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by

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INTRODUCTION

In certain problems of analytic geometry, especially in duality and in deformation theory, the need naturally arises to consider invariants $\mathrm{Ext}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$, $\mathrm{Ext}_c^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$, $\mathrm{Tor}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$ or $\mathrm{Tor}_c^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$ whose arguments are no longer sheaves but complexes of sheaves with coherent cohomology. Thus, in the absolute duality theorems of Ramis and Ruget [13], there appear invariants of the form $\mathrm{Ext}^q(X;\mathcal{F},K_X^\cdot)$ and $\mathrm{Ext}_c^q(X;\mathcal{F},K_X^\cdot)$ where $\mathcal{F}$ is a coherent sheaf on the complex space X and $K_X^\cdot$ is the dualizing complex of X which is a complex in $D^b_{coh}(X)$. Other important examples are the tangential cohomology groups introduced by Palamodov [12] . In order to solve problems of deformation theory, Palamodov has defined for every morphism of complex spaces $f:X\longrightarrow Y$ a complex $L^\cdot_{X/Y}\in D^-_{coh}(X)$ called the cotangent complex and has associated to every coherent $\mathcal{O}_X$-module $\mathcal{F}$ its tangential cohomology groups with respect to f:

$$T^i(X/Y,\mathcal{F})=\mathrm{Ext}^i(X;L^\cdot_{X/Y},\mathcal{F})$$

It should be remarked that, unlike $K_X^\cdot$, the cotangent complex is not, in general, a bounded complex.

The aim of this paper is to define natural topologies on cohomological invariants associated to complexes with coherent cohomology and to prove duality theorems for them. The essential technical tool that is used is the notion of semi-simplicial system of sheaves (s.s.s), introduced by Verdier in [14] and Forster and Knorr in [7].

If X is an analytic space and $\mathcal{F} \in Coh(X)$ it is well known that there exist natural topologies of type QFS (quotient of a Fréchet--Schwarz space) on $H^q(X, \mathcal{F})$ and of type QDFS (quotient of a strong dual of a FS space) on $H^q_c(X, \mathcal{F})$, deduced from the usual topology on the spaces of sections of $\mathcal{F}$ by means of Čech computations. Using resolutions with s.s.s. one can introduce natural topologies of the same type on the spaces $H^q(X, \mathcal{F}^\cdot)$ and $H^q_c(X, \mathcal{F}^\cdot)$ for every $\mathcal{F}^\cdot \in D_{coh}(X)$ (see proposition 4.1). The problem was first solved by Verdier in [14] for bounded complexes. The general case treated here is a direct extension of Verdier's method.

If the complexes $\mathcal{E}^\cdot$ and $\mathcal{F}^\cdot$ are in one of the situations: a) $\mathcal{E}^\cdot \in D^-_{coh}(X)$, $\mathcal{F}^\cdot \in D^+_{coh}(X)$ or b) $\mathcal{E}^\cdot \in D_{coh}(X)$, $\mathcal{F}^\cdot \in D^b_{coh}(X)$ has finite injective dimension (f.i.d.) or c) $\mathcal{E}^\cdot \in D^b_{coh}(X)$ has finite tor dimension (f.t.d.), $\mathcal{F}^\cdot \in D_{coh}(X)$, then the complex $R\mathcal{H}om^\cdot(\mathcal{E}^\cdot, \mathcal{F}^\cdot)$ has coherent cohomology (see propositions 1.8 and 3.7) and one can introduce a natural topology on the hyperext spaces $Ext^q(X; \mathcal{E}^\cdot, \mathcal{F}^\cdot)$ and $Ext^q_c(X; \mathcal{E}^\cdot, \mathcal{F}^\cdot)$ (see proposition 4.4). The method consists in computing these invariants in a category of s.s.s. by means of resolutions with free s.s.s. (in the sense of [7] or [6]).

The main duality result is:

<u>Theorem 5.1.</u> Let $\mathcal{E}^\cdot, \mathcal{F}^\cdot$ be complexes of sheaves such that a) $\mathcal{E}^\cdot \in D^-_{coh}(X)$, $\mathcal{F}^\cdot \in D^+_{coh}(X)$, or b) $\mathcal{E}^\cdot \in D_{coh}(X)$, $\mathcal{F}^\cdot \in D^b_{coh}(X)$ has f.i.d., or c) $\mathcal{E}^\cdot \in D^b_{coh}(X)$ has f.t.d., $\mathcal{F}^\cdot \in D_{coh}(X)$. Then there exists a natural pairing which induces a duality between the separated vector spaces associated to the spaces:

$\operatorname{Ext}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$ and $\operatorname{Tor}_c^q(X;D(\mathcal{F}^\cdot),\mathcal{E}^\cdot)$, respectively

$\operatorname{Ext}_c^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$ and $\operatorname{Tor}^q(X;D(\mathcal{F}^\cdot),\mathcal{E}^\cdot)$

considered with their natural topologies. Moreover, $\operatorname{Ext}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$, respectively $\operatorname{Ext}_c^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$, is separated iff $\operatorname{Tor}_c^{q-1}(X;D(\mathcal{F}^\cdot),\mathcal{E}^\cdot)$, respectively $\operatorname{Tor}^{q-1}(X;D(\mathcal{F}^\cdot),\mathcal{E}^\cdot)$, is.

In particular if one takes $\mathcal{E}^\cdot = \mathcal{O}_X$ in case c) one obtains the classical result of Ramis and Ruget [13], formulated for a complex of sheaves $\mathcal{F}^\cdot \in D_{coh}(X)$:

<u>Theorem 5.2</u>. For every complex of sheaves $\mathcal{F}^\cdot \in D_{coh}(X)$ there exists a natural pairing which induces a duality between the separated vector spaces associated to the spaces:

$H^q(X,\mathcal{F}^\cdot)$ and $\operatorname{Ext}_c^{-q}(X,\mathcal{F}^\cdot,K_X^\cdot)$, respectively

$H_c^q(X,\mathcal{F}^\cdot)$ and $\operatorname{Ext}^{-q}(X,\mathcal{F}^\cdot,K_X^\cdot)$,

considered with their natural topologies. Moreover, $H^q(X,\mathcal{F}^\cdot)$, respectively $H_c^q(X,\mathcal{F}^\cdot)$, is separated iff $\operatorname{Ext}_c^{1-q}(X,\mathcal{F}^\cdot,K_X^\cdot)$, respectively $\operatorname{Ext}_c^{1-q}(X,\mathcal{F}^\cdot,K_X^\cdot)$, is.

Since in the original paper of Ramis and Ruget the result is proved for coherent sheaves and in [5] it is mentioned only for a complex $\mathcal{F}^\cdot \in D_{coh}^b(X)$, and since the proof of theorem 1 relies on this result, for the sake of completeness we give a proof of theorem 2.

Imposing stronger conditions on $\mathcal{E}^\cdot$ and $\mathcal{F}^\cdot$ one obtains dualities in which only Ext invariants are involved:

<u>Theorem 5.3</u>. Let $\mathcal{E}^\cdot, \mathcal{F}^\cdot$ be complexes of sheaves such that d) $\mathcal{E}^\cdot \in D_{coh}^b(X)$ has f.t.d., $\mathcal{F}^\cdot \in D_{coh}^-(X)$ or e) $\mathcal{E}^\cdot \in D_{coh}^-(X)$, $\mathcal{F}^\cdot \in D_{coh}^b(X)$ has f.t.d. Then there exists a natural pairing which induces a duality between the separated vector spaces associated to the spaces:

$\mathrm{Ext}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$ and $\mathrm{Ext}_c^{-q}(X;\mathcal{F}^\cdot,\mathcal{E}^\cdot \underline{\underline{\otimes}}\ K_X^\cdot)$, respectively

$\mathrm{Ext}_c^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$ and $\mathrm{Ext}^{-q}(X;\mathcal{F}^\cdot,\mathcal{E}^\cdot \underline{\underline{\otimes}}\ K_X^\cdot)$,

considered with their natural topologies. Moreover, $\mathrm{Ext}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$, respectively $\mathrm{Ext}_c^q(X,\mathcal{E}^\cdot,\mathcal{F}^\cdot)$, is separated iff $\mathrm{Ext}_c^{1-q}(X;\mathcal{F}^\cdot,\mathcal{E}^\cdot \underline{\underline{\otimes}}\ K_X^\cdot)$, respectively $\mathrm{Ext}^{1-q}(X;\mathcal{F}^\cdot,\mathcal{E}^\cdot \underline{\underline{\otimes}}\ K_X^\cdot)$, is.

In particular, since every complex $\mathcal{F}^\cdot \in D^b_{coh}(X)$ has f.t.d. if X is a manifold (corollary 3.6) we get:

Corollary 5.4. If X is a complex manifold of dimension n and d) $\mathcal{E}^\cdot \in D^b_{coh}(X)$, $\mathcal{F}^\cdot \in D^-_{coh}(X)$ or e) $\mathcal{E}^\cdot \in D^-_{coh}(X)$, $\mathcal{F}^\cdot \in D^b_{coh}(X)$ then there exists a natural pairing which induces a duality between the separated vector spaces associated to the spaces:

$\mathrm{Ext}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$ and $\mathrm{Ext}_c^{n-q}(X;\mathcal{F}^\cdot,\mathcal{E}^\cdot \otimes \omega_X)$, respectively

$\mathrm{Ext}_c^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$ and $\mathrm{Ext}^{n-q}(X;\mathcal{F}^\cdot,\mathcal{E}^\cdot \otimes \omega_X)$,

considered with their natural topologies. Moreover, $\mathrm{Ext}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$, respectively $\mathrm{Ext}_c^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$, is separated iff $\mathrm{Ext}_c^{n-q+1}(X;\mathcal{F}^\cdot,\mathcal{E}^\cdot \otimes \omega_X)$, respectively $\mathrm{Ext}^{n-q+1}(X;\mathcal{F}^\cdot,\mathcal{E}^\cdot \otimes \omega_X)$, is.

A particular case of the corollary (X smooth projective variety and $\mathcal{E},\mathcal{F} \in \mathrm{Coh}(X)$ was considered by Drezet and Le Potier in [4]).

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1. COMPLEXES OF SHEAVES

In this paragraph we remind briefly some of the properties of the hypercohomology, hyperext and hypertor for complexes of sheaves on a complex space. As a general reference we follow Hartshorne [11], chapters I and II.

1.1. Unless otherwise stated, $(X, \mathcal{O}_X)$ will be throughout the paper a finite dimensional analytic space with countable topology. One denotes by Mod(X) the abelian category of $\mathcal{O}_X$-modules, by K(X) the triangulated category of complexes of objects in Mod(X) and by D(X) the derived category of Mod(X). The superscripts +,-,b will mean complexes bounded to the left, respectively to the right, respectively bounded in both directions. For every $\mathcal{F}^\cdot \in K(X)$ and $q \in Z$ we define the truncations of $\mathcal{F}^\cdot$:

$$^q\mathcal{F}^\cdot : \ldots \longrightarrow \mathcal{F}^{q-1} \longrightarrow \mathcal{F}^q \longrightarrow \operatorname{Im} d^q \longrightarrow 0 \longrightarrow \ldots$$

$$_q\mathcal{F}^\cdot : \ldots \longrightarrow 0 \longrightarrow \operatorname{Im} d^{q-1} \longrightarrow \mathcal{F}^q \longrightarrow \mathcal{F}^{q+1} \longrightarrow \ldots$$

Obviously, there exist natural morphisms of complexes $^q\mathcal{F}^\cdot \longrightarrow \mathcal{F}^\cdot$ and $\mathcal{F}^\cdot \longrightarrow {}_q\mathcal{F}^\cdot$.

As it is well known, Mod(X) has enough injective objects and consequently there exist right derived functors for the extensions of $\Gamma(X,\cdot)$ and $\Gamma_c(X,\cdot)$ to $K^+(X)$ (corollary I.5.3.α in [11]). In other words, every $\mathcal{F}^\cdot \in K^+(X)$ admits an injective resolution $\mathcal{F}^\cdot \longrightarrow \mathcal{J}^\cdot$ (i.e. there exist a complex of injective sheaves $\mathcal{J}^\cdot \in K^+(X)$ and a quasiisomorphism $\mathcal{F}^\cdot \longrightarrow \mathcal{J}^\cdot$) and $\Gamma(X,\mathcal{J}^\cdot)$, respectively $\Gamma_c(X,\mathcal{J}^\cdot)$ are representatives for $R\Gamma(X,\mathcal{F}^\cdot)$, respectively $R\Gamma_c(X,\mathcal{F}^\cdot)$. As usual, one denotes by $H^q(X,\mathcal{F}^\cdot)$, $H^q_c(X,\mathcal{F}^\cdot)$ the cohomology of $\Gamma(X,\mathcal{J}^\cdot)$, respectively $\Gamma_c(X,\mathcal{J}^\cdot)$.

Since X has finite topological dimension, $\Gamma(X,\cdot)$, $\Gamma_c(X,\cdot)$ have finite cohomological dimension and according to corollary I.5.3.γ in [11] there exist right derived functors for the extensions of $\Gamma(X,\cdot)$ and $\Gamma_c(X,\cdot)$ to K(X). This is because every complex $\mathcal{F}^\cdot \in K(X)$ admits a resolution $\mathcal{F}^\cdot \longrightarrow \mathcal{J}^\cdot$ whose components are $\Gamma(X,\cdot)$- and $\Gamma_c(X,\cdot)$-acyclic (see Lemma I.4.6 in [11]). As above $\Gamma(X,\mathcal{J}^\cdot)$

and $\Gamma_c(X,\mathcal{J}^\cdot)$ are representatives for $R\Gamma(X,\mathcal{F}^\cdot)$ and $R\Gamma_c(X,\mathcal{F}^\cdot)$ and their cohomology is denoted by $H^q(X,\mathcal{F}^\cdot)$, respectively $H^q_c(X,\mathcal{F}^\cdot)$.

We say that a sheaf is acyclic if it is $\Gamma(U,\cdot)$-acyclic for every open set U in X. It is easy to see that every $\mathcal{F}^\cdot \in D(X)$ admits a resolution with acyclic sheaves (see lemma I.4.6 in [11]).

Remark 1. Let $\mathcal{F}^\cdot \in D(X)$ be a complex of acyclic sheaves, exact in degrees $\geq p$. Then, if dim X=n, it is easy to show, by splitting $\mathcal{F}^\cdot$ into short exact sequences, that in degrees $\geq p+n$ the sheaves of boundaries of $\mathcal{F}^\cdot$ are acyclic sheaves. Consequently, every $\mathcal{F}^\cdot \in D^-(X)$ admits a resolution $\mathcal{F}^\cdot \longrightarrow \mathcal{J}^\cdot$ with acyclic sheaves such that $\mathcal{J}^\cdot \in D^-(X)$.

Remark 2. If $\mathcal{F}^\cdot \in D_{coh}(X)$ (i.e. $\mathcal{F}^\cdot$ has coherent cohomology) and has components acyclic on Stein open sets, one can show, using an exact sequence argument, that the sheaves of cycles and the sheaves of boundaries of $\mathcal{F}^\cdot$ are also acyclic on Stein open sets (here again the condition dim $X<\infty$ is essential).

1.2. For every $\mathcal{F}^\cdot \in D(X)$ one has a regular spectral sequence $E_2^{pq}=H^p(X,\mathcal{H}^q(\mathcal{F}^\cdot))$ converging to $H^{p+q}(X,\mathcal{F}^\cdot)$ ($\mathcal{H}^q(\mathcal{F}^\cdot)$ is the q-th cohomology sheaf of $\mathcal{F}^\cdot$).

If X is a Stein space and $\mathcal{F}^\cdot \in D_{coh}(X)$ then the spectral sequence is degenerate and the edge morphisms:

$$H^q(X,\mathcal{F}^\cdot) \longrightarrow \Gamma(X,\mathcal{H}^q(\mathcal{F}^\cdot))$$

are isomorphisms.

1.3. The functors Hom:Mod(X) x Mod(X) $\longrightarrow$ Ab and $\mathcal{H}om$:Mod(X) x Mod(X) $\longrightarrow$ Mod(X) can be extended to bi-∂-functors $\text{Hom}^\cdot$:K(X) x K(X) $\longrightarrow$ K(Ab), respectively $\mathcal{H}om^\cdot$:K(X) x K(X) $\longrightarrow$ K(X), which associate to every $\mathcal{E}^\cdot, \mathcal{F}^\cdot \in K(X)$ the complex of components

$\mathrm{Hom}^n(\mathcal{E}^\bullet,\mathcal{F}^\bullet)=\prod_{p\in Z}\mathrm{Hom}(\mathcal{E}^p,\mathcal{F}^{n+p})$ and differentials $d^n=\prod_{p\in Z}d_{\mathcal{E}^\bullet}^{p-1}+(-1)^{n+1}d_{\mathcal{F}^\bullet}^p$,

respectively $\mathcal{H}om^n(\mathcal{E}^\bullet,\mathcal{F}^\bullet)=\prod_{p\in Z}\mathcal{H}om(\mathcal{E}^p,\mathcal{F}^{n+p})$ and differentials of the same form.

