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MOVABLE SINGULARITIES AND GALOIS THEORY OF DIFFERENTIAL FIELDS

By ALEXANDRU BUIUM

The aim of this paper is to extend to several variables the "one variable theory" developed in [11]. Results will be stated and discussed in Section 0 below.

We shall follow Kolchin's terminology from [6]. A differential field will mean a field $\mathcal{F}$ of characteristic zero together with pairwise commuting derivatiohs $\delta_1, \ldots, \delta_m$ on it; $\mathcal{F}_0$ with denote the field of constants of $\mathcal{F}$. Unless otherwise stated, all differential fields will be assumed to be differential subfields of a fixed universal differential field $\mathcal{U}$.

For any field K a K-variety will mean here a geometrically integral quasi-projective K-scheme; K(V) will denote the function field of the K-variety V. If $L=K(V)$ and K^* is an extension of K we shall put $V_{K^*}=V\times_{Spec(K)}Spec(K^*)$ and $K^*(V)$ $K^*(L)=K^*(V_{K^*})$; furthermore $V(K^*)$ denotes as usual the set of K^*-points of V. By a K-group we mean a group scheme of finite type over K.

0. Introduction. An extension $\mathcal{F} \subset \mathcal{G}$ of differential fields will be called a Fuchs extension if there exists a nonsingular projective $\mathcal{F}$-variety V whose function field is $\mathcal{G}$ and all of whose local rings are differential subrings of $\mathcal{G}$; a V as above will be called a Fuchs model for $\mathcal{F} \subset \mathcal{G}$. Fuchs extensions were introduced in [11] only for the case tr.deg. $\mathcal{G}/\mathcal{F}=1$ and they were called there "differential algebraic function fields of one variable with no movable singularity". They are however quite classical objects cf. [12] (see also [11] for further discussion).

In Matsuda's book [11] a classification was given for Fuchs extensions with tr.deg. $\mathcal{G}/\mathcal{F}=1$ (and under the assumption that m=1 i.e. in the case of ordinary differential fields). In the present paper we shall provide such a classification for arbitrary tr.deg. $\mathcal{G}/\mathcal{F} \geqslant 1$ (and arbitrary $m \geqslant 1$) by bringing Fuchs extensions into the setting of Kolchin's Galois theory of differential fields. We should note that methods from [11] apply only to the "case of curves" so new tools are required for the case we are treating here. Such a tool will be provided by our previous paper [4]. Here is our main conclusion:

THEOREM 1. Let $\mathcal{F} \subset \mathcal{G}$ be a Fuchs extension with $\mathcal{F}_0 = \mathcal{G}_0$ and $\mathcal{F}_0$ algebraically closed. Then there exist an extension $\mathcal{E}$ of $\mathcal{G}$ with $\mathcal{E}_0 = \mathcal{F}_0$ and a finite extension $\mathcal{F}^*$ of $\mathcal{F}$ contained in $\mathcal{E}$ such that $\mathcal{F}^* \subset \mathcal{E}$ is a G-primitive extension (G being some connected $\mathcal{F}_0$-group); in particular $\mathcal{F}^* \subset \mathcal{E}$ is strongly normal.

Now by Kolchin's Galois theory [6] subextensions of strongly normal extensions are well understood; in our case $\mathcal{F}^* \subset \mathcal{F}^*(\mathcal{G})$ is a suextension of a strongly normal extension so we see from Theorem 1 that Fuchs extensions turn out to have a quite precise Galois-theoretic description "modulo finite base change". In the converse direction we shall prove:

THEOREM 2. Let $\mathcal{F} \subset \mathcal{E}$ be a full G-primitive extension with $\mathcal{F}_0 = \mathcal{E}_0$, $\mathcal{F}_0$ algebraically closed and G a connected $\mathcal{F}_0$-group which is either complete or linear. Then for any intermediate differential field $\mathcal{G}$ between $\mathcal{F}$ and $\mathcal{E}$, the extension $\mathcal{F} \subset \mathcal{G}$ is a Fuchs extension.

Recall that since G is connected, the property of $\mathcal{F} \subset \mathcal{E}$ being full simply means that tr.deg. $\mathcal{E}/\mathcal{F} = \dim(G)$. We expect Theorem 2 to hold in fact for arbitrary connected G not necessarily complete or linear.

