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FUNCTIONS HAVING THE GRAPH IN $\mathcal{E}^n$

by Şerban BUZEŢEANU and Cristian CALUDE

We characterize the number-theoretic functions having the graph in the nth Grzegorczyk class $\mathcal{E}^n$. Tradeoffs between the complexity of a function and the complexity of its graph are analysed. As an application we prove that some well-known recursive but non-primitive recursive functions of Ackermann's type have elementary graphs.

1. PRELIMINARIES

Let $\mathbb{N} = \{0,1,2,\ldots\}$ be the set of natural numbers. All functions will be concerned with are number-theoretic, i.e. they are defined on $\mathbb{N}^k (k \in \mathbb{N} \setminus \{0\})$, and they take only natural values.

Given a function $f : \mathbb{N}^k \longrightarrow \mathbb{N}$, we put: $\mathrm{Graph}(f) = \{(x_1, x_2, \ldots, x_k, y) \in \mathbb{N}^{k+1} \mid f(x_1, x_2, \ldots, x_k) = y\}$, $\mathrm{range}(f) = \{f(x_1, x_2, \ldots, x_k) \mid (x_1, x_2, \ldots, x_k) \in \mathbb{N}^k\}$. For every subset $X \subset \mathbb{N}^k$ we denote by $\chi_X : \mathbb{N}^k \longrightarrow \{0,1\}$ its characteristic function.

Denote by $\mathcal{E}^n$ the nth class in Grzegorczyk's hierarchy [8]. Each class $\mathcal{E}^n$ contains the functions $sg : \mathbb{N} \longrightarrow \mathbb{N}$, $sg(0)=0$, $sg(x)=1$, for $x > 0$, $\dot{-} : \mathbb{N}^2 \longrightarrow \mathbb{N}$, $x \dot{-} y = \max(x-y, 0)$, $eq : \mathbb{N}^2 \longrightarrow \{0,1\}$, $eq(x,x)=1$, $eq(x,y)=0$, for all $x, y \in \mathbb{N}$, $x \neq y$, and it is closed under limited minimization. The family of predicates in $\mathcal{E}^n$, denoted by $\mathcal{E}^n_*$, is closed under Boolean operations and limited quantifiers. The third class $\mathcal{E}^3$ coincides with the class of Kalmár elementary functions; it contains the functions $pn : \mathbb{N} \longrightarrow \mathbb{N}$, $pn(x)$ = the xth prime ($pn(0)=2$, $pn(1)=3, \ldots$), $exp : \mathbb{N}^2 \longrightarrow \mathbb{N}$, $exp(n,x)$ = the exponent of the nth prime in the prime factor decomposition of x, $long : \mathbb{N} \longrightarrow \mathbb{N}$, $long(x)$ = the greatest index of a prime which divides x, and a pairing functions ($J:\mathbb{N}^2 \longrightarrow \mathbb{N}$, $K,L:\mathbb{N} \longrightarrow \mathbb{N}$), and it is closed under limited summation and multiplication.

Finally, for each function $f:\mathbb{N}^k \longrightarrow \mathbb{N}$ we denote by $\mathcal{E}^n(f)$ the smallest class of number-theoretic functions which contains $\mathcal{E}^n \cup \{f\}$ and which is closed under limited primitive recursion and functional composition. We say that the function F is elementary in f in case $F \in \mathcal{E}^3(f)$.

For these well-known facts see [8],[13],[18].

2. MAIN RESULTS

We begin with a necessary and sufficient condition enssuring that a number-theoretic function has the graph in $\mathcal{E}^n$.

Theorem 1. Let $F:\mathbb{N} \longrightarrow \mathbb{N}$ be an arbitrary function and n a natural number. The following assertions are equivalent:

a) There exist three functions $p,b:\mathbb{N} \longrightarrow \mathbb{N}$ and $R:\mathbb{N}^2 \longrightarrow \{0,1\}$ in $\mathcal{E}^n$ such that for every natural x one has:

(1) $\mu y[R(x,y) = 1] \leq b(F(x))$,

(2) $F(x) = p(\mu y_{b(F(x))} [R(x,y) = 1])$.

b) The predicate $\chi_{Graph(F)}$ is in $\mathcal{E}^n_*$.

