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DUALITY FOR SHEAVES CATEGORIES

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Duality for Sheaves Categories

Introduction. In this paper we describe the dual of a Giraud -Grothendieck topos by means of uniform algebras. We do not reach here the whole generality of the categories $\mathcal{C}$ for which the duality theorem respectively the method applies to. As immediate consequences Stone duality, Oberst duality and Linton duality theorems are obtained.

This study serves, in my opinion, more as an example of a method to obtain dualities by means of injective cogenerators. As a duality theory it unifies the Stone duality and Gabriel-Roos-Oberst duality theories. It also starts a desired duality theory for locally noetherian toposes([19]).

What the main theorem essentially asserts is that for any Giraud-Grothendieck topos $\mathcal{C}$ there exists an algebraic category Σ^* over a finitary algebraic theory Σ such that $\mathcal{C}^o$ is equivalent to a category $\mathcal{D}$ of separated and complete uniform Σ-algebras.

In order to describe $\mathcal{D}$ we need the following definitions:

1) a uniform Σ-algebra $(A,\mathcal{U})$ is an algebra A endowed with a uniformity ([2],[10]) $\mathcal{U}$ such that all operations are uniform continuous maps in the product uniformity; a c-uniform Σ-algebra is a uniform Σ-algebra which has a fundamental system of entourages consisting of congruences. If Σ is the trivial theory then $\Sigma^* = \underline{SET}$ and we call c-uniform sets p-uniform spaces (this distinction from the nontrivial algebraic case must be done since

not any p-uniform algebra — i.e. the carrier set is a p-uniform space and all operations are uniform continuous in the product uniformity — is a c-uniform algebra)

2) a c-structure Γ on Σ/Σ^* is a functor

$$\Gamma : \Sigma \longrightarrow \underline{(\text{c-unif. } \Sigma\text{-algebras})}$$

for which the diagram

$$\begin{tikzcd}
\Sigma \arrow[rr, "\Gamma"] \arrow[dr, "\text{free algebra}"'] & & \underline{(\text{c-unif. } \Sigma\text{-algebras})} \arrow[dl, "\text{forget the uniformity}"] \\
& \Sigma^* &
\end{tikzcd}$$

commutes. A c-uniform Σ-algebra over Γ is a c-uniform Σ-algebra $(A,\mathcal{U})$ so that for each finitely generated free algebra L all algebra morphisms $L \to A$ are uniform continuous maps from $\Gamma(L)$ to $(A,\mathcal{U})$. Γ^* denotes the category of all c-uniform Σ-algebras over Γ and uniform continuous algebra morphisms between them.

3) For a c-structure Γ on Σ/Σ^*, $\underline{\text{Dis}\,\Gamma^*}$ denotes the category of all discrete uniform Σ-algebras which are in Γ^*. A coherent object of $\underline{\text{Dis}\,\Gamma^*}$ is an object A of Dis Γ^* which is of finite type in $\underline{\text{Dis}\,\Gamma^*}$ and so that the congruence of any algebra morphism $B \to A$ is of finite type in $\underline{\text{Dis}\,\Gamma^*}$ provided B is of finite type in $\underline{\text{Dis}\,\Gamma^*}$.

4) If $(A,\mathcal{U}) \in \text{ob}\,\Gamma^*$ then a special entourage R of $(A,\mathcal{U})$ is any algebraic congruence (i.e. equivalence relation compatible with the algebraic structure) on A so that $R \in \mathcal{U}$ and $A/_R$ is a coherent object in $\underline{\text{Dis}\,\Gamma^*}$. A special congruence on $(A,\mathcal{U})$ is any algebraic congruence R on A which is an intersection of special entourages and $A/_R$ is a coherent object of $\underline{\text{Dis}\,\Gamma^*}$.

5) a separated and complete c-uniform Σ-algebra $(A,\mathcal{U})$ which is in Γ^* is a strict object of Γ^* if each special congruence is a special entourage and the family of all special entourages

of $(A,\mathfrak{U})$ represents a base for $\mathfrak{U}$.

Finally $\mathcal{D}$ is the category of all strict objects of Γ^* for a suitable c-structure Γ on Σ/Σ^* (Theorem 2.7.5).

Some remarks on the above theorem are in order:

(a) the category of Stone spaces ([12]), is isomorphic to the category of all compact p-uniform spaces. From this point of view separated (Hausdorff) and complete p-uniform spaces are a natural generalization of Stone spaces; topologically a p-uniform space (separated and complete) is like a Stone space (or Boolean space [9]) minus quasi-compacity.

(b) if we replace the finitary rank of the algebraic theory Σ by a convenient greater rank then all uniformities involved are discrete; consequently we get a particular case of Linton contravariant representation theorem [15]. On this line 2.8.2 proves a topological complement of [15] from which by enlarging the rank we obtain the original Linton theorem.

(c) a very general consequence of our theorem is that for any category $\mathcal{C}$ which is well embedded in a Giraud-Grothendieck topos, the dual of $\mathcal{C}$ can be realised as a category of uniform algebras over a finitary theory. This leads to a duality theory of finitely presentable categories .

By combining the ideas of (b) and (c), a duality theorem very like the Isbell duality theorem on algebraic categories can be obtained.

(d)

The terminology used is that of [6],[18]. We point out that we denote the canonical projections (injections) of a product (coproduct) by π_i (resp. ∂_i). For a family $\{f_i : A_i \to A\}_i$ ($\{f_i : A \to A_i\}_i$) the morphism $f : \coprod_i A_i \to A$ (resp. $f : A \to \prod_i A_i$) uniquely defined by $f\partial_i = f_i$ (resp $\pi_i f = f_i$) for all i, is denoted by $[f_i]_i$ (resp. $\langle f_i \rangle_i$). $h : C \to A$ (resp. $h : B \to C$) is an equalizer (coequalizer) of a pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ if: (a) $fh=gh$ (resp. $hf=hg$) and (b) a morphism $u : X \to A$ (resp. $u : B \to X$) factors through h provided $fu=gu$ (resp. $uf=ug$). A mono (epi) is regular if it is an equalizer (coequalizer) of a suitable pair of morphisms. We shall sometimes denote a mono (regular epi) by $\rightarrowtail$ ($\twoheadrightarrow$). In presheaves or sheaves categories all monos and epis are regular. In an algebraic category regular epimorphisms are exactly the surjections. A pair $\{u,v\}$ of morphisms is a kernel pair (cokernel pair) of a morphism f if $fu=fv$ (resp. $uf=vf$) is a pullback (pushout). If $R \underset{v}{\overset{u}{\rightrightarrows}} A$ is a kernel pair of $f : A \twoheadrightarrow B$ then the subobject of $A \prod A$ defined by the monomorphism $\langle u,v \rangle : R \rightarrowtail A \prod A$ is called the congruence of f. A congruence on A is a subobject $R \overset{r}{\rightarrowtail} A \prod A$ of $A \prod A$ so that $\{\pi_1 r, \pi_2 r\}$ is a kernel pair of a suitable morphism defined on A. If in this case $f : A \twoheadrightarrow B$ is a coequalizer of $\{\pi_1 r, \pi_2 r\}$ then the quotient object of A defined by f is denoted by A/R and $f : A \twoheadrightarrow A/R$ is called the canonical projection of A onto its regular quotient A/R. In presheaves categories or algebraic categories congruences are equivalence relations compatible with the structure and regular quotients are the sets of cosets endowed with the induced structure. A sequence $R \underset{v}{\overset{u}{\rightrightarrows}} A \xrightarrow{f} B$ is a left (right) exact sequence if $\{u,v\}$ is a kernel pair of f (resp. f is a coequalizer of $\{u,v\}$); the sequence is short exact sequence if it is both left and right exact. A sequence $R \xrightarrow{f} A \underset{h}{\overset{g}{\rightrightarrows}} B$ is a left (right) exact sequence if f is an equalizer of $\{g,h\}$ (resp. $\{g,h\}$ is a cokernel pair of f); the sequence is a short exact sequence if it is both left and right exact.

A commutative diagram
$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \\ & C \arrow[ur, "q"'] & \end{tikzcd}$$
is (or defines) a coequalizer de-

composition of f if q is a mono and p is a regular epi; in this case the subobject (C,q) of B defined by q is called the image of f. If $\mathscr{C}$ is a regular category ([6]), if $f \in \mathrm{Hom}_{\mathscr{C}}(A;B)$ and if $X \overset{x}{\hookrightarrow} A$ is a subobject of A then $f(X)$ denotes the image (defined by a coequalizer decomposition) of fx. If $Y \overset{y}{\hookrightarrow} B$ is a subobject of B then $f^{-1}(Y)$ denotes the subobject of A defined by pulling back y along f.

Concerning filters in a lattice we point out that if L is a lattice then we shall say that a nonvoid subset F of L is a filter in L if (a) $\{x,y\} \subseteq F$ implies $x \wedge y \in F$, and (b) $y \in F$ provided y is greater than a suitable element of F. A nonvoid subset $\mathfrak{G}$ of L is a filter base in L if for each x,y in $\mathfrak{G}$ there exists a z in $\mathfrak{G}$ so that $x \geqslant z$ and $y \geqslant z$; the filter generated by $\mathfrak{G}$ in L is $\overline{\mathfrak{G}} = \{u \in L \mid \exists\, x \in \mathfrak{G} \text{ with } u \geqslant x\}$. For a morphism $f: A \to B$ in a regular category $\mathscr{C}$ one has a meet preserving map $f^{-1}: \mathscr{S}_{\mathscr{C}}(B) \longrightarrow \mathscr{S}_{\mathscr{C}}(A)$ between the lattices of all subobjects in $\mathscr{C}$ of B resp. A. If F is a filter in $\mathscr{S}_{\mathscr{C}}(B)$ then $f^{-1}[F]$ denotes the filter in $\mathscr{S}_{\mathscr{C}}(A)$ generated by $\{f^{-1}(X) \mid X \in F\}$. If F is a filter in $\mathscr{S}_{\mathscr{C}}(A)$ then $f[F]$ denotes the filter in $\mathscr{S}_{\mathscr{C}}(B)$ generated by $\mathfrak{G} = \{X \hookrightarrow B \mid f^{-1}(X) \in F\}$ (since $A \in F$ and since $f^{-1}(B) = A$, $\mathfrak{G}$ is nonempty).

1. p-uniform spaces & p-uniform objects

1.1 Uniform spaces Let X be a set . We recall that a uniformity on X is a filter $\mathfrak{U}$ of reflexive subsets of $X\times X$ so that

$$\begin{cases} (1) \text{ if } U\in\mathfrak{U} \text{ then } U^{-1}\in\mathfrak{U} \quad (U^{-1} \text{ denotes } \{(y,x)\mid(x,y)\in U\}) \\ (2) \text{ if } U\in\mathfrak{U} \text{ then there exists } V\in\mathfrak{U} \text{ so that } V\circ V\subseteq U \end{cases}$$

The pair $(X,\mathfrak{U})$ is called a uniform space and the elements of $\mathfrak{U}$ are called entourages of X. A base for the uniformity $\mathfrak{U}$ (or a fundamental system of entourages) is any subset $\mathfrak{S}$ of $\mathfrak{U}$ so that the filter of subsets of $X\times X$ generated by $\mathfrak{S}$ is $\mathfrak{U}$. $\mathfrak{S}\subseteq\mathfrak{U}$ is a base for $\mathfrak{U}$ iff

(1) $\mathfrak{S}$ is cofiltered with the inclusions (i.e. if A,B are in $\mathfrak{S}$ then there exists a $C\in\mathfrak{S}$ so that both A and B contain C)

(2) any $U\in\mathfrak{U}$ contains a suitable element of $\mathfrak{S}$.

Consequently a family $\mathfrak{S}$ of reflexive subsets of $X\times X$ is a base for a uniformity on X iff

$$\begin{cases} (1) \ \mathfrak{S} \text{ is cofiltered with the inclusions} \\ (2) \text{ if } A\in\mathfrak{S} \text{ then there exists a } B\in\mathfrak{S} \text{ so that } B\subseteq A^{-1} \\ (3) \text{ if } A\in\mathfrak{S} \text{ then there exists a } C\in\mathfrak{S} \text{ so that } C\circ C\subseteq A \end{cases}$$

In this case the uniformity on X for which $\mathfrak{S}$ is a fundamental system of entourages (or the uniformity generated by $\mathfrak{S}$) is

$$\bar{\mathfrak{S}} = \{U\subseteq X\times X \mid (\exists) A\in\mathfrak{S} \text{ so that } A\subseteq U\}$$

A uniformity $\mathfrak{U}$ on X is finer than another uniformity $\mathfrak{U}'$ on X if $\mathfrak{U}'\subseteq\mathfrak{U}$. The set of all uniformities on X is thus a poset. The join of a family $\{\mathfrak{U}_i\}_{i\in I}$ of uniformities on X is the uniformity for which a fundamental system of entourages is

$$\left\{\bigcap_{i\in J} A_i \;\middle|\; J\subseteq I \text{ finite and } A_i\in\mathfrak{U}_i,\ i\in J\right\}.$$

The meet of the family $\{\mathfrak{U}_i\}_{i\in I}$ is the final uniformity for the family $\{(X,\mathfrak{U}_i)\xrightarrow{1_X} X\}_{i\in I}$. Hence the set of all uniformities on X is a lattice.

If $f:X \to Y$ is a map then for any $A \subseteq X \times X$ and any $B \subseteq Y \times Y$ we shall denote $(f \times f)(A)$ by $f(A)$ and $(f \times f)^{-1}(B)$ by $f^{-1}(B)$. If $\mathcal{U}$ is a uniformity on Y then $f^{-1}(\mathcal{U}) \overset{\text{def}}{=\!=\!=} \{ f^{-1}(U) \mid U \in \mathcal{U} \}$ is a base for a uniformity on X , which we denote by $f^{-1}[\mathcal{U}]$. $f^{-1}[\mathcal{U}]$ is called the initial uniformity on X for the map $f:X \to (Y,\mathcal{U})$. If $\mathcal{U}'$ is a uniformity on X then $f:X \to Y$ is uniform continuous in the uniformities $\mathcal{U}'$ and $\mathcal{U}$ if $f^{-1}[\mathcal{U}]$ is less fine than $\mathcal{U}'$ i.e. $f^{-1}(U) \in \mathcal{U}'$ for any $U \in \mathcal{U}$; in this case we shall write $f:(X,\mathcal{U}') \to (Y,\mathcal{U})$.

If $\{(X_i, \mathcal{U}_i)\}_{i \in I}$ is a family of uniform spaces and $\{f_i : X \to X_i\}_{i \in I}$ is a family of maps then the initial uniformity $\mathcal{U}$ on X for (or induced by) the family $\{f_i : X \to (X_i, \mathcal{U}_i)\}_{i \in I}$ is the coarsest uniformity on X in which all functions f_i are uniform continuous. If for each i , $\mathfrak{S}_i$ is a base for $\mathcal{U}_i$ then a base of $\mathcal{U}$ is

$$\Big\{ \bigcap_{i \in J} f_i^{-1}(U_i) \;\Big|\; J \subseteq I \text{ finite and } U_i \in \mathfrak{S}_i,\ i \in J \Big\}.$$

If $\mathcal{U}$ is the initial uniformity on X for the family

$$\{ f_i : X \longrightarrow (X_i, \mathcal{U}_i) \}_{i \in I}$$

then for any uniform space $(Y,\mathcal{V})$ a map $f:Y \to X$ is uniform continuous in $\mathcal{V}$ and $\mathcal{U}$ iff all the maps $f_i f$ are uniform continuous. Consequently in the category <u>UNIF</u> of all uniform spaces a limit is the corresponding limit of sets endowed with the initial uniformity induced by the family of all canonical projections.

If $\{(X_i, \mathcal{U}_i)\}_{i \in I}$ is a family of uniform spaces and $\{f_i : X_i \to X\}_i$ is a family of maps then the final uniformity $\mathcal{U}$ on X induced by the family $\{f_i : (X_i, \mathcal{U}_i) \longrightarrow X\}_{i \in I}$ is the finest uniformity on X in which all functions f_i are uniform continuous. In this case for any uniform space $(Y,\mathcal{V})$ a map $f:X \to Y$ is uniform continuous in $\mathcal{U}$ and $\mathcal{V}$ iff all the maps ff_i are uniform continuous.

Consequently a colimit in UNIF is the corresponding colimit of sets endowed with the final uniformity induced by the family of all all canonical injections.

If $(X,\mathcal{U})$ is a uniform space and if Y is a subset of X then the uniformity induced by $\mathcal{U}$ on Y is $\mathcal{U}_{|Y}=\{U\cap(Y\times Y) \mid U\in\mathcal{U}\}$ $\mathcal{U}_{|Y}$ is the initial uniformity on Y induced by the inclusion $Y \hookrightarrow (X,\mathcal{U})$. $(Y,\mathcal{U}_{|Y})$ is called a uniform subspace of $(X,\mathcal{U})$; more generally we say that a uniform space $(Z,\mathcal{V})$ is a uniform subspace of $(X,\mathcal{U})$ if there exists a uniform continuous injection $f:Z\longrightarrow X$ so that $\mathcal{V}=f^{-1}[\mathcal{U}]$. A uniform continuous map $f:(Y,\mathcal{V})\longrightarrow(X,\mathcal{U})$ is a mono in UNIF iff f is an injection. Hence the notions of "uniform subspace" and "subobject" are not the same. In fact the uniform subspacea of $(X,\mathcal{U})$ are exactly the regular subobjects (in UNIF) of $(X,\mathcal{U})$.

A uniform space $(X,\mathcal{U})$ is:

a) discrete if $\Delta_X\in\mathcal{U}$

b) separated if $\bigcap\mathcal{U}=\Delta_X$

c) complete if each Cauchy filter in $(X,\mathcal{U})$ has a limit (we recall that a filter F of subsets of X is a Cauchy filter in $(X,\mathcal{U})$ if for each $U\in\mathcal{U}$ there exists an $A\in F$ so that $A\times A\subseteq U$)

The topology of a uniform space $(X,\mathcal{U})$ can be described by means of the neighbourhood filters of the points of X; for $x\in X$ the family $\mathcal{U}(x)=\{U(x)=\{y\in X \mid (x,y)\in U\} \mid U\in\mathcal{U}\}$ is the filter of all neighbourhoods of x. The topology $\mathcal{T}$ which arises in this way is called the topology induced by $\mathcal{U}$. We recall from [2], p.185 that:

(i) if A is a subset of $X\times X$ then UAU is a neighbourhood of A in $(X,\mathcal{T})\times(X,\mathcal{T})$ provided U is a symmetric entourage of X (here AU means $\left\{(x,x')\in X\times X \,\middle|\, (\exists)x''\in X \text{ so that } (x',x'')\in U \text{ and } (x'',x)\in A\right\}$

The closure of A in $(X,\mathfrak{T})\times(X,\mathfrak{T})$ is $\bigcap\{UAU \mid U$ symmetric entourage of X$\}$

(ii) if B is a subset of X and if U is a symmetric entourage of X then $U(B) \overset{def}{=\!=\!=\!=\!=} \bigcup_{x\in B} U(x)$ is a neighbourhood of B in $(X,\mathfrak{T})$. The closure of B in $(X,\mathfrak{T})$ is $\bigcap\{U(B) \mid U$ symmetric entourage of X$\}$

(iii) a uniform spaces is separated iff it is a Hausdorff topological space in the induced topology.

(iv) the correspondence $(X,\mathfrak{U}) \rightsquigarrow (X,\mathfrak{T})$, $f \rightsquigarrow f$, is a functor UNIF ⟶ TOP which preserves all limits.

(v) a Cauchy filter F in a uniform space $(X,\mathfrak{U})$ converges iff $\bigcap_{A\in F}$ (clA in $(X,\mathfrak{T})$) is nonempty.

1.2 p-Uniform spaces

Definition 1.2.1 A uniformity $\mathfrak{U}$ on X is a p-uniformity ("structure uniforme des partitions sur X") if $\mathfrak{U}$ has a fundamenta system of entourages consisting of equivalence relations on X. Topologically a p-uniformity is given by coverings which are partitions of X [10]. A uniform space $(X,\mathfrak{U})$ is a p-uniform space (p-u space for short) if $\mathfrak{U}$ is a p-uniformity on X. p-UNIF denotes the category of all p-u spaces and uniform continuous maps.

Remarks. Let X be a set. Let L(X) be the lattice of all equivalence relations on X. If $\mathfrak{U}$ is a uniformity on X then $\mathfrak{S}_{\mathfrak{U}} = \mathfrak{U}\cap L(X)$ is a filter in L(X).

1.2.2 A uniformity $\mathfrak{U}$ on X is a p-uniformity iff $\mathfrak{S}_{\mathfrak{U}}$ is a base for $\mathfrak{U}$.

1.2.3 A subset $\mathfrak{S}$ of L(X) is a base for a p-uniformity on X iff $\mathfrak{S}$ is a filter base in L(X). In this case if $\mathfrak{U}$ is the uniformity generated by $\mathfrak{S}$ then $\mathfrak{S}$ is a base for the filter $\mathfrak{S}_{\mathfrak{U}}$.

1.2.4 The map $\mathfrak{U} \longmapsto \mathfrak{S}_{\mathfrak{U}}$ is a bijection from the set of all p-uniformities on X to the set of all filters in L(X).

$\mathfrak{G}_{\mathfrak{U}} = \bigcap_i \mathfrak{G}_{\mathfrak{U}_i}$ and the isomorphism follows.

<u>1.2.7</u> Any map $f: X \longrightarrow Y$ induces an increasing map $f^{-1}: L(Y) \to L(X)$, $f^{-1}(R) = \{ (x,x') \in X \times X \mid (f(x), f(x')) \in R \}$. If $\mathfrak{G}$ is a filter base in $L(Y)$ then $f^{-1}(\mathfrak{G}) \overset{\text{def}}{=\!=\!=} \{ f^{-1}(R) \mid R \in \mathfrak{G} \}$ is a filter base in $L(X)$. The inverse image of a filter $\mathfrak{G}$ in $L(Y)$ under f is the filter $f^{-1}[\mathfrak{G}]$, in $L(X)$, generated by $f^{-1}(\mathfrak{G})$.

Now if $(X, \mathfrak{U})$ and $(Y, \mathfrak{V})$ are p-u spaces then $f: X \longrightarrow Y$ is uniform continuous iff $f^{-1}[\mathfrak{G}_{\mathfrak{V}}]$ is less fine than $\mathfrak{G}_{\mathfrak{U}}$ (i.e. $f^{-1}(R) \in \mathfrak{G}_{\mathfrak{U}}$ for each $R \in \mathfrak{G}_{\mathfrak{V}}$). Hence we can describe <u>p-UNIF</u> as the category whose

- <u>objects</u> are pairs $(X, \mathfrak{X})$ where X is a set and $\mathfrak{X}$ is a filter in $L(X)$
- <u>morphisms</u> from $(X, \mathfrak{X})$ to $(Y, \mathfrak{Y})$ are those maps $f: X \longrightarrow Y$ for which $f^{-1}[\mathfrak{Y}]$ is less fine than $\mathfrak{X}$

<u>1.28</u> The inclusion functor <u>p-UNIF</u> $\longrightarrow$ <u>UNIF</u> has a left adjoint $(X, \mathfrak{U}) \rightsquigarrow (X, p\&\mathfrak{U})$ where $p\&\mathfrak{U}$ is the uniformity on X which has $\mathfrak{G}_{\mathfrak{U}}$ as a fundamental system of entourages.