Now returning to Theorem 1, it follows from [6] p.427 that there exist intermediate differential fields

$$\mathcal{F}^* \subset \mathcal{F}^0 \subset \mathcal{A} \subset \mathcal{E}$$

such that $\mathcal{F}^* \subset \mathcal{F}^0$ is a finite extension, $\mathcal{F}^0 \subset \mathcal{A}$ is an A-primitive extension (A being an abelian variety) and $\mathcal{A} \subset \mathcal{E}$ is a Picard-Vessiot extension. This combined with our Theorem 1 and with Kolchin's analytic description [8] of A-primitive extensions may be viewed as a differential-algebraic answer to Poincaré's problem (formulated at the end of [12]) concerning the nature of transcendental functions which may appear from integration of systems S of algebraic (partial) differential equations with "no movable singularity": roughly speaking, if

S has meromorphic coefficients in some region in $\mathbb{C}^m$ then integration of S may be proved to reduce to integration of a linear system whose coefficients may be expressed in terms of algebraic functions, abelian functions and functions which are primitive (cf. [6] p.404) over the differential field generated by the coefficients of S. We do not enter into details (compare also with [3]).

In connection with splittings into abelian and linear parts we shall prove:

THEOREM 3. Let $\mathcal{F} \subset \mathcal{G}$ be a Fuchs extension with $\mathcal{F}_o = \mathcal{G}_o$, $\mathcal{F}_o$ being the complex field. Then there exist an extension $\mathcal{H}$ of $\mathcal{G}$ with $\mathcal{H}_o = \mathcal{F}_o$ and intermediate differential fields

$$\mathcal{F} \subset \mathcal{F}^+ \subset \mathcal{B} \subset \mathcal{B}^+ \subset \mathcal{H}$$

such that $\mathcal{B} \subset \mathcal{F}^+(\mathcal{G})$ and such that $\mathcal{F} \subset \mathcal{F}^+$ and $\mathcal{B} \subset \mathcal{B}^+$ are finite extensions, $\mathcal{F}^+ \subset \mathcal{B}$ is an A-primitive extension (A being an abelian variety) and $\mathcal{B}^+ \subset \mathcal{H}$ is a Picard-Vessiot extension.

Note that Theorem 3 does not follow from Theorem 1 and from the splitting $\mathcal{F}^* \subset \mathcal{F}^o \subset \mathcal{A} \subset \mathcal{E}$ by just putting $\mathcal{F}^+ = \mathcal{F}^o$, $\mathcal{B} = \mathcal{B}^+ = \mathcal{A}$, $\mathcal{H} = \mathcal{E}$; indeed one won't have in general that $\mathcal{A} \subset \mathcal{F}^o(\mathcal{G})$!

As an application of our results we shall give a generalisation to the case of simple abelian varieties of Kolchin's theorem [7] p.809 (see also [11] p.70) on elliptic differential function fields possessing infinitely many differential automorphisms:

THEOREM 4. Let $\mathcal{F} \subset \mathcal{G}$ be an extension of differential fields with $\mathcal{F}_o = \mathcal{G}_o$, $\mathcal{F}_o$ algebraically closed and $\mathcal{G} = \mathcal{F}(V)$ with V a simple abelian $\mathcal{F}$-variety. Suppose there are infinitely many differential automorphisms of $\mathcal{G}$ over $\mathcal{F}$. Then there exists a finite extension $\mathcal{F}^*$ of $\mathcal{F}$ such that the extension $\mathcal{F}^* \subset \mathcal{F}^*(\mathcal{G})$ is an A-primitive extension (A being an abelian $\mathcal{F}_o$-variety).

Trivial examples show that the assumption on V being simple cannot be removed.

Let's make a final remark on the case tr.deg. $\mathcal{G}/\mathcal{F}$=m=1; in this case our results were essentially known from [11] although connection with strongly normal extensions is not explicitely made there). For instance our Theorem 1 follows in this particular case from Theorems 3,8,14 in [11] pp. 13,37 and 91 and from the well known fact that the Riccati equation "reduces" to a (second order) linear equation. Conversely (as the reader will realize) our method can be used to give alternative proofs of the three theorems quoted from [11].