Proof. "a) $\Longrightarrow$ b)". Assume the existence of the functions p, b and R in $\mathcal{E}^n$ satisfying (1) and (2). We shall prove that for all naturals x and z one has:

(3) $\chi_{Graph(F)}(x,z)=eq(z,p(\mu y_{b(z)} [R(x,y)=1]))\cdot(\bigvee_{y=0}^{b(z)} R(x,y))$.

Suppose first that $\chi_{Graph(F)}(x,z)=1$. In view of the hypothesis and (2) one has: $z=F(x)=p(\mu y_{b(F(x))} [R(x,y)=1])=p(\mu y_{b(z)} [R(x,y)=1])$ i.e. $eq(z,p(\mu y_{b(z)} [R(x,y)=1]))=1$. Furthermore, again by hypothesis and (1), $\mu y[R(x,y)=1] \leq b(F(x))=b(z)$, i.e. $R(x,y)=1$, for some natural

$0 \leq y \leq b(z)$. So, $\bigvee_{y=0}^{b(z)} R(x,y)=1$.

Conversely, assume that the predicate in the right-hand side of (3) is valid, i.e.

(4) $z = p(\mu y_{b(z)} [R(x,y)=1])$,

and

(5) $R(x,y)=1$, for some $0 \leq y \leq b(z)$.

In view of (5) and (1) we have: $\mu y_{b(z)} [R(x,y)=1] = \mu y [R(x,y)=1] = \mu y_{b(F(x))} [R(x,y)=1]$. Hence, by (2) and (4) we obtain: $F(x) = p(\mu y_{b(F(x))} [R(x,y)=1]) = p(\mu y_{b(z)} [R(x,y)=1]) = z$.

To finish we notice that formula (3) guarantees that $\chi_{Graph(F)}$ is in $\mathcal{E}^n_*$.

"b) $\Longrightarrow$ a)". Take $p(z)=b(z)$ [=z], and $R(x,y)= \chi_{Graph(F)}(x,y)$, for all naturals z, x and y. □

Remarks. i) Condition a) in Theorem 1 merely says that in case we write F in Kleene's Normal Form (see [18],[5]), the minimization operator is bounded by a function in $\mathcal{E}^n(F)$. In other words, in a suitable Blum space ([5]), the complexity of F is bounded by a function in $\mathcal{E}^n(F)$; see also [4].

ii) Clearly, if one of the equivalent conditions a) or b) holds, then F is recursive.

iii) Theorem 1 can be easily extended to functions $F:\mathbb{N}^k \longrightarrow \mathbb{N}$ or to appropriate complexity classes.

Theorem 2. Let $F:\mathbb{N} \longrightarrow \mathbb{N}$ be an arbitrary bounded function, and n a natural number. Then, $F \in \mathcal{E}^n$ iff $\chi_{Graph(F)} \in \mathcal{E}^n_*$.

Proof. If range $(F)=\{a_0,a_1,\ldots,a_m\}$, then we consider the partition $(A_i)_{0 \leq i \leq m}$, $A_i=\{(x,a_i) \mid (x,a_i) \in Graph(F)\}$ of Graph(F).

If $F \in \mathcal{E}^n$, then $\chi_{Graph(F)}(x,z)=eq(F(x),z)$ is in $\mathcal{E}^n_*$. Converseky, if $\chi_{Graph(F)} \in \mathcal{E}^n_*$, then $F(x)=\mu_a y[\bigvee_{i=0}^{m} \chi_{A_i}(x,y)=1]$, where $a=\max(a_0,a_1,\ldots,a_m)$. Clearly, each predicate $\chi_{A_i}(x,y)=$ $=eq(y,a_i)\cdot\chi_{Graph(F)}(x,y)$ is in $\mathcal{E}^n_*$, and $F \in \mathcal{E}^n$. □

Corollary 3. Every bounded function $F:\mathbb{N} \longrightarrow \mathbb{N}$ having a primitive recursive graph is itself primitive recursive.