Lemma 3. Let $\mathcal{J}^\bullet\in K^+(X)$ be a complex of injective sheaves and $\mathcal{E}^\bullet\in K(X)$. Assume either a) $\mathcal{E}^\bullet$ is exact or b) $\mathcal{J}^\bullet$ is exact. Then $\mathrm{Hom}^\bullet(\mathcal{E}^\bullet,\mathcal{J}^\bullet)$ and $\mathcal{H}om^\bullet(\mathcal{E}^\bullet,\mathcal{J}^\bullet)$ are exact.

The proof can be found for example in [11], lemmas I.6.2 and II.3.1.

Since Mod(X) has enough injective objects, it follows from lemma 3 that one can define right derived functors:
$R\mathrm{Hom}^\bullet:D(X)^o\times D^+(X)\longrightarrow D(Ab)$, respectively $R\mathcal{H}om^\bullet:D(X)^o\times D^+(X)\longrightarrow D(X)$ for $\mathrm{Hom}^\bullet$ and $\mathcal{H}om^\bullet$, obtained by "deriving" first in the second variable and then in the first one. If $\mathcal{E}^\bullet\in D(X)$, $\mathcal{F}^\bullet\in D^+(X)$ and $\mathcal{F}^\bullet\longrightarrow\mathcal{J}^\bullet$ is an injective resolution for $\mathcal{F}^\bullet$, then $\mathrm{Hom}^\bullet(\mathcal{E}^\bullet,\mathcal{J}^\bullet)$ and $\mathcal{H}om^\bullet(\mathcal{E}^\bullet,\mathcal{J}^\bullet)$ are representatives for $R\mathrm{Hom}^\bullet(\mathcal{E}^\bullet,\mathcal{F}^\bullet)$, respectively $R\mathcal{H}om^\bullet(\mathcal{E}^\bullet,\mathcal{F}^\bullet)$. As usual, we denote by $\mathrm{Ext}^q(X;\mathcal{E}^\bullet,\mathcal{F}^\bullet)$ the cohomology groups of $R\mathrm{Hom}^\bullet(\mathcal{E}^\bullet,\mathcal{F}^\bullet)$ and by $\mathcal{E}xt^q(\mathcal{E}^\bullet,\mathcal{F}^\bullet)$ the cohomology sheaves of $R\mathcal{H}om^\bullet(\mathcal{E}^\bullet,\mathcal{F}^\bullet)$.

Remark 4. The complex $\mathcal{H}om^\bullet(\mathcal{E}^\bullet,\mathcal{J}^\bullet)$ has flabby components (this follows easily from the well known fact that for every $\mathcal{E}^\bullet\in\mathrm{Mod}(X)$, $\mathcal{H}om(\mathcal{E},\mathcal{J})$ is a flabby sheaf if $\mathcal{J}$ is an injective sheaf).

1.4. If A is an abelian category, for every two complexes in the derived category of A, $X^\bullet$, $Y^\bullet\in D(A)$, the q-th hyperext of $X^\bullet$ and $Y^\bullet$ is defined to be:

$$\mathrm{Ext}^q(X^\bullet,Y^\bullet)=\mathrm{Hom}_{D(A)}(X^\bullet,T^q(Y^\bullet))$$

where T is the translation functor.

If B is another abelian category and $F:D(A)\longrightarrow B$ is a covariant additive functor then one clearly has a pairing, called the Yoneda pairing: $\mathrm{Ext}^q(X^\bullet,Y^\bullet)\times F^p(X^\bullet)\longrightarrow F^{p+q}(Y^\bullet)$ where F^p denotes $F\circ T^p$. Of course, for a contravariant functor a similar pairing holds.

As in the case of Mod(X), the functor Hom on A extends to a bi-∂-functor $\mathrm{Hom}^\bullet:K(A)\times K(A)\longrightarrow K(Ab)$ for which a result similar to lemma 3 holds. Consequently, if A has enough injective objects one can define, as at 1.3, a right derived functor $R\mathrm{Hom}^\bullet:D(A)^o\times D^+(A)\longrightarrow D(Ab)$ for the functor $\mathrm{Hom}^\bullet$.

According to a theorem due to Yoneda ([11], theorem I.6.4) there is a canonical isomorphism:

$$H^i(R\mathrm{Hom}^\bullet(X^\bullet,Y^\bullet))\xrightarrow{\sim}\mathrm{Ext}^i(X^\bullet,Y^\bullet).$$

1.5. The connection between the functors $R\mathrm{Hom}^\bullet$ and $R\mathcal{H}om^\bullet$ is given by the natural isomorphism:

$$R\mathrm{Hom}^\bullet(\mathcal{E}^\bullet,\mathcal{F}^\bullet)\xrightarrow{\sim}R\Gamma(X,R\mathcal{H}om^\bullet(\mathcal{E}^\bullet,\mathcal{F}^\bullet)) \quad (1)$$

The above isomorphism implies there exists a spectral sequence of term $E_2^{pq}=H^p(X,\mathcal{E}xt^q(\mathcal{E}^\bullet,\mathcal{F}^\bullet))$ converging to $\mathrm{Ext}^{p+q}(X;\mathcal{E}^\bullet,\mathcal{F}^\bullet)$.

By analogy with the isomorphism (1) we define $\mathrm{Ext}_c^q(X;\mathcal{E}^\bullet,\mathcal{F}^\bullet)$ to be $H^q(R\Gamma_c(X,R\mathcal{H}om^\bullet(\mathcal{E}^\bullet,\mathcal{F}^\bullet)))$. Using remark 4 it follows that $\mathrm{Ext}_c^q(X;\mathcal{E}^\bullet,\mathcal{F}^\bullet)$ represents the cohomology of the complex $\Gamma_c(X,\mathcal{H}om^\bullet(\mathcal{E}^\bullet,\mathcal{J}^\bullet))$, where $\mathcal{J}^\bullet$ is an injective resolution of $\mathcal{F}^\bullet$.

1.6. A complex $\mathcal{F}^\bullet \in D^b(X)$ is said to have finite injective dimension (f.i.d.) if it admits an injective resolution of finite length (for equivalent definitions see [11], proposition I.7.6).

The most significant example of a complex having f.i.d. is the dualizing complex of X, $K_X^\bullet$ (for the construction of $K_X^\bullet$ see [13]; a construction based on a different approach can be found in [9]). One can actually prove the following more precise result:

<u>Lemma 5</u>. If X is an analytic space with Zariski dimension n, then its dualizing complex $K_X^\bullet$ admits an injective resolution which is zero outside the interval $[-n,0]$.

<u>Proof</u>. Since any locally injective sheaf is actually injective, it follows from [11], proposition I.7.6 that the statement is local. Hence we can suppose X is a closed subspace of an open set $U \subset \mathbb{C}^n$. In [8] Golovin has proved that $\mathcal{O}_U$ admits an injective resolution of length n. Since $\mathcal{O}_U \simeq \omega_U$ and since $K_U^\bullet$ is a resolution of $T^n(\omega_U)$, where T is the translation functor, it follows that $K_U^\bullet$ admits an injective resolution $K_U^\bullet \longrightarrow \mathcal{J}^\bullet$ which is zero outside $[-n,0]$. It is now easy to verify that $\mathcal{H}om^\bullet_{\mathcal{O}_U}(\mathcal{O}_X, \mathcal{J}^\bullet)$ is an injective resolution in $K(X)$ for $K_X^\bullet = \mathcal{H}om^\bullet_{\mathcal{O}_U}(\mathcal{O}_X, K_U^\bullet)|X$ (for the properties of the dualizing complex used in the proof see [13], proposition 1 or [2], theorem 7.2.6).

1.7. In paragraph 4, in order to introduce the natural topology on the hyperext we shall use the following truncation results.

<u>Lemma 6</u>. If $\mathcal{E}^\bullet \in D^-(X)$, $\mathcal{F}^\bullet \in D^+(X)$ and $p \in \mathbb{Z}$ then the natural morphisms $\mathcal{E}xt^q(\mathcal{E}^\bullet, {}^p\mathcal{F}^\bullet) \longrightarrow \mathcal{E}xt^q(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$, $\mathrm{Ext}^q(X; \mathcal{E}^\bullet, {}^p\mathcal{F}^\bullet) \longrightarrow \mathrm{Ext}^q(X; \mathcal{E}^\bullet, \mathcal{F}^\bullet)$, $\mathrm{Ext}^q_c(X; \mathcal{E}^\bullet, {}^p\mathcal{F}^\bullet) \longrightarrow \mathrm{Ext}^q_c(X; \mathcal{E}^\bullet, \mathcal{F}^\bullet)$ are isomorphisms for $q \leq p$.

Lemma 7. Let $\mathcal{E}^\bullet \in D(X)$. If $\mathcal{F}^\bullet \in D^b(X)$ has f.i.d., then for every $q_o \in Z$ there exists $p \in Z$ such that the natural morphisms $\mathcal{E}xt^q(\mathcal{E}^\bullet,\mathcal{F}^\bullet) \longrightarrow \mathcal{E}xt^q({}^p\mathcal{E}^\bullet,\mathcal{F}^\bullet)$, $Ext^q(X;\mathcal{E}^\bullet,\mathcal{F}^\bullet) \longrightarrow Ext^q(X;{}^p\mathcal{E}^\bullet,\mathcal{F}^\bullet)$, $Ext^q_c(X;\mathcal{E}^\bullet,\mathcal{F}^\bullet) \longrightarrow Ext^q_c(X;{}^p\mathcal{E}^\bullet,\mathcal{F}^\bullet)$ are isomorphisms for $q \geqslant q_o$.

The proof of the two lemmas is straightforward. In particular, if in Lemma 7 $\mathcal{F}^\bullet$ admits an injective resolution which is zero in all positive degrees (e.g. $\mathcal{F}^\bullet = K^\bullet_X$), then one can take $p \geqslant -q_o$.

1.8. We give now sufficient conditions on $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ which ensure the coherence of the sheaves $\mathcal{E}xt^q(\mathcal{E}^\bullet,\mathcal{F}^\bullet)$.

Proposition 8. Let either a) $\mathcal{E}^\bullet \in D^-_{coh}(X)$, $\mathcal{F}^\bullet \in D^+_{coh}(X)$ or b) $\mathcal{E}^\bullet \in D_{coh}(X)$, $\mathcal{F}^\bullet \in D^b_{coh}(X)$ and has f.i.d. Then $R\mathcal{H}om^\bullet(\mathcal{E}^\bullet,\mathcal{F}^\bullet) \in D_{coh}(X)$.

Proof. A general argument can be found in [11], proposition II.3.3 but we prefer here a direct proof.

a) One can suppose that $\mathcal{E}^\bullet$ has acyclic components (see remark 1). The proposition is clearly local, so that we can work in a neighbourhood of a Stein compact set.

By using a standard argument of descendent induction, one can show that $\mathcal{E}^\bullet$ admits a resolution $\mathcal{L}^\bullet \longrightarrow \mathcal{E}^\bullet$ with free sheaves of finite rank (in the neighbourhood of the fixed compact set). Let now $\mathcal{F}^\bullet \longrightarrow \mathcal{J}^\bullet$ be an injective resolution of $\mathcal{F}^\bullet$. Since $\mathcal{H}om^\bullet(\mathcal{L}^\bullet,\mathcal{J}^\bullet)$ is a representative for $R\mathcal{H}om^\bullet(\mathcal{E}^\bullet,\mathcal{F}^\bullet)$, it is sufficient to show that it has coherent cohomology. As $\mathcal{H}om^\bullet(\mathcal{L}^\bullet,\mathcal{J}^\bullet)$ is the simple complex associated to the first quadrant double complex $\mathcal{H}om(\mathcal{L}^p,\mathcal{J}^q)_{(p,q)}$, case a) follows studying the spectral sequence associated to this double complex.

Case b) follows now immediately from case a) and lemma 7.

If X is a Stein space and $\mathcal{E}^\cdot$, $\mathcal{F}^\cdot$ are in one of the situations in proposition 8 then the spectral sequence $E_2^{pq}=H^p(X, \mathcal{E}xt^q(\mathcal{E}^\cdot,\mathcal{F}^\cdot))$ is degenerate and the edge morphisms: $\mathrm{Ext}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot) \longrightarrow \Gamma(X, \mathcal{E}xt^q(\mathcal{E}^\cdot,\mathcal{F}^\cdot))$ are isomorphisms.

Remark 9. If $\mathcal{E}^\cdot$ and $\mathcal{F}^\cdot$ are as in proposition 8 then the natural morphism:

$$R\mathcal{H}om^\cdot(\mathcal{E}^\cdot,\mathcal{F}^\cdot)_x \longrightarrow R\mathrm{Hom}^\cdot(\mathcal{E}^\cdot_x,\mathcal{F}^\cdot_x)$$

is a quasiisomorphism and consequently one has natural isomorphisms: $\mathcal{E}xt^q(\mathcal{E}^\cdot,\mathcal{F}^\cdot)_x \longrightarrow \mathrm{Ext}^q_{\mathcal{O}_{X,x}}(\mathcal{E}^\cdot_x,\mathcal{F}^\cdot_x)$ (to see this, let $\mathcal{F}^\cdot \longrightarrow \mathcal{J}$ be an injective resolution, respectively an injective resolution of finite length, for $\mathcal{F}^\cdot$; the morphism above is given by the natural morphism of double complexes: $\mathcal{H}om(\mathcal{E}^\cdot,\mathcal{J}^\cdot)_x \longrightarrow \mathrm{Hom}(\mathcal{E}^\cdot_x,\mathcal{J}^\cdot_x)$ which induces isomorphisms on the second drawer of the spectral sequences associated to them: $\mathcal{E}xt^p(\mathcal{H}^q(\mathcal{E}^\cdot),\mathcal{F}^\cdot)_x \longrightarrow \mathrm{Ext}^p_{\mathcal{O}_{X,x}}(\mathcal{H}^q(\mathcal{E}^\cdot_x),\mathcal{F}^\cdot_x)$).

Remark 10. Let $D:D(X)\longrightarrow D(X)$ be the functor $R\mathcal{H}om^\cdot(\cdot,K^\cdot_X)$. Since $K^\cdot_X$ has coherent cohomology it follows from proposition 8 and lemma 5 that D sends $D_{coh}(X)$ in $D_{coh}(X)$.

Remark 11. By a projective limit reasoning and using case a) of proposition 8 one can actually show that the result of case b) still holds under a weaker condition on $\mathcal{F}^\cdot$, namely that $\mathcal{F}^\cdot \in D^b_{coh}(X)$ has f.i.d. uniformely on fibers. Since $K^\cdot_X$ has by construction this property, it follows that it is possible to show for instance that $D(\mathcal{E}^\cdot)\in D_{coh}(X)$ if $\mathcal{E}^\cdot \in D_{coh}(X)$ without resorting to Golovin's result (see the proof of lemma 5).

1.9. The tensor product in Mod(X) extends to a bi-∂-functor $K(X) \times K(X) \longrightarrow K(X)$ which takes the complexes $\mathcal{E}^\cdot$, $\mathcal{F}^\cdot \in K(X)$ in the complex $(\mathcal{E}^\cdot \otimes \mathcal{F}^\cdot)^\cdot$ which is the simple complex associated to the double complex $(\mathcal{E}^p \otimes \mathcal{F}^q)_{(p,q)}$.

Mod(X) has enough flat sheaves (see [11], proposition II.1.2) and one can prove the following:

Lemma 12. Let $\mathcal{E}^\bullet \in D^-(X)$ be a complex of flat sheaves and $\mathcal{F}^\bullet \in D(X)$. Suppose that either a) $\mathcal{E}^\bullet$ is exact or b) $\mathcal{F}^\bullet$ is exact. Then $\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet$ is exact.

Lemma 12 and the fact that there are enough flat sheaves implies that the tensor product functor admits a left derived functor "$\underline{\underline{\otimes}}$": $D^-(X) \times D(X) \longrightarrow D(X)$ obtained by "deriving" first in the first variable. If $\mathcal{P}^\bullet \longrightarrow \mathcal{E}^\bullet$ is a flat resolution for $\mathcal{E}^\bullet$ then the complex $\mathcal{P}^\bullet \otimes \mathcal{F}^\bullet$ is a representative for $\mathcal{E}^\bullet \underline{\underline{\otimes}} \mathcal{F}^\bullet$ ([11], II § 4).

Remark 13. Of course, if the second complex $\mathcal{F}^\bullet$ is bounded to the right, one can "derive" first in $\mathcal{F}^\bullet$ and then in $\mathcal{E}^\bullet$. The derived functor thus obtained coincides on $D^-(X) \times D^-(X)$ with the one previously defined.

If $\mathcal{E}^\bullet \in D^-(X)$, $\mathcal{F}^\bullet \in D(X)$ or if $\mathcal{E}^\bullet \in D(X)$, $\mathcal{F}^\bullet \in D^-(X)$ we define the local hypertor: $\mathcal{T}or^i(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \mathcal{H}^{-i}(\mathcal{E}^\bullet \underline{\underline{\otimes}} \mathcal{F}^\bullet)$ and the global hypertor: $Tor^i(X; \mathcal{E}^\bullet, \mathcal{F}^\bullet) = H^{-i}(X, \mathcal{E}^\bullet \underline{\underline{\otimes}} \mathcal{F}^\bullet)$, respectively $Tor^i_c(X; \mathcal{E}^\bullet, \mathcal{F}^\bullet) = H_c^{-i}(X, \mathcal{E}^\bullet \underline{\underline{\otimes}} \mathcal{F}^\bullet)$.