1. Proof of Theorem 1. We start by recalling a basic result from [4]. If K is an algebraically closed field (of characteristic zero as usual) and V is a K-variety then a subfield k of K is called a field of definition for V if there exists a k-variety X such that V is K-isomorphic to X_K. One defines:

$$\Delta(V)=\{\delta\in \mathrm{Der}_{\mathbb{Q}}(K(V),K(V));\ \delta(\mathcal{O}_{V,p})\subset \mathcal{O}_{V,p} \text{ for all } p\in V\}$$

$$K^{\Delta(V)}=\{\lambda\in K;\ \delta\lambda=0 \text{ for all } \delta\in\Delta(V)\}$$

THEOREM [4]. If V is nonsingular and projective then:

1) $K^{\Delta(V)}$ is a field of definition for V.

2) Any other algebraically closed field of definition for V must contain $K^{\Delta(V)}$.

For the proof of this theorem we send to [4] ; we should say that the proof involves deformation of polarized varieties and an analytic argument from Kodaira-Spencer theory. The hard part is 1); we shall need the following consequence of 1):

COROLLARY. If $\mathcal{F}\subset\mathcal{G}$ is a Fuchs extension with Fuchs model V and $\mathcal{F}_0$ algebraically closed, there exists a finite extension $\mathcal{F}^*$ of $\mathcal{F}$ and a nonsingular projective $\mathcal{F}_0$-variety Z such that we have an $\mathcal{F}^*$-isomorphism $V_{\mathcal{F}^*}\simeq Z_{\mathcal{F}^*}$.

Proof. Apply Theorem above to V_K (K=the algebraic closure of $\mathcal{F}$) and note that $K_0=\mathcal{F}_0$ and that

$$K^{\Delta(V_K)}\subset K_0.$$

Now suppose we are in the hypothesis of Theorem 1 and let $\mathcal{F}^*$ and Z be as in the Corollary above. Moreover let $\mathcal{C}=\mathcal{F}_0$ and $\sigma:\mathcal{F}^*(\mathcal{G})\longrightarrow\mathcal{F}^*(Z)$ be the isomorphism deduced from the isomorphism in the Corollary. Define on $\mathcal{F}^*(Z)$ the derivations $\delta_{1,Z},\dots,\delta_{m,Z}$ by the rule $\delta_{j,Z}(\alpha\otimes\beta)=\alpha\otimes\delta_j\beta$

($\alpha \in \mathcal{C}(Z)$, $\beta \in \mathcal{F}^*$). Now consider the $\mathcal{F}^*$-derivations of $\mathcal{F}^*(Z)$ into itself defined by

$$d_{j,Z} = \sigma \delta_j \sigma^{-1} - \delta_{j,Z}$$

Clearly $d_{j,Z}$ maps each local ring of $Z_{\mathcal{F}^*}$ into itself so

$$d_{j,Z} \in H^0(Z_{\mathcal{F}^*}, T_{Z_{\mathcal{F}^*}/\mathcal{F}^*}) = H^0(Z, T_{Z/\mathcal{C}}) \otimes_{\mathcal{C}} \mathcal{F}^*$$

where T is the dual of the sheaf Ω of relative differentials. Put $G = \mathrm{Aut}^0(Z/\mathcal{C})$ = identity component of the group scheme of $\mathcal{C}$-automorphisms of Z; as well known G is a $\mathcal{C}$-group. We claim that Z has a Zariski open G-orbit X (hence Z is almost homogenous in the sense of [1]). Indeed one can construct (using the method of Chow points as in [10] or [13] p.406) a dominant morphism $\varphi : X \longrightarrow Y$ from a G-invariant Zariski open subset X of Z to a variety Y such that for any $x \in X$ we have $\varphi^{-1}(\varphi(x)) = Gx$. Clearly, for any vector field $\theta \in H^0(Z, T_{Z/\mathcal{C}})$ the restriction $\theta|_X$ is tangent to the fibres of φ and consequently, viewing θ as a derivation on $\mathcal{C}(Z)$ we get $\theta(\varphi^*(\mathcal{C}(Y))) = 0$. Since $\delta_{j,Z}$ vanishes on $\varphi^*(\mathcal{C}(Y)) \otimes 1$ the same will hold for $\sigma \delta_j \sigma^{-1}$. But now the equality $(\mathcal{F}^*(\mathcal{G}))_0 = \mathcal{C}$ forces $\mathcal{C}(Y)$ to be equal to $\mathcal{C}$ hence Y is a point and X is an orbit. Now fix a point $p \in X(\mathcal{C})$; the map $\pi_p : G \longrightarrow Z$, $\pi_p(g) = gp$, permits to embed $\mathcal{C}(Z)$ into $\mathcal{C}(G)$ and hence to embed $\mathcal{F}^*(Z)$ into $\mathcal{F}^*(G)$. Let $\mathcal{L}_{\mathcal{C}}(G)$ be the Lie algebra of right invariant vector fields on G and let $\mathcal{L}_{\mathcal{F}^*}(G)$ be the image of the natural embedding

