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REPRESENTATION OF MONOTONE OPERATORS AS SUBGRADIENTS OF CONVEX FUNCTIONS
(preliminary version)

by

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1. Introduction

Let X and Y be reflexive Banach spaces with duals X^*, Y^*. By the Asplund [1] renorming theorem we may assume them to be strictly convex.

Consider $K : X \times X \to [-\infty, +\infty]$ be a closed, proper, saddle function satisfying

(1.1) $cl_2\, cl_1\, K = K$.

For fundamentals of convex analysis we quote the monographs of V.Barbu, Th.Precupanu [2], R.T.Rockafellar [7], I.Ekeland, R.Temam [3].

The following result will be important in the sequel

Theorem 1. The formulae

(1.2) $L(x,y^*) = \sup \{ (y,y^*)-K(x,y);\ y \in X \}$

(1.3) $K(x,y) = \sup \{ (y,y^*)-L(x,y^*);\ y^* \in X^* \}$

define a one-to-one correspondence between convex, lower semicontiuous, proper functions L on $X \times X^*$ and closed, proper saddle functions satisfying (1.1). Moreover, we have

(1.4) $[x^*,y^*] \in \partial K(x,y) \Leftrightarrow [x^*,y] \in \partial L(x,y^*)$.

In a sequence of papers [4], [5], [6], E.Krauss defined and studied the representation of arbitrary monotone operators via subgradients of saddle functions by the formula

(1.5) $x^* \in T_K x \Leftrightarrow [x^*,x^*] \in \partial K(x,x)$

It is proved that the operator $T_K \subset X \times X^*$ is maximal monotone and that every maximal monotone operator may be represented in this way. Moreover function K in (1.5) may be assumed skew-symmetric, i.e.

(1.6) $cl_2 K(x,y) = -cl_1 K(y,x) \quad , \forall x,y \in X.$

The purpose of this paper is to obtain similar results using convex functions instead of saddle functions. However, we underline that generally, our method depends heavily on the results of E.Krauss.

2. Representation results

Let $L : X \times X^* \to]-\infty, +\infty]$ be a convex, lower semicontinuous function.

Definition 1. Function L is called skew-symmetric if $L(x,y^*) = L^*(y^*,x)$ for any $y^* \in X^*$, $x \in X$.

Here $L^* : X^* \times X \to]-\infty, +\infty]$ is the conjugate of L.

Let K be the saddle function associated with L by (1.3).

Theorem 2 L is skew-symmetric iff K is skew-symmetric.

Proof.

Denote $(.,.)$ the pairing between X and X^*:

$$L^*(y^*,x) = \sup_{y,x^*} \{(y^*,y) + (x,x^*) - L(y,x^*)\} =$$

$$= \sup_{y,x^*} \{(y^*,y)+(x,x^*) - \sup_z \{(x^*,z)-cl_2 K(y,z)\}\}$$

by (1.2). Then

$$L^*(y^*,x) = \sup_{y,x^*} \{(y^*,y)+(x,x^*) - \sup_z \{(x^*,z)+cl_1 K(z,y)\}\}$$ since K is skew-symmetric.

Let $\varphi(z) = -cl_1 K(z,y)$. It is convex, lower semicontinuous, proper if $y \in \text{dom}\, K$. It is enough to work with $y \in \text{dom}\, K$ since we take sup after all the values $y \in X$. We have

$$L^*(y^*,x)= \sup_{y,x^*} \left\{ (y^*,y)+(x,x^*)-\varphi^*(x^*)\right\} =$$

$$= \sup_{y} \sup_{x^*} \left\{ (x,x^*)-\varphi^*(x^*) + (y^*,y)\right\} =$$

$$= \sup_{y} \left\{(y^*,y)+\varphi(x)\right\} = \sup_{y} \left\{(y,y^*)-cl_1K(x,y)\right\} =$$