For every $\mathcal{E}^\bullet, \mathcal{F}^\bullet \in D^-(X)$ it is easy to prove that there is a natural isomorphism $\mathcal{E}^\bullet \underline{\underline{\otimes}} \mathcal{F}^\bullet \xrightarrow{\sim} \mathcal{F}^\bullet \underline{\underline{\otimes}} \mathcal{E}^\bullet$.

Note also the isomorphism: $\mathcal{T}or^i(\mathcal{E}^\bullet, \mathcal{F}^\bullet)_x \xrightarrow{\sim} Tor^i_{\mathcal{O}_{X,x}}(\mathcal{E}^\bullet_x, \mathcal{F}^\bullet_x)$.

1.10. As for the hyperext we are interested in sufficient conditions such that $\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet \in D_{coh}(X)$.

We start with a definition. A complex $\mathcal{E}^\bullet \in D^b(X)$ is said to have finite tor dimension (f.t.d.) if it admits a flat resolution of finite length (for equivalent definitions see [11], proposition II.4.2).

Proposition 14. Let either a) $\mathcal{E}^\cdot, \mathcal{F}^\cdot \in D^-_{coh}(X)$ or b) $\mathcal{E}^\cdot \in D^b_{coh}(X)$ having f.t.d. and $\mathcal{F}^\cdot \in D_{coh}(X)$. Then $\mathcal{E}^\cdot \underline{\underline{\otimes}} \mathcal{F}^\cdot \in D_{coh}(X)$.

Proof. The statement follows easily by analyzing one of the spectral sequences of the double complex $(\mathcal{P}^i \otimes \mathcal{F}^j)_{(i,j)}$, where $\mathcal{P}^\cdot \longrightarrow \mathcal{E}^\cdot$ is a flat resolution for $\mathcal{E}^\cdot$.

2. SEMI-SIMPLICIAL SYSTEMS OF SHEAVES

The category of coherent sheaves on the complex space X does not have enough projective objects. Forster and Knorr in [7] and Verdier in [13] have introduced an abelian category larger than Mod(X), the category of semi-simplicial systems of sheaves (s.s.s.), having a subcategory with enough projective objects which contains Coh(X). In this paragraph we remind some basic facts on s.s.s. and compute the Ext invariants for complexes of sheaves in $D_{coh}(X)$ by using free resolutions (in the sense of [7] or [6]) for the first term.

2.1. Let $\mathcal{U}=(U_i)_{i\in I}$ be an open covering of X and let $\mathcal{N}$ be the nerve of $\mathcal{U}$ (one considers only alternated simplexes). A semi--simplicial system of sheaves (s.s.s.) relative to $\mathcal{U}$ consists of a family of sheaves $(\mathcal{A}_\alpha)_{\alpha\in\mathcal{N}}$, where $\mathcal{A}_\alpha \in \mathrm{Mod}(U_\alpha)$, and a family of connecting morphisms $(\rho_{\beta\alpha})_{\alpha\subset\beta}$, $\rho_{\beta\alpha} : \mathcal{A}_\alpha|U_\beta \longrightarrow \mathcal{A}_\beta$ such that for every α, $\rho_{\alpha\alpha}=\mathrm{id}$ and for every $\alpha\subset\beta\subset\gamma$, $\rho_{\gamma\beta}\circ(\rho_{\beta\alpha}|U_\gamma)=\rho_{\gamma\alpha}$.

A morphism between two such s.s.s. relative to the same covering $\mathcal{U}$, $\varphi : \mathcal{A} \longrightarrow \mathcal{A}'$, consists of a family of morphisms $(\varphi_\alpha)_{\alpha\in\mathcal{N}}$,

$\varphi_\alpha : \mathcal{A}_\alpha \longrightarrow \mathcal{A}'_\alpha$, which commute with the connecting morphisms.

If $\mathcal{W}=(W_i)_{i\in I}$ is a covering of X with closed sets, one can define in the same way as above, s.s.s. relative to $\mathcal{W}$. In this case the sheaves $\mathcal{A}_\alpha$ are defined in a neighbourhood of W_α, and the connecting morphisms, respectively the morphisms between two such s.s.s. consist actually of germs of morphisms.

The s.s.s. relative to a covering $\mathcal{U}$ form an abelian category that we shall denote by Mod$(\mathcal{U})$. The corresponding category of complexes and the derived category will be denoted by $K(\mathcal{U})$, respectively $D(\mathcal{U})$.

2.2. If $\mathcal{U}=(U_i)_{i\in I}$, $\mathcal{V}=(V_j)_{j\in J}$ are two coverings of X, $\mathcal{V}\prec\mathcal{U}$ ($\mathcal{V}$ finer than $\mathcal{U}$) then any refinement function $\tau : J\longrightarrow I$ gives rise to an obvious restriction functor Mod$(\mathcal{U})\longrightarrow$Mod$(\mathcal{V})$ and we denote by $\mathcal{A}|\mathcal{V}$ the image through this functor of $\mathcal{A}\in$ Mod$(\mathcal{U})$.

If $\mathcal{A}\in$Mod$(\mathcal{U})$ is a s.s.s. on X, we define its restriction to an open set $U\subset X$, $\mathcal{A}|U\in$Mod$(\mathcal{U}\cap U)$ as the s.s.s. of components $(\mathcal{A}_\alpha|U_\alpha\cap U)_\alpha$ and connecting morphisms $(\rho_{\beta\alpha}|U_\beta\cap U)_{\alpha\subset\beta}$.

For every s.s.s. $\mathcal{A},\mathcal{B}\in$Mod$(\mathcal{U})$ one verifies easily that the presheaf $U\longmapsto$Hom$(\mathcal{A}|U,\mathcal{B}|U)$ is actually a sheaf which shall be denoted $\mathcal{H}om(\mathcal{A},\mathcal{B})$.

To every $\mathcal{A}\in$Mod$(\mathcal{U})$ one can associate Čech complexes with components $C^p(\mathcal{U},\mathcal{A})=\prod_{|\alpha|=p+1}\mathcal{A}_\alpha(U_\alpha)$, respectively $C^p_c(\mathcal{U},\mathcal{A})=\bigoplus_{|\alpha|=p+1}\mathcal{A}_\alpha(U_\alpha)$ and with differentials defined in the usual way. Obviously, if $\mathcal{V}\prec\mathcal{U}$ then any refinement function gives rise to mappings $C^q(\mathcal{U},\mathcal{A})\longrightarrow$ $\longrightarrow C^q(\mathcal{V},\mathcal{A}|\mathcal{V})$, respectively $C^q_c(\mathcal{U},\mathcal{A})\longrightarrow C^q_c(\mathcal{V},\mathcal{A}|\mathcal{V})$.

Remark 1. If $\mathcal{U}$ is a locally finite covering of X and $\mathcal{A}\in$Coh$(\mathcal{U})$ (i.e. the components of $\mathcal{A}$ are coherent sheaves) then the spaces in the complex $C^\bullet(\mathcal{U},\mathcal{A})$ carry natural topologies of type F.S. (Fréchet--Schwarz). If $\mathcal{A},\mathcal{B}\in$Coh$(\mathcal{U})$ then Hom$(\mathcal{A},\mathcal{B})$ carries a natural topology of type F.S. (it is a closed subspace in $\prod_{\alpha\in\mathcal{N}}\Gamma(U_\alpha,\mathcal{H}om(\mathcal{A}_\alpha,\mathcal{B}_\alpha))$). Similarly, if $\mathcal{K}$ is a locally finite covering of X with compact sets and $\mathcal{A}\in$Coh$(\mathcal{K})$ then the spaces in the complex $C^\bullet_c(\mathcal{K},\mathcal{A})$ carry natural topologies of type D.F.S. (strong dual of a F.S. space).

If $\mathcal{A}, \mathcal{B} \in \mathrm{Coh}(\mathcal{K})$ then $\Gamma_c(X, \mathcal{H}om(\mathcal{A},\mathcal{B}))$ carries a natural topology of type D.F.S. (it is a closed subspace in $\bigoplus_{\alpha \in \mathcal{N}} \Gamma(K_\alpha, \mathcal{H}om(\mathcal{A}_\alpha, \mathcal{B}_\alpha)))$.

2.3. For the sake of simplicity we shall work with open coverings of X but the definitions and the results hold also for closed coverings.

We say that a s.s.s is acyclic, respectively acyclic on Stein open sets if all its components are (see 1.1 for the definition of acyclic sheaves).

According to [6] lemma I.2.2., $\mathrm{Mod}(\mathcal{U})$ has enough injective objects. As at 1.1 it follows from [11], lemma I.4.6 that every complex in $K^+(\mathcal{U})$ admits an injective resolution. Moreover, since X is finite dimensional, we get that every complex in $K(\mathcal{U})$ admits a resolution with acyclic s.s.s.

There is a natural, exact, fully faithful inclusion functor $\mathrm{Mod}(X) \longrightarrow \mathrm{Mod}(\mathcal{U})$ which takes every $\mathcal{O}_X$-module $\mathcal{F}$ in the s.s.s. $\mathcal{F}|\mathcal{U} = (\mathcal{F}|U_\alpha)_\alpha$. In general we shall identify $\mathcal{F}$ with its image in $\mathrm{Mod}(\mathcal{U})$ and, if no confusion is likely, we shall write $\mathcal{F}$ for $\mathcal{F}|\mathcal{U}$

If $\mathcal{U}$ is a locally finite covering of X then the inclusion functor $\mathrm{Mod}(X) \longrightarrow \mathrm{Mod}(\mathcal{U})$ has a right inverse (see Belkilani [3]). Every $\mathcal{A} \in \mathrm{Mod}(\mathcal{U})$ defines the presheaf $U \longmapsto \mathcal{A}_{\alpha(U)}(U)$, where $\alpha(U) \in \mathcal{N}$ is the largest simplex such that $U_{\alpha(U)} \supset U$. If one denotes by $\hat{\mathcal{A}}$ the associated sheaf, then one can easily see that for every $x \in X$, $\hat{\mathcal{A}}_x \simeq \mathcal{A}_{\alpha(x),x}$, where $\alpha(x) \in \mathcal{N}$ is the largest simplex such that $x \in U_{\alpha(x)}$. Consequently the association $\mathcal{A} \longmapsto \hat{\mathcal{A}}$ gives rise to an exact functor $\mathrm{Mod}(\mathcal{U}) \longrightarrow \mathrm{Mod}(X)$ such that for every $\mathcal{F} \in \mathrm{Mod}(X)$ one has a natural isomorphism $\mathcal{F} \xrightarrow{\sim} (\mathcal{F}|\mathcal{U})^\wedge$. Moreover, for every $\mathcal{A} \in \mathrm{Mod}(\mathcal{U})$ one can define a natural morphism $\mathcal{A} \longrightarrow \hat{\mathcal{A}}$, the component $\mathcal{A}_\alpha \longrightarrow \hat{\mathcal{A}}|U_\alpha$ being given by $\rho_{\alpha(U)\alpha}$ for every open set U in U_α (remember that we write $\hat{\mathcal{A}}$ for $\hat{\mathcal{A}}|\mathcal{U}$).

For every $\mathcal{F} \in \mathrm{Mod}(X)$ and every $\mathcal{A} \in \mathrm{Mod}(\mathcal{U})$ one has natural isomorphisms:

$$\mathrm{Hom}(\mathcal{A},\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}(\hat{\mathcal{A}},\mathcal{F}) \text{ and } \mathcal{H}om(\mathcal{A},\mathcal{F}) \xrightarrow{\sim} \mathcal{H}om(\hat{\mathcal{A}},\mathcal{F}).$$

Remark 2. The isomorphisms above imply that if $\mathcal{J}\in \mathrm{Mod}(X)$ is an injective sheaf then $\mathcal{J}$ is an injective object in $\mathrm{Mod}(\mathcal{U})$.

Remark 3. Let $\mathcal{F}^{\bullet}\in K(X)$ and $\mathcal{A}^{\bullet}\in K(\mathcal{U})$. Every quasiisomorphism $\mathcal{A}^{\bullet}\longrightarrow\mathcal{F}^{\bullet}$ factors, via the morphism $\mathcal{A}^{\bullet}\longrightarrow\hat{\mathcal{A}}^{\bullet}$, yielding a quasiisomorphism $\hat{\mathcal{A}}^{\bullet}\longrightarrow\mathcal{F}^{\bullet}$. On the other hand, it is easy to see that every quasiisomorphism $\mathcal{F}^{\bullet}\longrightarrow\mathcal{A}^{\bullet}$ in $K(\mathcal{U})$ can be completed to a quasiisomorphism $\mathcal{F}^{\bullet}\longrightarrow\mathcal{A}^{\bullet}\longrightarrow\hat{\mathcal{A}}^{\bullet}$ in $K(X)$. Consequently, the inclusion functor $D(X)\longrightarrow D(\mathcal{U})$ is fully faithful ([11], proposition I.3.3).

2.4. For every $\alpha\in\mathcal{N}$ there is a natural functor $\mathrm{Mod}(U_{\alpha})\longrightarrow \mathrm{Mod}(\mathcal{U})$ which takes $\mathcal{G}\in \mathrm{Mod}(U_{\alpha})$ in the s.s.s. $\tilde{\mathcal{G}}$ with $\tilde{\mathcal{G}}_{\beta}=\mathcal{G}|U_{\beta}$ if $\beta\supset\alpha$ and $\tilde{\mathcal{G}}_{\beta}=0$ otherwise. For every $\mathcal{B}\in \mathrm{Mod}(\mathcal{U})$ and $\mathcal{G}\in \mathrm{Mod}(U_{\alpha})$ one has natural isomorphisms:

$$\mathrm{Hom}(\tilde{\mathcal{G}},\mathcal{B}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{G},\mathcal{B}_{\alpha}) \xrightarrow{\sim} \Gamma(U_{\alpha},\mathcal{H}om(\mathcal{G},\mathcal{B}_{\alpha})) \text{ and}$$

$$\mathcal{H}om(\tilde{\mathcal{G}},\mathcal{B}) \xrightarrow{\sim} i_{\alpha *}\mathcal{H}om(\mathcal{G},\mathcal{B}_{\alpha}),$$

where $i_{\alpha}: U_{\alpha}\longrightarrow X$ is the inclusion.

A direct sum $\mathcal{P}=\bigoplus_{\alpha\in\mathcal{N}}\tilde{\mathcal{P}}_{\alpha}$ where each $\mathcal{P}_{\alpha}$ is $\mathcal{O}_{U_{\alpha}}$-free of finite rank is called a free s.s.s. (see [7] or [6]).

Remark 4. Let $\mathcal{U}$ be a covering of X with Stein open sets. If $\mathcal{P}$ is a free s.s.s., then it is a projective object in $\mathrm{Coh}(\mathcal{U})$ but it is no longer projective in $\mathrm{Mod}(\mathcal{U})$. However, if $\mathcal{A}\xrightarrow{\varphi}\mathcal{B}\longrightarrow 0$ is an exact sequence in $\mathrm{Mod}(\mathcal{U})$ and if $\ker\varphi$ is a s.s.s. acyclic on Stein open sets, then the morphism $\mathrm{Hom}(\mathcal{P},\mathcal{A})\longrightarrow \mathrm{Hom}(\mathcal{P},\mathcal{B})$ induced by φ is surjective.

Remark 5. Let $\mathcal{U}$, $\mathcal{V}$ be coverings of X such that $\mathcal{V} < \mathcal{U}$ and let τ be a refinement function. If $\mathcal{L} \in \mathrm{Mod}(\mathcal{U})$ is a free s.s.s. then $\mathcal{L}|\mathcal{V}$ may no longer be free. However, if τ is injective then $\mathcal{L}|\mathcal{V}$ is also free.

We want now to give sufficient conditions for a complex of s.s.s. to admit a free resolution.

Let $\mathcal{K} = (K_i)_{i \in I}$ be a locally finite covering of X with Stein compact sets.

Lemma 6. Every $\mathcal{A}^\bullet \in K^-_{coh}(\mathcal{K})$ (i.e. with cohomology in $\mathrm{Coh}(\mathcal{K})$) having components acyclic on Stein open sets admits a resolution with free s.s.s. in $K(\mathcal{K})$.

Proof. According to [6] lemma I.2.2., $\mathrm{Coh}(\mathcal{K})$ has enough projective objects and, more precisely, every s.s.s. in $\mathrm{Coh}(\mathcal{K})$ is the quotient of a free s.s.s.

The free resolution $\mathcal{L}^\bullet \longrightarrow \mathcal{A}^\bullet$ will be constructed by descendent induction.

For i sufficiently large one takes $\mathcal{L}^i = 0$. Suppose that we have constructed free s.s.s. $\mathcal{L}^j$ for $j > i$ and morphisms such that the following diagram commutes:

$$\begin{array}{ccccccc}
\mathcal{L}^i & \overset{\delta^i}{\dashrightarrow} & \mathcal{L}^{i+1} & \xrightarrow{\delta^{i+1}} & \mathcal{L}^{i+2} & \xrightarrow{\delta^{i+2}} & \\
\Big\downarrow f^i & & \Big\downarrow f^{i+1} & & \Big\downarrow f^{i+2} & & \\
\xrightarrow{d^{i-1}} \mathcal{A}^i & \xrightarrow{d^i} & \mathcal{A}^{i+1} & \xrightarrow{d^{i+1}} & \mathcal{A}^{i+2} & \xrightarrow{d^{i+2}} &
\end{array}$$

Moreover, suppose that f^j induces isomorphism on cohomology for $j \geq i+2$ and that f^{i+1} induces an epimorphism $\ker \delta^{i+1} \longrightarrow \mathcal{H}^{i+1}(\mathcal{A}^\bullet)$.

