H^1(\Omega);$$

hence we obtained (3.26). From (3.14) and (4.60) follows immediately (3.25) and the others conditions that (u,v) be a weak solution for (3.18-3.23) are obvious.

Conversely let us consider a weak solution (u,v) for (3.18-3.23). According to Proposition 2.3, we have

(4.64) $v(t,\cdot)=v_1(t)\psi_1(\cdot)+\dots+v_M(t)\psi_M(\cdot)+v_{M+1}(t)\phi_{M+1}(\cdot)+\dots ,\ t\geq 0$

with

(4.65)
$$\mathrm{col}\left[v_1(t),\dots,v_M(t)\right] = Q^{-1}\mathrm{col}\left[(v(t,\cdot),\phi_1),\dots,(v(t,\cdot),\phi_M)\right]$$

$$v_n(t)=(v(t,\cdot),\phi_n)-\left[(\psi_1,\phi_n),\dots,(\psi_M,\phi_n)\right]\cdot \mathrm{col}\left[v_1(t),\dots,v_M(t)\right], \qquad n>M.$$

From (3.26) we have that:

$$\frac{d}{dt}(v(t,\cdot),\phi_n) = (\frac{\partial v}{\partial t}(t,\cdot),\phi_n) = -(v(t,\cdot),\phi_n)_1+(1+\hat{q})(v(t,\cdot),\phi_n)+$$
$$+ (\hat{\ell},\phi_n)h(u(t,\cdot)-v(t,\cdot)) + (kg,\phi_n)_S(v(t,\cdot),\rho), \quad t>0, \quad n\geq 1,$$

and by (2.2')

(4.66)
$$\frac{d}{dt}(v(t,\cdot),\phi_n)=\lambda_n(v(t,\cdot),\phi_n)+(\hat{\ell},\phi_n)h(u(t,\cdot)-v(t,\cdot)) +$$
$$+ (kg,\phi_n)_S (v(t,\cdot),\rho), \qquad t>0, \quad n\geq 1;$$

Since, from (3.6) we have that $\hat{\ell} = \ell - \delta$, we obtain

(4.67)
$$(\hat{\ell},\phi_n) = \begin{cases} \ell^n & \text{if } 1\leq n\leq M, \\ 0 & \text{if } M<n\leq m, \\ -\ell_n & \text{if } n>m. \end{cases}$$

Let us take $W(t) = \mathrm{col}\left[v_1(t),\dots,v_M(t)\right]$ and $\hat{W}(t) = \mathrm{col}\left[(v(t,\cdot),\phi_1),\dots,(v(t,\cdot),\phi_M)\right]$. From (4.65), (4.66), (4.67) we have that:

$$W(t) = Q^{-1}\hat{W}(t),$$

$$\frac{d\hat{W}(t)}{dt} = \Lambda \hat{W} + Lh(u(t,\cdot)-v(t,\cdot)) + G \cdot (v(t,\cdot),\rho)$$

and since

$$(v(t,\cdot),\rho) = R\cdot Q\cdot W(t)$$

we obtain

$$\frac{dW(t)}{dt} = Q^{-1} (\Lambda + GR)Q\, W(t) + Q^{-1} Lh(u(t,\cdot) - v(t,\cdot))$$

or

$$\frac{dW(t)}{dt} = \mathrm{diag}(\mu_1,\dots,\mu_M)W(t) + Q^{-1} Lh(u(t,\cdot)-v(t,\cdot)) \tag{4.68}$$

For $n > M$ we have from (4.65), (4.66), (4.67), (4.68), (4.13) that:

$$\frac{dv_n(t)}{dt} = \frac{d}{dt}(v(t,\cdot),\phi_n) - [(\psi_1,\phi_n),\dots,(\psi_M,\phi_n)]\frac{dW(t)}{dt} =$$

$$= \lambda_n(v(t,\cdot),\phi_n)+(\hat{\ell},\phi_n)h(u(t,\cdot)-v(t,\cdot))+(kg,\phi_n)_S\, R\, Q\, W(t) -$$

$$-[(\psi_1,\phi_n),\dots,(\psi_M,\phi_n)]\Big\{\mathrm{diag}(\mu_1,\dots,\mu_M)W(t) +$$

$$+ Q^{-1} Lh(u(t,\cdot)-v(t,\cdot))\Big\} = \lambda_n v_n(t) + \Big\{(\hat{\ell},\phi_n) -$$

$$- [(\psi_1,\phi_n),\dots,(\psi_M,\phi_n)]\, Q^{-1} L\Big\}\, h(u(t,\cdot) - v(t,\cdot)) .$$

From (4.67) and (3.4) we obtain:

$$\frac{dv_n}{dt} = \lambda_n v_n(t)+\ell_n\, h(u(t,\cdot)-v(t,\cdot)), \quad t>0,\ M< n\leq m \tag{4.69}$$

and

$$\frac{dv_n(t)}{dt} = \lambda_n v_n(t), \qquad t>0, \quad n>m. \tag{4.70}$$

Since $v(0+,\cdot) = [\psi_1,\dots,\psi_M,\phi_{M+1},\dots,\phi_m]\cdot V_o$, from (4.64) we obtain that

$$v_n(0+) = 0,\ n>m, \quad \mathrm{col}[v_1(0+),\dots,v_m(0+)] = V_o \tag{4.71}$$

and by (4.70) we get that $v_n(t) = 0$, $t>0$, $n>m$. Hence

$$v(t,\cdot)=v_1(t)\psi_1(\cdot)+\dots+v_M(t)\psi_M(\cdot)v_{M+1}(t)\phi_{M+1}(\cdot)+\dots+v_m(t)\phi_m(\cdot)$$

and if we put

$$V(t) = \mathrm{col}\,[v_1(t),\dots,v_m(t)]$$

from (4.68), (4.69), (4.71) remarking that (4.62) holds, one obtains easily that $V(t)$ will verify (3.10) and (3.13). The condition (3.14) follows immediately from (3.25) if we remark that (4.60) holds and we conclude that (u,V) is a weak solution for (3.9-3.13) because all the other conditions are fulfilled obviously.

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