<u>Proof</u>. The map $1_X: (X, \mathfrak{U}) \longrightarrow (X, p\&\mathfrak{U})$ is uniform continuous since $p\&\mathfrak{U}$ is less fine than $\mathfrak{U}$. If f is any uniform continuous map from $(X, \mathfrak{U})$ to a p-u space $(Y, \mathfrak{V})$ then $f^{-1}(R) \in p\&\mathfrak{U}$ for any $R \in \mathfrak{G}_{\mathfrak{V}}$. Hence $f: X \longrightarrow Y$ is uniform continuous in $p\&\mathfrak{U}$ and $\mathfrak{V}$. Obviously $f: (X, p\&\mathfrak{U}) \longrightarrow (Y, \mathfrak{V})$ is the unique morphism of <u>p-UNIF</u> for which the diagram

$$\begin{tikzcd}
(X, \mathfrak{U}) \arrow[r, "f"] \arrow[dr, "1_X"'] & (Y, \mathfrak{V}) \\
& (X, p\&\mathfrak{U}) \arrow[u, "f"']
\end{tikzcd}$$

commutes. Hence <u>p-UNIF</u> is a reflective full subcategory of <u>UNIF</u> with the reflector $(X, \mathfrak{U}) \rightsquigarrow (X, p\&\mathfrak{U})$.

<u>1.2.5</u> Let $\{\mathcal{U}_i\}_{i\in I}$ be a family of p-uniformities on X. The coarsest uniformity on X which is finer than each $\mathcal{U}_i$ is just the initial uniformity $\mathcal{U}$ on X induced by the family $\{1_X : X \longrightarrow (X, \mathcal{U}_i)\}_i$. Since a base for $\mathcal{U}$ is

$$\{\cap J \mid J \text{ is a finite subset of } \bigcup_i \mathfrak{S}_{\mathcal{U}_i}\}$$

$\mathcal{U}$ is a p-uniformity. Remark also that $\mathfrak{S}_{\mathcal{U}} = \bigvee_i \mathfrak{S}_{\mathcal{U}_i}$ in the lattice of all filters in L(X). As a consequence the inclusion functor <u>p-UNIF</u> $\longrightarrow$ <u>UNIF</u> reflects all limits. Hence <u>p-UNIF</u> is a complete category and any limit of p-u spaces (made in <u>UNIF</u>) is a p-u space. If $F : \mathcal{I} \longrightarrow$ <u>p-UNIF</u> is a small functor and if

$$\left(X \xrightarrow{f_i} |F(i)|\right)_i = \varprojlim |F|$$

where $|\cdot| :$ <u>p-UNIF</u> $\longrightarrow$ <u>SET</u> denotes the forgetfull functor, then

$$\varprojlim F = \left((X, \mathcal{U}) \xrightarrow{f_i} F(i)\right)_i$$

where $\mathcal{U}$ is the initial uniformity on X for the family

$$\{f_i : X \longrightarrow F(i)\}_i$$

<u>1.2.6</u> The set <u>p-UNIF</u>(X) of all p-uniformities on X is a complete lattice isomorphic to the lattice of all filters in L(X). The isomorphism is $\mathcal{U} \longmapsto \mathfrak{S}_{\mathcal{U}}$.

<u>Proof</u>. <u>p-UNIF</u>(X) is a poset with the fineness relation. Moreover it is a subposet in the poset <u>UNIF</u>(X) of all uniformities on X. By <u>1.2.5</u> , <u>p-UNIF</u>(X) is closed in <u>UNIF</u>(X) at arbitrary joins. If $\{\mathcal{U}_i\}_i$ is a family of uniformities on X then the finest uniformity on X which is less fine than each $\mathcal{U}_i$ is the final uniformity on X induced by $\{1_X : (X, \mathcal{U}_i) \longrightarrow X\}_i$. This final uniformity on X does not always coincide with the intersection of the family $\{\mathcal{U}_i\}_i$,([2]). On the contrary if all $\mathcal{U}_i$ are p-uniformities then $\mathcal{U} = \bigcap_i \mathcal{U}_i$ is a p-uniformity on X. In this case $\mathcal{U}$ is the finest p-uniformity on X which is less fine than each $\mathcal{U}_i$. Hence <u>p-UNIF</u>(X) has arbitrary meets. Obviously

<u>1.2.9</u> If f is a function from a p-u space $(Y,\mathcal{V})$ to a set X then $\mathfrak{G} = \{R \in L(X) \mid f^{-1}(R) \in \mathcal{V}\}$ is a filter in $L(X)$. Indeed if R,S are in $\mathfrak{G}$ then $f^{-1}(R \cap S) = f^{-1}(R) \cap f^{-1}(S)$ is in $\mathcal{V}$ since both $f^{-1}(R)$ and $f^{-1}(S)$ are in $\mathcal{V}$. Hence $\mathfrak{G}$ is closed under finite intersections. If $R \in \mathfrak{G}$ and if $S \in L(X)$ so that $S \supseteq R$ then $f^{-1}(S) \supseteq f^{-1}(R)$ and $S \in \mathfrak{G}$.

The p-uniformity generated by $\mathfrak{G}$ is denoted $f[\mathcal{V}]$.

<u>1.2.10</u> Let $\{f_i : (X_i, \mathcal{U}_i) \longrightarrow X\}_i$ be a family of functions. Let $\mathcal{U}$ be the final uniformity on X induced by $\{f_i\}_i$. If all $(X_i, \mathcal{U}_i)$ are p-u spaces then $p\&\mathcal{U}$ is the finest p-uniformity on X which is less fine than each $f_i[\mathcal{U}_i]$. Moreover $\mathfrak{G}_{p\&\mathcal{U}} = \{R \in L(X) \mid f_i^{-1}(R) \in \mathcal{U}_i \text{ for all } i\}$.

<u>Proof</u>. Since for each $R \in \mathfrak{G}_{p\&\mathcal{U}}$ and each i, $f_i^{-1}(R) \in \mathcal{U}_i$ it follows that $\mathfrak{G}_{p\&\mathcal{U}}$ is contained in each $f_i[\mathcal{U}_i]$. Hence $p\&\mathcal{U}$ is less fine than each $f_i[\mathcal{U}_i]$. If $\mathcal{U}'$ is a p-uniformity on X less fine than each $f_i[\mathcal{U}_i]$ then $f_i^{-1}(A) \in \mathcal{U}_i$ for any $A \in \mathcal{U}'$ and any i. Hence all f_i are uniform con--tinuous in $\mathcal{U}_i$ and $\mathcal{U}'$. Since $\mathcal{U}$ is the finest unifor-mity on X for which all f_i are uniform continuous it follows that $\mathcal{U}' \subseteq \mathcal{U}$. Hence $\mathfrak{G}_{\mathcal{U}'} \subseteq p\&\mathcal{U}$ and $\mathcal{U}' = \mathfrak{G}_{\mathcal{U}'} \subseteq p\&\mathcal{U}$.

Actualy $p\&\mathcal{U}$ is the meet (in <u>p-UNIF</u>(X)) of the family $\{f_i[\mathcal{U}_i]\}_i$. According to <u>1.2.6</u>, $p\&\mathcal{U} = \bigcap_i f_i[\mathcal{U}_i]$ and

$$\mathfrak{G}_{p\&\mathcal{U}} = \bigcap_i \mathfrak{G}_{f_i[\mathcal{U}_i]} = \{R \in L(X) \mid f_i^{-1}(R) \in \mathcal{U}_i \text{ for all } i\}$$

<u>Definition 1.2.11</u> Let $\{(X_i, \mathcal{U}_i)\}_i$ be a family of p-u spaces and let $\{f_i : X_i \longrightarrow X\}_i$ be a family of functions. The final p-uniformity on X induced by $\{f_i : (X_i, \mathcal{U}_i) \longrightarrow X\}_i$ is $p\&\mathcal{U}$ where $\mathcal{U}$ is the final uniformity on X for $\{f_i\}_i$. By <u>1.2.10</u> the final p-uniformity for $\{f_i\}_i$ can be equally be

defined as being $\bigcap_i f_i[\mathcal{U}_i]$. Also a base for the final p-uniformity induced by $\{f_i\}_i$ is $\{R \in L(X) \mid f_i^{-1}(R) \in \mathcal{U}_i$ for all $i\}$.

1.2.12 Let $\{f_i:(X_i,\mathcal{U}_i) \longrightarrow X\}_i$ be a family of functions so that each $(X_i,\mathcal{U}_i)$ is a p-uniform space. Let $\mathcal{U}$ be the final p-uniformity on X induced by $\{f_i\}_i$. Then for any p-u space $(Y,\mathcal{V})$, a map $f:X \to Y$ is uniform continous in $\mathcal{U}$ and $\mathcal{V}$ iff all ff_i are uniform continuous.

Proof. Let $R \in \mathfrak{S}_{\mathcal{V}}$. Then $f^{-1}(R) \in \mathfrak{S}_{\mathcal{U}}$ iff $f_i^{-1}(f^{-1}(R)) \in$ $\in \mathfrak{S}_{\mathcal{U}_i}$ for all i. Since $f_i^{-1}(f^{-1}(R))=(ff_i)^{-1}(R)$ the proof follows.

Corollary 1.2.13 (colimits in p-UNIF) By 1.2.8 p-UNIF is a complete and cocomplete category. Moreover if $F:I \to$ p-UNIF is a small functor and if

$$\left(F(i) \xrightarrow[f_i]{} X\right)_i = \varinjlim \left(I \xrightarrow[F]{} \text{p-UNIF} \xrightarrow[\text{forget}]{} \text{SET}\right)$$

then $\varinjlim F = \left(F(i) \xrightarrow[f_i]{} (X,\mathcal{U})\right)_i$ where $\mathcal{U}$ is the final p-uniformity on X induced by $\left\{F(i) \xrightarrow[f_i]{} X\right\}_i$

1.2.14 (coproducts in p-UNIF) Let $\{(X_i,\mathcal{U}_i)\}_{i\in I}$ be a family of p-u spaces. For each i let $\mathfrak{S}_i \subseteq L(X_i)$ be a base for $\mathcal{U}_i$. The coproduct in p-UNIF of $\{(X_i,\mathcal{U}_i)\}_i$ is $\coprod_i X_i$, ($\coprod$ means coproduct in SET; the canonical injection are ∂_i) endowed with the final p-uniformity for the family $\{\partial_i:(X_i,\mathcal{U}_i) \to \coprod_i X_i\}_i$.

A fundamental system of entourages of $\coprod_i X_i$ is

$$\mathfrak{S} = \{R \in L(\coprod_i X_i) \mid \partial_i^{-1}(R) \in \mathcal{U}_i \text{ for all } i\}.$$

If we choose for each i an $R_i \in \mathfrak{S}_i$ then

$$R = \{((j,x),(k,y)) \in (\coprod_i X_i)^2 \mid j=k \text{ and } (x,y) \in R_j\}$$

is an equivalence relation on $\coprod_i X_i$. Also $\partial_i^{-1}(R)=R_i$ for all i. Hence $R \in \mathfrak{S}$. Actually any $S \in \mathfrak{S}$ contains such an R namely that which is given by $\left\{\partial_i^{-1}(S)\right\}_i$. Hence $\coprod_i (X_i, \mathcal{U}_i) = (\biguplus_i X_i, \mathcal{U})$ where a base for $\mathcal{U}$ consists of those $R \in L(\biguplus_i X_i)$ for which there exists a family $(R_i)_i \in \prod_i \mathfrak{S}_i$ so that $((j,x),(k,y)) \in R$ iff $j=k$ and $(x,y) \in R_j$.

Remark that each $\mathcal{U}_i = \mathcal{U}|_{X_i}$. Indeed if $A \in \mathcal{U}_i$ then there exists an $R \in \mathfrak{S}_{\mathcal{U}_i}$ so that $R \subseteq A$. Then one has

$$R' \cap (\{i\} \times X_i) \times (\{i\} \times X_i)) = \partial_i^{-1}(R') = R$$

where R' is the equivalence relation on $\biguplus_i X_i$ given by the family $\{R_j\}_{j \in I}$

$$R_j = \begin{cases} R, & \text{if } j=i \\ \Delta_{X_j} & \text{if } j \neq i \end{cases}$$

Hence A contains a $\partial_i^{-1}(R')$ with $R' \in \mathcal{U}$. Consequently each $(X_i, \mathcal{U}_i)$ is a uniform subspace in $\coprod_i (X_i, \mathcal{U}_i)$. Therefore all canonical injections of a coproduct in p-UNIF are regular monos.

1.2.15 (regular monomorphisms) A morphism $f:(X,\mathcal{U}) \to (Y,\mathcal{V})$ of p-UNIF is a mono in p-UNIF iff f is an injection. f is a regular mono in p-UNIF iff f is an injective map and $\mathcal{U} = f^{-1}[\mathcal{V}]$ (we can equally well say that f is an injection and $\mathcal{U}$ is the subspace uniformity — necessarily a p-uniformity — on X induced by $\mathcal{V}$). The above assertions are obvious by 1.2.5. Consequently the equalizer of a pair $(X,\mathcal{U}) \overset{f}{\underset{g}{\rightrightarrows}} (Y,\mathcal{V})$ of morphisms of p-UNIF is $h:(Z,\mathcal{W}) \to (X,\mathcal{U})$ where $Z = \{x \in X \mid f(x) = g(x)\}$, h is the inclusion and $\mathcal{W} = \mathcal{U}|_Z$. We recall that if $\mathfrak{S} \subseteq L(X)$ is a base of $\mathcal{U}$ then $\{R \cap (Z \times Z) \mid R \in \mathfrak{S}\}$ is a base for $\mathcal{W}$.

Hence we shall say that a p-u space $(X,\mathcal{U})$ is a p-u subspace of a p-u space $(Y,\mathcal{V})$ if $X \subseteq Y$ and $\mathcal{U} = \mathcal{V}|_X$. Also if $(Y,\mathcal{V})$ is a p-u space and if X is a subset of Y then we shall say that $\mathcal{V}|_X$ is the subspace p-uniformity on X induced by $(Y,\mathcal{V})$

<u>1.2.16</u> The product in <u>p-UNIF</u> of a family $\{(X_i,\mathcal{U}_i)\}_{i\in I}$ of p-u spaces is $(\prod_i X_i,\mathcal{U})$ where $\mathcal{U}$ is the initial uniformity on $\prod_i X_i$ induced by $\{\prod_i X_i \xrightarrow{\pi_i} (X_i,\mathcal{U}_i)\}_i$. If for each i, $\mathfrak{S}_i \subseteq L(X_i)$ is a base for $\mathcal{U}_i$ then a base for $\mathcal{U}$ is

$$\left\{\bigcap_{j\in J}\pi_j^{-1}(R_j)\ \middle|\ J\subseteq I \text{ finite and } R_j\in\mathfrak{S}_j \text{ for all } j\in J\right\}.$$

Remark that $\bigcap_{j\in J}\pi_j^{-1}(R_j)=\left\{((x_i)_{i\in I},(y_i)_{i\in I})\in(\prod_i X_i)^2\ \middle|\ (x_j,y_j)\in R_j \text{ for each } j\in J\right\}$

Also if $(Y,\mathcal{V})$ is a p-u space and if $\mathfrak{S}\subseteq L(Y)$ is a base for $\mathcal{V}$ then a base for $(Y,\mathcal{V})^I$ consists of those equivalence relations R on Y^I for which there exists an $S\in\mathfrak{S}$ and a finite $J\subseteq I$ so that

$$R=\left\{((y_i)_i,(y_i')_i)\in Y^I\times Y^I\ \middle|\ (y_j,y_j')\in S \text{ for all } j\in J\right\}$$

<u>1.2.17</u> The kernel pair of a morphism $f:(X,\mathcal{U})\longrightarrow(Y,\mathcal{V})$ of <u>p-UNIF</u> is $(R,\mathcal{W}) \underset{r_2}{\overset{r_1}{\rightrightarrows}} (X,\mathcal{U})$ where :

$$\begin{cases} R=\{(x,x')\in X^2 \mid f(x)=f(x')\} \\ r_1,\ r_2 \text{ are the canonical projections of } R \text{ onto } X \\ \mathcal{W} \text{ is the subspace p-uniformity on } R \text{ induced by} \\ \qquad (X,\mathcal{U})\prod(X,\mathcal{U}) \end{cases}$$

If $\mathfrak{S}\subseteq L(X)$ is a base for $\mathcal{U}$ then by <u>1.2.16</u> a base for $\mathcal{W}$ consists of

$$r_1^{-1}(S)\cap r_2^{-1}(S)=\{((x,y),(x',y'))\in R\times R \mid (x,x')\in S \text{ and } (y,y')\in S\}$$

when S runs over $\mathfrak{S}$.

A congruence on a p-u space $(X,\mathcal{U})$ is any $(R,\mathcal{W})$ where $R\in L(X)$ and $\mathcal{W}$ is the subspace p-uniformity on R induced by the product uniformity on $X\times X$. Such an $(R,\mathcal{W})$ is the congruence of $q:(X,\mathcal{U})\longrightarrow(X/_R,\mathcal{U}')$ where $q:X\longrightarrow X/_R$ is the canonical projection of X onto its quotient set and $\mathcal{U}'$ is the final p-uniformity on $X/_R$ induced by $\left\{ (X,\mathcal{U}) \xrightarrow{q} X/_R \right\}$.

1.2.18 Let $(X,\mathcal{U})$ be a p-u space. Let R be an equivalence relation on X. If $q: X \to X/_R$ is the canonical surjection then we shall say that $q[\mathcal{U}]$ is the quotient p-uniformity on $X/_R$, and we shall denote it by $\mathcal{U}/_R$. Also we shall denote $(X/_R, \mathcal{U}/_R)$ by $(X,\mathcal{U})/_R$ and we shall say that $(X,\mathcal{U})/_R$ is the quotient p-u space of $(X,\mathcal{U})$ by R. Since $\mathcal{U}/_R$ is the final p-uniformity induced by q, $\mathfrak{S}_{\mathcal{U}/R} = \left\{ S \in L(X/_R) \mid q^{-1}(S) \in \mathcal{U} \right\}$. Since $q(q^{-1}(S)) = S$, $\mathfrak{S}_{\mathcal{U}/R} = \left\{ q(T) \mid T \in \mathfrak{S}_{\mathcal{U}} \text{ and } T \supseteq R \right\}$. Obviously the congruence of q $q:(X,\mathcal{U}) \longrightarrow (X/_R; \mathcal{U}/_R)$ is $(R, \mathcal{W})$ where $\mathcal{W}$ is the subspace p-uniformity on R induced by the product p-uniformity on $X \times X$. By 1.2.13, $q:(X,\mathcal{U}) \longrightarrow (X,\mathcal{U})/_R$ is a coequalizer of $(R,\mathcal{W}) \overset{r_1}{\underset{r_2}{\rightrightarrows}} (X,\mathcal{U})$ where r_1, r_2 are the projections of R onto X.

1.2.19 A morphism $f:(X,\mathcal{U}) \longrightarrow (Y,\mathcal{V})$ of p-UNIF is a regular epi iff f is a surjection and $\mathcal{V} = f[\mathcal{U}]$

Proof. If f is a regular epi in p-UNIF then it is a coequalizer of its kernel pair. A kernel pair of f is $(R,\mathcal{W}) \overset{r_1}{\underset{r_2}{\rightrightarrows}} (X,\mathcal{U})$ where $R = \left\{ (x,x') \in X^2 \mid f(x) = f(x') \right\}$ and $\mathcal{W}$ is the subspace p-uniformity on R induced by the product uniformity on $X \times X$. Since if $q: X \to X/_R$ is the canonical surjection then $q:(X,\mathcal{U}) \longrightarrow (X,\mathcal{U})/_R$ is also a coequalizer of $\{r_1, r_2\}$ it follows that there exists a commutative diagram

$$\begin{tikzcd} (X,\mathcal{U}) \arrow[rr, "f"] \arrow[dr, "q"'] & & (Y,\mathcal{V}) \\ & (X,\mathcal{U})/_R \arrow[ur] & \end{tikzcd}$$

<u>1.2.20</u> A morphism $f:(X;\mathfrak{U}) \longrightarrow (Y;\mathfrak{V})$ of <u>p-UNIF</u> is an epi iff f is a surjection. In this case $\mathfrak{V}$ is less fine than $f[\mathfrak{U}]$.

<u>Proof</u> If f is an epi in <u>p-UNIF</u> then a cokernel pair of f is $(Y;\mathfrak{V}) \overset{1_Y}{\underset{1_Y}{\rightrightarrows}} (Y;\mathfrak{V})$. By <u>1.2.13</u> $\{1_Y, 1_Y\}$ is a cokernel pair of f in <u>SET</u>. Hence f is a surjection.

<u>1.2.21</u> <u>p-UNIF</u> has coequalizer decompositions. If $f:(X;\mathfrak{U}) \longrightarrow (Y;\mathfrak{V})$ is a morphism of <u>p-UNIF</u> then a coequalizer decomposition of f is $(X;\mathfrak{U}) \xrightarrow{\ q\ } (X/_R, \mathfrak{U}/_R) \xleftrightarrow{\ g\ } (Y;\mathfrak{V})$ where R is the equivalence relation on X defined by f and $g(q(x))=f(x)$ for any x. There is also a canonical decomposition of f

$$\begin{tikzcd}
(X;\mathfrak{U}) \arrow[r, "f"] \arrow[d, "q"'] & (Y;\mathfrak{V}) \\
(X;\mathfrak{U})/_R \arrow[r, "f'"] & (f(X);\mathfrak{V}|_{f(X)}) \arrow[u, "q'"']
\end{tikzcd}$$

where q is a regular epi , q' is a regular mono and f' is a bimorphism in <u>p-UNIF</u>. Hence the image of f is $(f(X), \mathfrak{U}/_R)$ and the coimage of f is $(f(X);\mathfrak{V}|_{f(X)})$. Obviously f' is an iso iff for any $S \in \mathfrak{G}_{\mathfrak{U}}$ with $S \supseteq R$ there exists a $T \in \mathfrak{G}_{\mathfrak{V}}$ so that $f(S) = T \cap (f(X) \times f(X))$.

<u>1.2.22</u> The forgetful functor <u>p-UNIF</u> $\longrightarrow$ <u>SET</u> has a fully faithful left adjoint <u>SET</u> $\longrightarrow$ <u>p-UNIF</u>, $X \rightsquigarrow (X;L)$, and a fully faithful right adjoit <u>SET</u> $\longrightarrow$ <u>p-UNIF</u> , $X \rightsquigarrow (X;\{X \times X\})$.

1.3. Discreteness, completeness and separation for p-u spaces

1.3.1 If $(X, \mathcal{U})$ is a descrete uniform space then $\Delta_X \in \mathcal{U}$. In this case $\{\Delta_X\}$ is a base for $\mathcal{U}$ and $\mathfrak{S}_{\mathcal{U}} = L(X)$. Hence any discrete uniform space is a p-u space and the category DIS of all discrete p-u spaces is isomorphic to SET. We shall denote a discrete p-u space $(X, \mathcal{U})$ simply by $(X;L)$. Moreover the inclusion DIS $\hookrightarrow$ p-UNIF is just the left adjoint of p-UNIF $\xrightarrow{|\cdot|}$ SET.