Proof. Directly from Theorem 2. □

Corollary 4. Let $P:\mathbb{N} \longrightarrow \{0,1\}$ be an arbitrary predicate. Then:

a) For each $n \geq 1$, $P \in \mathcal{E}^n_* \setminus \mathcal{E}^{n-1}_*$ iff $\chi_{Graph(P)} \in \mathcal{E}^n_* \setminus \mathcal{E}^{n-1}_*$.

b) The predicate P is recursive and non-primitive recursive iff $\chi_{Graph(P)}$ is recursive and non-primitive recursive.

Proof. a) By Theorem 2; b) is a consequence of a). □

Remarks. i) Following [8],[10], for every natural $n \geq 3$, $\mathcal{E}^n_* \setminus \mathcal{E}^{n-1}_* \neq \emptyset$. It is an open problem whether the strict inclusion lies between the three smallest Grzegorczyk classes (see [10] ,[7]).

ii) Examples of recursive and non-primitive recursive predicates can be obtained using Rabin's Theorem ([14],[5]). See also Remark iv) following Corollary 9.

iii) The predicates occurring in Corollary 4 do not satisfy the hypothesis of Theorem 1.

3. APPLICATIONS

We use Theorem 1 to prove that the graphs of some well-known rapidly growing functions of Ackermann's type are elementary.

The Ackermann-Péter function $A:\mathbb{N}^2 \longrightarrow \mathbb{N}$ is defined by the equations ([1],[13],[5]):

(6) $A(0,x) = x+1$,

(7) $A(n+1,0) = A(n,1)$,

(8) $A(n+1,x+1) = A(n,A(n+1,x))$.

We recall the following well-known fact, which can be easily proved by induction on x and n.

Lemma 5. For all naturals n and x, the expression $A(n,x)$ can be evaluated to a unique natural number by means of a finite number of rightmost applications of equations (6)-(8). □

Theorem 6. There exists an elementary predicate $R:\mathbb{N}^3 \longrightarrow \{0,1\}$ such that for all naturals n and x, we have:

(9) $A(n,x) = \exp(0, \mu y[R(n,x,y) = 1])$.

Proof. Following [8], we shall encode and decode the computation of $A(n,x)$ by means of the number

$$pn(0)^{A(n,x)} \cdot \prod pn(J(i,j))^{A(i,j)},$$

where the product runs over all pairs $(i,j) \neq (0,0)$, such that $A(i,j) \leq A(n,x)$. To this aim we shall consider the elementary functions $f:\mathbb{N}^3 \longrightarrow \mathbb{N}$, $f(n,x,y)=\exp(J(n,x),y)$, and $P:\mathbb{N}^3 \longrightarrow \{0,1\}$,

(10) $P(n,x,y) = \overline{sg}(n)\cdot eq(f(n,x,y),x+1)+sg(n)\cdot\overline{sg}(x)\cdot sg(f(n\dot{-}1, x+1,y))\cdot eq(f(n,x,y),f(n\dot{-}1,x+1,y))+sg(n)\cdot sg(x)\cdot sg(f(n,x\dot{-}1,y))\cdot sg(f(n\dot{-}1,f(n,x\dot{-}1,y),y))\cdot eq(f(n,x,y),f(n\dot{-}1,f(n,x\dot{-}1,y),y))$.

Finally, put

$$R(n,x,y) = sg(f(n,x,y))\cdot eq(f(0,0,y),f(n,x,y))\cdot \prod_{t=1}^{long(y)} sg(\overline{sg}(f(K(t),L(t),y)) + P(K(t),L(t),y)).$$

Intermediate step. Assume that $R(n,x,y) = 1$, for some naturals n,x, and y. Then, for every natural $1 \leq t \leq long(y)$, if $f(K(t),L(t),y) \neq 0$, then $A(K(t),L(t))=f(K(t),L(t),y)$.