$$= \sup_{y} \left\{(y,y^*)-cl_2cl_1K(x,y)\right\} = \sup_{y} \left\{(y,y^*)-K(x,y)\right\} =$$

$$= L(x,y^*).$$

Conversely, from (1.3) we see that $cl_2K=K$, therefore

$$cl_2K(y,x)=\sup_{y^*} \left\{(x,y^*)-\sup_{z^*,z} \left\{(z^*,y)+(z,y^*)-L(z,z^*)\right\}\right\} =$$

$$= \sup_{y^*} \left\{(x,y^*)-\sup_{z} \left\{(z,y^*)+\sup_{z^*} \left\{(z^*,y)-L(z,z^*)\right\}\right\}\right\} =$$

$$= \sup_{y^*} \left\{(x,y^*)-\sup_{z} \left\{(z,y^*)+K(z,y)\right\}\right\} =$$

$$= \sup_{y^*} \left\{(x,y^*)-\sup_{z} \left\{(z,y^*)-(-cl_1K(z,y))\right\}\right\}.$$

Returning to the notation with $\varphi(z)$, we get:

$$cl_2K(y,x)= \sup_{y^*} \left\{(x,y^*)-\varphi^*(y^*)\right\} =\varphi(x)=-cl_1K(x,y).$$

This finishes the proof.

Taking into account (1.4) and (1.5), we state

Definition 2. The operator T_L associated with L is given by:

(2.1) $[x^*,x] \in T_L \Longleftrightarrow [x^*,x] \in \partial L(x,x^*)$.

Proposition 1. Let L be a convex, lower semicontinuous , proper function on X x X*. Then T_L is monotone.

Proof.

For all $[x^*,x]$, $[y^*,y] \in T_L$ we have

$L(x,x^*) \leq L(z,z^*)+(x^*,x-z)+(x,x^*-z)$

$L(y,y^*) \leq L(w,w^*)+(y^*,y-w)+(y,y^*-w^*)$

for any $[z,z^*]$, $[w,w^*] \in X \times X^*$. Fix $[z,z^*]=$ $=[y,y^*]$ and $[w,w^*]=[x,x^*]$; By adding the two inequalities we obtain the conclusion.

Examples

1) Let $\varphi: X \to]-\infty,+\infty]$ be convex, lowersemicontinuous, proper

and $L(x,y^*)=\varphi(x)+\varphi^*(y^*)$. Then $T_L=\partial\varphi$.

2) $L(x,y^*) = \frac{1}{2}|x|^2 + \frac{1}{2}|y^*|^2$

Then $T_L=F$, the duality mapping on X.

3) $T_{L^*}=T_L^{-1}$.

Theorem 3. If $L:X \times X^* \to]-\infty,+\infty]$ is convex, lower semicontinous proper, skew-symmetric, then T_L is maximal monotone.

Proof.

Let $K: X \times X \to [-\infty,+\infty]$ be the skew-symmetric, saddle function defined by (1.3). Then (1.4), (2.1) give $x^*\in T_Lx \iff [x^*,x^*]\in \partial K(x,x)$ and by Krauss [5] ,p.9, T_L is maximal monotone.

Theorem 4. Let $A\subset X \times X^*$ be a maximal monotone operator. There exists $L:X \times X^* \to]-\infty,+\infty]$ convex proper, lower semicontinuous, skew-symmetric, such that $A=T_L$.

Proof.

By Corollary 1, p.13, Krauss [5] , there exists $K:X \times X \to [-\infty,+\infty]$ closed, proper, skew-symmetric saddle function, such that $A=T_K$. Take L given by (1.2). Then it satisfies all the conditions of the Theorem.

3. Final remarks.

First we give a new and purely variational (without duality arguments) proof of the Theorem of Krauss used in Theorem 3.

Theorem 5. Let $K:X \times X \to [-\infty, +\infty]$ be a saddle, closed, proper skew-symmetric function. Then the operator T_K defined by (1.5) is maximal monotone.

Proof.