$$\mathcal{L}_{\mathcal{C}}(G) \otimes_{\mathcal{C}} \mathcal{F}^* \longrightarrow \mathrm{Der}_{\mathcal{F}^*}(\mathcal{F}^*(G), \mathcal{F}^*(G))$$ cf. [6] p.325.

We claim that the restriction map

$$\mathrm{Der}_{\mathcal{C}}(\mathcal{F}^*(G), \mathcal{F}^*(G)) \to \mathrm{Der}_{\mathcal{C}}(\mathcal{F}^*(Z), \mathcal{F}^*(G))$$

induces an isomorphism of $\mathcal{F}^*$-vector spaces and Lie algebras over $\mathcal{C}$:

$$\rho : \mathcal{L}_{\mathcal{F}^*}(G) \longrightarrow H^0(Z_{\mathcal{F}^*}, T_{Z_{\mathcal{F}^*}/\mathcal{F}^*})$$

Indeed it is sufficient to prove that the restriction map $\mathrm{Der}_{\mathcal{C}}(\mathcal{C}(G), \mathcal{C}(G)) \to \mathrm{Der}_{\mathcal{C}}(\mathcal{C}(Z), \mathcal{C}(G))$ induces an isomorphism of Lie algebras over $\mathcal{C}$: $\mathcal{L}_{\mathcal{C}}(G) \longrightarrow H^0(Z, T_{Z/\mathcal{C}})$. But there is a well known identification of $H^0(Z, T_{Z/\mathcal{C}})$ and the tangent space T_1G which associates to any tangent vector $t \in T_1G$ the vector field $\theta_Z \in H^0(Z, T_{Z/\mathcal{C}})$ defined by $\theta_Z(q) = (T_1\pi_q)(t)$ for any $q \in Z(\mathcal{C})$ (where $T_1\pi_q : T_1G \longrightarrow T_qZ$ is the tangent map of $\pi_q : G \longrightarrow Z$, $\pi_q(g) = gq$). Let θ_G be the right invariant vector field on G with $\theta_G(1) = t$. It is sufficient to see that for any $x \in G$ we have $(T_x\pi_p)(\theta_G(p)) = \theta_Z(xp)$; but this follows from the equality $\pi_{xp} = \pi_p \circ R_x$ where $R_x : G \longrightarrow G$ is the right translation $R_x(g) = gx$. Our claim about ρ being an isomorphism of Lie algebras is proved.

Now let $\delta_{1,G}, \ldots, \delta_{m,G}$ be the derivations on $\mathcal{F}^*(G)$ defined by $\delta_{j,G}(\alpha \otimes \beta) = \alpha \otimes \delta_j \beta$ $(\alpha \in \mathcal{C}(G), \beta \in \mathcal{F}^*)$. It is trivial to check that for any $\theta \in \mathcal{L}_{\mathcal{F}^*}(G)$ and for any j we have $[\delta_{j,G}, \theta] \in \mathcal{L}_{\mathcal{F}^*}(G)$ and $\rho([\delta_{j,G}, \theta]) = [\delta_{j,Z}, \rho(\theta)]$ (where $[\,,\,]$ denotes the Poisson bracket on $\mathrm{Der}_{\mathcal{C}}(\mathcal{F}^*(G), \mathcal{F}^*(G))$ and $\mathrm{Der}_{\mathcal{C}}(\mathcal{F}^*(Z), \mathcal{F}^*(Z))$ respectively). Finally let