Consider the exact sequence:

$$0 \longrightarrow \operatorname{Im} d^{i-1} \longrightarrow \ker d^{i} \longrightarrow \mathcal{H}^{i}(\mathcal{A}^{\cdot}) \longrightarrow 0$$

and let $\varphi : \mathcal{P}_1 \longrightarrow \mathcal{H}^i(\mathcal{A}^{\cdot})$ be an epimorphism, where $\mathcal{P}_1$ is a free s.s.s. It is easy to show that the cycles and the boundaries of $\mathcal{A}^{\cdot}$ are acyclic on Stein open sets, and this, together with remark 4, implies that φ can be lifted to a morphism $\tilde{\varphi} : \mathcal{P}_1 \longrightarrow \ker d^i$. On the other hand let $\Psi : \mathcal{P}_2 \longrightarrow \ker \delta^{i+1} \cap \ker f^{i+1}$ be an epimorphism such that $\mathcal{P}_2$ is a free s.s.s. Now if one takes $\mathcal{L}^i = \mathcal{P}_1 \oplus \mathcal{P}_2$, $\delta^i = (0, \Psi)$ and $f^i = (\tilde{\varphi}, 0)$ it is easy to check that the induction hypothesis are satisfied.

Let now $\mathcal{U} = (U_i)_{i \in I}$ be a locally finite covering of X with Stein open sets such that for every $i \in I$, $\overline{U}_i$ is a Stein compact set, and let $\overline{\mathcal{U}} = (\overline{U}_i)_{i \in I}$.

Lemma 7. If $\mathcal{A}^{\cdot} \in K^-_{coh}(\overline{\mathcal{U}})$ has components acyclic on Stein open sets then $\mathcal{A}^{\cdot}|\mathcal{U}$ admits a free resolution in $K^-(\mathcal{U})$.

Proof. One has $\mathcal{U} < \overline{\mathcal{U}}$ with a natural bijective refinement function. So the lemma follows from lemma 6 and remark 5.

In particular, for complexes of sheaves one has:

Corollary 8. If $\mathcal{F}^{\cdot} \in K^-_{coh}(X)$ has components acyclic on Stein open sets then $\mathcal{F}^{\cdot}$ has a free resolution in $K^-(\mathcal{U})$.

A direct proof of this result can be found in [1], lemma 0.4.

2.5. Let $\mathcal{U}$ be a locally finite covering of X. As in the case of sheaves, one has bi-∂-functors $\operatorname{Hom}^{\cdot} : K(\mathcal{U}) \times K(\mathcal{U}) \longrightarrow K(Ab)$ and $\mathcal{H}om^{\cdot} : K(\mathcal{U}) \times K(\mathcal{U}) \longrightarrow K(X)$. Since $\operatorname{Mod}(\mathcal{U})$ has enough injective objects, these functors admit right derived functors $R\operatorname{Hom}^{\cdot} : D(\mathcal{U})^{o} \times D^{+}(\mathcal{U}) \longrightarrow D(Ab)$ and $R\mathcal{H}om^{\cdot} : D(\mathcal{U})^{o} \times D^{+}(\mathcal{U}) \longrightarrow D(X)$ obtained by "deriving" first in the second variable. If $\mathcal{A}^{\cdot} \in D(\mathcal{U})$, $\mathcal{B}^{\cdot} \in D^{+}(\mathcal{U})$ we denote the cohomology of $R\operatorname{Hom}^{\cdot}(\mathcal{A}^{\cdot}, \mathcal{B}^{\cdot})$ by $\operatorname{Ext}^{q}(\mathcal{U}; \mathcal{A}^{\cdot}, \mathcal{B}^{\cdot})$ and that of $R\mathcal{H}om^{\cdot}(\mathcal{A}^{\cdot}, \mathcal{B}^{\cdot})$

by $\mathcal{E}xt^q(\mathcal{A}^{\cdot},\mathcal{B}^{\cdot})$. By definition, $\mathrm{Ext}^q_c(\mathcal{U};\mathcal{A}^{\cdot},\mathcal{B}^{\cdot})$ is $H^q(R\Gamma_c(X,R\mathcal{H}om^{\cdot}(\mathcal{A}^{\cdot},\mathcal{B}^{\cdot})))$

Proposition 9. For every complexes of sheaves $\mathcal{E}^{\cdot}\in D(X)$ and $\mathcal{F}^{\cdot}\in D^+(X)$ there exist natural isomorphisms $\mathrm{Ext}^q(X;\mathcal{E}^{\cdot},\mathcal{F}^{\cdot})\simeq \mathrm{Ext}^q(\mathcal{U};\mathcal{E}^{\cdot},\mathcal{F}^{\cdot})$ and $\mathrm{Ext}^q_c(X;\mathcal{E}^{\cdot},\mathcal{F}^{\cdot})\simeq \mathrm{Ext}^q_c(\mathcal{U};\mathcal{E}^{\cdot},\mathcal{F}^{\cdot})$.

Proof. The proposition follows from remark 2 or from remark 3 and the theorem of Yoneda.

In order to define the Ext by means of free resolutions we need two lemmas.

Lemma 10. Let $\mathcal{L}^{\cdot}\in K^-(\mathcal{U})$ be a complex of free s.s.s. and $\mathcal{B}^{\cdot}\in K(\mathcal{U})$. Assume that either a) $\mathcal{L}^{\cdot}$ is exact or b) $\mathcal{B}^{\cdot}$ is exact and has components acyclic on Stein open sets. Then $\mathrm{Hom}^{\cdot}(\mathcal{L}^{\cdot},\mathcal{B}^{\cdot})$ and $\mathcal{H}om^{\cdot}(\mathcal{L}^{\cdot},\mathcal{B}^{\cdot})$ are exact.

Proof. We deal first with $\mathrm{Hom}^{\cdot}$. a) Since the components of $\mathcal{L}^{\cdot}$ are projective objects in $\mathrm{Coh}(\mathcal{U})$ it follows that $\mathcal{L}^{\cdot}$ is split exact and so, every morphism from $\mathcal{L}^{\cdot}$ in an arbitrary complex is homotopic to 0. Consequently, $\mathrm{Hom}^{\cdot}(\mathcal{L}^{\cdot},\mathcal{B}^{\cdot})$ is exact.

b) Since $\mathcal{B}^{\cdot}$ is exact and has components acyclic on Stein open sets, it follows easily that the cycles and the boundaries of $\mathcal{B}^{\cdot}$ are acyclic on Stein open sets (see remak 1.1). Now, using remark 4, it follows that any morphism of complexes of given degree from $\mathcal{L}^{\cdot}$ to $\mathcal{B}^{\cdot}$ is homotopic to 0 (the proof goes exactly as if $\mathcal{L}^{\cdot}$ would have projective components).

Now we consider $\mathcal{H}om^{\cdot}$. By a) and b) above, for every U in X Stein open set, $\mathrm{Hom}^{\cdot}(\mathcal{L}^{\cdot}|U,\mathcal{B}^{\cdot}|U)$ is exact and we conclude that $\mathcal{H}om^{\cdot}(\mathcal{L}^{\cdot},\mathcal{B}^{\cdot})$ is also exact.

Lemma 11. Let $\mathcal{L}^{\bullet} \in K^{-}(\mathcal{U})$ be a complex of free s.s.s. and $\mathcal{B}^{\bullet} \in K(\mathcal{U})$ a complex with components acyclic on Stein open sets. Then the complex $\mathcal{H}om^{\bullet}(\mathcal{L}^{\bullet},\mathcal{B}^{\bullet})$ has components $\Gamma(X,\cdot)$- and $\Gamma_c(X,\cdot)$-acyclic.

Proof. We remark first that if $\mathcal{G} \in \mathrm{Mod}(U_\alpha)$ is $\mathcal{O}_{U_\alpha}$-free of finite rank and if $\mathcal{B} \in \mathrm{Mod}(\mathcal{U})$ is acyclic on Stein open sets then $\mathcal{H}om(\tilde{\mathcal{G}},\mathcal{B})$ is acyclic on Stein open sets and on every open set containing U_α.

Every component of $\mathcal{L}^{\bullet}$ is of the form $\mathcal{L}^p = \bigoplus_{\alpha\in\mathcal{N}} \tilde{\mathcal{P}}_\alpha^p$ with $\mathcal{P}_\alpha^p$ an $\mathcal{O}_{U_\alpha}$-free sheaf of finite rank. One has the identifications:

$$\mathcal{H}om^n(\mathcal{L}^{\bullet},\mathcal{B}^{\bullet}) \simeq \prod_{p\in Z}\prod_{\alpha\in\mathcal{N}} i_{\alpha *}(\mathcal{H}om(\mathcal{P}_\alpha^p,\mathcal{B}_\alpha^{n+p}))$$

Let $\mathcal{J}^{\bullet}_{\alpha p}$ be a flabby resolution for $\mathcal{H}om(\mathcal{P}_\alpha^p,\mathcal{B}_\alpha^{n+p})$. It follows easily, using the remark at the beginning of the proof, that $\prod_{p\in Z}\prod_{\alpha\in\mathcal{N}} i_{\alpha *}(\mathcal{J}^{\bullet}_{\alpha p})$ is a flabby resolution for $\mathcal{H}om^n(\mathcal{L}^{\bullet},\mathcal{B}^{\bullet})$. Applying $\Gamma(X,\cdot)$ and $\Gamma_c(X,\cdot)$ one obtains the exact complexes:

$$\prod_{p\in Z}\prod_{\alpha\in\mathcal{N}}\Gamma(U_\alpha,\mathcal{H}om(\mathcal{P}_\alpha^p,\mathcal{B}_\alpha^{n+p})) \hookrightarrow \prod_{p\in Z}\prod_{\alpha\in\mathcal{N}}\Gamma(U_\alpha,\mathcal{J}^{\bullet}_{\alpha p})$$

$$\bigoplus_{\alpha\in\mathcal{N}}\prod_{p\in Z}\Gamma(U_\alpha,\mathcal{H}om(\mathcal{P}_\alpha^p,\mathcal{B}_\alpha^{n+p})) \hookrightarrow \bigoplus_{\alpha\in\mathcal{N}}\prod_{p\in Z}\Gamma(U_\alpha,\mathcal{J}^{\bullet}_{\alpha p})$$

Let now $\mathcal{U}=(U_i)_{i\in I}$ be a locally finite covering of X with Stein open sets such that for every $i\in I$, $\overline{U_i}$ is a Stein compact set and let $\overline{\mathcal{U}}=(\overline{U_i})_{i\in I}$. We denote by $K^{-}_{coh}(\overline{\mathcal{U}},\mathcal{U})$ and $D^{-}_{coh}(\overline{\mathcal{U}},\mathcal{U})$ the subcategories of $K^{-}(\mathcal{U})$, $D^{-}(\mathcal{U})$ consisting of restrictions of complexes from $K^{-}_{coh}(\overline{\mathcal{U}})$ and $D^{-}_{coh}(\overline{\mathcal{U}})$.

Lemmas 7 and 10 ensure that there exist right derived functors $R\mathrm{Hom}^{\bullet}: D^{-}_{coh}(\overline{\mathcal{U}},\mathcal{U})^{o} \times D(\mathcal{U}) \longrightarrow D(Ab)$ and $R\mathcal{H}om^{\bullet}: D^{-}_{coh}(\overline{\mathcal{U}},\mathcal{U})^{o} \times D(\mathcal{U}) \longrightarrow D(X)$ for $\mathrm{Hom}^{\bullet}$ and $\mathcal{H}om^{\bullet}$, obtained by "deriving" first in the first variable.

Let $\mathcal{A}^\bullet \in D^-_{coh}(\overline{\mathcal{U}},\mathcal{U})$, $\mathcal{B}^\bullet \in D(\mathcal{U})$, $\mathcal{A}^\bullet \longrightarrow \mathcal{I}^\bullet$ be a resolution of $\mathcal{A}^\bullet$, where $\mathcal{I}^\bullet \in D^-_{coh}(\overline{\mathcal{U}},\mathcal{U})$ is the restriction of a complex in $D^-_{coh}(\overline{\mathcal{U}})$, acyclic on Stein open sets; let $\mathcal{L}^\bullet \longrightarrow \mathcal{I}^\bullet$ be a free resolution of $\mathcal{I}^\bullet$ in $D(\mathcal{U})$ (see lemma 7) and let $\mathcal{B}^\bullet \longrightarrow \mathcal{J}^\bullet$ be a resolution of $\mathcal{B}^\bullet$ with components acyclic on Stein open sets. The complexes $\mathrm{Hom}^\bullet(\mathcal{L}^\bullet,\mathcal{J}^\bullet)$ and $\mathcal{H}om^\bullet(\mathcal{L}^\bullet,\mathcal{J}^\bullet)$ are representatives for $\mathrm{RHom}^\bullet(\mathcal{A}^\bullet,\mathcal{B}^\bullet)$ and $R\mathcal{H}om^\bullet(\mathcal{A}^\bullet,\mathcal{B}^\bullet)$. Moreover, according to lemma 11, $\Gamma_c(X,\mathcal{H}om^\bullet(\mathcal{L}^\bullet,\mathcal{J}^\bullet))$ is a representative for $R\Gamma_c(X,R\mathcal{H}om^\bullet(\mathcal{A}^\bullet,\mathcal{B}^\bullet))$. The cohomology of $\mathrm{RHom}^\bullet(\mathcal{A}^\bullet,\mathcal{B}^\bullet)$ is denoted $\mathrm{Ext}^q(\mathcal{U};\mathcal{A}^\bullet,\mathcal{B}^\bullet)$, that of $R\mathcal{H}om^\bullet(\mathcal{A}^\bullet,\mathcal{B}^\bullet)$, $\mathcal{E}xt^q(\mathcal{A}^\bullet,\mathcal{B}^\bullet)$ and that of $R\Gamma_c(X,R\mathcal{H}om^\bullet(\mathcal{A}^\bullet,\mathcal{B}^\bullet))$, $\mathrm{Ext}^q_c(\mathcal{U};\mathcal{A}^\bullet,\mathcal{B}^\bullet)$.

Lemma 11 implies there exists a natural isomorphism:

$$\mathrm{RHom}^\bullet(\mathcal{A}^\bullet,\mathcal{B}^\bullet) \xrightarrow{\sim} R\Gamma(X,R\mathcal{H}om^\bullet(\mathcal{A}^\bullet,\mathcal{B}^\bullet))$$

which gives rise to a spectral sequence $E_2^{pq}=H^p(X,\mathcal{E}xt^q(\mathcal{A}^\bullet,\mathcal{B}^\bullet))$ converging to $\mathrm{Ext}^{p+q}(\mathcal{U};\mathcal{A}^\bullet,\mathcal{B}^\bullet)$.

If $\mathcal{B}^\bullet \in D^+(\mathcal{U})$ and $\mathcal{J}^\bullet$ is an injective resolution for $\mathcal{B}^\bullet$, the sequences of quasiisomorphisms:

$$\mathrm{Hom}^\bullet(\mathcal{L}^\bullet,\mathcal{J}^\bullet) \longleftarrow \mathrm{Hom}^\bullet(\mathcal{I}^\bullet,\mathcal{J}^\bullet) \longrightarrow \mathrm{Hom}^\bullet(\mathcal{A}^\bullet,\mathcal{J}^\bullet)$$

$$\mathcal{H}om^\bullet(\mathcal{L}^\bullet,\mathcal{J}^\bullet) \longleftarrow \mathcal{H}om^\bullet(\mathcal{I}^\bullet,\mathcal{J}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{A}^\bullet,\mathcal{J}^\bullet)$$

show that the functors $\mathrm{RHom}^\bullet$ and $R\mathcal{H}om^\bullet$ defined with injective resolutions and with free resolutions, coincide on $D^-_{coh}(\overline{\mathcal{U}},\mathcal{U})^o \times D^+(\mathcal{U})$. In particular, if $\mathcal{E}^\bullet \in D^-_{coh}(X)$ and $\mathcal{F}^\bullet \in D^+(X)$ one can compute $\mathrm{Ext}^q(X;\mathcal{E}^\bullet,\mathcal{F}^\bullet)$ by means of free resolutions for $\mathcal{E}^\bullet$.

<u>Propostion 12</u>. If $\mathcal{E}^\bullet \in D^-_{coh}(X)$, $\mathcal{F}^\bullet \in D_{coh}(X)$ then $\mathrm{Ext}^q(\mathcal{U};\mathcal{E}^\bullet,\mathcal{F}^\bullet)$, $\mathrm{Ext}^q_c(\mathcal{U};\mathcal{E}^\bullet,\mathcal{F}^\bullet)$ and $\mathcal{E}xt^q(\mathcal{E}^\bullet,\mathcal{F}^\bullet)$ are independent of the covering $\mathcal{U}$.