1.3.2 A p-u space $(X, \mathcal{U})$ is separated iff $\bigcap \mathfrak{S}_{\mathcal{U}} = \Delta_X$. The functor sep.p-UNIF $\hookrightarrow$ p-UNIF has a left adjoint $(X, \mathcal{U}) \rightsquigarrow (X; \mathcal{U})/\bigcap\mathfrak{S}_{\mathcal{U}}$.

Proof Let $(X; \mathcal{U})$ be a p-u space. If $H = \bigcap \mathfrak{S}_{\mathcal{U}}$ then $\mathfrak{S}_{\mathcal{U}/H} = \{S/_H \mid S \in \mathfrak{S}_{\mathcal{U}}\}$ where $S/_H$ denotes $p_H(S)$, $p_H : X \to X/_H$ being the canonical surjection. Hence $(p_H(x), p_H(y)) \in \bigcap \mathfrak{S}_{\mathcal{U}/H}$ iff $(x,y) \in p_H^{-1}(S/_H)$ for all $S \in \mathfrak{S}_{\mathcal{U}}$. Since any $S \in \mathfrak{S}_{\mathcal{U}}$ contains H, $(x,y) \in H$. Hence $\bigcap \mathfrak{S}_{\mathcal{U}/H} = \Delta_{X/_H}$ and $(X; \mathcal{U})/_H$ is separated.

If $(Y; \mathcal{V})$ is a p-u separated space and if $f : (X; \mathcal{U}) \to (Y; \mathcal{V})$ is a uniform continuous map then $f^{-1}(S) \in \mathfrak{S}_{\mathcal{U}}$ for any $S \in \mathfrak{S}_{\mathcal{V}}$. Then : (the equivalence relation on X defined by f) $= f^{-1}(\Delta_Y) =$ $= f^{-1}(\bigcap_{S \in \mathfrak{S}_{\mathcal{V}}} S) = \bigcap_{S \in \mathfrak{S}_{\mathcal{V}}} f^{-1}(S) \supseteq \bigcap \mathfrak{S}_{\mathcal{U}} = H$. There exists then a unique map $g : X/_H \to Y$ so that $gp_H = f$. Since gp_H is uniform continuous in $\mathcal{U}$ and $\mathcal{V}$ it follows that g is uniform continuous in $\mathcal{U}/_H$ and $\mathcal{V}$. Hence there exists a unique morphism $g : (X; \mathcal{U})/_H \to (Y; \mathcal{V})$ of p-UNIF so that $gp_H = f$. Consequently sep.p-UNIF is a full reflective subcategory of p-UNIF with the reflection $(X, \mathcal{U}) \rightsquigarrow (X; \mathcal{U})/\bigcap\mathfrak{S}_{\mathcal{U}}$. We shall denote $\bigcap \mathfrak{S}_{\mathcal{U}}$ by H or H^X.

1.3.3 A p-u space $(X; \mathcal{U})$ is complete if any Cauchy filter in X, [2], is convergent. Since for any subset A of X the closure of A in X is $\bigcap_{R \in \mathfrak{S}_{\mathcal{U}}} R(A)$, (here $R(A) = \{y \in X \mid (\exists) a \in A \text{ with } (y,a) \in R\}$)

it follows that a Cauchy filter F of subsets of X converges iff $\bigcap_{A\in F}\left(\bigcap_{R\in\mathfrak{S}_{\mathfrak{U}}} R(A)\right)$ is nonempty.

Proposition Let $(X;\mathfrak{U})$ be a p-u space. For each $R\in\mathfrak{S}_{\mathfrak{U}}$ let $p_R: X\longrightarrow X/_R$ be the canonical surjection. Also for any R, S in $\mathfrak{S}_{\mathfrak{U}}$ with $R\subseteq S$ let $p_{RS}: X/_R\longrightarrow X/_S$ be the unique map such that $p_{RS}p_R=p_S$.

If $(X;\mathfrak{U})$ is complete then the map

$$p=(\varprojlim p_R):(X;\mathfrak{U})\longrightarrow (X^*,\mathfrak{U}^*)\ ,$$

where $(X^*,\mathfrak{U}^*)=\varprojlim((X/_R;L),p_{RS})_{R,S\in\mathfrak{S}_{\mathfrak{U}}}$ is a regular epi in p-UNIF.

Proof Suppose that $(X;\mathfrak{U})$ is a complete p-u space. Let ξ be an element of $X^*=\varprojlim(X/_R;p_{RS})_{R,S\in\mathfrak{S}_{\mathfrak{U}}}$. Then $\xi=(\xi_R)_R\in\prod_{R\in\mathfrak{S}_{\mathfrak{U}}}(X/_R)$ so that $p_{RS}(\xi_R)=\xi_S$ for any R,S in $\mathfrak{S}_{\mathfrak{U}}$ with $R\subseteq S$. Now $F=\{p_R^{-1}(\xi_R)\mid R\in\mathfrak{S}_{\mathfrak{U}}\}$ is a base for a Cauchy filter in X. Indeed for any $R_i\in\mathfrak{S}_{\mathfrak{U}}$, $i=1,2$, $\bigcap_i^2 p_{R_i}^{-1}(\xi_{R_i})\supseteq p^{-1}_{R_1\cap R_2}(\xi_{R_1\cap R_2})$ since for each $x\in p^{-1}_{R_1\cap R_2}(\xi_{R_1\cap R_2})$ we have $p_{R_i}(x)=p_{R_1\cap R_2;R_i}p_{R_1\cap R_2}(x)$ $=p_{R_1\cap R_2;R_i}(\xi_{R_1\cap R_2})=\xi_{R_i}$, $i=1,2$. Obviously $p_R^{-1}(\xi_R)\times p_R^{-1}(\xi_R)\subseteq R$ for any R in $\mathfrak{S}_{\mathfrak{U}}$.

By the completeness of X, $\lim(F)=\bigcap_{R\in\mathfrak{S}_{\mathfrak{U}}}\bigcap_{S\in\mathfrak{S}_{\mathfrak{U}}} R(p_S^{-1}(\xi_S))$ is nonempty. If $x\in\lim(F)$ then $p_T(x)=\xi_T$ for each $T\in\mathfrak{S}_{\mathfrak{U}}$ since $x\in T(p_T^{-1}(\xi_T))$. Hence $p(x)=\xi$ and p is a surjection. Also $p[\mathfrak{U}]$ is finer than $\mathfrak{U}^*$. A base for $\mathfrak{U}^*$ consists of the entourages $f_S^{-1}(\Delta_{X/_S})$ where $S\in\mathfrak{S}_{\mathfrak{U}}$ and $f_S: X^*\longrightarrow X/_S$ are the canonical projections of the limit. Since $p^{-1}(f_S^{-1}(\Delta_{X/S})=p_S^{-1}(\Delta_{X/S})=S$, $p(S)=$ $=f_S^{-1}(\Delta_{X/S})$ for any $S\in\mathfrak{S}_{\mathfrak{U}}$. But any $S\in\mathfrak{S}_{\mathfrak{U}}$ contains the equivalence relation of p. Hence $\{p(S)\mid S\in\mathfrak{S}_{\mathfrak{U}}\}$ is a base for $p[\mathfrak{U}]$. Hence $p[\mathfrak{U}]=\mathfrak{U}^*$.

1.3.4 For any p-u space $(X;\mathcal{U})$ let us consider the co-filtered systems $\lambda=(X/_R;p^X_{RS})_{R,S\in\mathfrak{S}_{\mathcal{U}}}$ and $\Lambda=((X;\mathcal{U})/_R;p^X_{RS})_{R,S\in\mathfrak{S}_{\mathcal{U}}}$ where for each $R\in\mathfrak{S}_{\mathcal{U}}$, $p^X_R:X\longrightarrow X/_R$ or simply p_R is the canonical surjection of X onto its quotient set and for each R,S in $\mathfrak{S}_{\mathcal{U}}$ with $R\subseteq S$, $p^X_{RS}:X/_R\longrightarrow X/_S$ is the map uniquely defined by $p^X_{RS}p^X_R=p^X_S$. Since $R\in\mathcal{U}$, $(X;\mathcal{U})/_R=(X/_R;L)$ is a discrete space. Hence by the identification $\underline{DIS}\simeq\underline{SET}$ the above systems are the same, but as they were written the first is a system of sets and the second is a system of p-u spaces.

Let $\left((X^*,\mathcal{U}^*)\xrightarrow{f_R}(X,\mathcal{U})/R\right)_{R\in\mathfrak{S}_{\mathcal{U}}}$ be the limit of Λ. Hence $(X^*\xrightarrow[f_R]{}X/R)_{R\in\mathfrak{S}_{\mathcal{U}}}=\underleftarrow{\lim}\,\lambda$ and $\mathcal{U}^*$ is the initial p-uniformity on X induced by $\{f_R:X^*\longrightarrow(X/_R;L)\}_{R\in\mathfrak{S}_{\mathcal{U}}}$.
The family $\{p^X_R:(X,\mathcal{U})\longrightarrow(X;\mathcal{U})/_R\}_{R\in\mathfrak{S}_{\mathcal{U}}}$ induces a uniform continuous map $p^X:(X;\mathcal{U})\longrightarrow(X^*;\mathcal{U}^*)$.

Remark that $(X^*;\mathcal{U}^*)$ is a separated and complete p-u space as a projective limit of separated and complete uniform (discrete) spaces.

Proposition (i) The equivalence relation on X defined by p^X is $H=\bigcap\mathfrak{S}_{\mathcal{U}}$.

(ii) $p^X(X)$ is a dense subset of $(X^*;\mathcal{U}^*)$.

(iii) $(X,\mathcal{U})$ is separated iff p^X is a mono. In this case p^X is a regular mono and $(X,\mathcal{U})$ is a dense p-u subspace of $(X^*,\mathcal{U}^*)$.

(iv) $\underline{sep.p\text{-}UNIF}$ is the full subcategory of $\underline{p\text{-}UNIF}$ co-separated by $\underline{DIS}$.

Proof (i) Since $X^*=\left\{(\xi_R)_R\in\prod_{R\in\mathfrak{S}_{\mathcal{U}}}(X/_R)\;\middle|\;p^X_{RS}(\xi_R)=\xi_S \text{ for all } R\subseteq S \text{ in } \mathfrak{S}_{\mathcal{U}}\right\}$ and $f_S((\xi_R)_R)=\xi_S$ it follows that $p^X(x)=(p^X_R(x))_R$. Hence $p^X(x)=p^X(y)$ iff $p^X_R(x)=p^X_R(y)$ for all R in $\mathfrak{S}_{\mathcal{U}}$ iff $(x,y)\in\bigcap\mathfrak{S}_{\mathcal{U}}=H$.

(ii) A base for $\mathcal{U}^*$ is $\mathfrak{S}=\left\{f_R^{-1}(\Delta_{X/R})\;\middle|\;R\in\mathfrak{S}_{\mathcal{U}}\right\}$. Indeed a base for $\mathcal{U}^*$ certainly is the family of all finite intersections of elements of $\mathfrak{S}$. But $f_R^{-1}(\Delta_{X/R})=\{((\xi_S)_S;(\zeta_S)_S)\in X^*\times X^* \mid \xi_R=\zeta_R\}$. Actually $f_{R_1}^{-1}(\Delta_{X/R_1})\cap f_{R_2}^{-1}(\Delta_{X/R_2})\supseteq f_{R_1\cap R_2}^{-1}(\Delta_{X/(R_1\cap R_2)})$ by the fact that if $\xi_{R_1\cap R_2}=\zeta_{R_1\cap R_2}$ then $\xi_{R_i}=p^X_{R_1\cap R_2,R_i}(\xi_{R_1\cap R_2})=$ $=p^X_{R_1\cap R_2,R_i}(\zeta_{R_1\cap R_2})=\zeta_{R_i}$,$i=1,2$. Hence $\mathfrak{S}$ is cofiltered with the inclusions and any fundamental entourage of X^* contains a suitable entourage from $\mathfrak{S}$.

Now if $\xi=(\xi_R)_R\in X^*$ then the sets $A_S=\{(\zeta_R)_R \mid \zeta_S=\xi_S\}=$ $=(f_S^{-1}(\Delta_{X/S}))(\xi)$, $S\in\mathfrak{S}_{\mathfrak{U}}$, form a fundamental system of neighbourhoods of ξ . For $x\in X$ so that $\xi_S=p^X_S(x)$, $p^X(x)\in A_S$. Hence any neighbourhood of ξ contains points of $p^X(X)$.

(iii) $(X;\mathfrak{U})$ is separated iff $\bigcap\mathfrak{S}_{\mathfrak{U}}=\Delta_X$ iff p^X is an injection . In this case $(p^X)^{-1}[\mathfrak{U}^*]$ is less fine than $\mathfrak{U}$. Also for any $R\in\mathfrak{S}_{\mathfrak{U}}$, $(p^X)^{-1}(f_R^{-1}(\Delta_{X/R}))=(f_Rp^X)^{-1}(\Delta_{X/R})=(p^X_R)^{-1}(\Delta_{X/R})=$ $=R$. Hence $(p^X)^{-1}[\mathfrak{U}^*]$ contains a base of $\mathfrak{U}$. Consequently $(p^X)^{-1}[\mathfrak{U}^*]=\mathfrak{U}$ and p^X is a regular mono in p-UNIF

(iv) By (iii) a p-u space is separated iff it is a p-u subspace in a product of discrete p-u spaces. Hence if $f,g:(X;\mathfrak{U})\to(Y;\mathfrak{V})$ are distinct and if $(Y;\mathfrak{V})$ is a separated p-u space then there exists a uniform continuous map h on Y taking values in a discrete p-u space so that $hf\neq hg$. Hence sep.p-UNIF is coseparated by DIS. Conversely let $(X;\mathfrak{U})$ be a p-u space so that for any two distinct morphisms $\bullet \underset{g}{\overset{f}{\rightrightarrows}} (X;\mathfrak{U})$ of p-UNIF there exists an $h:(X;\mathfrak{U})\to D$ with $D\in\mathrm{ob}\,\mathrm{DIS}$ and $hf\neq hg$. Then for any two distinct points x,y of X , for the morphisms $(\{*\};L)\underset{*\mapsto y}{\overset{*\mapsto x}{\rightrightarrows}}(X;\mathfrak{U})$ there exists a discrete space D and a uniform continuous map $f:(X;\mathfrak{U})\to(D;L)$ so that $f(x)\neq f(y)$. The equivalence relation R defined by f is an entourage of X by the uniform continuity of f. Since $(x,y)\notin R$ it follows that (x,y) cannot be in $\bigcap\mathfrak{S}_{\mathfrak{U}}$. Hence $(X;\mathfrak{U})$ is separated.

<u>Proposition 1.3.5</u> (i) The correspondence $(X;\mathfrak{U}) \rightsquigarrow (X^*,\mathfrak{U}^*)$ represents a left adjoint for the inclusion in <u>p-UNIF</u> of the full subcategory of all separated and complete p-u spaces. We can equally well say that $(X^*;\mathfrak{U}^*)$ also denoted $(X,\mathfrak{U})^*$, is the p-u separated completion of $(X;\mathfrak{U})$. Moreover $(X^*,\mathfrak{U}^*)$ is isomorphic in <u>UNIF</u> to the separated completion of $(X;\mathfrak{U})$ as a uniform space.

(ii) A p-u space $(X;\mathfrak{U})$ is separated and complete iff p^X is a bijection. Obviously in this case p^X is an iso in <u>p-UNIF</u>

(iii) The separated and complete p-u spaces are exactly the projective limits of discrete uniform spaces (pro-discrete uniform spaces)

<u>Proof</u> (i) Let f be a uniform continuous map from the p-u space $(X;\mathfrak{U})$ to a separated and complete p-u space $(Y;\mathfrak{V})$ The equivalence relation on X defined by f is $f^{-1}(\Delta_Y)=f^{-1}(\bigcap\mathfrak{S}_{\mathfrak{V}}) = \bigcap\{f^{-1}(S) \mid S\in\mathfrak{S}_{\mathfrak{V}}\} \supseteq \bigcap\mathfrak{S}_{\mathfrak{U}} = H^X$. Hence there exists a unique $g:(X;\mathfrak{U})/_H \longrightarrow (Y;\mathfrak{V})$ so that $gp_H=f$. Since $(X;\mathfrak{U})/H$ is separated it is a dense subspace in $((X/_H)^*;(\mathfrak{U}/_H)^*)=(X/_H;\mathfrak{U}/_H)^*$ By continuity we can extend g at an $h:(X/_H;\mathfrak{U}/_H)^* \longrightarrow (Y;\mathfrak{V})$. The diagram

$$\begin{tikzcd}
(X;\mathfrak{U}) \arrow[r, "p_H"] \arrow[dr, "f"'] & (X;\mathfrak{U})/_H \arrow[r, hook] \arrow[d, "g"] & (X/H;\mathfrak{U}/H)^* \arrow[dl, "h"] \\
& (Y;\mathfrak{V}) &
\end{tikzcd}$$

obviously commutes.

On the other hand $\mathfrak{U}/_H=p_H[\mathfrak{U}] = \{p_H(R) \mid R\in\mathfrak{S}_{\mathfrak{U}}, R\supseteq H\} = \{p_H(R) \mid R\in\mathfrak{S}_{\mathfrak{U}}\}$ since $H=\bigcap\mathfrak{S}_{\mathfrak{U}}$. $(X/H)^* = \varprojlim_{\mathfrak{S}_{\mathfrak{U}/H}} (X/H)/_{p_H(R)}$.

But for any $R\subseteq S$ in $\mathfrak{S}_{\mathfrak{U}}$, $(X/H)/_{p_H(R)}$ is in bijection to X/R and the diagram

$$\begin{tikzcd}
(X/H)/_{p_H(R)} \arrow[r, "can"] \arrow[d, dash, "bij."'] & (X/H)/_{p_H(S)} \arrow[d, dash, "bij."] \\
X/_R \arrow[r, "p_{RS}"] & X/_S
\end{tikzcd}$$

commutes. Hence $(X/H)^*$ is canonically in bijection with X^*. For the same reason $(X/_H;\mathcal{U}/_H)^*$ is isomorphic to $(X;\mathcal{U})^*$ and $p:(X;\mathcal{U})\longrightarrow(X;\mathcal{U})^*$ equals the composition

$$(X;\mathcal{U}) \xrightarrow[p_H]{} (X;\mathcal{U})/_H \xhookrightarrow[can]{} (X/_H;\mathcal{U}/_H)^* \xrightarrow[iso]{} (X;\mathcal{U})^*$$

Hence for the given f there exists a $\varphi:(X;\mathcal{U})^*\longrightarrow(Y;\mathcal{V})$ so that $\varphi p=f$. By the density of p(X) in $(X;\mathcal{U})^*$ and by the completion of $(Y;\mathcal{V})$ this φ is uniquely defined by $\varphi p=f$.

Let $(X';\mathcal{U}')$ be the separated completion of $(X;\mathcal{U})$ in UNIF. Let $q:(X;\mathcal{U})\longrightarrow(X';\mathcal{U}')$ be the canonical uniform continuous map. In UNIF one has $p^X=fq$. The equivalence relation on X defined by q is $Q=q^{-1}(\Delta_{X'})=q^{-1}(\bigcap_{U'\in\mathcal{U}'} U')=\bigcap_{U'\in\mathcal{U}'} q^{-1}(U')$. Since each $q^{-1}(U')$ contains a suitable $R\in\mathfrak{S}_{\mathcal{U}}$, $H=H^X$ is contained in Q and there exists a uniform continuous map $g:(X;\mathcal{U})/_H\longrightarrow(X';\mathcal{U}')$ so that $gp_H=q$. Since $(X;\mathcal{U})/H$ is a uniform subspace of $(X^*;\mathcal{U}^*)$ g can be extended by density to a uniform continuous map $h:X^*\to X'$ and $hp=q$. Then $hfq=q=1_{X'}q$ and $fhp=fq=p=1_{X^*}p$. By the density of q(X) in X' and that of p(X) in X^* it follows that $hf=1_{X'}$ and $fh=1_{X^*}$. Hence $(X^*;\mathcal{U}^*)\xrightarrow{\sim}(X';\mathcal{U}')$ in UNIF.

(ii) If $(X;\mathcal{U})$ is separated and complete then p^X is in p-UNIF both a regular mono and an epi. Hence p^X is an iso. Conversely if p is an iso then $(X;\mathcal{U})$ is separated and complete as a limit of discrete uniform spaces.

1.4 The topology of p-u spaces, Stone spaces.

1.4.1 We recall that there exists a functor UNIF $\to$ TOP $(X;\mathfrak{U}) \to (X;\mathfrak{T})$ and $f \mapsto f$. This functor commutes with limits, monos, coequalizers and epimorphisms. If $x \in X$ then

$$\mathfrak{U}(x)=\{\, U(x) \mid U(x)=\{y\in X \mid (x,y)\in U\} \text{ and } U\in\mathfrak{U}\,\}$$

is the system of all neighbourhoods of x in $(X;\mathfrak{T})$. Obviously if $\mathfrak{S}$ is a fundamental system of entourages of X then $\mathfrak{S}(x)=\{A(x) \mid A\in\mathfrak{S}\}$ is a fundamental system of neighbourhoods of x.

From now on we suppose that $(X;\mathfrak{U})$ is a p-u space. Let $\mathfrak{S}\subseteq L(X)$ be a base for $\mathfrak{U}$. For $R\in L(X)$ and $x\in X$, $R(x)=\{y\in X \mid (x,y)\in R\}$

1.4.2 For each $x\in X$, $\mathfrak{S}(x)$ is a fundamental system of neighbourhoods of x consisting of clopen subsets of X. Consequently $(X;\mathfrak{T})$ is a zero-dimensional space (i.e. the clopen subsets of X form a base for $\mathfrak{T}$).

Proof If $R\in\mathfrak{S}$ then $R(y)=R(x)$ for all $y\in R(x)$. Hence any $R(x)$ is open in $(X,\mathfrak{T})$. If $y\notin R(x)$ then $R(y)\cap R(x)=\emptyset$. Hence $R(x)$ is also closed in $(X;\mathfrak{T})$.

1.4.3 Any $R\in\mathfrak{S}_{\mathfrak{U}}$ is clopen in $(X;\mathfrak{T})\times(X;\mathfrak{T})$.

Proof The closure R' of R in $(X;\mathfrak{T})\times(X;\mathfrak{T})$ is $\bigcap\{SRS \mid S\in\mathfrak{S}_{\mathfrak{U}}\}$ (SRS denotes the composition of relations). Since $R\subseteq R'$ and $R'\subseteq RRR=R$, $R'=R$. Hence R is closed in $(X;\mathfrak{T})\times(X;\mathfrak{T})$. On the other hand for each $x\in X$, $R(x)$ is open in $(X;\mathfrak{T})$. Hence $R(x)\times R(x)$ is open in $(X;\mathfrak{T})\times(X;\mathfrak{T})$. Consequently $R=\bigcup_{x\in X}(R(x)\times R(x))$ is open in $(X;\mathfrak{T})\times(X;\mathfrak{T})$.

1.4.4 $H=\bigcap\mathfrak{S}_{\mathfrak{U}}$ is closed in $(X;\mathfrak{T})\times(X;\mathfrak{T})$. The equivalence classes of X mod.H are exactly the connected components of $(X;\mathfrak{T})$.