Suppose that $R(n,x,y)=1$, and $f(K(t),L(t),y) \neq 0$, for some

$1 \leq t \leq long(y)$. In view of the construction of R, $sg(\overline{sg}(f(K(t),L(t),y)) + P(K(T),L(T),y)) = 1$. It follows that $P(K(t),L(t),y) = 1$, because $f(K(t),L(t),y) \neq 0$. We shall prove, in a double inductive way the relation $A(K(t),L(t)) = f(K(t),L(t),y)$. To this aim we shall analyse three cases (corresponding to equations (6)-(8)):

i) If $K(t)=0$, then $1=P(K(t),L(t),y)=eq(f(K(t),L(t),y), L(t)+1)$, i.e. $f(K(t),L(t),y)=L(t)+1=A(K(t),L(t))$, in view of (6).

ii) If $K(t)\neq 0$ and $L(t)=0$, then (according to (10)) one has: $f(K(t)\dot{-}1,L(t)+1,y) > 0$, and $f(K(t),L(t),y)=f(K(t)\dot{-}1,L(t)+1,y)$. Using the induction hypothesis and (7) we have: $A(K(t),L(t)) = A(K(t)\dot{-}1, L(t)+1)=f(K(t)\dot{-}1,L(t)+1,y)=f(K(t),L(t),y)$.

iii) If $K(t)\cdot L(t) \neq 0$, then (by (10)) we have: $f(K(t),L(t)\dot{-}1,y) > 0$, $f(K(t)\dot{-}1,f(K(t),L(t)\dot{-}1,y),y) > 0$, and $f(K(t),L(t),y)= f(K(t)\dot{-}1, f(K(t),L(t)\dot{-}1,y),y)$. Using the induction hypothesis and (8) we get: $A(K(t),L(t))=A(K(t)\dot{-}1,A(K(t),L(t)\dot{-}1))=A(K(t)\dot{-}1,f(K(t),L(t)\dot{-}1,y), y))=f(K(t)\dot{-}1,f(K(t),L(t)\dot{-}1,y),y)=f(K(t),L(t),y)$, thus finishing the proof of the Intermediate step.

Finally, we notice that formula (9) follows from the Intermediate step and the fact that for all naturals n and x, the number

$$(11) \quad \overline{y}_{n,x} = pn(0)^{A(n,x)} \cdot \prod pn(J(i,j))^{A(i,j)} ,$$

where the product runs over all pairs $(i,j) \neq (0,0)$ for which $A(i,j)$ is used in the rightmost computation of $A(n,x)$, is the least natural satisfying $R(n,x,\overline{y}_{n,x})=1$. (Notice that the above product is finite in view of Lemma 5, and that for every pair $(n,x) \neq (0,0)$, $A(0,0)$ is not used in the rightmost computation of $A(n,x)$.) □

Remark. Formula (9) enssures the recursiveness of A. See also Remark ii) following Corollary 9 below.

Theorem 7. There exists an elementary function $b:\mathbb{N}\longrightarrow\mathbb{N}$ such that the function A can be represented as

$$(12)\quad A(n,x) = \exp(0, \mu y_{b(A(n,x))} [R(n,x,y) = 1]).$$

Proof. We shall establish an elementary bound $b:\mathbb{N}\longrightarrow\mathbb{N}$ such that for all naturals n and x one has:

$$(13)\quad \mu y[R(n,x,y) = 1] \leq b(A(n,x)).$$

To this aim we construct for each pair (n,x) the natural number

$$y_{n,x} = pn(0)^{A(n,x)} \cdot \prod_{(i,j)\in I_{n,x}} pn(J(i,j))^{A(i,j)},$$

where $I_{n,x}=\{(i,j)\in\mathbb{N}^2\setminus\{0,0\} \mid A(i,j)\leq A(n,x)\}$. In view of the monotonicity properties of the Ackermann-Péter function ($A(n,x)>\max(n,x)$, and A is increasing in both arguments), one has $y_{n,x}\geq\bar{y}_{n,x}$ ($\bar{y}_{n,x}$ comes from (11)). Furthermore, the inequality