<u>Proof</u>. We can suppose that $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ have components acyclic on Stein open sets and let $\mathcal{L}^\bullet \longrightarrow \mathcal{E}^\bullet$ be a free resolution for $\mathcal{E}^\bullet$ in $K(\mathcal{U})$.

Since $\mathcal{F}^{\cdot}$ has coherent cohomology, ${}_p\mathcal{F}^{\cdot}$ has also components acyclic on Stein open sets and so, $\mathrm{Hom}^{\cdot}(\mathcal{L}^{\cdot},{}_p\mathcal{F}^{\cdot})$ is a representative for $\mathrm{RHom}^{\cdot}(\mathcal{E}^{\cdot},{}_p\mathcal{F}^{\cdot})$. Since $\mathcal{L}^{\cdot}$ has free components, it is easy to check that for every q, $\mathrm{Hom}^q(\mathcal{L}^{\cdot},\mathcal{F}^{\cdot}) \simeq \text{proj lim } \mathrm{Hom}^q(\mathcal{L}^{\cdot},{}_p\mathcal{F}^{\cdot})$ and that $(\mathrm{Hom}^q(\mathcal{L}^{\cdot},{}_p\mathcal{F}^{\cdot}))_p$ and $(\mathrm{Ext}^q(\mathcal{U};\mathcal{E}^{\cdot},{}_p\mathcal{F}^{\cdot}))_p$ are surjective projective systems. This means, according to [10], Ch. 0_{III}, proposition 13.2.3, that $\mathrm{Ext}^q(\mathcal{U};\mathcal{E}^{\cdot},\mathcal{F}^{\cdot}) \simeq \text{proj lim } \mathrm{Ext}^q(\mathcal{U};\mathcal{E}^{\cdot},{}_p\mathcal{F}^{\cdot})$. Since $\mathrm{Ext}^q(\mathcal{U};\mathcal{E}^{\cdot},{}_p\mathcal{F}^{\cdot})$ is canonically isomorphic with $\mathrm{Ext}^q(X;\mathcal{E}^{\cdot},{}_p\mathcal{F}^{\cdot})$ (the second complex is bounded to the left) it follows that $\mathrm{Ext}^q(\mathcal{U};\mathcal{E}^{\cdot},\mathcal{F}^{\cdot}) \simeq \text{proj lim } \mathrm{Ext}^q(X;\mathcal{E}^{\cdot},{}_p\mathcal{F}^{\cdot})$. The statement concerning $\mathrm{Ext}^q_c(\mathcal{U};\mathcal{E}^{\cdot},\mathcal{F}^{\cdot})$ and $\mathcal{E}xt^q(\mathcal{E}^{\cdot},\mathcal{F}^{\cdot})$ can be proved in a similar way.

Proposition 12 justifies the use of the notations $\mathrm{Ext}^q(X;\mathcal{E}^{\cdot},\mathcal{F}^{\cdot})$ and $\mathrm{Ext}^q_c(X;\mathcal{E}^{\cdot},\mathcal{F}^{\cdot})$ when $\mathcal{F}^{\cdot}$ is not bounded to the left.

Remark 13. If the complex on the second place in $\mathrm{Ext}^q(X;\mathcal{E}^{\cdot},\mathcal{F}^{\cdot})$ is not bounded to the left it does not seem easy to prove that the Yoneda isomorphism still holds.

Using lemmas 6 and 10, one can define, for every locally finite covering $\mathcal{K}$ of X with Stein compact sets, right derived functors $\mathrm{RHom}^{\cdot}: D^-_{coh}(\mathcal{K})^o \times D(\mathcal{K}) \longrightarrow D(Ab)$ and $R\mathcal{H}om^{\cdot}: D^-_{coh}(\mathcal{K})^o \times D(\mathcal{K}) \longrightarrow D(X)$. As above, the functors $R\mathcal{H}om^{\cdot}$ and $\mathrm{RHom}^{\cdot}$ defined with injective resolutions coincide on $D^-_{coh}(\mathcal{K})^o \times D^+(\mathcal{K})$ with those defined with free resolutions and it is easy to see that proposition 12 holds also for compact converings of X.

3. COMPLEXES WITH FINITE TOR-DIMENSION

In this paragraph we prove that complexes in $D^b_{coh}(X)$ having f.t.d. admit free resolutions of finite length and, using this fact, study the properties of the hyperext when the first complex has f.t.d.

3.1. We start with a general property which holds in the context of ringed spaces:

Proposition 1. Let $(X, \mathcal{O}_X)$ be a ringed space of finite topological dimension and let $\mathcal{G}$ be a locally free $\mathcal{O}_X$-module of finite rank. Then there exists a covering $\mathcal{U}$ of X such that $\mathcal{G}$ admits a free resolution of finite length in $K(\mathcal{U})$.

Proof. We can choose $\mathcal{U} = (U_i)_{i \in I}$ to be any finite dimensional covering of X such that $\mathcal{G}$ is free on every U_i. Let n be the dimension of $\mathcal{U}$ (i.e. the length of the largest simplex in the nerve $\mathcal{N}$ of $\mathcal{U}$) and let $(\varphi_i)_{i \in I}$ be the family of commuting automorphisms of $\mathcal{G}$, where $\varphi_i = \mathrm{id}$ for every i. Then, for $\alpha \in \mathcal{N}$ one has the free resolution of $\mathcal{G}|U_\alpha$ given by the Koszul complex associated to the automorphisms $(\varphi_i | U_\alpha)_{i \in \alpha}$.

$$0 \longrightarrow \mathcal{G}|U_\alpha^{\binom{|\alpha|}{|\alpha|}} \longrightarrow (\mathcal{G}|U_\alpha)^{\binom{|\alpha|}{|\alpha|-1}} \longrightarrow \dots \longrightarrow (\mathcal{G}|U_\alpha)^{\binom{|\alpha|}{1}} \longrightarrow \mathcal{G}|U_\alpha \longrightarrow 0$$

If one takes now the free s.s.s. $\mathcal{P}^0 = \bigoplus_{i \in I} (\mathcal{G}|U_i)$, $\mathcal{P}^1 = \bigoplus_{(i,j) \in \mathcal{N}} (\mathcal{G}|U_{ij})$, ... etc., one can verify that the above resolutions give in $K(\mathcal{U})$ the free resolution:

$$0 \longrightarrow \mathcal{P}^n \longrightarrow \mathcal{P}^{n-1} \longrightarrow \dots \longrightarrow \mathcal{P}^0 \longrightarrow \mathcal{G} \longrightarrow 0$$

Corollary 2. Let $(X, \mathcal{O}_X)$ be a ringed space of finite topological dimension and $\mathcal{U}$ a finite dimensional covering of X. Then $\mathcal{O}_X$ has a free resolution of finite length in $K(\mathcal{U})$.

Proposition 3. Let $(X, \mathcal{O}_X)$ be a finite dimensional analytic space and $\mathcal{F}^{\cdot} \in K^b_{coh}(X)$ a complex acyclic on Stein open sets and having f.t.d. Then there exists a locally finite covering of X with Stein open sets, $\mathcal{U}$, such that $\mathcal{F}^{\cdot}$ has a free resolution of finite length in $K(\mathcal{U})$.

Proof. Let $\mathcal{U}=(U_i)_{i\in I_1}$, $\mathcal{V}=(V_i)_{i\in I_2}$, $\mathcal{K}=(K_i)_{i\in I_3}$ be locally finite coverings of X such that:

1) $\mathcal{U}$ is a covering with Stein open sets.

2) $\mathcal{V}$ is an open covering such that any intersection of sets in $\mathcal{V}$ is contractible.

3) $\mathcal{K}$ is a finite dimensional covering with Stein compact sets.

4) $\mathcal{U}\prec\mathcal{V}\prec\mathcal{K}$ and the refinement functions between the coverings are injective.

(to construct the coverings one starts by choosing $\mathcal{K}$; the existence of $\mathcal{V}$ finer than $\mathcal{K}$ follows from the fact that any analytic space has a triangulation; the refinement functions can be made injective by "repeating" the sets in $\mathcal{V}$ and then in $\mathcal{K}$).

Let $\mathcal{L}^\cdot \longrightarrow \mathcal{F}^\cdot$ be a free resolution of $\mathcal{F}^\cdot$ in $K^-(\mathcal{K})$ (see lemma 2.6) and let $r\in Z$ be strictly smaller than the degree of the first non-zero component of a bounded flat resolution of $\mathcal{F}^\cdot$. We define inductively a new complex $\mathcal{P}^\cdot\in K^-(\mathcal{K})$. In degrees $i>r$ we take $\mathcal{P}^\cdot$ identical with $\mathcal{L}^\cdot$. If $i\leq r$ let $\mathcal{P}^i=\bigoplus_{\alpha\in\mathcal{N}(\mathcal{K})}\widetilde{\ker d_\alpha^{i+1}}$ and d^i be the composite of the morphisms:

$$\mathcal{P}^i=\bigoplus_{\alpha\in\mathcal{N}(\mathcal{K})}\ker d_\alpha^{i+1}\longrightarrow \ker d^{i+1}\longrightarrow \mathcal{P}^{i+1}$$

It follows immediately from the construction of $\mathcal{P}^\cdot$ that $\mathcal{P}_\alpha^{r-j}=0$ for every $j\geq 2$ and α with $|\alpha|\leq j-1$. Since the covering $\mathcal{K}$ is finite dimensional, it follows that $\mathcal{P}^\cdot$ is bounded.

It is easy to see that $\mathcal{P}^\cdot$ is still quasiisomorphic to $\mathcal{F}^\cdot$.

According to the proof of proposition II.4.2 in [11], it follows that for every $i\leq r$, $\ker d^i$ has flat components. Since ker $d^i \in Coh(\mathcal{K})$ one gets that its components are locally free sheaves.

To finish the proof we remark that, since V_α is contractible for each α in the nerve of $\mathcal{V}$, $\mathcal{P}^i|\mathcal{V}$ has free components. Furthermore, taking into account the form of the s.s.s $\mathcal{P}^i$ and the fact that the refinement functions between the three coverings are injective, we obtain that $\mathcal{P}^\cdot|\mathcal{V}$ and $\mathcal{P}^\cdot|\mathcal{U}$ are free resolutions of finite length for $\mathcal{F}^\cdot$ in $K(\mathcal{V})$, respectively $K(\mathcal{U})$.

<u>Corollary 4</u>. Let X be a finite dimensional analytic space and $\mathcal{F}^\cdot \in K^b(X)$ a complex of locally free sheaves of finite rank. Then there exists a locally finite covering of X with Stein open sets such that $\mathcal{F}^\cdot$ has a free resolution of finite length in $K(\mathcal{U})$.

<u>Remark 5</u>. If X is a ringed space of finite topological dimension, one can prove, using proposition 1, a result similar to corollary 4.

<u>Corollary 6</u>. Let X be a complex manifold and $\mathcal{F}^\cdot \in K^b_{coh}(X)$ a complex of sheaves acyclic on Stein open sets (in particular $\mathcal{F}$ can be a coherent sheaf on X). Then $\mathcal{F}^\cdot$ has a free resolution of finite length in some $K(\mathcal{U})$.

<u>Proof</u>. Hilbert's syzygy theorem implies that $\mathcal{F}^\cdot$ has f.t.d. and the statement follows immediately from proposition 3.

3.2. <u>Proposition 7</u>. If $\mathcal{E}^\cdot \in D^b_{coh}(X)$ has f.t.d. and $\mathcal{F}^\cdot \in D_{coh}(X)$ then $R\mathcal{H}om^\cdot(\mathcal{E}^\cdot, \mathcal{F}^\cdot) \in D_{coh}(X)$.

<u>Proof</u>. Using proposition 1.8 the result follows from:

<u>Lemma 8</u>. If $\mathcal{E}^\cdot \in D^b_{coh}(X)$ has f.t.d. and $\mathcal{F}^\cdot \in D_{coh}(X)$ then for every $q_o \in Z$ there exists $p \in Z$ such that the natural morphisms

$\mathrm{Ext}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)\longrightarrow \mathrm{Ext}^q(X;\mathcal{E}^\cdot,{}_p\mathcal{F}^\cdot)$, $\mathrm{Ext}^q_c(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)\longrightarrow \mathrm{Ext}^q_c(X;\mathcal{E}^\cdot,{}_p\mathcal{F}^\cdot)$, $\mathcal{E}xt^q(\mathcal{E}^\cdot,\mathcal{F}^\cdot)\longrightarrow \mathcal{E}xt^q(\mathcal{E}^\cdot,{}_p\mathcal{F}^\cdot)$ are isomorphisms for $q\geq q_o$.

Proof. We can suppose that $\mathcal{E}^\cdot$, $\mathcal{F}^\cdot$ have components acyclic on Stein open sets. Since $\mathcal{F}^\cdot$ has coherent cohomology, for every $p\in Z$, ${}_p\mathcal{F}^\cdot$ has also components acyclic on Stein open sets.

Now let $\mathcal{U}$ be a covering of X such that $\mathcal{E}^\cdot$ admits a free resolution of finite length in $K(\mathcal{U})$, $\mathcal{L}^\cdot\longrightarrow\mathcal{E}^\cdot$(see proposition 3). The natural morphism $\mathrm{RHom}^\bullet(\mathcal{E}^\cdot,\mathcal{F}^\cdot)\longrightarrow \mathrm{RHom}^\bullet(\mathcal{E}^\cdot,{}_p\mathcal{F}^\cdot)$ is given by the morphism $\mathrm{Hom}^\bullet(\mathcal{L}^\cdot,\mathcal{F}^\cdot)\longrightarrow \mathrm{Hom}^\bullet(\mathcal{L}^\cdot,{}_p\mathcal{F}^\cdot)$. Since, in this case, $\mathrm{Hom}^\bullet(\mathcal{L}^\cdot,\mathcal{F}^\cdot)$ and $\mathrm{Hom}^\bullet(\mathcal{L}^\cdot,{}_p\mathcal{F}^\cdot)$ are the simple complexes associated to the double complexes $\mathrm{Hom}(\mathcal{L}^r,\mathcal{F}^q)_{(r,q)}$ and $\mathrm{Hom}(\mathcal{L}^r,{}_p\mathcal{F}^q)_{(r,q)}$, the result follows easily.

The other morphisms in the lemma can be dealt with in exactly the same way.

Remark 9. If $\mathcal{E}^\cdot$, $\mathcal{F}^\cdot$ are as in proposition 7 then the natural morphism $R\mathcal{H}om^\bullet(\mathcal{E}^\cdot,\mathcal{F}^\cdot)_x\longrightarrow \mathrm{RHom}^\bullet(\mathcal{E}^\cdot_x,\mathcal{F}^\cdot_x)$ is an isomorphism and consequently, for every $q\in Z$ one has a natural isomorphism: $\mathcal{E}xt^q(\mathcal{E}^\cdot,\mathcal{F}^\cdot)_x\xrightarrow{\sim} \mathrm{Ext}^q_{\mathcal{O}_{X,x}}(\mathcal{E}^\cdot_x,\mathcal{F}^\cdot_x)$ (to see this, use a truncation of $\mathcal{F}^\cdot$, ${}_p\mathcal{F}^\cdot$, and remark 1.9).

Remark 10. If X is a Stein space and $\mathcal{E}^\cdot\in D^b_{coh}(X)$ has f.t.d., $\mathcal{F}^\cdot\in D_{coh}(X)$, the spectral sequence of term $E_2^{pq}=H^p(X,\mathcal{E}xt^q(\mathcal{E}^\cdot,\mathcal{F}^\cdot))$ is degenerate and the edge morphisms: $\mathrm{Ext}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)\longrightarrow\Gamma(X,\mathcal{E}xt^q(\mathcal{E}^\cdot,\mathcal{F}^\cdot))$ are isomorphisms.

4. THE NATURAL TOPOLOGY

Proposition 1. 1) For every $\mathcal{F}^\cdot\in D_{coh}(X)$ there is a topology of type QFS on $H^q(X,\mathcal{F}^\cdot)$, called the natural topology, such that:

i) if $\mathcal{F}^\cdot,\mathcal{G}^\cdot\in D_{coh}(X)$ and $u:\mathcal{F}^\cdot\longrightarrow\mathcal{G}^\cdot$ is a morphism then the induced mapping $\tilde{u}:H^q(X,\mathcal{F}^\cdot)\longrightarrow H^q(X,\mathcal{G}^\cdot)$ is continuous. Moreover, if

$\tilde{u}$ is an algebraic isomorphism then it is a topological isomorphism;

ii) if $U \subset X$ is an open set then the restriction $H^q(X,\mathcal{F}^\cdot) \longrightarrow H^q(U,\mathcal{F}^\cdot)$ is continuous;

iii) if X is a Stein space then the natural morphism $H^q(X,\mathcal{F}^\cdot) \longrightarrow \Gamma(X,\mathcal{H}^q(\mathcal{F}^\cdot))$ is a topological isomorphism when on the right hand side term one takes the usual topology on the sections of a coherent sheaf.

2) For every $\mathcal{F}^\cdot \in D_{coh}(X)$ there is a topology of type QDFS on $H^q_c(X,\mathcal{F}^\cdot)$ called the natural topology such that:

i) same as 1i) for $H^q_c(X,\mathcal{F}^\cdot)$;

ii) if $U \subset X$ is an open set then the extension morphism $H^q_c(U,\mathcal{F}^\cdot) \longrightarrow H^q_c(X,\mathcal{F}^\cdot)$ is continuous.