Proof H is closed as intersection of closed subsets of $X\times X$. Let A be a connected subset of X. If a is any element of A then $H(a) \supseteq A$. Indeed for each $R \in \mathfrak{S}_{\mathcal{U}}$, $R(a) \cap A$ is a clopen subset of A in the subspace topology. Hence $R(a) \cap A$ is empty or equals the whole of A. Since $a \in A$ it remains that $R(a) \supseteq A$ for all $R \in \mathfrak{S}_{\mathcal{U}}$.

On the other hand each $H(x)$ is connected since for every $y \in X$ and every $R \in \mathfrak{S}_{\mathcal{U}}$, $R(y) \cap H(x)$ is $\emptyset$ or $H(x)$. Indeed if $z \in R(y) \cap H(x)$ then from yRz and zHx it follows that $(y,x) \in R$. Hence $R(y) \cap H(x) = R(x) \cap H(x) = H(x)$ since $H \subseteq R$.

Finally the maximal connected subsets of X are exactly the classes mod.H .

Corollary 1.4.5 A p-u space is separated iff the corresponding topological space is totally disconnected (i.e. singletons are the only connected subsets of X)

Corollary 1.4.6 A p-u space is separated iff it is a totally separated space in the induced topology (i.e. any two distinct points can be separated by disjoint clopen sets).

Proposition 1.4.7 A separated p-u space $(X;\mathcal{U})$ is compact iff there exists a base $\mathfrak{S} \subseteq L(X)$ of $\mathcal{U}$ so that all X/R with $R \in \mathfrak{S}$ are finite sets. In this case X/S is a finite set for each $S \in \mathfrak{S}_{\mathcal{U}}$.

Proof A separated uniform space $(X,\mathcal{U})$ is compact iff for each entourage U of X there exists a finite open covering $\{A_i\}_i$ of X so that $\bigcup_i (A_i \times A_i) \subseteq U$, [2] . Consequently $(X;\mathcal{U})$ is compact iff for each $R \in \mathfrak{S}_{\mathcal{U}}$ there exists a finite open covering $\{A_i\}_i$ of X so that $\bigcup_i (A_i \times A_i) \subseteq R$. From $\bigcup_{i=1}^{n} (A_i \times A_i) \subseteq R$ we infer that each A_i is contained in a class mod.R since if $x \in A_i$ then $\{(x,y) \mid y \in A_i\} \subseteq R$ and $A_i \subseteq R(x)$. From $\bigcup_{i=1}^{n} A_i = X$ it follows that each class mod.R contains an A_i. Hence there exists a partition $\{I_k\}_{k=1}^{m}$ of $\{1,\dots,n\}$ so that

$\left\{ \bigcup_{i \in I_k} A_i \right\}_{k=1}^{m} = \{R(x)\}_{x \in X}$. Hence X/R is finite.

Conversely if there exists a base $\mathfrak{S} \subseteq L(X)$ for $\mathcal{U}$ so that all X/R, $R \in \mathfrak{S}$, are finite sets then each $S \in \mathfrak{S}_{\mathcal{U}}$ contains a suitable $R \in \mathfrak{S}$. Hence $S \supseteq \bigcup_{x \in X} (R(x) \times R(x))$ and $\{R(x)\}_{x \in X}$ is a finite open covering of $\mathfrak{X}$.

Proposition 1.4.8 The category of all compact p-u spaces is isomorphic to the category of all Stone spaces.

Proof A Stone space is a compact totally separated space or, equivalently, a zero-dimensional compact space [12]. By 1.4.6 and 1.4.7 any compact p-u space is totally separated. Hence any compact p-u space is a Stone space in the induced topology. Conversely, a Stone space X is uniformisable in a unique way since it is a compact space ([2],[10]). Its uniformity has as fundamental system of entourages the family

$$\left\{ \bigcup_i (A_i \times A_i) \;\middle|\; \{A_i\}_i \text{ is a finite open covering of } X \right\}$$

By zero-dimensionality the family

$$\mathfrak{S} = \left\{ \bigcup_i (A_i \times A_i) \;\middle|\; \{A_i\}_i \text{ is a finite clopen covering of } X \right\}$$

is a fundamental system of entourages of X, too. Actually each such finite clopen covering can be refined to a finite partition of X by clopen subsets. Indeed, for the finite clopen covering $\{A_1, \ldots, A_n\}$ of X let us consider $Y = \bigcup_2^n A_i$ and $B_1 = A_1 - Y$. B_1 is clopen in X and $X - B_1 = Y$. Also Y is a Stone space in the subspace topology, and $\{A_2, \ldots, A_n\}$ is a finite clopen covering of Y. By an induction hypothesis on the number n there exists a partition $\{B_2, \ldots, B_k\}$ of Y by clopen subsets in Y so that $\{B_2, \ldots, B_k\}$ is a refinement of the covering $\{A_2, \ldots, A_n\}$ (i.e. each B_j is contained in a suitable A_i). Since Y is clopen in X, any clopen subset in Y is clopen in X too. Hence $\{B_1, B_2, \ldots, B_k\}$ is a partition of X by clopen subsets and each B_j, $j=1, \ldots, k$, is contained in a suitable

A_i . Hence for any entourage $V \in \mathfrak{S}$ there exists an entourage $V' \in \mathfrak{S}$ so that $V' \subseteq V$ and V' is an equivalence relation on X. Consequently the uniformity of X is a p-uniformity.

Finally by the fact that a map between compact spaces is uniform continuous iff it is continuous, the proof follows.

1.5 c-uniform objects in a category

Let $\mathcal{C}$ be a complete, cocomplete and regular category. Leading examples of $\mathcal{C}$ which we keep in mind are presheaves categories, Giraud toposes and algebraic categories over finitary theories. The key property of $\mathcal{C}$ is given by the usual calculus of images, inverse images and congruences [6].

For an object A of $\mathcal{C}$ we shall denote by L(A) or $L_{\mathcal{C}}(A)$ the lattice of all congruences on A.

Definition 1.51 A uniformity $\mathcal{U}$ on A is any filter $\mathcal{U}$ of reflexive subobjects of $A \pi A$ (reflexive means "contains the diagonal Δ_A") so that for each $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ with $V \circ V^{-1} \subseteq U$ (for V^{-1}, $V \circ W$, relations,... see [6]). The elements of $\mathcal{U}$ are called entourages of A. A fundamental system of entourages of A (a base for $\mathcal{U}$) is any filter base for $\mathcal{U}$. The pair $(A;\mathcal{U})$ is called a uniform object in $\mathcal{C}$. If $(A;\mathcal{U})$ and $(B;\mathcal{V})$ are uniform objects in $\mathcal{C}$ then we shall say that a morphism $f:A \to B$ is uniform continuous in $\mathcal{U}$ and $\mathcal{V}$ if $f^{-1}[\mathcal{V}] \subseteq \mathcal{U}$; here $f^{-1}[\mathcal{V}]$ denotes the subobjects filter on $A \pi A$ generated by $\{(f \pi f)^{-1}(V) \mid V \in \mathcal{V}\}$.

Finally u-$\mathcal{C}$ denotes the category of uniform objects in $\mathcal{C}$ and uniform continuous morphisms between them

Definition 1.5.2 A uniformity $\mathcal{U}$ on A is a c-uniformity if $\mathcal{U}$ has a base consisting of congruences on A. Obviously any filter base $\mathfrak{S}$ in L(A) is a base for a c-uniformity $\bar{\mathfrak{S}}$ on A.

A uniform object $(A;\mathcal{U})$ in $\mathcal{C}$ is a c-uniform object (c-u object) in $\mathcal{C}$ if $\mathcal{U}$ is a c-uniformity.

c-U-$\mathcal{C}$ denotes the category of all c-u objects in $\mathcal{C}$ and all uniform continuous morphisms between them.

1.5.3 As in 1.2.7 the category c-U-$\mathcal{C}$ can be described as the category whose

$$\begin{cases} \text{objects are pairs } (A;\mathcal{A}) \text{ with } A\in\mathcal{C} \text{ and } \mathcal{A}\in \text{Filt } L_{\mathcal{C}}(A) \\ \text{morphisms from } (A;\mathcal{A}) \text{ to } (B;\mathcal{B}) \text{ are those } f\in \text{Hom}_{\mathcal{C}}(A;B) \text{ for} \\ \text{which } f^{-1}[\mathcal{B}]\subseteq\mathcal{A}. \end{cases}$$

In this context we shall refer to the filter $\mathcal{A}$ as to the c-uniformity on A.

1.5.4 Examples :

a) c-U-Ab is the category of all topological abelian groups which has a fundamental system of neighbourhoods of o consisting of subgroups

b) c-U-Rg is the category of all linear topological rings [4],[12],[9],i.e. topological rings which has a base of neighbourhoods of o consisting of bilateral ideals.

c) c-U-R.Mod is the category of all topological R-modules which has a base of neighbourhoods of o consisting of submodules. A c-uniformity on R considered as an object of R.Mod is just a "left" linear topology on R([13],[9]); (i.e. not all module morphism $R\to R$ are continuous, see also 4.6.

d) c-U-SET is just p-UNIF

e) c-U-K.Mod where K is a field is the category of all linearly topologised vectorial spaces over K. ([14])

1.5.5 Let $(A;\mathcal{U})$ be a c-u object in $\mathcal{C}$. Let $\mathfrak{S}\subseteq L(A)$ be a base for $\mathcal{U}$. Since the functor $h^B=\text{Hom}_{\mathcal{C}}(B,-):\mathcal{C}\to \underline{\text{SET}}$ preserves all monomorphisms, all intersections of subobjects and all limits it follows that the family $\{h^B(R)\}_{R\in\mathfrak{S}}$ is naturally a filter base in $L_{\underline{\text{SET}}}(\text{Hom}_{\mathcal{C}}(B;A))$. We point that $h^B(R)$ is the set

$$\{(u,v)\in h^B(A)\times h^B(A) \mid \langle u,v\rangle \text{ factors through } \langle r_1;r_2\rangle\}$$

where $\langle r_1;r_2\rangle:R\to A\Pi A$ is the inclusion. The p-uniformity on

$Hom_{\mathcal{C}}(B;A)$ generated by $\{h^B(R)\}_{R\in\mathfrak{S}}$ is denoted by $h^B[\mathcal{U}]$ and the corresponding p-u space is denoted by $(h^B(A);h^B[\mathcal{U}])$.

If $f:(A;\mathcal{U})\longrightarrow(A';\mathcal{U}')$ is a uniform continuous morphism between c-u objects then the map $h^B(f):h^B(A)\longrightarrow h^B(A')$, $x\mapsto fx$ is uniform continuous in $h^B[\mathcal{U}]$ and $h^B[\mathcal{U}']$, since for each R' $(h^B(f))^{-1}(h^B(R'))=h^B(f^{-1}(R'))$. Hence any covariant hom-functor on $\mathcal{C}$ produces a covariant functor $\underline{\text{c-u-}\mathcal{C}}\rightarrow\underline{\text{p-UNIF}}$.

On the other hand for each $R\in\mathfrak{S}$, h_R is a congruence on $h_A=$ $=Hom_{\mathcal{C}}(-,A)$, namely $h_R(B)=\{(x,y)\in h_A(B)\times h_A(B)\mid\langle x,y\rangle$ factors through $\langle r_1,r_2\rangle\}$. Since $h_R\cap h_S=h_{R\cap S}$ the family $\{h_R\}_{R\in\mathfrak{S}}$ is a base for a c-uniformity on h_A, and h_A is canonically a c-u object in $\mathcal{C}^\wedge$. For any morphism $f:B\rightarrow B'$ and any $R\in\mathfrak{S}$ one has $(h_A(f))^{-1}(h_R(B))=$ $=\{\langle u,v\rangle\in Hom_{\mathcal{C}}(B';A\Pi A)\mid\langle u,v\rangle f$ factors through $\langle r_1,r_2\rangle\}\supseteq h_R(B')$. Since $h_A(f)=(h^f)_A$ and $h_R(X)=h^X(R)$ it results that

$$(h^f)_A:h^{B'}(A)\longrightarrow h^B(A)$$

is a uniform continuous map in the uniformities $h^{B'}[\mathcal{U}]$ and $h^B[\mathcal{U}]$. Hence the canonical c-u object h_A associated to $(A;\mathcal{U})$ is just the p-u space valued cofunctor on $\mathcal{C}$ arising from the family

$$\{(h^B(A),h^B[\mathcal{U}])\}_{B\in ob\,\mathcal{C}}$$

Also each uniform continuous morphism $f:(A;\mathcal{U})\longrightarrow(A';\mathcal{U}')$ between c-u objects in $\mathcal{C}$ produces a uniform continuous functorial morphism $h_f:h_A\longrightarrow h_{A'}$ since $(h_f)^{-1}(h_{R'})=h_{f^{-1}(R')}$ for all $R'\in\mathcal{U}'$. In this way the inclusion $\mathcal{C}\rightarrow\mathcal{C}^\wedge$ canonically produces two functors

$$\underline{\text{c-u-}\mathcal{C}}\xrightarrow{\alpha}\underline{\text{c-u-}\mathcal{C}^\wedge};\ (A;\mathcal{U})\rightsquigarrow h_A$$

and

$$\underline{\text{c-u-}\mathcal{C}}\xrightarrow{\beta}[\mathcal{C}^o,\underline{\text{p-UNIF}}];(A;\mathcal{U})\rightsquigarrow\{(h^B(A),h^B[\mathcal{U}])\}_{B\in ob\,\mathcal{C}}$$

By what was said above the diagram

$$\begin{tikzcd} \underline{c\text{-}u\text{-}\mathscr{C}} \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & \underline{c\text{-}u\text{-}\mathscr{C}}^{\wedge} \arrow[dl, "(F;\mathcal{U}) \rightsquigarrow \{(F(X);\{R(X)\}_{R\in\mathcal{U}})\}_{X\in ob\mathscr{C}}"] \\ & [\mathscr{C}^{o}, \underline{p\text{-UNIF}}] & \end{tikzcd}$$

commutes, and all these three functors are faithful and full functors but none of them is an equivalence.

1.5.6 All remarks and definitions from 1.2.8 to 1.2.2o except 1.2.14 , can be extended word by word to c-u objects replacing "surjection" by "regular epimorphism". Also 1.2.14 is true if convenient hypotheses on $\mathscr{C}$ are made. We still point out that :

1.5.6.1 In order to study the intrinsic properties of $\underline{c\text{-}u\text{-}\mathscr{C}}$ the most convenient point of view is that of $\underline{c\text{-}u\text{-}\mathscr{C}}$ as the category of all pairs $(A;\mathcal{A})$ with $A\in ob\mathscr{C}$ and $\mathcal{A}\in \text{Filt}\, L(A)$. In this case we shall refer to $\mathcal{A}$ as the c-u structure on A or simply the c-uniformity on A.

From this point of view the initial c-u structure $\mathcal{A}$ on A induced by the family $\{f_i:A \to (A_i;\mathcal{A}_i)\}_{i\in I} \subseteq \text{Morf}\,\mathscr{C}$ is is the filter in L(A) generated by all finite intersections of the family $\{f_i^{-1}(R_i) \mid i\in I \text{ and } R_i\in\mathcal{A}_i\}$. $\mathcal{A}$ is the coarsest c-u structure on A for which all f_i are uniform continuous. Moreover for a c-u object $(B;\mathcal{B})$ a morphism $f:B\to A$ is uniform continuous in $\mathcal{B}$ and $\mathcal{A}$ iff all $f_i f$ are uniform continuous. Consequently $\underline{c\text{-}u\text{-}\mathscr{C}}$ is a complete category. If $F:I \to \underline{c\text{-}u\text{-}\mathscr{C}}$ is a small functor, say $F(i)=(A_i,\mathcal{A}_i)$, then $\varprojlim F = (\varprojlim A_i , \mathcal{A})$ where $\mathcal{A}$ is the initial c-u structure on $\varprojlim A_i$ induced by the family of all canonical projections $\varprojlim A_i \to A_i$. Particularly if $f:(A;\mathcal{A})\to(B;\mathcal{B})$ is a morphism of $\underline{c\text{-}u\text{-}\mathscr{C}}$ and if $R \overset{u}{\underset{v}{\rightrightarrows}} A$ is a kernel pair of $f:A\to B$ then $(R;\mathfrak{S}) \overset{u}{\underset{v}{\rightrightarrows}} (A;\mathcal{A})$ is a kernel pair of f in $\underline{c\text{-}u\text{-}\mathscr{C}}$ provided $\mathfrak{S}$ is the initial c-u structure on R induced by $u,v : R \rightrightarrows (A;\mathcal{A})$. Also $f:(A;\mathcal{A})\to(B;\mathcal{B})$ is a mono (regular mono) iff $f:A\to B$ is a mono (regular mono and $\mathcal{A}=f^{-1}[\mathcal{B}]$).

If f is a mono and $\mathcal{A}=f^{-1}[\mathcal{B}]$ then we shall say that $(A,\mathcal{A})$ is a strong subobject of $(B,\mathcal{B})$.

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1.5.6.2 The final c-u structure on A induced by the family $\{f_i:(A_i,\mathcal{A}_i)\longrightarrow A\}_i$ is $\mathcal{A}=\{R\in L(A)\mid f_i^{-1}(R)\in\mathcal{A}_i$ for all $i\}$. As for p-u spaces:

(i) $\mathcal{A}$ is the finest c-u structure on A so that all f_i are uniform continuous

(ii) $\mathcal{A}=\bigcap_i f_i[\mathcal{A}_i]$ where $f_i[\mathcal{A}_i]=\{R\in L(A)\mid f_i^{-1}(R)\in\mathcal{A}_i\}$

(iii) for any c-u object $(B;\mathcal{B})$, a morphism $f:A\to B$ is uniform continuous in $\mathcal{A}$ and $\mathcal{B}$ iff all ff_i are uniform continuous

(iv) the colimit of a functor $F:I\longrightarrow$ c-u-$\mathcal{C}$ is $(\varinjlim|F|,\mathcal{A})$ where c-u-$\mathcal{C}\xrightarrow{|\,|}\mathcal{C}$ forgets the c-u structure and $\mathcal{A}$ is the final c-u structure on $\varinjlim|F|$ induced by the family of all canonical morphisms of $\varinjlim|F|$

(v) for a regular epi $f:A\to B$ and for a c-u structure $\mathcal{A}$ on A the final c-u structure on B induced by f is

$$f[\mathcal{A}]=\{f(R)\mid R\in\mathcal{A} \text{ and } R\supseteq \text{the congruence of } f\}$$

In this case $f:(A,\mathcal{A})\longrightarrow(B;f[\mathcal{A}])$ is a regular epi in c-u-$\mathcal{C}$.

(VI) a sequence $(A;\mathcal{A})\overset{u}{\underset{v}{\rightrightarrows}}(B;\mathcal{B})\xrightarrow{w}(B';\mathcal{B}')$ of c-u-$\mathcal{C}$ is a right exact sequence iff w is a coequalizer of $\{u,v\}$ in $\mathcal{C}$, and $\mathcal{B}'=w[\mathcal{B}]$.

(vii) a sequence $(A;\mathcal{A})\overset{u}{\underset{v}{\rightrightarrows}}(B;\mathcal{B})\xrightarrow{w}(B';\mathcal{B}')$ of c-u-$\mathcal{C}$ is a short exact sequence iff $A\overset{u}{\underset{v}{\rightrightarrows}}B\xrightarrow{w}B'$ is a short exact sequence in $\mathcal{C}$, $\mathcal{A}=(\langle u,v\rangle)^{-1}[\mathcal{B}]$ and $\mathcal{B}'=w[\mathcal{B}]$.

Definitions: 1.5.7 A c-u object $(A;\mathcal{A})$ is separated if $\bigcap\mathcal{A}=\Delta_A$. Obviously if $\mathfrak{S}\subseteq L_{\mathcal{C}}(A)$ is a base for $\mathcal{A}$ then $(A;\mathcal{A})$ is separated iff $\bigcap\mathfrak{S}=\Delta_A$. Also $(A;\mathcal{A})$ is separated iff all $(h^B(A);h^B[\mathcal{A}])$, $B\in\operatorname{ob}\mathcal{C}$, are separated p-u spaces.

1.5.8 A c-u object $(A;\mathcal{A})$ is discrete if $\Delta_A\in\mathcal{A}$. Obviously $(A;\mathcal{A})$ is discrete iff $\mathcal{A}=L_{\mathcal{C}}(A)$. A discrete c-u object $(A;\mathcal{A})$ will be denoted simply by $(A;L)$. Remark that the category

<u>Dis.c-u-$\mathcal{C}$</u> of all discrete c-u objects in $\mathcal{C}$ is on one hand isomorphic to $\mathcal{C}$ and on the other hand is closed in <u>c-u-$\mathcal{C}$</u> at finite limits and subobjects.

<u>1.5.9</u> Let $(A;\mathcal{A})$ be a separated c-u object. Let $\mathfrak{S}$ be a base for $\mathcal{A}$. For each $R\in\mathfrak{S}$ let $p_R^A:A \to A/R$ be the canonical regular epimorphism from A to its regular quotient A/R. Also for $R,S\in\mathfrak{S}$ with $R\subseteq S$ let $p_{RS}^A:A/_R \to A/_S$ be the morphism uniquely defined by $p_{RS}^A p_R^A = p_S^A$.

We shall say that $(A;\mathcal{A})$ is a separated and complete c-u object if $\left(A \xrightarrow[p_R^A]{} A/R\right)_{R\in\mathfrak{S}} = \varprojlim_{\mathfrak{S}}(A/R;p_{RS}^A)$. Obviously this is equivalent to $\left((A,\mathcal{A}) \xrightarrow[p_R^A]{} (A,\mathcal{A})/_R\right)_{R\in\mathfrak{S}} = \varprojlim_{\mathfrak{S}}\left((A,\mathcal{A})/_R ; p_{RS}^A\right) = \varprojlim_{\mathfrak{S}}\left((A/_R,L), p_{RS}^A\right)$.

<u>1.5.9.1</u> Suppose that $\mathcal{C}$ has a set $\mathcal{M}$ of special Γ-generators (i.e. $f\in\mathrm{Mor}\,\mathcal{C}$ is a regular epi iff all $h^M(f)$, $M\in\mathcal{M}$, are surjections) Let $(A;\mathcal{A})$ be a c-u object in $\mathcal{C}$ and let $\mathfrak{S}\subseteq L_{\mathcal{C}}(A)$ be a base for $\mathcal{A}$. For each $M\in\mathcal{M}$, $(h^M(A);h^M[\mathcal{A}])$ is a p-u space and $\{h^M(R)\mid R\in\mathfrak{S}\}$ is a base for $h^M[\mathcal{A}]$. By <u>1.3.5 (ii)</u> $(h^M(A),h^M[\mathcal{A}])$ is a separated and complete p-u space iff the canonical map

$$\mathrm{Hom}_{\mathcal{C}}(M;A) \longrightarrow \varprojlim_{R\in\mathfrak{S}}\left(\mathrm{Hom}_{\mathcal{C}}(M;A)/\mathrm{Hom}_{\mathcal{C}}(M;R)\right)$$

is a bijection. Since $\mathcal{M}$ is a set of special Γ-generators

$$h^M(R) \underset{h^M(r_2)}{\overset{h^M(r_1)}{\rightrightarrows}} h^M(A) \xrightarrow{\;h^M(p_R)\;} h^M(A/R)$$

is a short exact sequence of sets for each $M\in\mathcal{M}$. Consequently $\varprojlim_{\mathfrak{S}}\left(h^M(A)/h^M(R)\right)$ is canonically isomorphic to $\varprojlim_{\mathfrak{S}} h^M(A/R)$ which in turn is canonically isomorphic to $h^M(\varprojlim_{\mathfrak{S}}(A/_R))$. Now if $p:A\to\varprojlim_{\mathfrak{S}}(A/_R)$ denotes the canonical morphism induced by $\{p_R\}_{R\in\mathfrak{S}}$ then one has a canonically commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(M,A) \arrow[r] \arrow[d, equal] & \varprojlim_{\mathcal{G}}\left(\mathrm{Hom}_{\mathcal{C}}(M;A)/\mathrm{Hom}_{\mathcal{C}}(M;R)\right) \arrow[d, "\wr", "\mathrm{can}"'] \\
\mathrm{Hom}_{\mathcal{C}}(M;A) \arrow[r, "h^M(p)"] & \mathrm{Hom}_{\mathcal{C}}\left(M; \varprojlim_{\mathcal{G}}(A/_R)\right)
\end{tikzcd}$$

Since p is an iso iff all $h^M(p)$ are bijections, by the above diagram it results that $(A;\mathcal{A})$ is separated and complete iff all $(h^M(A);h^M[\mathcal{A}])$, $M\in\mathcal{M}$, are separated and complete p-u spaces. Consequently if $\mathcal{C}$ is a presheaves category or an algebraic category over a finitary theory then the definition 1.5.9 is in concordance with the separation and completeness of all underlying spaces.