$d_{1,G},\dots,d_{m,G}\in\mathcal{L}_{\mathcal{F}^*}(G)$ be such that $\rho(d_{j,G})=d_{j,Z}$ for all j and define derivations $D_1,\dots,D_m$ on $\mathcal{F}^*(G)$ by the formulae $D_j=\delta_{j,G}+d_{j,G}$. We claim that $[D_j,D_k]=0$ for all j and k. Indeed we have

$$[D_j,D_k]=[\delta_{j,G},d_{k,G}]+[d_{j,G},\delta_{k,G}]+[d_{j,G},d_{k,G}]$$

and the image of the above sum via ρ equals

$$[\delta_{j,Z},d_{k,Z}]+[d_{j,Z},\delta_{k,Z}]+[d_{j,Z},d_{k,Z}]=$$

$$=[\sigma\delta_j\sigma^{-1},\sigma\delta_k\sigma^{-1}]=0$$

So $\mathcal{F}^*(G)$ together with $D_1,\dots,D_m$ becomes a differential field (not embedded in the universal field $\mathcal{U}$); moreover $\mathcal{F}^*(Z)$ becomes a differential subfield of it and σ becomes an isomorphism of differential fields. Now let Spec(R) be any affine Zariski open subset of G. Put $R^*=R\otimes_{\mathcal{C}}\mathcal{F}^*$; clearly R^* is a differential subring of $\mathcal{F}^*(G)$. Write $\mathcal{C}(Z)=\mathcal{C}(g_1,\dots,g_N)$, $g_i=h_i/f_i$, $h_i\in R$, $f_i\in R$; replacing R by R_f ($f=f_1\dots f_N$) we may suppose that $g_i\in R$ for all i. It is clear that $\mathcal{F}^*(Z)[R^*]=$ $=S^{-1}R^*$ where $S=\mathcal{F}^*[g_1,\dots,g_N]\setminus\{0\}$ so $S^{-1}R^*$ is a finitely generated $\mathcal{F}^*(Z)$-algebra. By results of Kolchin [6] pp.142-143 the isomorphism $\tau=\sigma^{-1}:\mathcal{F}^*(Z)\longrightarrow\mathcal{F}^*(\mathcal{G})\subset\mathcal{U}$ can be extended to a morphism of differential rings $\tau:S^{-1}R^*\longrightarrow\mathcal{U}$ such that if $\mathcal{E}$ denotes the field generated by the image of τ then $\mathcal{E}_o$ is algebraic over $(\mathcal{F}^*(\mathcal{G}))_o$; in our case we get $\mathcal{E}_o=\mathcal{C}$. Put $P^*=\ker(\tau)\cap R^*\in \mathrm{Spec}(R^*)$, $P=P^*\cap R\in\mathrm{Spec}(R)$; we shall still denote by τ the induced morphism $R^*_{P^*}\longrightarrow\mathcal{U}$. We claim that the point $\alpha\in G(\mathcal{U})$ given by $\alpha^*:\mathcal{O}_{G,P}=R_P\subset R^*_{P^*}\xrightarrow{\tau}\mathcal{U}$

is a G-primitive of $\mathcal{F}^* \subset \mathcal{E}$ (Note that this would close the proof, since $\mathcal{F}^*_o = \mathcal{E}_o$, cf. [6] p. 419). Indeed we have $\mathcal{F}^*(\alpha) = \mathcal{F}^*(\tau(R)) = \mathcal{E}$. Moreover for any $\lambda \in R_P$ we have

$$\delta_j \alpha^*(\lambda) = \tau(D_j \lambda) = \tau(d_{j,G}\lambda) = \alpha^*_{\mathcal{U}}(d_{j,G}\lambda)$$

where $\alpha^*_{\mathcal{U}} : \mathcal{O}_{G_{\mathcal{U}}, P_{\mathcal{U}}} \longrightarrow \mathcal{U}$ is the $\mathcal{U}$-morphism induced by α. By the very definition of the logarithmic derivative $\ell\delta_j(\alpha)$ [6] p.349 we must have then $\ell\delta_j(\alpha) = d_{j,G}$ so α is a G-primitive of $\mathcal{F}^* \subset \mathcal{E}$ and Theorem 1 is proved.