<u>Proof.</u> 1. Let $\mathcal{U} = (U_i)_{i \in I}$ be a locally finite covering of X with Stein open sets such that for every $i \in I$, $\overline{U}_i$ is a Stein compact set.

We treat first the case $\mathcal{F}^\cdot \in D^-_{coh}(X)$. Let $\mathcal{F}^\cdot \longrightarrow \mathcal{J}^\cdot$ be a resolution of $\mathcal{F}^\cdot$ with sheaves acyclic on Stein open sets and let $\mathcal{L}^\cdot \longrightarrow \mathcal{J}^\cdot$ be a resolution of $\mathcal{J}^\cdot$ with s.s.s in Coh$(\mathcal{U})$ (in particular a free resolution; for the existence of $\mathcal{L}^\cdot$ see corollary 2.8). Taking Čech complexes we obtain a morphism $C^\cdot(\mathcal{U},\mathcal{L}^\cdot) \longrightarrow C^\cdot(\mathcal{U},\mathcal{J}^\cdot)$ which induces a quasiisomorphism between the associated simple complexes. The left hand side complex is a complex of FS spaces (see remark 2.1) and the cohomology of the right hand side complex is $H^q(X,\mathcal{F}^\cdot)$. The topology of type QFS thus induced on $H^q(X,\mathcal{F}^\cdot)$ is the natural topology.

The natural topology is of course independent of the different choices we have made (to prove the independence of the resolutions and covering, one refines two choices by a third one and one uses the well known fact that a continuous algebraic quasiisomorphism between two Fréchet complexes induces topological isomorphisms on cohomology).

i) let $u:\mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet$ be a morphism of complexes, with $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D^-_{coh}(X)$ and let $\mathcal{F}^\bullet \longrightarrow \mathcal{I}^\bullet$ be a resolution with sheaves acyclic on Stein open sets for $\mathcal{F}^\bullet$, $\mathcal{I}^\bullet \in D^-_{coh}(X)$. There exist $\mathcal{J}^\bullet \in D^-_{coh}(X)$ with components acyclic on Stein open sets, a quasiisomorphism v and a morphism u′ such that the first square of the diagram below commutes. If $\mathcal{L}^\bullet \longrightarrow \mathcal{J}^\bullet$ is a resolution with components in $Coh(\mathcal{U})$ for $\mathcal{J}^\bullet$ then there exist $\mathcal{M}^\bullet \in D^-(\mathcal{U})$ with components in $Coh(\mathcal{U})$, a quasi-isomorphism w and a morphism u" such that the second square of the diagram below commutes:

$$\begin{array}{ccccc}
\mathcal{F}^\bullet & \longleftarrow & \mathcal{I}^\bullet & \xrightarrow{\ w\ } & \mathcal{M}^\bullet \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u''} \\
\mathcal{G}^\bullet & \xleftarrow{\ v\ } & \mathcal{J}^\bullet & \longrightarrow & \mathcal{L}^\bullet
\end{array}$$

It is now obvious that u" induces a continuous morphism between the Čech complexes which give the topology on the cohomology of $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ and consequently, that the induced morphism $\tilde{u}:H^q(X,\mathcal{F}^\bullet) \longrightarrow H^q(X,\mathcal{G}^\bullet)$ is continuous. Moreover, if $\tilde{u}$ is an algebraic isomorphism then, according to a variant of the topological result referred to above, u" induces a topological isomorphism on the cohomology.

ii) Let $U \subset X$ be an open set and let $\mathcal{V}$ be a locally finite covering of U with relatively compact Stein open sets, $\mathcal{V} \prec \mathcal{U} \cap U$. If $\mathcal{F}^\bullet \longrightarrow \mathcal{I}^\bullet$ is a resolution for $\mathcal{F}^\bullet$ as above and $\mathcal{L}^\bullet \longrightarrow \mathcal{I}^\bullet$ a resolution of $\mathcal{I}^\bullet$ with objects in $Coh(\mathcal{U})$, one can verify that the composite of the continuous morphisms:

$$C^\bullet(\mathcal{U}, \mathcal{L}^\bullet) \longrightarrow C^\bullet(\mathcal{U} \cap U, \mathcal{L}^\bullet|U) \longrightarrow C^\bullet(\mathcal{V}, \mathcal{L}^\bullet|U|\mathcal{V})$$

induces the natural restriction on the cohomology.

iii) We use the following vector spaces lemma:

Lemma 2. Let A be a Fréchet algebra and $u:E \longrightarrow F$ a continuous morphism of Fréchet A-modules. Suppose that $G=\operatorname{coker} u$ is a finitely generated A-module and that there exists on G a separated topology of topological A-module (on G one considers the algebraic structure induced by F). Then the quotient topology induced by F on G is separated and consequently Fréchet.

Proof. One can write the commutative diagram:

$$\begin{array}{ccc} A^n \oplus E & \xrightarrow{(\Psi,u)} & F \\ \big\downarrow{\scriptstyle (\varphi,0)} & & \big\downarrow \\ G & \xrightarrow{\ \mathrm{id}\ } & G \end{array}$$

where $\varphi:A^n \longrightarrow G$ is a surjective A-module morphism and $\Psi:A^n \longrightarrow F$ is a lifting of φ. Since (Ψ,u) is continuous and surjective it is open and consequently the identity is open when on the left hand side G one takes the separated topology in the hypothesis.

Now let $U \subset X$ be a relatively compact Stein open set. Since $H^q(U,\mathcal{F}^\bullet) \xrightarrow{\sim} \Gamma(U, \mathcal{H}^q(\mathcal{F}^\bullet))$ it follows that $H^q(U,\mathcal{F}^\bullet)$ is finitely generated as $\mathcal{O}_X(U)$-module and consequently, according to lemma 2, its topology is separated and coincides with that of $\Gamma(U, \mathcal{H}^q(\mathcal{F}^\bullet))$.

The following commutative diagram solves the problem for X, since Ψ is a topological isomorphism and i is continuous:

$$\begin{array}{ccc} H^q(X,\mathcal{F}^\bullet) & \xrightarrow{\ \sim\ } & \Gamma(X, \mathcal{H}^q(\mathcal{F}^\bullet)) \\ \big\downarrow{\scriptstyle i} & & \big\downarrow \\ \prod_{i\in I} H^q(U_i,\mathcal{F}^\bullet) & \underset{\Psi}{\xrightarrow{\ \sim\ }} & \prod_{i\in I} \Gamma(U_i, \mathcal{H}^q(\mathcal{F}^\bullet)) \end{array}$$

Finally we treat the case $\mathcal{F}^\bullet \in D_{coh}(X)$. If $p \in \mathbb{Z}$ then for every $q \geq p$ one has a natural algebraic isomorphism:

$$H^p(X, {}^q\mathcal{F}^\cdot) \xrightarrow{\sim} H^p(X, \mathcal{F}^\cdot).$$

By definition we take the natural topology on $H^p(X,\mathcal{F}^\cdot)$ to be the image through this isomorphism of the natural topology on $H^p(X,{}^q\mathcal{F}^\cdot)$. Since for every $p\leq q\leq r$ one has algebraic and consequently topological isomorphisms: $H^p(X,{}^q\mathcal{F}^\cdot) \xrightarrow{\sim} H^p(X,{}^r\mathcal{F}^\cdot)$, it follows that the natural topology is independent of the truncation of $\mathcal{F}^\cdot$.

As the trucation ${}^q\mathcal{F}^\cdot$ is a functorial operation, the natural topology on $H^q(X,\mathcal{F}^\cdot)$ has properties i), ii) and iii).

2. To obtain the natural QDFS topology on $H^q_c(X,\mathcal{F}^\cdot)$ one repetes the above considerations, replacing the open covering $\mathcal{U}$ by a locally finite covering $\mathcal{K}$ of X with Stein compact sets and the Čech complexes $C^\cdot(\mathcal{U},\cdot)$ by $C^\cdot_c(\mathcal{K},\cdot)$.

i) is proved exactly as 1 i).

ii) If $\mathcal{K}=(K_i)_{i\in N}$ is a compact covering of X and $K\subset X$ is a compact set we define the sets of indexes: $I(\mathcal{K},K)=\{i\in N \mid K_i\subset K\}$ and $I'(\mathcal{K},K)=\{i\in N \mid K_i\cap K\neq\emptyset\}$.

Let $U\subset X$ be an open set and let $(L_i)_{i\in N}$ be a sequence of compact sets such that $L_i\subset \mathring{L}_{i+1}$ and $\bigcup_{i\in N} L_i=U$. It is not difficult to construct a sequence of coverings $(\mathcal{K}_i)_{i\in N}$ for X, $\mathcal{K}_i=(K_i^j)_{j\in N}$ and a covering $\mathcal{R}=(R_j)_{j\in N}$ for U such that:

a) $\mathcal{K}_i$ and $\mathcal{R}$ are locally finite coverings of X, respectively U, with Stein compact sets whose interiors cover X, respectively U.

b) for every $i\in N$, $I'(\mathcal{K}_i,L_i)= I'(\mathcal{R},L_i)$, and furthermore, for every $j\in I'(\mathcal{K}_i,L_i)$, $K_i^j=R_j$ (i.e. the coverings $\mathcal{R}$ and $\mathcal{K}_i$ coincide on a neighbourhood of L_i).

c) for every $i\in N$, $\mathcal{K}_{i+1}\prec\mathcal{K}_i$ and $\mathcal{R}\prec\mathcal{K}_i\cap U$.

We can obviously suppose that $\mathcal{F}^\bullet \in D^-_{coh}(X)$ and has acyclic components. Let $\mathcal{L}^\bullet \longrightarrow \mathcal{F}^\bullet$ be a resolution of $\mathcal{F}^\bullet$ with objects in $Coh(\mathcal{K}_1)$. For every $i \in N$, denote by $C^\bullet(I(\mathcal{K}_i, L_i))$, respectively $C^\bullet(I(\mathcal{R}, L_i))$ the subcomplexes of $C^\bullet_c(\mathcal{K}_i, \mathcal{L}^\bullet|\mathcal{K}_i)$, respectively $C^\bullet_c(\mathcal{R}, \mathcal{L}^\bullet|\mathcal{R})$ determined by the finite sets of indexes $I(\mathcal{K}_i, L_i)$, respectively $I(\mathcal{R}, L_i)$ and by u_i, respectively v_i the inclusion morphisms.

One has $C^\bullet(I(\mathcal{K}_i, L_i) = C^\bullet(I(\mathcal{R}, L_i))$. Taking inductive limits we obtain the diagram:

$$\begin{array}{ccc} \text{inj lim } C^\bullet_c(\mathcal{K}_i, \mathcal{L}^\bullet|\mathcal{K}_i) & \overset{u}{\longleftarrow\!-\,-\,-} & C^\bullet_c(\mathcal{R}, \mathcal{L}^\bullet|\mathcal{R}) \\ \uparrow \scriptstyle{\text{inj lim } u_i} & & \uparrow \scriptstyle{\text{inj lim } v_i} \\ \text{inj lim } C^\bullet(I(\mathcal{K}_i, L_i)) & = & \text{inj lim } C^\bullet(I(\mathcal{R}, L_i)) \end{array}$$

Since all the spaces in the diagram are of type DFS it follows that inj lim v_i is a topological isomorphism and consequently that u is continuous. On the other hand, u induces on the cohomology the usual extension morphism: $H^\bullet_c(U, \mathcal{F}^\bullet) \longrightarrow H^\bullet_c(X, \mathcal{F}^\bullet)$.

<u>Remark 3</u>. Let $\mathcal{F}^\bullet \in D^-_{coh}(X)$ with components acyclic on Stein open sets and let $\mathcal{K}_1 \prec \mathcal{U} \prec \mathcal{K}_2$ be locally finite coverings of X, $\mathcal{K}_1$ and $\mathcal{K}_2$ with Stein compact sets and $\mathcal{U}$ with Stein open sets. If $\mathcal{L}^\bullet \longrightarrow \mathcal{F}^\bullet$ is a resolution of $\mathcal{F}^\bullet$ with components in $Coh(\mathcal{K}_2)$, then the commutative diagram of continuous quasiisomorphisms shows that one can obtain the natural topology on $H^q_c(X, \mathcal{F}^\bullet)$ by working with open coverings of X:

$$\begin{array}{ccc} C^\bullet_c(\mathcal{K}_2, \mathcal{L}^\bullet) & \longrightarrow & C^\bullet_c(\mathcal{K}_1, \mathcal{L}^\bullet|\mathcal{K}_1) \\ & \searrow \quad \swarrow & \\ & C^\bullet_c(\mathcal{U}, \mathcal{L}^\bullet|\mathcal{U}) & \end{array}$$

Proposition 4. Let either a) $\mathcal{E}^\cdot \in D^-_{coh}(X)$, $\mathcal{F}^\cdot \in D^+_{coh}(X)$ or b) $\mathcal{E}^\cdot \in D_{coh}(X)$, $\mathcal{F}^\cdot \in D^b_{coh}(X)$ has f.i.d. or c) $\mathcal{E}^\cdot \in D^b_{coh}(X)$ has f.t.d., $\mathcal{F}^\cdot \in D_{coh}(X)$.

1. There exists a topology of type QFS on $Ext^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$, called the natural topology, such that:

i) if $\mathcal{E}^\cdot_1, \mathcal{E}^\cdot_2 \in D_{coh}(X)$ and $u: \mathcal{E}^\cdot_1 \longrightarrow \mathcal{E}^\cdot_2$ is a morphism, respectively if $\mathcal{F}^\cdot_1, \mathcal{F}^\cdot_2 \in D_{coh}(X)$ and $v: \mathcal{F}^\cdot_1 \longrightarrow \mathcal{F}^\cdot_2$ is a morphism then the induced morphisms between Ext^q-invariants $\tilde{u}$, respectively $\tilde{v}$, are continuous. Moreover, if $\tilde{u}$ or $\tilde{v}$ are algebraic isomorphisms, they are topological isomorphisms as well.

ii) if $U \subset X$ is an open set then the restriction $Ext^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot) \longrightarrow$ $\longrightarrow Ext^q(U;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$ is continuous.

iii) if X is a Stein space then the natural morphism $Ext^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot) \to$ $\to \Gamma(X, \mathcal{E}xt^q(\mathcal{E}^\cdot,\mathcal{F}^\cdot))$ is a topological isomorphism when on the right hand side term one takes the usual topology on the sections of a coherent sheaf.

2. There exists a topology of type QDFS on $Ext^q_c(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$, called the natural topology, such that:

i) same as 1 i) with compact supports.

ii) if $U \subset X$ is an open set then the extension $Ext^q_c(U;\mathcal{E}^\cdot,\mathcal{F}^\cdot) \longrightarrow$ $\longrightarrow Ext^q_c(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$ is continuous.

Proof. Let $\mathcal{U} = (U_i)_{i \in I}$ be a locally finite covering of X with Stein open sets such that for every $i \in I$, $\overline{U}_i$ is a Stein compact set.

We treat first the case $\mathcal{E}^\cdot \in D^-_{coh}(X)$, $\mathcal{F}^\cdot \in D^b_{coh}(X)$. Let $\mathcal{E}^\cdot \longrightarrow \mathcal{I}^\cdot$, $\mathcal{F}^\cdot \longrightarrow \mathcal{J}^\cdot$ be resolutions with sheaves acyclic on Stein open sets for $\mathcal{E}^\cdot$ and $\mathcal{F}^\cdot$, $\mathcal{I}^\cdot \in K^-(X)$, $\mathcal{J}^\cdot \in K^b(X)$; let $\mathcal{L}^\cdot \longrightarrow \mathcal{I}^\cdot$ be a free resolution for $\mathcal{I}^\cdot$ in $K^-(\mathcal{U})$ and $\mathcal{M}^\cdot \longrightarrow \mathcal{J}^\cdot$ be a resolution for $\mathcal{J}^\cdot$ with components in $Coh(\mathcal{U})$. As we have seen, $Hom^\cdot(\mathcal{L}^\cdot,\mathcal{M}^\cdot)$ is a representative for $R\mathcal{H}om^\cdot(\mathcal{E}^\cdot,\mathcal{F}^\cdot)$

and is a complex of FS spaces (since, according to remark 2.1, every space $Hom(\mathcal{L}^i, \mathcal{M}^j)$ is of type FS). The natural topology on $Ext^q(X; \mathcal{E}^\cdot, \mathcal{F}^\cdot)$ is by definition the topology induced by $\Gamma Hom^\cdot(\mathcal{L}^\cdot, \mathcal{M}^\cdot)$.

Lemma 5. If $\mathcal{E}^\cdot \in D^-_{coh}(X)$, $\mathcal{F}^\cdot \in D^b_{coh}(X)$, the natural topology on $Ext^q(X; \mathcal{E}^\cdot, \mathcal{F}^\cdot)$ coincides with that on $H^q(X, R\mathcal{H}om^\cdot(\mathcal{E}^\cdot, \mathcal{F}^\cdot))$.