Proposition 1.5.1o If any epi of $\mathcal{C}$ is a regular one and if $\mathcal{C}$ is a $(G\text{-}C)^o$-category then the category of all separated and complete c-u objects in $\mathcal{C}$ is equivalent to the category of all strict pro-objects of $\mathcal{C}$. ([18]: G-C condition means that the canonical morphisms of any monofiltered colimit are all monos)

Proof The category of all pro-objects of $\mathcal{C}$ is equivalent to the dual of the full subcategory of $[\mathcal{C}, \underline{SET}]$ consisting of all functors which are filtered colimits of representable functors. The category of all strict pro-objects is equivalent to the dual of the full subcategory of $[\mathcal{C}, \underline{SET}]$ consisting of all functors which are monofiltered colimits of representable functors, [7].

Each separated and complete c-u object $(A;\mathcal{A})$ associates with the functor $\widetilde{(A;\mathcal{A})}=\varinjlim\,(h^{A/R},h^{p_{RS}})_{\substack{R\subseteq S\\ R,S\in\mathcal{A}}}$. $\widetilde{(A;\mathcal{A})}(X)=$

$=\{[f] \mid$ there exists $R\in\mathcal{A}$ and $f:A/_R\longrightarrow X\}$ where $[f]=[g]$ iff $fp_{R\cap S;R}=gp_{R\cap S,S}$ for $f:A/_R\longrightarrow X$ and $g:A/_S\longrightarrow X$. Remark that if $[f:A/_R\to X]=[g:A/S\longrightarrow X]$ then $fp_R^A=gp_S^A$. Conversely if

$fp^A_R = gp^A_S$ then from the commutative diagram

$$\begin{tikzcd}
& & A/_R \arrow[dr, "f"] & \\
A \arrow[urr, bend left, "p_R"] \arrow[r, two heads, "p^A_{R\cap S}"] \arrow[drr, bend right, "p_S"'] & A/_{R\cap S} \arrow[ur, "p_{R\cap S,R}"] \arrow[dr, "p_{R\cap S,S}"] & & X \\
& & A/_S \arrow[ur, "g"'] &
\end{tikzcd}$$

it results $fp_{R\cap S,R} = gp_{R\cap S,S}$. Hence $[f]=[g]$ is equivalent to $fp_R = gp_S$.

For a morphism $u:Y\to X$ of $\mathscr{C}$, $\widetilde{(A;\mathscr{A})}(u)$ is the map $[f]\mapsto[uf]$.

Finally a uniform continuous morphism $(B,\mathscr{B}) \xrightarrow{a} (A,\mathscr{A})$ induces $\tilde{a}:\widetilde{(A;\mathscr{A})} \to \widetilde{(B;\mathscr{B})}$, $\tilde{a}_X([f:A/_R \to X]) = [fa_R]$, where $a_R: B/_{a^{-1}(R)} \to A/_R$ is uniquely defined by the commutative diagram

$$\begin{tikzcd}
B/_{a^{-1}(R)} \arrow[r, hook, "a_R"'] & A/_R \\
B \arrow[u, two heads, "p^B_{a^{-1}(R)}"] \arrow[r, "a"] & A \arrow[u, two heads, "p^A_R"']
\end{tikzcd}$$

In this way we have obtained a cofunctor $\widetilde{?}$ from <u>sep.compl.c-u$\mathscr{C}$</u> to <u>strict pro-$\mathscr{C}$</u>.

Let $a,b:(B;\mathscr{B}) \to (A;\mathscr{A})$ be two uniform continuous morphisms so that $\tilde{a}=\tilde{b}$. Since $\tilde{a}_{A/R}([1_{A/R}]) = \tilde{b}_{A/R}([1_{A/R}])$ for any $R\in\mathscr{A}$, the diagrams

$$\begin{tikzcd}
& B/_{a^{-1}(R)} \arrow[dr, hook, "a_R"] & \\
B/_{a^{-1}(R)\cap b^{-1}(R)} \arrow[ur, "can"] \arrow[dr, "can"'] & & A/_R \\
& B/_{b^{-1}(R)} \arrow[ur, hook, "b_R"'] &
\end{tikzcd}$$

commute for all $R \in \mathcal{A}$. Hence $a_R p^B_{a^{-1}(R)} = b_R p^B_{b^{-1}(R)}$ for all $R \in \mathcal{A}$. Then $p^A_R a = p^A_R b$ for all $R \in \mathcal{A}$. Since $(A;\mathcal{A})$ is separated, $a=b$.

Now let $F:\widetilde{(A;\mathcal{A})} \longrightarrow \widetilde{(B;\mathcal{B})}$ be a functorial morphism. For each $R \in \mathcal{A}$ let $g_R : B/_{R'} \rightarrow A/_R$ be so that $F_{A/R}([1_{A/R}]) = [g_R]$

The family $\{f_{(R)} = g_R\, p^B_{R'}\}_{R \in \mathcal{A}}$ produces a morphism $f: B \rightarrow A$. Indeed, if $R \subseteq S$ are in $\mathcal{A}$ then from the functoriality of F we obtain the commutative diagram

$$\begin{tikzcd}
\widetilde{(A;\mathcal{A})}(A/R) \arrow[r, "F_{A/R}"] \arrow[d, "\widetilde{(A;\mathcal{A})}(p^A_{R,S})"'] & \widetilde{(B;\mathcal{B})}(A/R) \arrow[d, "\widetilde{(B;\mathcal{B})}(p^A_{R,S})"] \\
\widetilde{(A;\mathcal{A})}(A/S) \arrow[r, "F_{A/S}"] & \widetilde{(B;\mathcal{B})}(A/S)
\end{tikzcd}$$

Since $p^A_{R,S}\, p^A_{R,R} = 1_{A/S}\, p^A_{R,S}$, $[p^A_{R,S}] = [1_{A/S}]$ and $[p^A_{R,S}\, g_R] =$ $= F_{A/S}([p^A_{R,S}]) = F_{A/S}([1_{A/S}]) = [g_S]$. Hence $p^A_{R,S}\, f_{(R)} = f_{(S)}$. Consequently there exists a morphism $f: B \rightarrow A$ uniquely defined by $p^A_R f = f_{(R)}$, $R \in \mathcal{A}$. Also for each $R \in \mathcal{A}$ there exists $R' \in \mathcal{B}$ (namely that one which appears in the definition of g_R) so that the diagram

$$\begin{tikzcd}
 & A \arrow[dr, "p^A_R"] & \\
B \arrow[ur, "f"] \arrow[dr, "p^B_{R'}"'] & & A/R \\
 & B/R' \arrow[ur, "g_R"'] &
\end{tikzcd}$$

commutes. Since $(p^A_R f)^{-1}(\Delta_{A/R}) \supseteq R'$ and since $R' \in \mathcal{B}$ it follows that all $p^A_R f$, $R \in \mathcal{A}$, are uniform continuous in $\mathcal{B}$ and $L_{\mathcal{C}}(A/R)$. Actually $f: B \rightarrow A$ is uniform continous in $\mathcal{B}$ and $\mathcal{A}$ because $\mathcal{A}$ is the initial c-u structure on A induced by the family $\{p^A_R\}_{R \in \mathcal{A}}$

Also $\tilde{f}_X([u:A/_R \longrightarrow X]) = \tilde{f}_X(((\widetilde{(A;\mathcal{A})})(u))([1_{A/R}]))=$
$=((\widetilde{(B;\mathcal{B})})(u))(\tilde{f}_{A/R}([1_{A/R}]))$. Since $f^{-1}(R) \supseteq R'$, $\tilde{f}_{A/R}([1_{A/R}])=$
$=[f_R]=[g_R]$ by the commutative diagram

$$\begin{tikzcd}
B \arrow[rr, "f"] \arrow[dr, "p^B_{f^{-1}(R)}"] \arrow[ddr, bend right, "p^B_{R'}"'] & & A \arrow[d, "p^A_R"] \\
& B/_{f^{-1}(R)} \arrow[r, hook, "f_R"'] & A/R \\
& B/R' \arrow[u, "p^B_{R',f^{-1}(R)}"] \arrow[ur, bend right, "g_R"'] &
\end{tikzcd}$$

Hence $\tilde{f}_X([u])=((\widetilde{(B;\mathcal{B})})(u))(F_{A/R}([1_{A/R}]))=F_X([u])$ for any $X \in ob\,\mathcal{C}$ and any $[u] \in \widetilde{(A;\mathcal{A})}(X)$. Hence $\tilde{f}=F$ and $\tilde{?}$ is a fully faithful cofunctor.

To end the proof let $\mathcal{F}$ be a strict pro-representable functor $\mathcal{C} \longrightarrow SET$. Then there exists a monofiltered system $(h^{X_i}, h^{x_{ij}})_{i \leq j}$ whose colimit is $\mathcal{F}$. Since each $h^{x_{ij}}$ is a mono each x_{ij} is an epi in $\mathcal{C}$. Hence $(X_i, x_{ij})_{i \leq j}$ is an epicofiltered system in $\mathcal{C}$. By $(G\text{-}C)^o$ the canonical projections $\underleftarrow{\lim}\,(X_i, x_{ij})_{i \leq j} \xrightarrow{x_i} X_i$ are epimorphisms in $\mathcal{C}$; moreover they are regular epimorphisms. If $(A;\mathcal{A})$ is the limit in <u>c-u-$\mathcal{C}$</u> of the system $((X_i;L), x_{ij})_{i \leq j}$ then a base for $\mathcal{A}$ is $\{R_i = \text{congruence of } x_i\}_i$. Also $A/_{R_i} \simeq X_i$ for each i. Since $(h^{X_i}, h^{x_{ij}})_{i \leq j}$ is a cofinal system in $(h^{A/R}, h^{p_{RS}})_{R \subseteq S}$ in $\mathcal{A}$, $\widetilde{(A;\mathcal{A})} = \underrightarrow{\lim}\,(h^{X_i}, h^{x_{ij}})_{i \leq j}$. Finally $\widetilde{(A;\mathcal{A})} \simeq \mathcal{F}$ and $\tilde{?}$ is an equivalence.

<u>1.6 The categories Γ^*</u> Let $\mathcal{J}$ be a full subcategory of $\mathcal{C}$. A c-structure Γ of $\mathcal{J}/\mathcal{C}$ is a functor $\Gamma : \mathcal{J} \longrightarrow \underline{c\text{-}u\text{-}\mathcal{C}}$, $\Gamma(X)=(X;\mathcal{X})$ so that the diagram

$$\begin{tikzcd}
\mathcal{J} \arrow[rr, "\Gamma"] \arrow[dr] & & \underline{c\text{-}u\text{-}\mathcal{C}} \arrow[dl, "1.1" ', "\text{forgets c-uniformity}"] \\
& \mathcal{C} &
\end{tikzcd}$$

commutes.

For a c-structure Γ of $\mathcal{J}/\mathcal{C}$ let us denote by Γ^* the full subcategory of $\underline{c\text{-}u\text{-}\mathcal{C}}$ consisting of all $(A;\mathcal{A})$ so that for each $X \in \mathrm{ob}\,\mathcal{J}$ all the morphisms $X \longrightarrow A$ are uniform continuous in $\mathcal{X}$ and $\mathcal{A}$. These data can be also described as follows: we endow each $X \in \mathrm{ob}\,\mathcal{J}$ with a c-u structure $\mathcal{X}$ so that all the morphisms of $\mathcal{J}$ are oniform continuous in the chosen uniformities; The functor Γ is just $\Gamma(X)=(X;\mathcal{X})$, $\Gamma(f)=f$. Γ^* is the full subcategory of $\underline{c\text{-}u\text{-}\mathcal{C}}$ of all $(A;\mathcal{A})$ so that for each $X \in \mathrm{ob}\,\mathcal{J}$ the canonical inclusion $\mathrm{Hom}_{\underline{c\text{-}u\text{-}\mathcal{C}}}((X;\mathcal{X}),(A;\mathcal{A})) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,A)$ is a bijection.

If $\mathcal{C}$ is the category of all left modules over the ring R then a left linear topology ([M7]) on R is simply a c-uniformity on the left module R such that all module morphisms $R \to R$ are uniform continuous. If Γ denotes the c-structure of $\mathcal{J}/\mathcal{C}$ obtained in this way then Γ^* is just the category of all left linear topological modules over the left linear topological ring R ([M7]). ($\mathcal{J}$ denotes the full subcategory of $\mathcal{C}$ whose class of objects is $\{R\}$)

In this way we can generalize almost all constructions of to algebraic categories or sheaves categories.

In what follows we shall fix a c-structure Γ of $\mathcal{J}/\mathcal{C}$, we shall suppose that $\mathcal{J}/A$ is nonempty for all $A \in \mathrm{ob}\,\mathcal{C}$ and we shall identify morphisms from objects of $\mathcal{J}$ to A with objects of $\mathcal{J}/A$

<u>Proposition 1.6.1</u> (i) Γ^* is a reflective subcategory of $\underline{c\text{-}u\text{-}\mathcal{C}}$. The reflection is $(A;\mathcal{A}) \rightsquigarrow (A;\mathcal{A}^{\Gamma})$ where $\mathcal{A}^{\Gamma}$

is the meet c-uniformity of $\mathcal{A}$ and of the final c-uniformity on A induced by $\{f:\Gamma(X) \to A \mid (X;f) \in \mathrm{ob}\ \mathcal{J}/A\}$.

(ii) Γ^* is closed in c-u-$\mathcal{C}$ under strong subobjects and limits (i.e. if $(A;\mathcal{A}) \in \Gamma^*$ and if $(B;\mathcal{B})$ is a strong subobject of $(A,\mathcal{A})$ in c-u-$\mathcal{C}$ then $(B;\mathcal{B}) \in \mathrm{ob}\ \Gamma^*$; closed under limits means that any limit (in c-u-$\mathcal{C}$) of objects of Γ^* is in Γ^*too) If $\{(A_i,\mathcal{A}_i)\}_i \subseteq \mathrm{ob}\ \Gamma^*$ and if $\mathcal{A}$ is the initial c-u structure on A induced by $\{f_i : A \to (A_i;\mathcal{A}_i)\}_i$ then $(A;\mathcal{A}) \in \mathrm{ob}\ \Gamma^*$. Moreover Γ^* is a complete and cocomplete category.

(iii) If $\mathcal{J}$ is a set of special generators in $\mathcal{C}$ then $\Gamma^*$is closed in c-u-$\mathcal{C}$ under regular quotients and $\Gamma^* \hookrightarrow$ c-u-$\mathcal{C}$ creates and preserves coequalizers.

Proof (i) Let $f:(A;\mathcal{A}) \longrightarrow (B;\mathcal{B})$ be a morphism of c-u-$\mathcal{C}$. The final c-uniformity on A induced by the family ob $\mathcal{J}/A$ is $\{R \in L_{\mathcal{C}}(A) \mid x^{-1}(R) \in \mathcal{X}$ for each $(X;x) \in \mathrm{ob}\ \mathcal{J}/A\}$. Hence $\mathcal{A}^\Gamma =$ $= \{R \in \mathcal{A} \mid x^{-1}(R) \in \mathcal{X}$ for each $(X;x) \in \mathrm{ob}\ \mathcal{J}/A\}$. Now if $R \in \mathcal{B}^\Gamma$ then for each $(X;x) \in \mathrm{ob}\ \mathcal{J}/A$, $x^{-1}(f^{-1}(R))=(fx)^{-1}(R)$ is an entourage of X,and hence $f^{-1}(R) \in \mathcal{A}^\Gamma$. Consequently $f:A \to B$ is uniform continuous in $\mathcal{A}^\Gamma$ and $\mathcal{B}^\Gamma$. For the reflection remark that if $(B;\mathcal{B}) \in \mathrm{ob}\ \Gamma^*$ then $f:A \to B$ is uniform continuous in $\mathcal{A}$ and $\mathcal{B}$ iff it is uniform continuous in $\mathcal{A}^\Gamma$ and $\mathcal{B}$.

Since $\Gamma^*$is a full reflective subcategory of c-u-$\mathcal{C}$ Γ^* is a cocomplete category.

(ii) Let $F:J \to \Gamma^*$ be a small functor which we denote by $F(j)=(A_j;\mathcal{A}_j)$, $F(\tau)=f_\tau$, $j \in \mathrm{ob}J$, $\tau \in \mathrm{Morph}\ J$. If $(g_j : A \to A_j)_j =$ $= \varprojlim_J (A_j;f_\tau)$ in $\mathcal{C}$ then $\varprojlim F$ in c-u-$\mathcal{C}$ is $(g_j:(A;\mathcal{A}) \to F(j))_j$ where $\mathcal{A}$ is the initial c-uniformity induced by $\{g_j : A \to F(j)\}_j$. Now for each $(X,x) \in \mathrm{ob}\ \mathcal{J}/A$ the uniform continuity of x is equivalent to the uniform continuity of all $g_j x$, which is just the case. Hence $(A;\mathcal{A}) \in \mathrm{ob}\ \Gamma^*$ and the proof follows.

(iii) We must prove that if $(A;\mathcal{A}) \in \mathrm{ob}\,\Gamma^*$ and if $(B;\mathcal{B}) \underset{v}{\overset{u}{\rightrightarrows}} (A;\mathcal{A})$ are in $\underline{\text{c-u-}\mathcal{C}}$ then the coequalizer p of $\{u,v\}$ is in Γ^*. Since p is the coequalizer of its kernel pair and since Γ^* is closed in $\underline{\text{c-u-}\mathcal{C}}$ under strong subobjects it suffices to prove that if $(A;\mathcal{A}) \in \mathrm{ob}\,\Gamma^*$ and if $R \in L_{\mathcal{C}}(A)$ then $(A;\mathcal{A})/_R \in \mathrm{ob}\,\Gamma^*$ i.e. $(\mathcal{A}/_R)^{\Gamma} = \mathcal{A}/_R$. Then let $p: A \twoheadrightarrow A/R$ be the canonical regular epi. $(A;\mathcal{A})/_R = (A/_R;\mathcal{A}/_R)$ and $\mathcal{A}/_R = \{p(S) \mid S \in \mathcal{A} \text{ and } S \supseteq R\}$. Since $\mathcal{J}$ is a set of special generators in $\mathcal{C}$, for each $(X;x) \in \mathrm{ob}\,\mathcal{J}/(A/_R)$ there exists an $(X,y) \in \mathrm{ob}\,\mathcal{J}/A$ so that $x=py$. Hence $x^{-1}(p(S)) = y^{-1}p^{-1}p(S) = y^{-1}(S) \in \mathcal{X}$ for any $S \in \mathcal{A}$ with $S \supseteq R$. Hence $(A;\mathcal{A})/_R \in \mathrm{ob}\,\Gamma^*$.

<u>1.6.2</u> Let $\underline{\mathrm{Dis}\,\Gamma^*}$ be the full subcategory of Γ^* consisting of all discrete c-u objects in $\mathcal{C}$ which are also in Γ^*. Hence $(A;L) \in \Gamma^*$ iff for each $(X;x) \in \mathrm{ob}\,\mathcal{J}/A$ the congruence of x is an entourage of X. Also $\underline{\mathrm{Dis}\,\Gamma^*}$ is a full subcategory of $\mathcal{C}$ with the inclusion $(A;L) \rightarrowtail A$.

<u>1.6.3</u> $\underline{\mathrm{Dis}\,\Gamma^*}$ is closed under subobjects both in Γ^* and $\mathcal{C}$.

<u>Proof</u> Let (D,L) be in $\underline{\mathrm{Dis}\,\Gamma^*}$ and let $(B;\mathcal{B})$ be a subobject of $(D;L)$. Then $(B;\mathcal{B})$ is discrete too since $\Delta_B = f^{-1}(\Delta_D)$ where f denotes the inclusion of B in D. Actually for each $(X;x) \in \mathrm{ob}\,\mathcal{J}/B$, $x^{-1}(\Delta_B) = (fx)^{-1}(\Delta_D)$ is an entourage of X. Hence $(B;\mathcal{B}) \in \mathrm{Dis}\,\Gamma^*$.

<u>1.6.4</u> $\underline{\mathrm{Dis}\,\Gamma^*}$ is closed under finite products both in Γ^* and $\mathcal{C}$.

<u>Proof</u> A finite product of discrete c-u objects is discrete too and Γ^* is closed in $\underline{\text{c-u-}\mathcal{C}}$ under finite limits. Hence $\underline{\mathrm{Dis}\,\Gamma^*}$ is closed in Γ^* under finite products. Finally if $A_1,\dots,A_n$ are in $\underline{\mathrm{Dis}\,\Gamma^*}$ and if $(X,x) \in \mathrm{ob}\,\mathcal{J}/(\prod_1^n A_i)$ then $x = \langle x_1,\dots,x_n\rangle$ and $x^{-1}(\Delta_{\prod_1^n A_i}) = \bigcap_1^n x_i^{-1}(\Delta_{A_i}) \in \mathcal{X}$. Hence the product of $\{A_i\}_{i=1}^n$ is in $\underline{\mathrm{Dis}\,\Gamma^*}$.

Corollary 1.6.5 $\mathrm{Dis}\,\Gamma^*$ is closed under finite limits both in Γ^* and $\mathcal{C}$.

1.6.6 If $\mathcal{J}$ is a set of special generators in $\mathcal{C}$ then $\mathrm{Dis}\,\Gamma^*$ is closed under regular quotients both in Γ^* and $\mathcal{C}$.

Proof We must prove that if $D \in \mathrm{Dis}\,\Gamma^*$ and if $p: D \to D'$ is a regular epi in Γ^* or in $\mathcal{C}$ then D' is in $\mathrm{Dis}\,\Gamma^*$. Since any regular quotient in c-u-$\mathcal{C}$ of a discrete c-u object is discrete too and since Γ^* is closed in c-u-$\mathcal{C}$ under regular quotients (1.6.1 (iii)) the proof follows.

1.6.7 If $\mathcal{J}$ is a set of generators in $\mathcal{C}$ which are finitely generated (finitely presentable) objects ([5]) in $\mathcal{C}$ then $\mathrm{Dis}\,\Gamma^*$ is closed in $\mathcal{C}$ under monofiltered (filtered) colimits.