$$\prod_{(i,j)\in I_{n,x}} pn(J(i,j)) \leq pn(J(A(n,x),A(n,x)))^{(A(n,x)+1)^2},$$

guarantees that the elementary function

$$b(z) = (2\cdot pn(J(z,z))^{(z+1)^2})^z,$$

satisfies (13). Accordingly, formula (12) is a consequence of (9) and (13). □

Corollary 8. Ackermann-Péter's recursive (and non-primitive recursive) function has an elementary graph.

Proof. It is seen that A satisfies conditions (1) and (2) in Theorem 1 (see (13) and (12)). □

Corollary 9. Ackermann-Péter's function has an elementary range.

<u>Proof.</u> We have: $\chi_{range(A)}(y) = \sum_{n=0}^{y} \sum_{x=0}^{y} \chi_{Graph(A)}(n,x,y)$. □

<u>Remarks.</u> i) Using Gödel's β function one can construct two functions $p^{*}: \mathbb{N} \longrightarrow \mathbb{N}$, $R^{*}: \mathbb{N}^3 \longrightarrow \{0,1\}$, in $\mathcal{E}^2$, such that $A(n,x) = p^{*}(\mu y[R^{*}(n,x,y) = 1])$. We are unable, however, to establish a bound b^{*} in $\mathcal{E}^2$, which would guarantee that A has the graph in $\mathcal{E}^2_{*}$.

Furthermore, a minor modification in the structure of the predicate R gives a singlefold exponentially diophantine representation of the graph of A (see [11]).

ii) Other well-known non-primitive recursive functions, e.g. Ackermann's original function [1], Sudan's function [16],[5], Grzegorczyk's function [8], Löb-Wainer's function [12] also satisfy condition a) in Theorem 1; they have elementary graphs and ranges as well.

iii) Corollaries 4 and 8 offer an example of tradeoff between the property of a function to be "honest" in the sense of Blum's theory of complexity [5], i.e. the rate of growth of the function "reflects" its complexity, and the property of a function to have "honest graph", i.e. the graph of the function has the same complexity as the function. A predicate in $\mathcal{E}^n_{*} \setminus \mathcal{E}^{n-1}_{*}$ is dishonest, but, in view of Corollary 4 has a honest graph. Conversely, Ackermann-Péter's function is honest, but has a dishonest graph (Corollary 8).

The difference between the complexity of a function and the complexity of its graph is discussed also in Warkentin's thesis [19], as credit in [10].

Further examples of functions having a dishonest graph are: $x+y$ is in $\mathcal{E}^1 \setminus \mathcal{E}^0$, but the predicate $z=x+y$ is in $\mathcal{E}^0_{*}$, xy is in $\mathcal{E}^2 \setminus \mathcal{E}^1$, but the predicate $z=xy$ is in $\mathcal{E}^0_{*}$ [10].

iv) In view of Corollary 4,b), it is more difficult to obtain an example of recursive and non-primitive recursive predicate than a corresponding example of function (see Corollaries 8 and 9). Furthermore, if we denote by g(n,x,y) the Grzegorczyk function [8], we claim that: α) the predicate $z \leq g(n,x,y)$ is in $\mathcal{E}^n_* \setminus \mathcal{E}^{n-1}_*$, for every fixed $n \geq 4$, β) the predicate $z \leq g(n,x,y)$ is not primitive recursive. Indeed, we use the fact that the graph of g is elementary and Grzegorczyk's Theorem of characterization of the predicates in $\mathcal{E}^n$ (Theorem 4.10 in [8]). Accordingly, there is a devasting difference between the predicates $z = g(n,x,y)$ and $z \leq g(n,x,y)$; the last predicate does not grow too rapidly, but it requires too many values to search.