Proof. $\mathcal{H}om^\cdot(\mathcal{L}^\cdot, \mathcal{M}^\cdot)$, which is a representative for $R\mathcal{H}om^\cdot(\mathcal{E}^\cdot, \mathcal{F}^\cdot)$ has coherent cohomology and components acyclic on Stein open sets. Since for every $p \in Z$, ${}^p\mathcal{H}om^\cdot(\mathcal{L}^\cdot, \mathcal{M}^\cdot)$ has the same properties (see remark 1.2), it admits a resolution $\mathcal{L}^\cdot_1 \longrightarrow {}^p\mathcal{H}om^\cdot(\mathcal{L}^\cdot, \mathcal{M}^\cdot)$ with s.s.s. in $Coh(\mathcal{U})$. Taking Čech complexes one obtains a continuous quasiisomorphism: $C^\cdot(\mathcal{U}, \mathcal{L}^\cdot_1) \longrightarrow C^\cdot(\mathcal{U}, {}^p\mathcal{H}om^\cdot(\mathcal{L}^\cdot, \mathcal{M}^\cdot))$ which, according to proposition 1, 1.i shows that for $p>q$ the topology induced on $Ext^q(X; \mathcal{E}^\cdot, \mathcal{F}^\cdot) = H^q(X, \mathcal{H}om^\cdot(\mathcal{L}^\cdot, \mathcal{M}^\cdot)) = H^q(X, {}^p\mathcal{H}om^\cdot(\mathcal{L}^\cdot, \mathcal{M}^\cdot))$ by $C^\cdot(\mathcal{U}, {}^p\mathcal{H}om^\cdot(\mathcal{L}^\cdot, \mathcal{M}^\cdot))$ coincides with the natural topology on $H^q(X, R\mathcal{H}om^\cdot(\mathcal{E}^\cdot, \mathcal{F}^\cdot))$.

On the other hand, the natural morphism: ${}^p Hom^\cdot(\mathcal{L}^\cdot, \mathcal{M}^\cdot) \longrightarrow C^\cdot(\mathcal{U}, {}^p\mathcal{H}om^\cdot(\mathcal{L}^\cdot, \mathcal{M}^\cdot))$ is continuous and induces isomorphism on the cohomology in degrees $\leq p$. Since ${}^p Hom^\cdot(\mathcal{L}^\cdot, \mathcal{M}^\cdot)$ induces the natural topology on $Ext^q(X; \mathcal{E}^\cdot, \mathcal{F}^\cdot)$ the statement follows.

Lemma 5 implies that the natural topology on $Ext^q(X; \mathcal{E}^\cdot, \mathcal{F}^\cdot)$ is independent of the different choices and has properties i, ii, iii.

Now in case a) the natural topology is introduced using lemma 1.6 in case b) lemma 1.7 and in case c) lemma 3.8.

Remark 6. If $\mathcal{E}^\cdot, \mathcal{F}^\cdot$ are as in proposition 4, it is easy to deduce from the proof of this proposition that one has topological isomorphisms: $Ext^q(X; \mathcal{E}^\cdot, \mathcal{F}^\cdot) \xrightarrow{\sim} H^q(X, R\mathcal{H}om^\cdot(\mathcal{E}^\cdot, \mathcal{F}^\cdot))$ and $Ext^q_c(X; \mathcal{E}^\cdot, \mathcal{F}^\cdot) \xrightarrow{\sim} H^q_c(X, R\mathcal{H}om^\cdot(\mathcal{E}^\cdot, \mathcal{F}^\cdot))$.

Corollary 7. If $\mathcal{G}^\cdot \in D_{coh}(X)$ then for every $q \in Z$ one has topological isomorphisms $H^q(X,\mathcal{G}^\cdot) \xrightarrow{\sim} \mathrm{Ext}^q(X;\mathcal{O}_X,\mathcal{G}^\cdot)$; $H^q_c(X,\mathcal{G}^\cdot) \xrightarrow{\sim} \mathrm{Ext}^q_c(X;\mathcal{O}_X,\mathcal{G}^\cdot)$.

<u>Proof</u>. According to corollary 3.2, there exists a covering $\mathcal{U}$ and a free resolution of finite length $\mathcal{L}^\cdot \longrightarrow \mathcal{O}_X$ in $K(\mathcal{U})$. Let $\mathcal{G}^\cdot \longrightarrow \mathcal{J}^\cdot$ be a resolution of $\mathcal{G}^\cdot$ with acyclic and $\Gamma_c(X,\cdot)$-acyclic sheaves (see 1.1). Using remark 3.9 it follows that the natural morphism $\mathcal{H}om^\cdot(\mathcal{O}_X,\mathcal{J}^\cdot) \longrightarrow \mathcal{H}om^\cdot(\mathcal{L}^\cdot,\mathcal{J}^\cdot)$ is a quasiisomorphism. Since $\mathcal{H}om^\cdot(\mathcal{O}_X,\mathcal{J}^\cdot)$ is isomorphic to $\mathcal{J}^\cdot$ and since both complexes have $\Gamma(X,\cdot)$- and $\Gamma_c(X,\cdot)$-acyclic components, the statement follows.

<u>Corollary 8</u>. Let X be a compact analytic space. If $\mathcal{G}^\cdot \in D_{coh}(X)$ then the spaces $H^q(X,\mathcal{G}^\cdot)$ are finite dimensional and therefore separate in the natural topology. If $\mathcal{E}^\cdot, \mathcal{F}^\cdot \in D_{coh}(X)$ are in one of the cases a), b) or c) in proposition 4, then the spaces $\mathrm{Ext}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$ are finite dimensional and therefore separate in the natural topology.

<u>Proof</u>. If $\mathcal{E}^\cdot, \mathcal{F}^\cdot$ are as in proposition 4, then $R\mathcal{H}om^\cdot(\mathcal{E}^\cdot,\mathcal{F}^\cdot) \in D_{coh}(X)$ and according to remark 6 it is sufficient to prove the statement on the hypercohomology. The second drawer terms of the regular spectral sequence $E_2^{pq} = H^p(X,\mathcal{H}^q(\mathcal{G}^\cdot))$ are finite dimensional spaces. This implies that $H^q(X,\mathcal{G}^\cdot)$ admits a finite filtration with finite dimensional vector spaces and so is itself finite dimensional.

<u>Remark 9</u>. Since the global hypertor spaces have been defined as hypercohomology spaces, it follows from proposition 1 that if $\mathcal{E}^\cdot, \mathcal{F}^\cdot$ are as in proposition 1.14, there exist natural topologies on $\mathrm{Tor}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$ and $\mathrm{Tor}^q_c(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$. These natural topologies can also be introduced using resolutions with s.s.s. for $\mathcal{E}^\cdot$ and $\mathcal{F}^\cdot$, as we did in the case of the hyperext in proposition 4.

5. DUALITY RESULTS

Since the cohomological invariants depend only on the quasi-isomorphism class of the complexes involved (being defined on the derived category) we shall suppose that all the complexes that appear have components acyclic on Stein open sets (see 1.1).

Proof of theorem 2 (see the introduction).

Making the identifications: $\mathrm{Ext}^q(X;\mathcal{O}_X,\mathcal{F}^\bullet)\simeq H^q(X,\mathcal{F}^\bullet)$, $\mathrm{Ext}_c^q(X;\mathcal{O}_X,K_X^\bullet)\simeq H_c^q(X,K_X^\bullet)$ and considering the functor $\mathrm{Ext}_c^{-q}(X;\bullet,K_X^\bullet)$ one obtains a Yoneda pairing (see 1.4):

$$H^q(X,\mathcal{F}^\bullet)\times \mathrm{Ext}_c^{-q}(X;\mathcal{F}^\bullet,K_X^\bullet)\longrightarrow H_c^0(X,K_X^\bullet)$$

The pairing in the theorem is given by the composition of the trace morphism $T_X:H_c^0(X,K_X^\bullet)\longrightarrow \mathbb{C}$ defined in [13], with this Yoneda pairing.

The proof has three steps: the first two are actually adaptations of the proofs of the absolute duality theorems of Ramis and Ruget (see [13] or [2]); the third one establishes the identity between the duality topologies defined at steps 1 and 2 and the natural topologies.

Step 1. There exists a topology of type QDFS on $\mathrm{Ext}_c^{-q}(X;\mathcal{F}^\bullet,K_X^\bullet)$ (not necessarily the natural one) such that the statement of the theorem is true when on $H^q(X,\mathcal{F}^\bullet)$ one considers the natural topology.

For this it is sufficient to suppose that $\mathcal{F}^\bullet\in D^-_{coh}(X)$. If $\mathcal{F}^\bullet$ is not bounded to the right one takes a truncation ${}^p\mathcal{F}^\bullet$ of $\mathcal{F}^\bullet$ with $p\geq q$. The natural morphism ${}^p\mathcal{F}^\bullet\longrightarrow\mathcal{F}^\bullet$ induces isomorphisms:

$$u:H^q(X,{}^p\mathcal{F}^\bullet)\longrightarrow H^q(X,\mathcal{F}^\bullet)$$

$$v:\mathrm{Ext}_c^{-q}(X;\mathcal{F}^\bullet,K_X^\bullet)\longrightarrow \mathrm{Ext}_c^{-q}(X;{}^p\mathcal{F}^\bullet,K_X^\bullet)$$

which are one the transposed of the other with respect to the

Yoneda pairings. Since u is a topological isomorphism for the natural topologies (see proposition 4.1), if we have found the required topology on $\mathrm{Ext}_c^{-q}(X;{}^{p}\mathcal{F}^\cdot,K_X^\cdot)$ then its image through v^{-1} is the required topology on $\mathrm{Ext}_c^{-q}(X;\mathcal{F}^\cdot,K_X^\cdot)$.

Let $\mathcal{U}=(U_i)_{i\in I}$ be a locally finite covering of X with relatively compact Stein open sets such that $\mathcal{F}^\cdot$ admits a resolution $\mathcal{L}^\cdot \longrightarrow \mathcal{F}^\cdot$ where $\mathcal{L}^\cdot \in D^-(\mathcal{U})$ and has components in $\mathrm{Coh}(\mathcal{U})$.

Let $K^{pq}=C^p(\mathcal{U},\mathcal{L}^q)=\prod_{|\alpha|=p+1}\Gamma(U_\alpha,\mathcal{L}_\alpha^q)$. As we have already seen, the simple complex associated to K^{pq} is a complex of FS spaces which induces on $H^q(X,\mathcal{F}^\cdot)$ the natural topology.

Let now $L^{pq}=\bigoplus_{|\alpha|=p+1}\mathrm{Ext}_c^0(U_\alpha;\mathcal{L}_\alpha^q,K_X^\cdot)$ be the double complex obtained by dualizing K^{pq}. To get the result it is sufficient to prove that the cohomology of L^{pq} is $\mathrm{Ext}_c^q(X;\mathcal{F}^\cdot,K_X^\cdot)$ (one can check that the duality induced on the cohomology of L^{pq} and K^{pq} is given by the Yoneda mapping and the trace).

Let $K_X^\cdot \longrightarrow \mathcal{J}^\cdot$ be an injective resolution for $K_X^\cdot$. We consider the complexes:

$$M^{pqr}=\bigoplus_{|\alpha|=p+1}\Gamma_c(U_\alpha,\mathcal{H}om(\mathcal{L}_\alpha^q,\mathcal{J}^r))$$

$$N^{pqr}=\bigoplus_{|\alpha|=p+1}\Gamma_c(U_\alpha,\mathcal{H}om(\mathcal{F}^q,\mathcal{J}^r))$$

According to the usual duality theorem for a Stein space, the cohomology of M^{pqr} along the r-direction is:

$$\begin{cases} 0 & \text{for } r\neq 0 \\ \bigoplus_{|\alpha|=p+1}\mathrm{Ext}_c^0(U_\alpha;\mathcal{L}_\alpha^q,K_X^\cdot)=L^{pq} & \text{for } r=0 \end{cases}$$

and consequently the simple complexes associated to L^{pq} and M^{pqr} have the same cohomology. It is easy to see that the quasiisomorphism $\mathcal{L}^{\cdot} \longrightarrow \mathcal{F}^{\cdot}$ induces a morphism $N^{pqr} \longrightarrow M^{pqr}$ which, if p and r are kept fixed, is a quasiisomorphism along the q-direction and so the simple complexes associated to M^{pqr} and N^{pqr} are quasiisomorphic. The cohomology of N^{pqr} in direction p is:

$$\begin{cases} 0 & \text{for } p\neq 0 \\ \Gamma_c(X, \mathcal{H}om(\mathcal{F}^q, \mathcal{J}^r)) & \text{for } p=0 \end{cases}$$

(this follows from lemma 7.3.9 in [2], by remarking that the sheaves $\mathcal{H}om(\mathcal{F}^q, \mathcal{J}^r)$ are flabby). It is now clear that the cohomology of N^{pqr} is $Ext_c^q(X;\mathcal{F}^{\cdot}, K_X^{\cdot})$, which ends the proof of step 1.

Moreover, it is easy to see that, according to a known result of duality, the separation result of the theorem is true when on $H^q(X,\mathcal{F}^{\cdot})$ one takes the natural topology and on $Ext_c^{1-q}(X;\mathcal{F}^{\cdot},K_X^{\cdot})$ the topology induced by L^{pq}.

In a similar way one can prove:

Step 2. There exists a topology of type QFS on $Ext^{-q}(X;\mathcal{F}^{\cdot},K_X^{\cdot})$ (not necessarily the natural one) such that the statement of the theorem is true when on $H_c^q(X,\mathcal{F}^{\cdot})$ one considers the natural topology.

Step 3. The "duality" topologies induced at steps 1 and 2 on $Ext^{-q}(X;\mathcal{F}^{\cdot},K_X^{\cdot})$ and $Ext_c^{-q}(X;\mathcal{F}^{\cdot},K_X^{\cdot})$ coincide with the natural topologies.

Indeed, according to step 1 one has a duality modulo separation:

$$H^q(X,\mathcal{F}^{\cdot}) \times Ext_c^{-q}(X;\mathcal{F}^{\cdot},K_X^{\cdot}) \longrightarrow H_c^0(X,K_X^{\cdot}) \longrightarrow \mathbb{C}$$

and according to step 2 a duality modulo separation:

$$H_c^{-q}(X,D(\mathcal{F}^{\cdot})) \times Ext^q(X;D(\mathcal{F}^{\cdot}),K_X^{\cdot}) \longrightarrow H_c^0(X,K_X^{\cdot}) \longrightarrow \mathbb{C}$$

where $H^q(X,\mathcal{F}^\cdot)$ and $H_c^{-q}(X,D(\mathcal{F}^\cdot))$ carry the natural topologies and $\mathrm{Ext}_c^{-q}(X;\mathcal{F}^\cdot,K_X^\cdot)$ and $\mathrm{Ext}^q(X;D(\mathcal{F}^\cdot),K_X^\cdot)$ the "duality" topologies.

Let $u:H^q(X,\mathcal{F}^\cdot)\longrightarrow \mathrm{Ext}^q(X;D(\mathcal{F}^\cdot),K_X^\cdot)$ be the isomorphism induced by the isomorphism $\mathcal{F}^\cdot \longrightarrow DD(\mathcal{F}^\cdot)$ (see [13], proposition 1) and $v:H_c^{-q}(X,D(\mathcal{F}^\cdot))\longrightarrow \mathrm{Ext}_c^{-q}(X;\mathcal{F}^\cdot,K_X^\cdot)$ the canonical identification. It is easy to verify that u and v are each the transposed of the other with respect to the Yoneda pairings above. According to lemma 1.4 in [13] this implies that v is continuous and consequently, that the topology on $\mathrm{Ext}_c^{-q}(X;\mathcal{F}^\cdot,K_X^\cdot)$ is weaker than the natural topology. On the other hand, since u is an isomorphism, it follows that v maps the closure of $\{0\}$ in $H_c^{-q}(X,D(\mathcal{F}^\cdot))$ bijectively on the closure of $\{0\}$ in $\mathrm{Ext}_c^{-q}(X;\mathcal{F}^\cdot,K_X^\cdot)$; this implies that v is a topological isomorphism and so that the topology on $\mathrm{Ext}_c^{-q}(X;\mathcal{F}^\cdot,K_X^\cdot)$ coincides with the natural one.

Similarly, since u is continuous and maps bijectively the closure of $\{0\}$ in $H^q(X,\mathcal{F}^\cdot)$ on the closure of $\{0\}$ in $\mathrm{Ext}^q(X;D(\mathcal{F}^\cdot),K_X^\cdot)$ it follows that the second space carries the natural topology. In particular, replacing $\mathcal{F}^\cdot$ with $D(\mathcal{F}^\cdot)$ and using once again the reflexivity of $\mathcal{F}^\cdot$ with respect to D, one gets that $\mathrm{Ext}^{-q}(X;\mathcal{F}^\cdot,K_X^\cdot)$ carries the natural topology, which ends the proof.

<u>Proof of theorem 1</u> (see the introduction)

If $\mathcal{G}^\cdot\in D^-(X)$ it is easy to see that the natural morphism:

$$\mathcal{G}^\cdot\otimes \mathcal{H}om^\cdot(\mathcal{G}^\cdot,K_X^\cdot)\longrightarrow K_X^\cdot$$

induces on D(X) a morphism $\mathcal{G}^\cdot \underline{\underline{\otimes}} D(\mathcal{G}^\cdot)\xrightarrow{\alpha} K_X^\cdot$. As in theorem 2, the pairing in theorem 1 is obtained by composing the Yoneda pairing for the invariants involved with a trace mapping. This trace mapping is given by the composition of mappings:

$$\mathrm{Tor}_c^0(X;D(\mathcal{G}^\cdot),\mathcal{G}^\cdot)\xrightarrow{H_c^0(\alpha)}H_c^0(X,K_X^\cdot)\xrightarrow{T_X}\mathbb{C}$$

where $\mathcal{G}^\cdot$ is the one of the complexes $\mathcal{E}^\cdot$ and $\mathcal{F}^\cdot$ which is bounded to the right.