Proof Let $((A_i;L),f_{ij})_{i\in J}$ be a monofiltered system in $\mathrm{Dis}\,\Gamma^*$ whose colimit in $\mathcal{C}$ is $(A_i \xrightarrow{f_i} A)_i$. If $(X,x) \in \mathrm{ob}\,\mathcal{J}/A$ then there exists an i so that x factors through f_i to an $y: X \to A_i$, say. Then the congruence of x contains the congruence of y which is an entourage of X. Hence $(A;L) \in \mathrm{Dis}\,\Gamma^*$. Also in c-u-$\mathcal{C}$ a colimit of discrete c-u objects is discrete too.

1.6.8 Let $\mathcal{J}$ be a set of special generators in $\mathcal{C}$. Then $X/_R \in \mathrm{Dis}\,\Gamma^*$ provided $X \in \mathrm{ob}\,\mathcal{J}$ and $R \in \mathcal{X}$. Moreover if $X \in \mathrm{ob}\,\mathcal{J}$ then $\mathcal{X}$ is the initial c-uniformity on X induced by $\{f: X \to (A;L) \mid A \in \mathrm{Dis}\,\Gamma^*$
Also $\mathcal{X} = \{R \in L_{\mathcal{C}}(X) \mid X/_R \in \mathrm{Dis}\,\Gamma^*\}$.

Proof Since $(X;\mathcal{X}) \in \Gamma^*$ and since $(X;\mathcal{X})/R$ is discrete for any $R \in \mathcal{X}$ it follows by 1.6.1(iii) that $X/_R \in \mathrm{Dis}\,\Gamma^*$. Conversely if $R \in L(X)$ and if $X/_R \in \mathrm{Dis}\,\Gamma^*$ then $R \in \mathcal{X}$ since it is the congruence of $X \xrightarrow{\text{can}} X/_R$.

The initial c-uniformity $\mathcal{X}'$ on X induced by $\{f: X \to (A;L) \mid A \in \mathrm{Dis}\,\Gamma^*\}$ has as fundamental system of entourages the congruences $\bigcap_{i=1}^{n} f_i^{-1}(\Delta_{A_i})$ where $n \in \mathbb{N}$, $A_i \in \mathrm{Dis}\,\Gamma^*$ and $f_i: X \to A_i$, $i=1\ldots n$, Since $\bigcap_{i=1}^{n} f_i^{-1}(\Delta_{A_i})$ is the congruence of $\langle f_1,\dots,f_n\rangle$ and since $\underline{\mathrm{Dis}\,\Gamma^*}$ is closed in $\mathcal{C}$ under finite products the family $\mathfrak{G}=\{ f^{-1}(\Delta_A) \mid f:X\to A \text{ and } A\in \underline{\mathrm{Dis}\,\Gamma^*}\}$ is a base for $\mathcal{X}'$. Since $\mathrm{Dis}\,\Gamma^*$ is closed under regular quotients $\mathfrak{G}$ is a filter in $L(X)$. Hence $\mathfrak{G}=\mathcal{X}'$. To end the proof remark that $\mathfrak{G}=\mathcal{X}$.

<u>1.6.9</u> By what was proved above results a generalization of a lemma of Gabriel concerning the set of all left linear topologies on a ring, namely :

<u>Proposition</u> Let $\mathcal{C}$ be a (G-C) category. Let $\mathcal{J}$ be a set of special generators in $\mathcal{C}$ closed under finite coproducts. Then the correspondence $\Gamma \rightsquigarrow \underline{\mathrm{Dis}\,\Gamma^*}$ is a bijection from the set of all c-structures of $\mathcal{J}/\mathcal{C}$ to the class of all closed subcategories of $\mathcal{C}$ (by closed subcategory in $\mathcal{C}$ we mean: full subcategory closed in $\mathcal{C}$ under subobjects, regular quotients, finite limits and monofiltered colimits)

<u>Proof</u> By <u>1.6.8</u> the correspondence is an injective one. Now, let $\mathcal{D}$ be a closed subcategory of $\mathcal{C}$. For each $X\in \mathrm{ob}\,\mathcal{J}$ $\mathcal{X}=\{R\in L_{\mathcal{C}}(X) \mid X/_R\in\mathcal{D}\}$ is a filter in $L_{\mathcal{C}}(X)$. Indeed if R,S are in $\mathcal{X}$ then $X/_{R\cap S}$ is a subobject of $X/_R\prod X/_S$ which is in $\mathcal{D}$. If $R\in\mathcal{X}$, if $T\in L_{\mathcal{C}}(X)$ and if $R\subseteq T$ then X/T is isomorphic to $(X/_R)/(T/_R)$ which is in $\mathcal{D}$ ($T/R=p(T)$ where $p:X\to X/_R$ is the canonical regular epi). Hence for each $X\in\mathrm{ob}\,\mathcal{J}$, $\mathcal{X}$ is a c-uniformity on X. Since $\mathcal{X}=\{f^{-1}(\Delta_A) \mid A\in\mathcal{D} \text{ and } f:X\to A\}$, $\mathcal{X}$ is the initial c-uniformity on X induced by the family $\{f:X\to(A;L) \mid A\in\mathcal{D} \text{ and } f:X\to A\}$

Consequently each morphism $g:X\to X'$ of $\mathcal{J}$ is uniform continuous in $\mathcal{X}$ and $\mathcal{X}'$. Hence $\Gamma(X)=(X;\mathcal{X})$, $\Gamma(g)=g$, is a c-structure of $\mathcal{J}/\mathcal{C}$. We shall prove now that $\mathrm{Dis}\,\Gamma^*=\mathcal{D}$. Indeed if $D\in\mathcal{D}$ then for any $(X;x)\in\mathrm{ob}\,\mathcal{J}/D$, $x^{-1}(\Delta_D)\in\mathcal{X}$ since $X/_{x^{-1}(\Delta_D)}$ is in $\mathcal{D}$ as a subobject of D. Hence $\mathcal{D}\subseteq\mathrm{Dis}\,\Gamma^*$. Conversely let A be in

Dis Γ^*. Since $\mathcal{C}$ is a G-C regular category and since $\mathcal{J}$ is a set of special generators in $\mathcal{C}$ closed under finite coproducts it follows ([5] [1.8.1]) that A is the monofiltered colimit of its subobjects which are regular quotients of objects of $\mathcal{J}$. But all these regular quotients are quotients by entourages; it follows that all the subobjects of A which are regular quotients of objects of $\mathcal{J}$ are also in $\mathcal{D}$. Finally $A \in \mathcal{D}$ since $\mathcal{D}$ is closed under monofiltered colimits.

Definition 1.6.10 Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. If $(A;\mathcal{A})$ is a c-u object in $\mathcal{C}$ then we shall say that:

(1) $R \in L_{\mathcal{C}}(A)$ is a $\mathcal{D}$-special entourage of $(A;\mathcal{A})$ if $R \in \mathcal{A}$ and $A/_R \in \mathcal{D}$.

(2) $R \in L_{\mathcal{C}}(A)$ is a $\mathcal{D}$-special congruence on $(A;\mathcal{A})$ if R is an intersection of $\mathcal{D}$-special entourages and $A/_R \in \mathcal{D}$.

(3) $(A;\mathcal{A})$ is $\mathcal{D}$-strict if

(a) $(A;\mathcal{A})$ is separated and complete
(b) the family of all $\mathcal{D}$-special entourages of $(A;\mathcal{A})$ is a base for $\mathcal{A}$.
(c) each $\mathcal{D}$-special congruence on $(A;\mathcal{A})$ is a $\mathcal{D}$-special entourage of $(A,\mathcal{A})$.

Remark that $(A;\mathcal{A}) \in \Gamma^*$ iff $\mathcal{A}$ has a base consisting of $\underline{\text{Dis}}\,\Gamma^*$-special entourages. Indeed if $R \in \mathcal{A}$ then $(A;\mathcal{A})/_R = (A/R;L)$ is in Γ^*. Hence $A/_R \in \underline{\text{Dis}}\,\Gamma^*$. Conversely if $\mathcal{A}$ has a base consisting of $\underline{\text{Dis}}\,\Gamma^*$-special entourages then for each $(X,x) \in \text{ob}\,\mathcal{J}/A$ and each $\underline{\text{Dis}}\,\Gamma^*$-special entourage R of $(A;\mathcal{A})$ the inverse image of R by x is $(p_R^A\, x)^{-1}(\Delta_{A/R})$ which is an entourage of X since $A/_R \in \text{Dis}\,\Gamma^*$.

Also in $\underline{\text{c-u-SET}} = \underline{\text{p-UNIF}}$ the $\underline{\text{SET}}_f$-strict p-u spaces are exactly the compact p-u spaces. ($\underline{\text{SET}}_f$ is the category of all finite sets).

1.6.11 Let Σ be a small category with finite coproducts and let Σ^* be the category of all set valued finite coproducts preserving cofunctors on Σ, i.e. the category of all Σ-algebras. By 1.6.9 the class of all closed subcategories of Σ^* is in bijection to the set of all c-structures of Σ/Σ^*. Let $\Gamma:\Sigma \longrightarrow \text{c-u-}\Sigma^*$, $\Gamma(X)=(X;\mathfrak{X})$ be a c-structure of Σ/Σ^*. $\mathrm{Dis}\,\Gamma^*$, which is the closed subcategory of Σ^* associated to Γ, is a regular Lawvere C_3 category ([6]) and the inclusion $\mathrm{Dis}\Gamma^* \hookrightarrow \Sigma^*$ preserves and reflects finite limits, coequalizers and filtered colimits. Also the set $\mathcal{M} = \{X/R \mid X \in \Sigma \text{ and } R \in \mathfrak{X}\}$ is a set of generators and separators in $\mathrm{Dis}\Gamma^*$. In addition $\mathcal{M}$ equals the set of all finitely generated (resp. finite type) objects of $\mathrm{Dis}\Gamma^*$. Under additional conditions on the c-structure Γ (e.g. as a topologically left linearly compact ring [47]) the above remarks are related to the following

Theorem Let $\mathfrak{D}$ be a full subcategory of Σ^* such that

(1) $\mathfrak{D}$ is closed in Σ^* under finite limits

(2) if $A \in \mathfrak{D}$ and if $R \in \mathcal{L}_{\mathfrak{D}}(A)$ then $A/_R \in \mathfrak{D}$

(3) for each $A \in \mathfrak{D}$, $\mathcal{S}_{\mathfrak{D}}(A)$ is closed in $\mathcal{S}_{\Sigma^*}(A)$ under cofiltered intersections (here and below $\mathcal{S}_{\mathfrak{D}}(A)$ and $\mathcal{L}_{\mathfrak{D}}(A)$ denote the sets of subobjects resp. congruences of A in Σ^* which are objects of $\mathfrak{D}$)

(4) if $A \in \mathfrak{D}$ and if $\mathfrak{S}$ is a filter base in $L_{\mathfrak{D}}(A)$ then the canonical algebra morphism $A \longrightarrow \varprojlim_{R \in \mathfrak{S}} A/R$ is a regular epi in Σ^*.

(5) if $F: I \to \mathfrak{D}$ is a small cofiltered functor and if $(\varprojlim F \text{ in } \Sigma^*) = (A \xrightarrow{f_i} F(i))_i$ then $f_i(A) = \bigcap_{(j,\tau) \in ob(I/i)}$ (image of $F(\tau)$) for each $i \in obI$.

(6) if $A \in \mathfrak{D}$ and if $B \in \mathcal{S}_{\mathfrak{D}}(A \sqcap A)$ so that $B \supseteq \Delta_A$ then $B \in L_{\mathfrak{D}}(A)$

(7) if $(A;\mathfrak{A}) \in \text{c-u-}\Sigma^*$ is separated and complete and

has a base consisting of $\mathcal{D}$-special entourages then the intersection of any cofiltered family of $\mathcal{D}$-special entourages of $(A;\mathcal{A})$ is a $\mathcal{D}$-special congruence on $(A,\mathcal{A})$

Then, under the conditions (1)-(7) on $\mathcal{D}$, the category $\mathcal{D}'$ of all $\mathcal{D}$-strict c-u objects in Σ^* has the following properties :

(a) $\mathcal{D}'$ has small limits, coequalizers and coequalizer decompositions

(b) $\mathcal{D}$ is a full subcategory of cogenerators in $\mathcal{D}'$. Moreover the objects of $\mathcal{D}$ are exactly the finitely cogenerated objects ([5]) of $\mathcal{D}'$

(c) regular epi-cofiltered limits of $\mathcal{D}'$ preserve regular epimorphisms

Moreover $\mathcal{D}'$ has finite colimits and any bimorphism of $\mathcal{D}'$ is an iso provided $\mathcal{D}$ has finite coproducts and any mono in $\mathcal{D}$ is an equalizer in $\mathcal{D}$. REmark that in this case $(\mathcal{D}')^0$ is a cocomplete and finite complete category in which

- all bimorphisms are isomorphisms
- monofiltered colimits are left exact.
- the class of all finitely generated objects is a set of generators and separators

2. Duality for presheaves categories

Let C be a small category. Let C^ be the category of set valued presheaves on C. The duality data are:

2.1 a pair $(\mathcal{G},\mathcal{E})$ so that

(1) $\mathcal{G}$ is a skeletally small full replete subcategory of C^

(2) $\mathcal{G}$ is a subcategory of dense generators in C^

(3) $\mathcal{G}$ is closed in C^ under subobjects, quotients and finite coproducts

(4) $\mathcal{E}$ is a small subcategory of cogenerators in C^ closed under finite products

(5) all objects of $\mathcal{E}$ are injective objects in C^ and for each G in $\mathcal{G}$ there exists an $E \in \mathcal{E}$ and a mono $g : G \rightarrowtail E$

2.2 Remarks A pair $(\mathcal{G},\mathcal{E})$ as above can be obtained as follows: we start with $\mathcal{G}_o = \{\hat{X} \mid X \in \mathrm{ob}C\}$ and in order to get $\mathcal{G}$ we inductively add subobjects, quotients and finite coproducts of objects of $\mathcal{G}_o$; for $\mathcal{E}$ we have more choises all based on the fact that C^ has enough injectives ([7],[11]). We can for instance extend each $G \in \mathcal{G}$ to an injective object G' and we can consider $\mathcal{E}$ as the collection of all finite products of these G'; another choise for $\mathcal{E}$ is to consider a skeleton Γ of $\mathcal{G}$ and to take E as an injective object which contains $\coprod\Gamma$, in this case $\mathcal{E}$ is the collection of all finite powers of E, hence a finitary algebraic theory.

Precisely we have

Lemma 2.2.1 Let Γ be a set of objects of C^. Let

(a) $\mathcal{G}_o$ be the collection of all finite coproducts of objects of Γ.

(b) $\mathcal{G}_1$ be the collection of all subobjects or quotients of objects of $\mathcal{G}_0$

(c) $\mathcal{G}_2$ be the collection of all subobjects and quotients of objects from $\mathcal{G}_1$.

Then $\mathcal{G}_2$ is closed under subobjects and quotients.

Proof. Let G_2 be an object of $\mathcal{G}_2$. If $f: A \hookrightarrow G_2$ is a subobject of G_2 then the following diagrams arise:

(i) $A \xhookrightarrow{f} G_2 \xhookrightarrow{g} G_1$ where G_1 is in $\mathcal{G}_1$. In this case $A \in \mathcal{G}_2$ as a subobject of an object from $\mathcal{G}_1$.

(ii)
$$\begin{tikzcd} A \arrow[r, hook, "f"] & G_2 \\ & G_1 \arrow[u, "g"] \end{tikzcd}$$
where G_1 is in $\mathcal{G}_1$.

For G_1 there are two alternatives either (ii.1) $G_0 \twoheadrightarrow G_1$ or (ii.2) $G_1 \hookrightarrow G_0$ both with G_0 in $\mathcal{G}_0$.

(ii.1)
$$\begin{tikzcd} A \arrow[r, hook, "f"] & G_2 \\ A' \arrow[u, "g'"] \arrow[r, hook, "f'"] & G_1 \arrow[u, "g"] \\ A'' \arrow[u, two heads] \arrow[r, hook, "f''"] & G_0 \arrow[u, two heads] \end{tikzcd}$$

(ii.2)
$$\begin{tikzcd} A \arrow[r, hook, "f"] & G_2 \arrow[r, hook, "h'"] & A_1 \\ & G_1 \arrow[u, "g"] \arrow[r, hook, "h"] & G_0 \arrow[u, two heads, "g'"] \end{tikzcd}$$

If (ii.1) is the case then A is a quotient of A'' the inverse image of A under $G_0 \twoheadrightarrow G_2$. A'' is in $\mathcal{G}_1$ as a subobject of G_0 so that A is in $\mathcal{G}_2$.

If (ii.2) arises let $g'h=h'g$ be a pushout. By the limit and colimit properties of C, h' is a monomorphism. Actually A_1 is in $\mathcal{G}_1$ as a quotient of G_0, and A is in $\mathcal{G}_2$ as a subobject in A.

So $\mathcal{G}_2$ is closed under subobjects.

If $G_2 \overset{f}{\twoheadrightarrow} A$ is a quotient of G_2 then the following diagrams arise:

(iii) $G_1 \twoheadrightarrow G_2 \overset{f}{\twoheadrightarrow} A$ where G_1 is in $\mathcal{G}_1$. In this case A is in $\mathcal{G}_2$ as a quotient of an object of $\mathcal{G}_1$.

$$\text{(iv)}\qquad \begin{tikzcd} G_2 \arrow[r, two heads, "f"] \arrow[d, hook, "g"] & A \\ G_1 \end{tikzcd} \qquad \text{where } G_1 \in \mathcal{G}_1 .$$

Again for G_1 there are two alternatives either (iv.1) $G_1 \xhookrightarrow{h} G_0$ or (iv.2) $G_0 \xrightarrow{h} G_1$ both with $G_0 \in \mathcal{G}_0$.

$$\text{(iv.1)} \quad \begin{tikzcd} G_2 \arrow[r, two heads, "f"] \arrow[d, hook, "g"] & A \arrow[dd, hook, "h'"] \\ G_1 \arrow[d, hook, "h"] & \\ G_0 \arrow[r, two heads, "f'"] & A' \end{tikzcd} \qquad\qquad \text{(iv.2)} \quad \begin{tikzcd} A' \arrow[r, two heads, "h'"] \arrow[d, hook, "g'"] & G_2 \arrow[r, two heads, "f"] \arrow[d, hook, "g"] & A \\ G_0 \arrow[r, two heads, "h"] & G_1 & \end{tikzcd}$$

In the situation (iv.1) let $f'(hg)=h'f$ be a pushout. Then A' is in $\mathcal{G}_1$ as a quotient of G_0 and $A \in \mathcal{G}_2$ as a subobject in A'. If (iv.2) is the case then let $gh'=hg'$ be a pullback. Actually $A' \in \mathcal{G}_1$ as a subobject in G_0 and $A \in \mathcal{G}_2$ as a quotient of A'. So $\mathcal{G}_2$ is closed under quotients .

<u>Corollary 2.2.2</u> Let Γ be a set of objects of $C^\wedge$. Let $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$ be as in <u>lemma 2.2.1</u>. For each $k \in \mathbb{N}$ let us define inductively the sets $\mathcal{G}_k$ so that $\mathcal{G}_k \subseteq \mathcal{G}_{k+1}$,and if $k=3p-1$ then $\mathcal{G}_{3p}$ is the closure in $C^\wedge$ of $\mathcal{G}_{3p-1}$ under finite coproducts, $\mathcal{G}_{3p+1}$ is the collection of all subobjects and quotients of objects of $\mathcal{G}_{3p}$ and finally $\mathcal{G}_{3p+2}$ consists of all subobjects and quotients of objects from $\mathcal{G}_{3p+1}$. Let us take $\mathcal{G}' = \bigcup_{k \in \mathbb{N}} \mathcal{G}_k$ and let $\mathcal{G}$ be the collection of all presheaves on C which are isomorphic to objects from $\mathcal{G}'$.

Then $\mathcal{G}$ is a skeletally small full replete subcategory of $C^\wedge$ closed under finite coproducts , subobjects and quotients. In addition, if Γ contains a set of generators of $C^\wedge$ then $\mathcal{G}$ is a set of dense generators in $C^\wedge$.

<u>Proof</u>. Obviously $\mathcal{G}$ is skeletally small. If A is a subobject

or a quotient of a $G \in \mathcal{G}$ then we can always consider G in a $\mathcal{G}_{3p+2}$ which is closed under subobjects and quotients by lemma 2.2.1, so that $A \in \mathcal{G}_{3p+2} \subseteq \mathcal{G}$. If G, G' are objects of $\mathcal{G}$ then by the inclusions $\mathcal{G}_k \subseteq \mathcal{G}_{k+1}$, $k \in \mathbb{N}$, we can always consider G, G' in a $\mathcal{G}_{3p+2}$ so that $G \amalg G' \in \mathcal{G}_{3(p+1)} \subset \mathcal{G}$.

Suppose now that Γ is a set of generators in $C^\wedge$. By the regularity properties of $C^\wedge$, $\mathcal{G}_1$ is a set of regular generators in $C^\wedge$, hence $\mathcal{G}$ is a subcategory of regular generators in $C^\wedge$. By definition this means that for each presheaf F the morphism $\coprod_{(G,f)\in \mathrm{ob}\,\mathcal{G}/F} G \xrightarrow{[f]_{(G,f)}} F$ is a regular epi. If for each $(G,F) \in \mathrm{ob}\,\mathcal{G}/F$, $G \underset{p_f}{\twoheadrightarrow} A_f \underset{q_f}{\hookrightarrow} F$ is a coequalizer decomposition of f then $\{ A_f \mid (G,f) \in \mathrm{ob}\,\mathcal{G}/F \}$ is a monofiltered system of subobjects of F (the filtration results from the closedness of $\mathcal{G}$ at finite coproducts). A colimit of this monofiltered system is $(A_f \xhookrightarrow{q_f} F)_{(G,f)\in \mathrm{ob}\,\mathcal{G}/F}$. Since $\mathcal{G}$ is closed under quotients, all A_f are in $\mathcal{G}$. The family $\{(A_f, q_f) \mid (G,f) \in \mathrm{ob}\,\mathcal{G}/F\}$ gives a cofinal subcategory of $\mathcal{G}/F$. It follows that $\varinjlim(\mathcal{G}/F \xrightarrow{U} C^\wedge) = (G = U(G,f) \xrightarrow{f} F)_{(G,f)\in \mathrm{ob}\,\mathcal{G}/F}$. On the other hand this equality is just the definition of the density of $\mathcal{G}$ as a subcategory in $C^\wedge$.

Once we have $\mathcal{G}$ we can take $\mathcal{E}_1$ to be the collection performed by extending each G of $\mathcal{G}$ to an injective object E_G. Finally $\mathcal{E}$ will be the full subcategory of $C^\wedge$ consisting of all finite products of objects from $\mathcal{E}_1$. The fact that $\mathcal{E}_1$ respectively $\mathcal{E}$ is a set of cogenerators in $C^\wedge$ is proved in SGA4, exp.I. We give it as

Lemma 2.2.3 Let $\mathcal{C}$ be a category with coequalizer decompositions and finite coproducts. Let $\mathcal{G}$ be a set of generators in $\mathcal{C}$ and $\mathcal{E}$ be a set of injective objects in $\mathcal{C}$ so that any quotient of a finite coproduct of objects of $\mathcal{G}$ can be embedded in a suitable object of $\mathcal{E}$. Then $\mathcal{E}$ is a set of cogenerators in $\mathcal{C}$.