The above results can also be deduced from the fact (credit in [10] , to [6]), that the predicate $z = g(n,x,y)$ is rudimentary for each fixed $n \geq 0$. In this respect it would be interesting to prove that the predicates $y \leq A(n,x)$ (or $y < A(n,x)$, $y > A(n,x)$) are not primitive recursive.

v) Particularly interesting is the result (credit in [10] to [3]) that the predicate $z = x^y$ is rudimentary, hence, in $\mathcal{E}^o_*$ [9] (x^y is in $\mathcal{E}^3 \setminus \mathcal{E}^2$). In view of a result in [2], it follows that the exponential function x^y has a graph polynomially computable in a variant of loop-programs. "Guessing" the correct value z, implies the possibility of polynomially computation of x^y, because the computation reduces to the verification of the relation $z = x^y$. A (weak) sort of the famous problem P = NP arises and motivates the plausibility of the conjecture $P \neq NP$. In fact, analoguous problems for each class $\mathcal{E}^n$ $(n \geq 0)$ can be similarly tackled.

vi) The context-sensitiveness (and, in particular, the primitive recursiveness) of the graph of Ackermann-Péter's function was proved in [15] and [17].

4. REFERENCES

1. W.ACKERMANN. Zum Hilbertschen Aufbau der reellen Zahlen, Math. Ann. 99 (1928), 118-133.

2. A.AMIHUD and Y.CHOUEKA. Loop-programs and polynomially computable functions, Internat. J. Computer Math. 9 (1981), 195-205.

3. J.H.BENNETT. On spectra, Ph.D. dissertation, Princeton, 1962 (quoted following [10]).

4. E.BÖRGER. Note on bounded diophantine representation of subrecursive sets, in I.KALANTARI (ed.), The 1982 Summer Institute on Recursion Theory, Abstracts, Cornell University, 1982, 18-22.

5. C.CALUDE. Theories of Computational Complexity, North-Holland, Amsterdam, New York, Oxford (in print).

6. S.FINKELSTEIN. On rudimentary languages, Unpublished manuscript, 1977 (quoted following [10]).

7. R.O.GANDY. Some relations between classes for low computational complexity, Bull. London Math. Soc. 16 (1984), 127-134.

8. A.GRZEGORCZYK. Some classes of recursive functions, Rozprawy Mathematyczne 4 (1953), 1-45.

9. K.HARROW. Small Grzegorczyk classes and limited minimum, Z.Math. Logik Grundlag. Math. 21 (1975), 417-426.

10. K.HARROW. Equivalence of some hierarchies of primitive recursive functions, Z.Math. Logik Grundlag. Math. 25 (1979), 411-418.

11. J.P.JONES and Yu.V.MATIJASEVIC. A new representation for symmetric binomial coefficient and its applications, Ann. Sci, Math. Québec 6 (1982), 81-97.

12. M.H.LÖB and S.S.WAINER. Hierarchies of number-theoretic functions I and II, Arch. Math. Logik Grundlag. 13 (1970), 39-51, 97-113.

13. R.PÉTER. Rekursive Funktionen, Akadémiai Kiadó, Budapest, 1957.

14. M.RABIN. Degree of difficulty of computing a function and a partial ordering of recursive sets, Hebrew University, Jerusalem, Technical Report Nr.2, 1960.

15. D.SIMOVICI. Context-sensitive languages and Ackermann's function, Found. Control Engrg. 5 (1980), 91-103.

16. G.SUDAN. Sur le nombre transfini ω^{ω}, Bull. Math. Soc.Roumaine Sci. 30 (1927), 11-30.

17. M.TĂTĂRÂM. Ackermann-Péter's function has primitive recursive graph and range, Found. Control. Engrg. (1984) (in print).

18. A.YASUHARA. Recursive Function Theory and Logic, Academic Press, New York, London, 1971.

19. J.WARKENTIN. Small classes of recursive functions and relations, Ph.D. dissertation, Waterloo, 1971 (quoted following [10]).

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