2. Proof of Theorm 2. By [6] p.419 $\mathcal{F} \subset \mathcal{E}$ is a strongly normal extension with differential Galois group G. By the Galois theory [6] p.398 (and by the remark at p.402) $\mathcal{G}$ is the field of invariants of the group $H(\mathcal{C})$ where H is some $\mathcal{C}$-subgroup of G ($\mathcal{C} = \mathcal{F}_o$). Now suppose $\alpha \in G(\mathcal{U})$, given by $\alpha^* : \mathcal{O}_{G,P} \longrightarrow \mathcal{U}$ ($P \in G$) is a full G-primitive of $\mathcal{F} \subset \mathcal{E}$. The equality tr.deg. $\mathcal{E}/\mathcal{F} = \dim(G)$ implies that P is the generic point of the $\mathcal{C}$-scheme G and moreover we get an $\mathcal{F}$-morphism $\alpha^*_{\mathcal{F}} : \mathcal{F}(G) \longrightarrow \mathcal{U}$ whose image is $\mathcal{E}$. Put $d_{j,G} = \ell\delta_j(\alpha)$; by hypothesis $d_{j,G} \in \mathcal{L}_{\mathcal{F}}(G)$. It immediately follows that if we let D_j be the derivation induced on $\mathcal{F}(G)$ via $\alpha^*_{\mathcal{F}}$ by δ_j (i.e. if $D_j = (\alpha^*_{\mathcal{F}})^{-1} \delta_j \alpha^*_{\mathcal{F}}$) then $D_j = \delta_{j,G} + d_{j,G}$ where $\delta_{j,G}(\gamma \otimes \beta) = \gamma \otimes \delta_j \beta$ ($\gamma \in \mathcal{C}(G), \beta \in \mathcal{F}$). Now we claim there exist a nonsingular projective $\mathcal{C}$-variety Z, a G-action on Z and a point $p \in Z(\mathcal{C})$ such that the orbit Gp is Zariski open and the isotropy group of p is H. If G is

complete this is clear by just taking $Z=G/H$. If G is linear then by Chevalley's theorem [5] p. there exist a G-action on a projective space $\mathbb{P}$ and a point $p \in \mathbb{P}(\mathcal{C})$ whose isotropy group is H; by Hironaka's work on equivariant resolutions (cf. [10]) one can G-equivariantly desingularize $Z \longrightarrow \overline{Gp}$ the closure of the orbit of p in $\mathbb{P}$ to obtain the desired Z. Since the field $\mathcal{C}(G)^{H(\mathcal{C})}$ of fixed elements of $\mathcal{C}(G)$ under $H(\mathcal{C})$ ($H(\mathcal{C})$ acting by right translations) equals $\mathcal{C}(Z)$ we get (by the computation in [13] pp.405-406) that

$$(\mathcal{F}(G))^{H(\mathcal{C})} = \mathcal{F}(Z)$$

It is straight forward to check that D_j takes each local ring of $Z_{\mathcal{F}}$ into itself. We conclude that $Z_{\mathcal{F}}$ is a Fuchs model of $\mathcal{F} \subset \mathcal{G}$ and we are done.

3. Proof of Theorem 3. Start with the following:

LEMMA. Let $\mathcal{F} \subset \mathcal{G}$ be a Fuchs extension with Fuchs model V and suppose we are given a dominant morphism of non-singular projective $\mathcal{F}$-varieties $f: V \longrightarrow W$ with $\mathcal{B} = \mathcal{F}(W)$ algebraically closed in $\mathcal{G} = \mathcal{F}(V)$. Then:

1) $\mathcal{B}$ is a differential subfield of $\mathcal{G}$ and $\mathcal{F} \subset \mathcal{B}$ is a Fuchs extension with Fuchs model W.

2) $\mathcal{B} \subset \mathcal{G}$ is a Fuchs extension with Fuchs model $V \times_W \mathrm{Spec}(\mathcal{B})$.

Proof. Let $T_{V/\mathbb{Q}} = \underline{\mathrm{Hom}}_{\mathcal{O}_V}(\Omega_{V/\mathbb{Q}}, \mathcal{O}_V)$ be the sheaf of $\mathbb{Q}$-derivations from $\mathcal{O}_V$ into itself. The map $f^*\Omega_{W/\mathbb{Q}} \to \Omega_{V/\mathbb{Q}}$

induces (via $\mathrm{Hom}_V(-,\mathcal{O}_V)$) a map

$$H^0(V,T_{V/\mathbb{Q}}) \longrightarrow \mathrm{Hom}_V(f^*\Omega_{W/\mathbb{Q}},\mathcal{O}_V)=\mathrm{Hom}_W(\Omega_{W/\mathbb{Q}},\mathcal{O}_W)=H^0(W,T_{W/\mathbb{Q}})$$

The existtence of this map already proves 1). Statement 2) follows trivially.