Proof. Let $A \overset{u}{\underset{v}{\rightrightarrows}} B$ be two distinct arrows of $\mathcal{C}$. There exists $G \in \mathcal{G}$ and $f: G \to A$ so that $uf \neq vf$. Let $G \amalg G \xrightarrow{p} Q \xrightarrow{q} B$ be a coequalizer decomposition of $[uf, vf]$. We can find $E \in \mathcal{E}$ and a mono $g: Q \hookrightarrow E$. Now g can be extended to B at a $g': B \to E$ with $g'q = g$, by the injectivity of E. It follows $g'u \neq g'v$, since otherwise we get $g'u = g'v \Longrightarrow gp_1 = g'uf = g'vf = gp_2$ (here $[p_1, p_2] = p$) which in turn implies $p_1 = p_2$ and $uf = qp_1 = qp_2 = vf$ which is nonsense.

2.3 Dual objects as algebras Let ($\mathcal{G}$, $\mathcal{E}$) be a pair as in 2.1 . Let $\mathcal{E}^*$ be the category of all finite products preserving functors $\mathcal{E} \longrightarrow \underline{SET}$. The assignments

$$\mathrm{ob}C^\wedge \ni F \longrightarrow \tilde{F} = \mathrm{Hom}_{C^\wedge}(F, -)|_{\mathcal{E}}$$

$$\mathrm{Morf}C^\wedge \ni \varphi \longrightarrow \tilde{\varphi} = \mathrm{Hom}(\varphi, -)|_{\mathcal{E}}$$

define a cofunctor $\tilde{?}: C^\wedge \longrightarrow \mathcal{E}^*$

Since $\mathcal{E}$ is a cogenerating subcategory in $C^\wedge$ we get that $\tilde{?}$ is a faithful cofunctor. For each E in $\mathcal{E}$ let $U_E: \mathcal{E}^* \to \underline{SET}$ be the functor $U_E(A) = A(E)$. Since $U_E \tilde{?} = \mathrm{Hom}_{C^\wedge}(-, E)$ for any $E \in \mathcal{E}$ it follows that $\tilde{?}$ sends colimits of $C^\wedge$ to limits of $\mathcal{E}^*$ and also $\tilde{?}$ maps epimorphisms of $C^\wedge$ to monomorphisms of $\mathcal{E}^*$. By the injectivity of E we get that $U_E \tilde{?}$ maps monomorphisms of $C^\wedge$ to surjections in $\underline{SET}$, so that $\tilde{?}$ maps monomorphisms to regular epimorphisms (i.e. componentwise surjections) in $\mathcal{E}^*$. As a consequence $\tilde{?}$ preserves coequalizer decompositions and sends short exact sequences $\cdot \hookrightarrow \cdot \rightrightarrows \cdot$ of $C^\wedge$ to short exact sequences $\cdot \rightrightarrows \cdot \twoheadrightarrow \cdot$ in $\mathcal{E}^*$.

By the fact that $\tilde{?}$ is faithfuly it follows that $\tilde{\varphi}$ is a regular epi in $\mathcal{E}^*$ iff φ is a mono in $C^\wedge$.

Finally remark that for any $E \in \mathcal{E}$ we have that $\tilde{E} = \mathrm{Hom}_{\mathcal{E}}(E, -)$. By the Yoneda lemma we get for any $A \in \mathrm{ob}\mathcal{E}^*$ the bijection

$\mathrm{Hom}_{\mathcal{E}^*}(\tilde{E},A) \longrightarrow A(E)$, $\mathrm{Hom}_{\mathcal{E}^*}(\tilde{E},A) \ni \Phi \rightsquigarrow \Phi_E(1_E) \in A(E)$

whose inverse is $A(E) \ni x \rightsquigarrow \tilde{x}:\tilde{E} \to A$, $\tilde{x}(\alpha)=A(\alpha)(x)$.

To keep the analogy with the algebraic case we shall use the term operation for a morphism of $\mathcal{E}$ and the term $\mathcal{E}$-algebra for an object of $\mathcal{E}^*$. An $\mathcal{E}$-algebra A consists in a family $\{A(E)\}_{E\in \mathrm{ob}\,\mathcal{E}}$ of sets and a family of operations on A, $\{\alpha_A\}_{\alpha\in \mathrm{Morf}\,\mathcal{E}}$ so that

(i) $A(\prod_1^k E_j)$ is canonically isomorphic in SET with $\prod_1^k A(E_j)$

(ii) if $\alpha:E \to E'$ is a morphism of $\mathcal{E}$ then α_A is a function from $A(E)$ to $A(E')$.

(iii) if $\pi_i:\prod_1^k(E_j) \longrightarrow E_i$ is the projection of a product in $\mathcal{E}$ onto the i^{th} factor then $(\pi_i)_A:A(\prod_1^k E_j) \to A(E_i)$ is also the projection onto the i^{th}-factor.

(iv) for any operations $\alpha_i:E_i \longrightarrow E'$, $i=1\ldots k$, and any operation $\alpha:E' \longrightarrow E''$ we have that $(\alpha \langle\alpha_i\rangle_{i=1}^k)_A = \alpha_A \langle(\alpha_i)_A\rangle_{i=1}^k$

For an operation $\alpha:E \to E'$ and for $x\in A(E)$ we shall write $\alpha_A(x)$ as $\alpha_A\cdot x$ or simply $\alpha\cdot x$.

If A, B are $\mathcal{E}$-algebras then an $\mathcal{E}^*$-morphism or an $\mathcal{E}$-algebras morphism Φ consists in a family $\{\Phi_E:A(E) \to B(E)\}_{E\in \mathrm{ob}\,\mathcal{E}}$ so that for any operation $\alpha:E \longrightarrow E'$ one has $\Phi_{E'}\alpha_A = \alpha_B\,\Phi_E$. Without danger of confusion we can omit the indices E,E',A,B and we shall write the above equality as $\Phi(\alpha\cdot x)=\alpha\cdot\Phi(x)$ for all $x\in A(E)$. Note that if $\varphi\in C^\infty$ then $\tilde{\varphi}(\xi)=\xi\varphi$.

From the above point of view for $E\in \mathrm{ob}\,\mathcal{E}$, $\tilde{E}$ is a "free" $\mathcal{E}$-algebra in the following sense:

for any $\mathcal{E}$-algebra A and any $x\in A(E)$ there exists a unique $\mathcal{E}^*$-morphism $\tilde{x}:\tilde{E} \longrightarrow A$ sending 1_E to x (precisely $\tilde{x}(\alpha)=\alpha\cdot x$) and all morphisms $\tilde{E} \longrightarrow A$ are obtained in this way. For $E_i\in \mathrm{ob}\,\mathcal{E}$ $i=1\ldots k$, and $x_i\in A(E_i)$ where $A\in \mathrm{ob}\,\mathcal{E}^*$ the unique $\mathcal{E}^*$-morphism $\widetilde{\prod_1^k E_i} \longrightarrow A$ sending each canonical projection π_i to x_i is in fact $\widetilde{(x_1,\dots,x_k)}$ where $(x_1,\dots,x_k)\in A(\prod_1^k E_i)$. It follows that the map

$$\prod_1^k \mathrm{Hom}_{\mathcal{E}^*}(\tilde{E}_i, A) \longrightarrow \mathrm{Hom}_{\mathcal{E}^*}(\widetilde{\textstyle\prod_1^k E_i}, A)$$

$$(\tilde{x}_1,\dots,\tilde{x}_k) \longmapsto \widetilde{(x_1,\dots,x_k)}$$

is a bijection. Since $\widetilde{(x_1,\dots,x_k)}\,\tilde{\pi}_j=\tilde{x}_j$ for each canonical projection $\pi_j:\prod_1^k E_i \to E_j$, we get that

$$\widetilde{\prod_1^k E_i}=\coprod_1^k \tilde{E}_i$$ in $\mathcal{E}^*$ with $\tilde{\pi}_j$, $j=1\dots k$, as canonical morphisms.

Finally note that :

<u>2.3.1</u> $\phi\tilde{x}=\widetilde{\phi(x)}$ for any $\mathcal{E}^*$-morphism $\phi:A\to B$ and any x in A

<u>2.3.2</u> $\tilde{x}\tilde{\alpha}=\widetilde{\alpha\cdot x}$ for any operation $\alpha:E\to E'$ and any $x\in A(E')$, $A\in ob\,\mathcal{E}^*$

<u>2.3.3</u> for any $F\in C^\wedge$, any operation $\alpha:E\to E'$ and any $\xi\in\tilde{F}(E)$ we have that $\alpha_{\tilde{F}}\cdot\xi=\alpha\xi$.

<u>2.3.4</u> $\tilde{\xi}(\alpha)=\alpha\xi$ for any $F\in C^\wedge$ and any $\xi\in\tilde{F}(E)$ with $E\in\mathcal{C}$ i.e. the unique $\mathcal{E}^*$-morphism $\tilde{E}\to\tilde{F}$ sending 1_E to ξ is $\tilde{\xi}$ which is just the value of $\tilde{?}$ on $\xi\in Morf\,C^\wedge$.

<u>2.3.5</u> $\tilde{F}(\prod_1^k E_i)$ is the product $\prod_1^k\tilde{F}(E_i)$ via the bijection

$$\begin{tikzcd}
\prod_1^k \tilde{F}(E_i) \arrow[rr, "(\xi_1,\dots,\xi_k)\rightsquigarrow\langle\xi_1,\dots,\xi_k\rangle"] \arrow[d, "\mathrm{proj.}"'] & & \tilde{F}(\prod_1^k E_i) \arrow[d, "\tilde{\pi}_j"] \\
\tilde{F}(E_j) \arrow[rr, equal] & & \tilde{F}(E_j)
\end{tikzcd}$$

<u>Proposition 2.3.6</u> If F is a presheaf so that there exists a mono $f:F\to E$ with $E\in ob\,\mathcal{E}$ then the map

$\tilde{?}:\mathrm{Hom}_{C^\wedge}(H,F) \longrightarrow \mathrm{Hom}_{\mathcal{C}^*}(\tilde{F},\tilde{H})$ is a bijection for any presheaf H.

<u>Proof.</u> Let $\Phi:\tilde{F}\rightarrow\tilde{H}$ be an $\mathcal{C}^*$-morphism. Let $E \underset{v}{\overset{u}{\rightrightarrows}} G$ be a cokernel pair of f. Since in $C^\wedge$ any mono is a regular one it follows that f is an equalizer of the pair $\{u,v\}$; in other words $F \xrightarrow{f} E \underset{v}{\overset{u}{\rightrightarrows}} G$ is a short exact sequence in $C^\wedge$.

Now f thought as an element of $\tilde{F}(E)$ produces an element $\Phi(f):H\rightarrow E$ of $\tilde{H}(E)$. For each $E'\in \mathrm{ob}\,\mathcal{C}$ and each $\xi\in\tilde{G}(E')$, ξu and ξv are operations, so that

$$\xi u\cdot\Phi(f)=(\xi u)_{\tilde{H}}\cdot\Phi(f)=\Phi((\xi u)_{\tilde{F}}\cdot f)=\Phi(\xi uf)=\Phi(\xi vf)=\ldots=\xi v\Phi(f)$$

by the fact that Φ is an $\mathcal{C}$-algebras morphism. Since $\mathcal{C}$ is a cogenerating set in $C^\wedge$ it follows that $u\Phi(f)=v\Phi(f)$ so that there exists $\varphi:H\rightarrow F$ with $f\varphi=\Phi(f)$.

The last equality can be written as $\tilde{\varphi}(f)=\Phi(f)$ .Now, by the injectivity of any $E'\in\mathrm{ob}\,\mathcal{C}$, for such an E' and any $\xi\in\tilde{F}(E')$ one gets an operation $\xi':E\rightarrow E'$ so that $\xi'\cdot f=\xi' f=\xi$ .Then

$$\tilde{\varphi}(\xi)=\tilde{\varphi}(\xi'\cdot f)=\xi'\cdot\tilde{\varphi}(f)=\xi'\cdot\Phi(f)=\Phi(\xi'\cdot f)=\Phi(\xi) \text{ and } \tilde{\varphi}=\Phi.$$

<u>2.4 Finitely generated dual objects</u> An $\mathcal{C}$-algebra A is a finitely generated object in $\mathcal{C}^*$ if $\mathrm{Hom}_{\mathcal{C}^*}(A,-):\mathcal{C}^*\longrightarrow$ <u>SET</u> preserves monofiltered colimits ([5]). By definition A is of finite type in $\mathcal{C}^*$ if for each monofiltered system $(A_i,\varphi_{ij})_{i\leq j}$ of $\mathcal{C}$-algebras whose colimit is A, at least one canonical morphism $A_i\longrightarrow A$ is an iso. It is easy to prove the equivalence of the following three assertions ([5]) :

<u>2.4.1</u> the $\mathcal{C}$-algebra A is finitely generated

<u>2.4.2</u> the $\mathcal{C}$-algebra A is of finite type

<u>2.4.3</u> the $\mathcal{C}$-algebra A is a regular quotient of an $\tilde{E}$ for a suitable $E\in\mathrm{ob}\,\mathcal{C}$.

Remember that if $\mathcal{C}$ is an algebraic theory then <u>2.4.3</u> says that A is a regular quotient of a free algebra over a finite set.

For the proof remark that $\mathrm{Hom}_{\mathscr{C}^*}(A,-)$ preserves monofiltered colimits iff for any monofiltered system $(A_i, \varphi_{ij})_{i \leq j}$ with the colimit $(A_i \xrightarrow{\Phi_i} B)_i$ and any $\mathscr{C}^*$-morphism $\Psi: A \to B$ there exists an i so that Ψ factors through $\Phi_i([a])$. Indeed, by the assumption that

$$\left(\mathrm{Hom}_{\mathscr{C}^*}(A,A_i) \xrightarrow[x \mapsto \Phi_i x]{\mathrm{Hom}(A,\Phi_i)} \mathrm{Hom}_{\mathscr{C}^*}(A,B)\right)_i$$

is the colimit of the monofiltered system

$$\begin{aligned}\Big(\mathrm{Hom}_{\mathscr{C}^*}(A,A_i),\ \mathrm{Hom}(A,\varphi_{ij}):\mathrm{Hom}_{\mathscr{C}^*}(A,A_i) &\longrightarrow \mathrm{Hom}_{\mathscr{C}^*}(A,A_j)\Big)_{i \leq j}\\ x &\longmapsto \varphi_{ij}x\end{aligned}$$

we get that for any $\Psi \in \mathrm{Hom}_{\mathscr{C}^*}(A,B)$ there exists a suitable i and a suitable $x \in \mathrm{Hom}_{\mathscr{C}^*}(A,A_i)$ so that $\Psi = \Phi_i x$. To prove the converse remark that in this case the map

$$\coprod_i \mathrm{Hom}_{\mathscr{C}^*}(A,A_i) \xrightarrow[f=[\mathrm{Hom}(A,\Phi_i)]_i]{} \mathrm{Hom}_{\mathscr{C}^*}(A,B)$$

is a sujective one. If $f(x)=f(y)$ with $x:A \to A_i$ and $y:A \to A_j$ then $\Phi_i x = \Phi_j y$. If we chose a k greater than i and j then

$$\Phi_k \varphi_{ik} x = \Phi_i x = \Phi_j y = \Phi_k \varphi_{jk} \quad \text{so that} \quad \varphi_{ik}x = \varphi_{jk}y, \text{ and}$$

the equivalence relation of the surjection f is just the equivalence relation on $\coprod_i \mathrm{Hom}_{\mathscr{C}^*}(A,A_i)$ given by the monofiltered system

$$\left(\mathrm{Hom}_{\mathscr{C}^*}(A,A_i),\mathrm{Hom}(A,\varphi_{ij})\right)_{i \leq j}$$

<u>2.4.1</u> $\Rightarrow$ <u>2.4.2</u> If $(A_i, \varphi_{ij})_{i \leq j}$ is a monofiltered system whose colimit is $(A_i \xrightarrow{\varphi_i} A)_i$ then for id_A there exists an i so that id_A factors through φ_i. Actually this φ_i is both a mono and a regular epi in $\mathscr{C}^*$, hence φ_i is an iso.

<u>2.4.2</u> $\Rightarrow$ <u>2.4.3</u> Since $\{\tilde{E} \mid E \in \mathrm{ob}\,\mathscr{C}\}$ is a set of regular generators in $\mathscr{C}^*$ closed at finite coproducts it follows that A is the monofiltered colimit of its subobjects which are regular quotients of $\tilde{E}$, $E \in \mathrm{ob}\,\mathscr{C}$. By the assumption <u>2.4.2</u> we get an $E \in \mathrm{ob}\,\mathscr{C}$ and a regular epi $\tilde{E} \to A$.

<u>2.4.3</u> $\Longrightarrow$ <u>2.4.2</u> If $\Phi : \tilde{E} \to A$ is a regular epi in $\mathcal{E}^*$ then Φ is a componentwise surjection. In this case if $\Phi(1_E)=a$ then $\Phi=\tilde{a}$ and for each $E' \in \mathrm{ob}\,\mathcal{E}$ and each $x \in A(E')$ there exists an operation $\alpha : E \to E'$ so that $\alpha_A \cdot a = x$. Now for a monofiltered system $(A_i, \Phi_{ij})_{i \leq j}$ whose colimit is $(A_i \xrightarrow{\Phi_i} A)_i$ since $(A_i(E), \Phi_{ij})_{i\leq j}$ is a monofiltered system of sets whose colimit is $(A_i(E) \xrightarrow[\Phi_i]{} A(E))_i$ there exists an i so that a is in the set $\Phi_i(A_i(E))$. In this case Φ_i is a componentwise surjection and consequently an iso.

<u>2.4.2</u> $\Longrightarrow$ <u>2.4.1</u> Let $(A_i, \Phi_{ij})_{i\leq j}$ be a monofiltered system of $\mathcal{E}$-algebras and let $(\Phi_i : A_i \to B)_i$ be its colimit. For an $\mathcal{E}^*$-morphism $\Psi : A \to B$ let us consider the pullbacks $\Psi \Phi'_i = \Phi_i \Psi'_i$, for each i,

$$\begin{tikzcd}
A \arrow[rrr, "\Psi"] & & & B \\
& A'_i \arrow[ul, "\Phi'_i"'] \arrow[dl, "\Phi'_{ij}"'] \arrow[r, "\Psi'_i"] & A_i \arrow[ur, "\Phi_i"] \arrow[dr, "\Phi_{ij}"] & \\
A'_j \arrow[uu, "\Phi'_j"] \arrow[rrr, "\Psi'_j"] & & & A_j \arrow[uu, "\Phi_j"']
\end{tikzcd}$$

which produce the monofiltered system $(A'_i, \Phi'_{ij})_{i\leq j}$. Its colimit is just $(A'_i \xrightarrow{\Phi'_i} A)_i$ by the fact that in $\mathcal{E}^*$ filtered colimits commutes with pullbacks. Then by <u>2.4.2</u> there exists an i so that Φ'_i is an iso, hence Ψ factors through Φ_i for at least one i.

<u>Proposition 2.4.4</u> If $F \in \mathrm{ob}\,C^\wedge$ then $\tilde{F}$ is a finitely generated $\mathcal{E}$-algebra iff F can be embedded in a suitable object of $\mathcal{E}$.

<u>Proof</u>. $\tilde{F}$ is a finitely generated $\mathcal{E}$-algebra iff there exists an $E \in \mathrm{ob}\,\mathcal{E}$ and a regular epi $\Phi : \tilde{E} \to \tilde{F}$. By <u>2.3.6</u> $\Phi = \tilde{\varphi}$ for a $\varphi : F \to E$. Finally $\tilde{\varphi}$ is a regular epi iff φ is a mono (by <u>2.3</u>

<u>Proposition 2.4.5</u> Let A be a finitely generated subalgebra of an $\tilde{F}$ with $F \in C^\wedge$. Then there exists an epi $\varphi: F \twoheadrightarrow F'$ in $C^\wedge$ and a commutative diagram

$$\begin{tikzcd} \tilde{F}' \arrow[r, hook, "\tilde{\varphi}"] & \tilde{F} \\ & A \arrow[ul, "\sim"] \arrow[u, hook] \end{tikzcd}$$

in $\mathcal{E}^*$.

<u>Proof.</u> Since A is a finitely generated $\mathcal{E}$-algebra there exists a regular epi $\Phi: \tilde{E} \twoheadrightarrow A$ with $E \in \mathrm{ob}\,\mathcal{E}$. By <u>2.3.6</u> the composition $\tilde{E} \xrightarrow{\Phi} A \hookrightarrow \tilde{F}$ has the form $\tilde{f}$ for a suitable $f: F \longrightarrow E$. If

$$\begin{tikzcd} F \arrow[rr, "f"] \arrow[dr, two heads, "\varphi"'] & & E \\ & F' \arrow[ur, hook, "\psi"'] & \end{tikzcd}$$

is a coequalizer decomposition of f in $C^\wedge$ then in $\mathcal{E}^*$, $\tilde{E} \xrightarrow{\tilde{\psi}} \tilde{F}' \xrightarrow{\tilde{\varphi}} \tilde{F}$ and $\tilde{E} \xrightarrow{\Phi} A \hookrightarrow \tilde{F}$ are coequalizer decompositions of f and the assertion follows.

<u>2.5 uniform structure on dual objects</u> Let us consider for any $F \in \mathrm{ob}C^\wedge$ the set $\mathcal{G}(F)$ consisting of those subobjects of F which are in $\mathcal{G}$. $\mathcal{G}(F)$ can be viewed as a partially ordered small category whose morphisms are the inclusions between subobjects of F. Since $\mathcal{G}$ is closed in $C^\wedge$ under finite coproducts, subobjects and quotients, $\mathcal{G}(F)$ is closed in the subobjects lattice of F at finite unions and arbitrary intersections. As a consequence $\mathcal{G}(F)$ is a filtered category or equally can be viewed as a monofiltered system of subobjects of F. By <u>2.1</u> the colimit of the monofiltered system $\mathcal{G}(F)$ is F with inclusions

$G \xrightarrow{g} F, (G,g) \in \mathcal{G}(F)$, as canonical morphisms.

If $N \xrightarrow{n} F$ is in $\mathcal{G}(F)$ let $D_F(N)$ be the congruence of $\tilde{n}: \tilde{F} \twoheadrightarrow \tilde{N}$. $(D_F(N))(E) = \{ (\xi, \zeta) \in \tilde{F}(E) \times \tilde{F}(E) \mid \xi_{|N} = \zeta_{|N} \}$ for each E in $\mathcal{E}$ (obviously $\xi_{|N}$ means ξn).

<u>Proposition 2.5.1</u> The family $D_F = \{ D_F(N) \mid N \in \mathcal{G}(F) \}$ is a base for a filter in $L(\tilde{F})$ Particularly :

(i) $D_F(N)\cap D_F(N')=D_F(N\cup N')$

(ii) $D_F(N)\subseteq D_F(N')$ iff $N'\subseteq N$

(iii) $\widetilde{\varphi}^{-1}(D_F(N))=D_H(\varphi(N))$ for any presheaf morphism $\varphi:F\to H$.