Obviously T_K is monotone. In order to be maximal, we consider the equation

(3.1) $x^*\in T_K x+Fx$

for all $x^*\in X^*$. F is as in Example 2.

Equivalently, we have:

(3.2) $[x^*,x^*]\in [T_Kx,T_Kx] + [Fx,Fx]$, (3.3) $[x^*,x^*] \in \partial K(x,x) + [Fx,Fx]$,

(3.4) $[0,0] \in \partial K(x,x) + [Fx,Fx] - [x^*,x^*]$.

Consider the skew-symmetric, proper, closed saddle function on X x X, $M(x,y)=K(x,y)+\frac{1}{2}|y|^2-\frac{1}{2}|x|^2-(x^*,x-y)$. We remark that: $\partial M(x,y)= \partial K(x,y)+ [Fx,Fy] - [x^*,x^*]$ and (3.4) is equivalent with $[0,0] \in \partial M(x,x)$

that is M has a saddle point of the form $[x,x]$.

The existence of a saddle point for M is obvious by the coercivity due to $\frac{1}{2}|y|^2-\frac{1}{2}|x|^2$.

Let $[x_0,y_0]$ be this point. Since M is closed, skew-symmetric $[y_0,x_0]$ is also a saddle point.

Since the space is strictly convex, also by the presence of $\frac{1}{2}|y|^2-\frac{1}{2}|x|^2$, M is strictly concave-convex, therefore the saddle point is unique and we conclude $x_0=y_0$. This finishes the proof.

Finally, in opposition with section 2, we study in detail the case of symmetric convex functions. Unfortunately, this condition is too restrictive and the result doesn't extend beyond the classical subdifferential.

Let $L: X \times X \to]-\infty,+\infty]$ be a convex, lower semicontinuous, proper function.

Definition 3. L is called symmetric if $L(x,y)=L(y,x)$ for all $x,y \in X$.

With L the following monotone operator is associated:

(3.5) $A:X \to X^*$

$x^* \in Ax \iff [x^*,x^*] \in \partial L(x,x)$.

Operator A is monotone without the symmetry assumption.

Theorem 6. If L is symmetric, A is maximal monotone.

Proof.

Consider the equation

(3.6) $x^* \in Ax+Fx$.

Equivalent, we have:

$[x^*,x^*] \in [Ax,Ax] + [Fx,Fx]$,

$[x^*,x^*] \in \partial L(x,x) + [Fx,Fx]$.

Let $M : X \times X \to]-\infty,+\infty]$ be the convex, lower semicontinuous function given by

$$M(x,y)=L(x,y)+\frac{1}{2}|x|^2+\frac{1}{2}|y|^2-(x^*,x+y)$$

Obviously M is symmetric and (3.6) becomes $[0,0] \in \partial M(x,x)$.

Since M is coercive, it has a minimum point, unique since $\frac{1}{2}|x|^2+\frac{1}{2}|y|^2$ is strictly convex. Let us denote it $[x_0,y_0]$. By the symmetry property $[y_0,x_0]$ is a minimum point for M too and we infer $x_0=y_0$.

Therefore equation (3.6) has solution for any $x^* \in X^*$ and the proof follows.

The following result shows that not every maximal monotone operator $A \subset X \times X^*$ may be represented in the form (3.5).

Consider $\varphi : X \to]-\infty,+\infty]$ the convex, lower semicontinous proper function

$$\varphi(x)=\frac{1}{2}L(x,x).$$

Proposition 2. The operator A defined by (3.5) coincides with $\partial\varphi$.

Proof.

It is enough to show that $A \subset \partial\varphi$. We have

$$x^* \in Ax \Leftrightarrow [x^*,x^*] \in \partial L(x,x) \Rightarrow L(x,x) \leq L(v,v)+(X^*,x-v)+(x^*,x-v),$$

$$\Rightarrow \varphi(x) \leq \varphi(v)+(x^*,x-v),\ \forall v \in X.$$

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