Now suppose we are in the hypothesis of Theorem 3 and put $\mathcal{C}=\mathcal{F}_0$. By the proof of Theorem 1, if V is a Fuchs model of $\mathcal{F}\subset\mathcal{G}$ we may find a finite extension $\mathcal{F}^+$ of $\mathcal{F}$ such that $V_{\mathcal{F}^+}\simeq Z_{\mathcal{F}^+}$ for some nonsingular projective $\mathcal{C}$-variety Z which is almost homogenous. By [1] p.55 the Albanese map $a:Z\longrightarrow A=\mathrm{Alb}(Z)$ is surjectiveand has connected fibres Z_s $(s\in A(\mathcal{C}))$ such that $H^i(Z_s,\mathcal{O}_{Z_s})=0$ for $i\geqslant 1$ (the information available in [1] concerning the Albanese morphism is far more precise but we don't need it). Putting $W=A_{\mathcal{F}^+}$ we get a morphism $f:V_{\mathcal{F}^+}\longrightarrow W$ satisfying the hypothesis in our Lemma so both $\mathcal{F}^+\subset\mathcal{B}=\mathcal{F}^+(A)$ and $\mathcal{B}\subset\mathcal{F}^+(\mathcal{G})$ are Fuchs extensions. It is apparent from the proof of Theorem 1 that $\mathcal{F}^+\subset\mathcal{B}$ is an A-primitive extension. On the other hand if $V_{\mathcal{B}}=V\times_{\mathcal{F}^+W}\mathrm{Spec}(\mathcal{B})$ then clearly $H^i(V_{\mathcal{B}},\mathcal{O}_{V_{\mathcal{B}}})=0$ for $i\geqslant 1$. By the proof of Theorem 1 there existsa finite extension $\mathcal{B}^+$ of $\mathcal{B}$ such that $(V_{\mathcal{B}})_{\mathcal{B}^+}\simeq X_{\mathcal{B}^+}$ for some $\mathcal{C}$-variety X; furthermore if $G=\mathrm{Aut}^0(X)$ then the extension $\mathcal{B}^+\subset\mathcal{B}^+(\mathcal{F}^+(\mathcal{G}))$ embeds into a G-primitive extension $\mathcal{B}^+\subset\mathcal{H}$ with $\mathcal{B}^+_0=\mathcal{H}_0$. The only thing to be proved is that G is linear. But this follows from the obvious equalities $H^i(X,\mathcal{O}_X)=0$, $i\geqslant 1$ (in fact one can prove that $X\simeq Z_s$) and from [1] p.55 or [10] p.161. Theorem 3 is proved.

REMARK. Let K be the algebraic closure of $\mathcal{F}$ in Theorem 3. The following additional facts can be said in the conclusion of Theorem 3:

1) If $K \subset K(\mathcal{G})$ is not ruled then one can take $\mathcal{B}=\mathcal{H}=\mathcal{F}^+(\mathcal{G})$.

2) If $K \subset K(\mathcal{G})$ has irregularity zero then one can take $\mathcal{F}^+=\mathcal{B}^+$

Indeed 1) follows from the fact that Z turns out to be non ruled and almost homogenous, hence an abelian variety by [10]. Finally in 2) Z will have irregularity zero and hence $\mathrm{Aut}^0(Z/\mathcal{C})$ will be linear.

4. Proof of Theorem 4. We can easily reduce ourselves to the case when $\mathcal{F}$ is algebraically closed. Now it can be easily checked (see also [2] p.58) that the set

$$U=\left\{p \in V \; ; \; \delta_j(\mathcal{O}_{V,p}) \subset \mathcal{O}_{V,p} \text{ for all } j\right\}$$

is the complement of a divisor D on V. If U=V then is a Fuchs extension and the proof of Theorem 1 immediately yelds the conclusion. Suppose $U \neq V$ and let's look for a contradiction. Since V is a simple abelian variety D must be ample. Now it is easy to check using [9] p.85 that the set of all $\mathcal{F}$-automorphisms σ of V (not necessarily preserving the zero element of V) such that $\sigma(D)=D$ is finite. On the other hand it is clear that any differential $\mathcal{F}$-automorphism σ of $\mathcal{G}$ induces an $\mathcal{F}$-automorphism σ of V with $\sigma(D)=D$, contradicting the infinity from hypothesis. Theorem 4 is proved.

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