The theorem is based on the following result:

Lemma 5. If $\mathcal{E}^\cdot$, $\mathcal{F}^\cdot \in D_{coh}(X)$ are in one of the cases a), b) or c) of theorem 1, then there exists a functorial isomorphism:

$$D(\mathcal{F}^\cdot)\underline{\otimes}\mathcal{E}^\cdot \xrightarrow{\sim} D(R\mathcal{H}om^\cdot(\mathcal{E}^\cdot,\mathcal{F}^\cdot))$$

Proof. Cases a) and b) are treated in [11], proposition V.2.6.b. For case c), let $K_X^\cdot \longrightarrow \mathcal{J}^\cdot$ be an injective resolution of $K_X^\cdot$, and $\mathcal{L}^\cdot \longrightarrow \mathcal{E}^\cdot$ a free resolution of finite length of $\mathcal{E}^\cdot$ in a suitable $D(\mathcal{U})$. According to remark 2.3, $\hat{\mathcal{L}}^\cdot \longrightarrow \mathcal{E}^\cdot$ is still a resolution for $\mathcal{E}^\cdot$ and since $\mathcal{L}^\cdot$ has free components, $\hat{\mathcal{L}}^\cdot$ is a flat resolution. The morphism in the lemma is given by:

$$\mathcal{H}om^\cdot(\mathcal{F}^\cdot,\mathcal{J}^\cdot)\otimes\hat{\mathcal{L}}^\cdot \longrightarrow \mathcal{H}om^\cdot(\mathcal{H}om^\cdot(\hat{\mathcal{L}}^\cdot,\mathcal{F}^\cdot),\mathcal{J}^\cdot)$$

which induces isomorphism on fibers (this follows from remark 3.9 and the fact that $\hat{\mathcal{L}}^\cdot$ has fibers free of finite rank).

If $\mathcal{E}^\cdot$, $\mathcal{F}^\cdot$ are in one of the cases a), b) or c) both $\mathcal{E}^\cdot\underline{\otimes}D(\mathcal{F}^\cdot)$ and $D(R\mathcal{H}om^\cdot(\mathcal{E}^\cdot,\mathcal{F}^\cdot))$ have coherent cohomology and so, the isomorphism in lemma 5 induces a topological isomorphism:

$$\beta:\mathrm{Tor}_c^q(X;D(\mathcal{F}^\cdot),\mathcal{E}^\cdot)\longrightarrow \mathrm{Ext}_c^{-q}(X;R\mathcal{H}om^\cdot(\mathcal{E}^\cdot,\mathcal{F}^\cdot),K_X^\cdot)\ .$$

when on the two spaces one considers the natural topologies.

Since in all three cases $R\mathcal{H}om^\cdot(\mathcal{E}^\cdot,\mathcal{F}^\cdot)$ has coherent cohomology, applying theorem 2, one gets that the Yoneda pairing and the trace morphism induce a topological duality between the separated vector spaces associated to the spaces:

$$H^q(X, R\mathcal{H}om^\cdot(\mathcal{E}^\cdot, \mathcal{F}^\cdot)) = \mathrm{Ext}^q(X; \mathcal{E}^\cdot, \mathcal{F}^\cdot) \quad \text{and} \quad \mathrm{Ext}_c^{-q}(X; R\mathcal{H}om^\cdot(\mathcal{E}^\cdot, \mathcal{F}^\cdot), K_X^\cdot) \quad ,$$

considered with their natural topologies.

Moreover, the separation result in theorem 2 together with the isomorphism β imply that $\mathrm{Ext}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$ is separated iff $\mathrm{Tor}_c^{q-1}(X;D(\mathcal{F}^\cdot),\mathcal{E}^\cdot)$ is.

In cases a) and c) the theorem follows from the commutative diagram:

$$\begin{array}{ccc}
H^q(X, R\mathcal{H}om^\cdot(\mathcal{E}^\cdot,\mathcal{F}^\cdot)) \times \mathrm{Ext}_c^{-q}(X; R\mathcal{H}om^\cdot(\mathcal{E}^\cdot,\mathcal{F}^\cdot), K_X^\cdot) & \longrightarrow & H_c^0(X, K_X^\cdot) \\
\wr\Big\uparrow (u,\beta) & & \Big\uparrow \\
\mathrm{Ext}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot) \times \mathrm{Tor}_c^q(X; D(\mathcal{F}^\cdot),\mathcal{E}^\cdot) & \longrightarrow & \mathrm{Tor}_c^0(X; D(\mathcal{E}^\cdot),\mathcal{E}^\cdot)
\end{array}$$

in which u is the canonical identification and the horizontal morphisms are the Yoneda pairings corresponding to the functors $\mathrm{Ext}_c^{-q}(X;\cdot,K_X^\cdot)$ respectively $\mathrm{Tor}_c^q(X;D(\cdot),\mathcal{E}^\cdot)$.

In case b) since $\mathcal{E}^\cdot$ is not bounded to the right, the space $\mathrm{Tor}_c^0(X;D(\mathcal{E}^\cdot),\mathcal{E}^\cdot)$ cannot be defined. In this case the theorem follows from a similar diagram, in which $\mathrm{Tor}_c^0(X;D(\mathcal{E}^\cdot),\mathcal{E}^\cdot)$ is replaced by $\mathrm{Tor}_c^0(X;D(\mathcal{F}^\cdot),\mathcal{F}^\cdot)$ and the last Yoneda pairing corresponds to the functor $\mathrm{Tor}_c^q(X,D(\mathcal{F}^\cdot),\cdot)$.

The duality between $\mathrm{Ext}_c^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$ and $\mathrm{Tor}^q(X;D(\mathcal{F}^\cdot),\mathcal{E}^\cdot)$ can be proved in a similar way.

<u>Proof of theorem 3</u> (see the introduction)

We shall use the following:

<u>Lemma 6</u>. If $\mathcal{E}^\cdot, \mathcal{F}^\cdot \in D_{coh}(X)$ are in one of the cases d) or e) of theorem 3 then there exists a functorial isomorphism:

$$R\mathcal{H}om^\cdot(\mathcal{F}^\cdot, K_X^\cdot) \underline{\underline{\otimes}} \mathcal{E}^\cdot \xrightarrow{\sim} R\mathcal{H}om^\cdot(\mathcal{F}^\cdot, \mathcal{E}^\cdot \underline{\underline{\otimes}} K_X^\cdot) \quad .$$

Proof. Case d) is treated in [11], proposition II.5.14.

e) Let $K_X^\bullet \longrightarrow \mathcal{J}^\bullet$ be an injective resolution for $K_X^\bullet$, $\mathcal{P}^\bullet \longrightarrow \mathcal{E}^\bullet$ a flat resolution for $\mathcal{E}^\bullet$, $\mathcal{P}^\bullet \otimes \mathcal{J}^\bullet \longrightarrow \mathcal{I}^\bullet$ a resolution with acyclic sheaves for $\mathcal{P}^\bullet \otimes \mathcal{J}^\bullet$ and $\mathcal{L}^\bullet \longrightarrow \mathcal{F}^\bullet$ a free resolution of finite length for $\mathcal{F}^\bullet$, in a suitable $D(\mathcal{U})$.

The morphism in the lemma is given by the compositionof morphisms:

$$\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{J}^\bullet) \otimes \mathcal{P}^\bullet \longrightarrow \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{J}^\bullet \otimes \mathcal{P}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$$

where obviously the extreme terms are representatives for $R\mathcal{H}om^\bullet(\mathcal{F}^\bullet, K_X^\bullet) \underline{\underline{\otimes}} \mathcal{E}^\bullet$ and $R\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{E}^\bullet \underline{\underline{\otimes}} K_X^\bullet)$.

Since $\mathcal{L}^\bullet$ has fibers free of finite rank, it follows applying remårk 3.9 that the above morphism induces quasiisomorphism on every fiber.

In particular, if $\mathcal{E}^\bullet \in D^b_{coh}(X)$ has f.t.d. then, according to lemma 6 one has a functorial isomorphism:

$$\delta : R\mathcal{H}om^\bullet(\mathcal{E}^\bullet, K_X^\bullet) \underline{\underline{\otimes}} \mathcal{E}^\bullet \longrightarrow R\mathcal{H}om^\bullet(\mathcal{E}^\bullet, \mathcal{E}^\bullet \underline{\underline{\otimes}} K_X^\bullet)$$

As in theorems 1 and 2, the pairing in theorem 3 is obtained by composing the Yoneda pairing for the invariants involved,with a trace morphism. This trace morphism will be in case a) the composition of mappings:

$$\mathrm{Ext}^0_c(X; \mathcal{E}^\bullet, \mathcal{E}^\bullet \underline{\underline{\otimes}} K_X^\bullet) \xrightarrow{H^0_c(\delta)^{-1}} \mathrm{Tor}^0_c(X; D(\mathcal{E}^\bullet), \mathcal{E}^\bullet) \longrightarrow H^0_c(X, K_X^\bullet) \xrightarrow{T_X} \mathbb{C}$$

and in case b) the same composition with $\mathcal{E}^\bullet$ replaced by $\mathcal{F}^\bullet$.

Since $R\mathcal{H}om^\bullet(\mathcal{F}^\bullet, K_X^\bullet) \underline{\underline{\otimes}} \mathcal{E}^\bullet$ and $R\mathrm{Hom}^\bullet(\mathcal{F}^\bullet, \mathcal{E}^\bullet \underline{\underline{\otimes}} K_X^\bullet)$ have both coherent cohomology it follows that the isomorphism in lemma 6 induces topological isomorphisms:

$$\gamma : \mathrm{Tor}^q_c(X; D(\mathcal{F}^\bullet), \mathcal{E}^\bullet) \longrightarrow \mathrm{Ext}^{-q}_c(X; \mathcal{F}^\bullet, \mathcal{E}^\bullet \underline{\underline{\otimes}} K_X^\bullet)$$

when on the two spaces one considers the natural topology. The separation result in theorem 2, together with the isomorphism γ imply that $\mathrm{Ext}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$ is separated iff $\mathrm{Ext}_c^{1-q}(X;\mathcal{F}^\cdot,\mathcal{E}^\cdot \underline{\underline{\otimes}}\, K_X^\cdot)$ is.

The duality statement in case d) follows from theorem 2 by reading the commutative diagram:

$$
\begin{array}{ccc}
H^q(X,R\mathcal{H}om^\cdot(\mathcal{E}^\cdot,\mathcal{F}^\cdot)) \times \mathrm{Ext}_c^{-q}(X;R\mathcal{H}om^\cdot(\mathcal{E}^\cdot,\mathcal{F}^\cdot),K_X^\cdot) & \longrightarrow & H_c^0(X,K_X^\cdot) \\
\uparrow \wr \; (\beta,u) & & \uparrow \\
\mathrm{Ext}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot) \times \mathrm{Tor}_c^q(X;D(\mathcal{F}^\cdot),\mathcal{E}^\cdot) & \longrightarrow & \mathrm{Tor}_c^0(X;D(\mathcal{E}^\cdot),\mathcal{E}^\cdot) \\
\downarrow \wr \; (id,\gamma) & & \downarrow H_c^0(\delta) \\
\mathrm{Ext}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot) \times \mathrm{Ext}_c^{-q}(X;\mathcal{F}^\cdot,\mathcal{E}^\cdot \underline{\underline{\otimes}}\, K_X^\cdot) & \longrightarrow & \mathrm{Ext}_c^0(X;\mathcal{E}^\cdot,\mathcal{E}^\cdot \underline{\underline{\otimes}}\, K_X^\cdot)
\end{array}
$$

(The upper square of the diagram has appeared in the proof of theorem 1; the last horizontal arrow is the Yoneda pairing corresponding to the functor $\mathrm{Ext}_c^{-q}(X;\bullet,\mathcal{E}^\cdot \underline{\underline{\otimes}}\, K_X^\cdot)$).

Case e) follows from a similar commutative diagram and the duality between the spaces $\mathrm{Ext}_c^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$ and $\mathrm{Ext}^{-q}(X;\mathcal{F}^\cdot,\mathcal{E}^\cdot \underline{\underline{\otimes}}\, K_X^\cdot)$ can be proved along the same lines.

In the compact case, theorems 1 and 3 give:

<u>Corollary 7</u>. Let X be a compact analytic space and a) $\mathcal{E}^\cdot \in D^-_{coh}(X)$, $\mathcal{F}^\cdot \in D^+_{coh}(X)$ or b) $\mathcal{E}^\cdot \in D_{coh}(X)$, $\mathcal{F}^\cdot \in D^b_{coh}(X)$ has f.i.d. or c) $\mathcal{E}^\cdot \in D^b_{coh}(X)$ has f.t.d., $\mathcal{F}^\cdot \in D_{coh}(X)$. Then the spaces $\mathrm{Ext}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot)$ and $\mathrm{Tor}^q(X;D(\mathcal{F}^\cdot),\mathcal{E}^\cdot)$ are finite dimensional and there exists a natural pairing which induces an algebraic duality between:

$$\mathrm{Ext}^q(X;\mathcal{E}^\cdot,\mathcal{F}^\cdot) \text{ and } \mathrm{Tor}^q(X;D(\mathcal{F}^\cdot),\mathcal{E}^\cdot).$$

Moreover, if d) $\mathcal{E}^\cdot \in D^b_{coh}(X)$ has f.t.d., $\mathcal{F}^\cdot \in D^-_{coh}(X)$ or e) $\mathcal{E}^\cdot \in D^-_{coh}(X)$,

$\mathcal{F}^{\cdot} \in D^{b}_{coh}(X)$ has f.t.d. then there exists a natural pairing which induces an algebraic duality between:

$$\mathrm{Ext}^{q}(X;\mathcal{E}^{\cdot},\mathcal{F}^{\cdot}) \text{ and } \mathrm{Ext}^{-q}(X;\mathcal{F}^{\cdot},\mathcal{E}^{\cdot}\underline{\underline{\otimes}}\, K^{\cdot}_{X}) .$$

Finally, in the Stein case we have:

Corollary 8. If X is a Stein space and a) $\mathcal{E}^{\cdot} \in D^{-}_{coh}(X)$, $\mathcal{F}^{\cdot} \in D^{+}_{coh}(X)$ or b) $\mathcal{E}^{\cdot} \in D_{coh}(X)$, $\mathcal{F}^{\cdot} \in D^{b}_{coh}(X)$ has f.i.d. or c) $\mathcal{E}^{\cdot} \in D^{b}_{coh}(X)$ has f.t.d., $\mathcal{F}^{\cdot} \in D_{coh}(X)$ then the spaces $\mathrm{Ext}^{q}(X;\mathcal{E}^{\cdot},\mathcal{F}^{\cdot})$, $\mathrm{Ext}^{q}_{c}(X;\mathcal{E}^{\cdot},\mathcal{F}^{\cdot})$, $\mathrm{Tor}^{q}(X;D(\mathcal{F}^{\cdot}),\mathcal{E}^{\cdot})$ and $\mathrm{Tor}^{q}_{c}(X;D(\mathcal{F}^{\cdot}),\mathcal{E}^{\cdot})$ are separated in the natural topology and there exists a natural pairing which induces topological dualities between:

$\mathrm{Ext}^{q}(X;\mathcal{E}^{\cdot},\mathcal{F}^{\cdot})$ and $\mathrm{Tor}^{q}_{c}(X;D(\mathcal{F}^{\cdot}),\mathcal{E}^{\cdot})$, respectively

$\mathrm{Ext}^{q}_{c}(X;\mathcal{E}^{\cdot},\mathcal{F}^{\cdot})$ and $\mathrm{Tor}^{q}(X;D(\mathcal{F}^{\cdot}),\mathcal{E}^{\cdot})$.

Moreover, if d) $\mathcal{E}^{\cdot} \in D^{b}_{coh}$ has f.t.d., $\mathcal{F}^{\cdot} \in D^{-}_{coh}(X)$, or e) $\mathcal{E}^{\cdot} \in D^{-}_{coh}(X)$ and $\mathcal{F}^{\cdot} \in D^{b}_{coh}(X)$ has f.t.d. then there exists a natural pairing which induces topological dualities between:

$\mathrm{Ext}^{q}(X;\mathcal{E}^{\cdot},\mathcal{F}^{\cdot})$ and $\mathrm{Ext}^{-q}_{c}(X;\mathcal{F}^{\cdot},\mathcal{E}^{\cdot}\underline{\underline{\otimes}}\, K^{\cdot}_{X})$, respectively

$\mathrm{Ext}^{q}_{c}(X;\mathcal{E}^{\cdot},\mathcal{F}^{\cdot})$ and $\mathrm{Ext}^{-q}(X;\mathcal{F}^{\cdot},\mathcal{E}^{\cdot}\underline{\underline{\otimes}}\, K^{\cdot}_{X})$.

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