Proof. By the fact that $\mathcal{G}$ is a subcategory of dense generators in $C^{\wedge}$ we get that $\mathcal{G}(F)$ is a nonvoid set for any presheaf F. Consequently D_F is a nonvoid subset of $L(\widetilde{F})$. It remains to show that D_F is cofiltered with the inclusions (a system $(X_i,x_{ij})_{i\leq j}$ indexed by a poset I is cofiltered if for any i,j in I there exists a k in I so that $k\leq i$ and $k\leq j$). This fact immediately follows from (i).

(i) Let $N\overset{n}{\hookrightarrow}F$ and $N'\overset{n'}{\hookrightarrow}F$ be in $\mathcal{G}(F)$. $N\cup N'\subset F$ is the image of the presheaf morphism $N\amalg N'\xrightarrow{[n,n']}F$. Since $\mathcal{G}$ is closed at finite coproducts and quotients it follows that $N\cup N'\in\mathcal{G}(F)$. Finally for each $E\in\mathrm{ob}\,\mathcal{E}$ and each pair $(\xi,\zeta)\in$ $\in\widetilde{F}(E)\times\widetilde{F}(E)$ one easily gets $\xi|_{N\cup N'}=\zeta|_{N\cup N'}$ iff $\xi|_N=\zeta|_N$ and $\xi|_{N'}=\zeta|_{N'}$. Hence $D_F(N)\cap D_F(N')=D_F(N\cup N')$.

(ii) Let $N\overset{n}{\hookrightarrow}F$ and $N'\overset{n'}{\hookrightarrow}F$ be in $\mathcal{G}(F)$. Obviously if $N\subseteq N'$ then $D_F(N')\subseteq D_F(N)$. On the other hand if $D_F(N')\subseteq D_F(N)$ then the coequalizer $\widetilde{n}$ of $D_F(N)$ factors through the coequalizer $\widetilde{n'}$ of $D_F(N')$ to a regular epi $\Phi:\widetilde{N'}\to\widetilde{N}$. Since N' is embeddable in an object of $\mathcal{E}$, by 2.3.6 we get a mono $\varphi:N\hookrightarrow N'$ so that $\widetilde{\varphi}=\Phi$. By the faithfulness of $\widetilde{?}$ it follows from $\widetilde{n}=\widetilde{\varphi}\,\widetilde{n'}$ that $n'\varphi=n$. Consequently $N\subseteq N'$ in $\mathcal{G}(F)$.

(iii) Let $\varphi:F\to H$ be a presheaf morphism and let $N\overset{n}{\hookrightarrow}F$ be in $\mathcal{G}(F)$. The image of N under φ is the presheaf $\varphi(N)$ defined by the coequalizer decomposition of φn namely

$$\begin{tikzcd} N \arrow[rr, "\varphi n"] \arrow[dr, "p"'] & & H \\ & \varphi(N) \arrow[ur, "q"'] & \end{tikzcd}$$

Now if $E\in\mathrm{ob}\,\mathcal{E}$ and $(\xi,\zeta)\in(D_H(\varphi(N)))(E)$ then from $\xi|_{\varphi(N)}=\zeta|_{\varphi(N)}$ it follows that $\xi\varphi n=\zeta\varphi n$. Since $\xi\varphi=\widetilde{\varphi}(\xi)$ one gets that $\widetilde{\varphi}(\xi)|_N=$ $=\widetilde{\varphi}(\zeta)|_N$.

Hence $D_H(\varphi(N)) \subseteq \tilde{\varphi}^{-1}(D_F(N))$. On the other hand if $E \in ob\,\mathcal{E}$ and $(\xi,\zeta) \in (\tilde{\varphi}^{-1}(D_F(N)))(E)$ then from $\xi\varphi n = \zeta\varphi n$ it follows that $\xi q = \zeta q$, (p is epi). Hence $(\xi,\zeta) \in (D_H(\varphi(N)))(E)$. When E runs over $\mathcal{E}$ one gets that $\tilde{\varphi}^{-1}(D_F(N)) \subseteq D_H(\varphi(N))$.

2.5.2 Remark. By 2.5.1 for each presheaf F the $\mathcal{E}$-algebra $\tilde{F}$ endowed with the filter $\bar{D}_F$ in $L_{\mathcal{E}^*}(\tilde{F})$ generated by D_F is a c-uniform object in $\mathcal{E}^*$. Also for each presheaf morphism $f:F \to H$ the algebra morphism $\tilde{f}:\tilde{H} \to \tilde{F}$ is uniform continuous in the c--uniformities $\bar{D}_H$ and $\bar{D}_F$. Consequently the cofunctor $\tilde{?}:C^\wedge \to \mathcal{E}^*$ arises to a faithful cofunctor $T:C^\wedge \to \underline{c\text{-}u\text{-}\mathcal{E}^*}$ defined by $T(F) = (\tilde{F},\bar{D}_F)$ and $T(f)=\tilde{f}$. The restriction $\Gamma:\mathcal{E} \to \underline{c\text{-}u\text{-}\mathcal{E}^*}$ of T

produces ,according to 1.6 , a full subcategory Γ^* of $\underline{c\text{-}u\text{-}\mathcal{E}^*}$. Γ^* consists of all objects $(A,\mathcal{A})$ of $\underline{c\text{-}u\text{-}\mathcal{E}^*}$ so that each algebra morphism $\tilde{E} \to A$, $E \in \mathcal{E}$, is uniform continuous in the c-uniformities $\bar{D}_E$ and $\mathcal{A}$, for all $E \in ob\,\mathcal{E}$. Since each $\mathcal{E}$-algebra morphism $\tilde{E} \to \tilde{F}$ has the form $\tilde{f}$ for a suitable $f:F \to E$, for any $E \in ob\,\mathcal{E}$, one gets that the cofunctor T induces a cofunctor , also denoted by T,

$$T:C^\wedge \to \Gamma^*,\ T(F)=(\tilde{F},\bar{D}_F),\ T(f)=\tilde{f}.$$

2.6 DisΓ^*. The category $\underline{\text{Dis}\Gamma^*}$ consists of all $\mathcal{E}$-algebras A so that for each $E \in ob\,\mathcal{E}$ and each $\mathcal{E}$-algebra morphism $\tilde{E} \xrightarrow{\Phi} A$ the congruence of Φ is in $\bar{D}_E$. Equally $\underline{\text{Dis}\Gamma^*}$ is the full subcategory of Γ^* of all $(A,\mathcal{A})$ for which $\Delta_A \in \mathcal{A}$. $\underline{\text{Dis}\Gamma^*}$ is also a full subcategory of $\mathcal{E}^*$. In what follows we shall denote an object of $\underline{\text{Dis}\Gamma^*}$ whose algebraic support is A by (A,L) if the object is thought in Γ^* and by A if the object is thought in $\mathcal{E}^*$. Also if $A \in ob\underline{\text{Dis}\Gamma^*}$ and B is a subalgebra of A then $B \in ob\underline{\text{Dis}\Gamma^*}$ too,since the congruence of any $\tilde{E} \to B$ equals the congruence of $\tilde{E} \to B \subseteq A$. Hence $\underline{\text{Dis}\Gamma^*}$ is closed under subobjects both in $\mathcal{C}^*$ and in Γ^*.

Proposition 2.6.1

(i) if $G \in \mathcal{G}$ then $T(G) \in \underline{\mathrm{Dis}}\Gamma^*$

(ii) $T(\mathcal{G})$ is a subcategory of generators in $\underline{\mathrm{Dis}}\Gamma^*$

Proof (i) If $G \in \mathcal{G}$ then $G \xrightarrow{1_G} G$ is in $\mathcal{G}(G)$. Since $D_G(G) = \Delta_{\tilde{G}}$ it follows that $\overline{D}_G = L(\tilde{G})$ and $T(G) \in \mathrm{ob}\underline{\mathrm{Dis}}\Gamma^*$.

(ii) Let $f,g:(A,L) \longrightarrow (B,L)$ be two distinct arrows of $\underline{\mathrm{Dis}}\Gamma^*$. Then $f \neq g$ in $\mathcal{C}^*$ so that there exists $E \in \mathrm{ob}\mathcal{C}$ and $x \in A(E)$ with $f\tilde{x} \neq g\tilde{x}$. Since $\tilde{x}:\tilde{E} \longrightarrow A$ is uniform continuous in $\overline{D}_E$ and $L(A)$ it follows that the congruence of $\tilde{x}$ contains a $D_E(N)$ for a suitable $N \xrightarrow{n} E$ of $\mathcal{G}(E)$. Hence $\tilde{x}$ factors through $\tilde{n}:\tilde{E} \longrightarrow \tilde{N}$ at a morphism $h:\tilde{N} \longrightarrow A$. By the fact that $\tilde{n}$ is a regular epi in $\mathcal{C}^*$ one gets that $fh \neq gh$.

Proposition 2.6.2 If $(A,L) \in \mathrm{ob}\ \underline{\mathrm{Dis}}\Gamma^*$ then the following assertions are equivalent:

(i) (A,L) is of finite type in $\underline{\mathrm{Dis}}\Gamma^*$

(ii) (A,L) is of finite type in $\underline{\text{c-u-}\mathcal{C}^*}$

(iii) A is a finitely generated $\mathcal{C}$-algebra

(iv) The algebra A is a regular quotient of an $\tilde{N}$ for a suitable $N \in \mathcal{G}$.

Proof. (i)$\Leftrightarrow$(ii) If $f:(B,\mathcal{B}) \longrightarrow (A,L)$ is a mono in $\underline{\text{c-u-}\mathcal{C}^*}$ then f is a mono in $\mathcal{C}^*$ too. Indeed for any $E \in \mathrm{ob}\mathcal{C}$ and any x,y in $B(E)$, $\tilde{x}$ and $\tilde{y}$ are in Γ^* and $f\tilde{x} = \widetilde{f(x)}$. Hence $f(x)=f(y)$ iff $f\tilde{x}=f\tilde{y}$ iff $\tilde{x}=\tilde{y}$ iff $x=y$.

Since f is a mono in $\mathcal{C}^*$, $\Delta_B = f^{-1}(\Delta_A) \in \mathcal{B}$ and $\overline{\mathcal{B}} = L(B)$. Hence if $(A,L) \in \mathrm{ob}\underline{\mathrm{Dis}}\Gamma^*$ then all subobjects of (A,L) from $\underline{\text{c-u-}\mathcal{C}^*}$ are in $\underline{\mathrm{Dis}}\Gamma^*$ too. By this assertion and the fact that the inclusion $\underline{\mathrm{Dis}}\Gamma^* \hookrightarrow \underline{\text{c-u-}\mathcal{C}^*}$ creates colimits, the equivalence (i)$\Leftrightarrow$(ii) is obvious.

(ii)$\Rightarrow$(iii) Since (A,L) is of finite type in $\underline{\text{c-u-}\mathcal{C}^*}$

it is of finite type in $\underline{Dis}\Gamma^{*}$ by the above proof. In $\mathcal{E}^{*}$ the algebra A is the filtered colimit of its finitely generated subalgebras. But each subalgebra of A is in $\underline{Dis}\Gamma^{*}$ too. Since the inclusion $\underline{Dis}\Gamma^{*} \hookrightarrow \mathcal{E}^{*}$ creates filtered colimits it follows that A must be isomorphic to one of its finitely generated subalgebra.

<u>(iii) $\Rightarrow$ (iv)</u> If A is a finitely generated $\mathcal{E}$-algebra then there exists a regular epi $\tilde{E} \xrightarrow{f} A$ for a suitable $E \in ob\mathcal{E}$. Since (A,L) is in $\underline{Dis}\Gamma^{*}$ the congruence of f is in $\tilde{D}_E$. Hence there exists an N in $\mathcal{G}(E)$ so that $D_E(N)$ is contained in the congruence of f. Consequently there exists an N in $\mathcal{G}$ so that f factors through $\tilde{E} \twoheadrightarrow \tilde{N}$ necessarily to a regular epi $g:\tilde{N} \twoheadrightarrow A$.

Remark that $g:T(N)=(\tilde{N},L) \twoheadrightarrow (A,L)$ is a regular epi in Γ^{*} too.

<u>(iv) $\Longrightarrow$ (i)</u> Since all $\tilde{N}$ with $N \in \mathcal{G}$ are regular quotients of algebras $\tilde{E}$ with $E \in ob\mathcal{E}$ it follows that A is a finitely generated algebra in $\mathcal{E}^{*}$. Hence A is of finite type in $\mathcal{E}^{*}$. Since the colimits in $\underline{Dis}\Gamma^{*}$ are performed as in $\mathcal{E}^{*}$, (A,L) is of finite type in $\underline{Dis}\Gamma^{*}$ too.

<u>Definition 2.6.3</u> $(A,L) \in ob\underline{Dis}\Gamma^{*}$ is coherent in $\underline{Dis}\Gamma^{*}$ (or coherent discrete uniform algebra over Γ) if the following two conditions hold:

(a) (A,L) is of finite type in $\underline{Dis}\Gamma^{*}$

(b) for any (B,L) of finite type in $\underline{Dis}\Gamma^{*}$ each morphism $(B,L) \longrightarrow (A,L)$ has its congruence of finite type in $\underline{Dis}\Gamma^{*}$, too.

<u>Proposition 2.6.4</u> Let (A,L) be of finite type in $\underline{Dis}\Gamma^{*}$. The following assertions are equivalent:

(1) $(A;L)$ is coherent in $\underline{Dis}\Gamma^{*}$

(2) there exists in $\mathcal{E}^{*}$ at least a sequence

$$\tilde{G}_2 \xrightarrow{\Phi_2} R \underset{v}{\overset{u}{\rightrightarrows}} \tilde{G}_1 \xrightarrow{\Phi_1} A$$

with G_i, $i=1,2$, in $\mathcal{G}$ and $R \overset{u}{\underset{v}{\rightrightarrows}} \tilde{G}_1$ a kernel pair of Φ_1.

(3) there exists an $N \in \mathcal{G}$ so that $T(N) \cong (A;L)$

(4) for each G in $\mathcal{G}$ the congruence of any $\mathcal{E}$-algebra morphism $\tilde{G} \longrightarrow A$ is of finite type

(5) $(A;L)$ is a subobject in a suitable $\tilde{N}$ with $N \in \mathcal{G}$.

Proof. (1)$\Rightarrow$(2) Since $(A;L)$ is of finite type in $\underline{\mathrm{Dis}\,\Gamma^*}$ one gets a G_1 in $\mathcal{G}$ and a regular epi $\Phi_1 : \tilde{G}_1 \longrightarrow A$. Let $R \overset{u}{\underset{v}{\rightrightarrows}} \tilde{G}_1$ be a kernel pair of Φ_1. R as a subobject of $\tilde{G}_1 \times \tilde{G}_1$, which is isomorphic to $\widetilde{G_1 \amalg G_1}$ is in $\underline{\mathrm{Dis}\,\Gamma^*}$. Since $(A;L)$ is coherent in $\underline{\mathrm{Dis}\,\Gamma^*}$ and $\tilde{G}_1$ is of finite type in $\underline{\mathrm{Dis}\,\Gamma^*}$ it follows that $(R;L)$ is of finite type in $\underline{\mathrm{Dis}\,\Gamma^*}$ too. Consequently there exists a regular epi $\Phi_2 : \tilde{G}_2 \longrightarrow R$ with G_2 in $\mathcal{G}$.

(2)$\Rightarrow$(3) Since $\tilde{?}$ preserves colimits $\tilde{G}_1 \times \tilde{G}_1$ is canonically isomorphic to $\widetilde{G_1 \amalg G_1}$. By 2.3.6 there exists a presheaf morphism $\varphi : G_1 \amalg G_1 \longrightarrow G_2$ so that $\tilde{\varphi} = \langle u,v \rangle \Phi_2$.

If $G_1 \amalg G_1 \xrightarrow{p=[p_1,p_2]} G \xleftarrow{q} G_2$ is a coequalizer decomposition of φ then

$$\tilde{G}_2 \xrightarrow{\tilde{q}} \tilde{G} \xrightarrow{\tilde{p}=\langle \tilde{p}_1, \tilde{p}_2 \rangle} \widetilde{G_1 \amalg G_1} = \tilde{G}_1 \times \tilde{G}_1$$

is a coequalizer decomposition of $\langle u,v \rangle \Phi_2$. Hence one gets an iso $\mu : R \longrightarrow \tilde{G}$ so that $\tilde{p}\mu = \langle u,v \rangle$. Finally we can suppose that for A one has a short exact sequence

$$\tilde{G} \overset{\tilde{p}_1}{\underset{\tilde{p}_2}{\rightrightarrows}} \tilde{G}_1 \xrightarrow{\Phi_1} A \quad \text{in} \quad \mathcal{S}^*.$$

Since $\{\tilde{p}_1, \tilde{p}_2\}$ is a kernel pair of Φ_1, $\tilde{p}_1$ and $\tilde{p}_2$ are regular epimorphisms and there exists a diagonal morphism $\delta : \tilde{G}_1 \longrightarrow \tilde{G}$ with $\tilde{p}_1 \delta = \tilde{p}_2 \delta = id_{\tilde{G}_1}$. Consequently $p_1, p_2 : G_1 \longrightarrow G$ are monomorphisms and there exists a presheaf morphism $d : G \longrightarrow G_1$, $\tilde{d} = \delta$, so that $dp_1 = dp_2 = id_{G_1}$. Also $[p_1, p_2] : G_1 \amalg G_1 \longrightarrow G$ is a regular epi in $C^{\wedge}$.

According to the following lemma one gets that $\{p_1, p_2\}$ is a cokernel pair of its equalizer (in $C^{\wedge}$).

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<u>Lemma 2.6.5</u> The pair $u,v:F \rightrightarrows H$ of $C^\frown$ is a cokernel pair of its equalizer iff $\begin{cases} [u,v] \text{ is a regular epi} \\ u \text{ and } v \text{ are monomorphisms} \\ \text{if } uf=vg \text{ then } f=g \end{cases}$

<u>Proof.</u> Since in $C^\frown$ limits and colimits are performed compo--nentwise and monic resp. epic means componentwise injection resp. surjection it suffices to prove the lemma in <u>SET</u>.

Let $u,v:F \rightrightarrows H$ be two injections in <u>SET</u> so that $[u,v]:F^{(2)} \to H$ is a surjection and for each x,y in F with $u(x)=v(y)$ one has $x=y$. Let $w:F' \hookrightarrow F$ be the equalizer of $\{u,v\}$. $F'=\{x \in F \mid u(x)=v(x)\}$ and w is the inclusion. If $f,g:F \rightrightarrows M$ are two maps with $fw=gw$ then we can define $h:H \to M$ by $h(z)=\begin{cases} f(x), & \text{if } z=u(x) \\ g(x), & \text{if } z=v(x) \end{cases}$

($[u,v]:F \amalg F \to H$ is a surjection and consequently its image which is $u(F) \cup v(F)$, equals H). Under the assumptions made on u,v the function h is well defined and $\begin{cases} hu=f \\ hv=g \end{cases}$

Now we turn back to <u>(2)⟹(3)</u>. If $w:N \hookrightarrow G_1$ is the equalizer of $\{p_1,p_2\}$ then $\{p_1,p_2\}$ is a cokernel pair of w. Hence

$$\tilde{G} \underset{\tilde{p}_2}{\overset{\tilde{p}_1}{\rightrightarrows}} \tilde{G}_1 \xrightarrow{\tilde{w}} \tilde{N}$$

is a short exact sequence in $\mathcal{E}^*$ and $N \in \mathcal{G}$ as a subpresheaf in G_1. Comsequently A is isomorphic to $\tilde{N}$.

<u>(3)⟹(4)</u> If A is isomorphic with an $\tilde{N}$ for a suitable N in $\mathcal{G}$ then for any G in $\mathcal{G}$ each morphism $\tilde{G} \to A$ can be viewed as an $\tilde{f}$ for a suitable $f:N \to G$. If $G \underset{v}{\overset{u}{\rightrightarrows}} H$ is the cokernel pair of f then H is in $\mathcal{G}$ and $\{\tilde{u},\tilde{v}\}$ is a kernel pair of $\tilde{f}$. Consequently the congruence R of $\tilde{G} \to A$ is isomorphic

to $\widetilde{H}$ with H in $\mathcal{G}$. It follows that R is of finite type in <u>Dis</u>Γ^*

<u>(4)$\Rightarrow$(1)</u> If (B;L) is of finite type in <u>Dis</u>Γ^* then there exists a regular epi $\Phi_1:\widetilde{G}\twoheadrightarrow B$ for a suitable G in $\mathcal{G}$. If $R \overset{u}{\underset{v}{\rightrightarrows}} B$ is a kernel pair of a morphism $\Phi:(B;L)\longrightarrow(A;L)$ then one has a mutative diagram of $\mathcal{C}$-algebras

$$\begin{tikzcd}
\Phi_1^{-1}(R) \arrow[r, shift left, "u'"] \arrow[r, shift right, "v'"'] \arrow[d, "\mu"'] & \widetilde{G} \arrow[r, "\Phi\Phi_1"] \arrow[d, "\Phi_1"] & A \arrow[d, equal] \\
R \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B \arrow[r, "\Phi"] & A
\end{tikzcd}$$

where $(\Phi_1\times\Phi_1)\langle u',v'\rangle=\langle u,v\rangle\mu$ is a pullback. It is easy to see that $\{u',v'\}$ is a kernel pair of $\Phi\Phi_1$. Consequently $\Phi_1^{-1}(R)$, which is in <u>Dis</u>Γ^* as a subobject of $\widetilde{G}\times\widetilde{G}$, is of finite type in <u>Dis</u>Γ^*. Hence R is of finite type in <u>Dis</u>Γ^*.

<u>(5)$\Leftrightarrow$(3)</u> If (A;L) is a subobject in a T(N) with N in $\mathcal{G}$ then by <u>2.4.5</u> there exists a regular quotient N' of N so that $\widetilde{N'}$ is isomorphic to A. Also N' is in $\mathcal{G}$ as a quotient of N.

<u>Corollary 2.6.7</u> $T:C^{\wedge}\longrightarrow\Gamma^*$ induces an equivalence between $\mathcal{G}^o$ and Coh<u>Dis</u>Γ^*, the full subcategory of coherent objects in <u>Dis</u>Γ^*.

<u>Proof</u> We know that T is fully faithful on $\mathcal{G}$, by <u>2.3.6</u>. For each G in $\mathcal{G}$, $\Gamma(G)=(\widetilde{G};L)$ is coherent in <u>Dis</u>Γ^* according to the sequence

$$\widetilde{G} \xrightarrow{\widetilde{I}_G} \widetilde{G} \overset{\widetilde{I}_G}{\underset{\widetilde{I}_G}{\rightrightarrows}} \widetilde{G} \xrightarrow{\widetilde{I}_G} \widetilde{G}$$

Finally <u>2.6.4</u> shows that each coherent object in DisΓ^* is isomorphic to an object T(G) for a suitable G in $\mathcal{G}$.

2.7 The duality theorem

Proposition 2.7.1 If $F \in \mathrm{ob}C^{\wedge}$ let $\mathcal{F}$ be the monofiltered system $(N, \nu_{NN'})_{N \subseteq N'}$ in $\mathcal{G}(F)$ whose colimit is $(N \xrightarrow{n} F)_{N \in \mathcal{G}(F)}$ (for an N in $\mathcal{G}(F)$ the inclusion of N in F is always denoted by n ; sometimes to be more precisely we shall denote elements of $\mathcal{G}(F)$ by (N,n)).

Then in Γ^* one has $\varprojlim T(\mathcal{F}) = (T(F) \xrightarrow{\tilde{n}} T(N))_{N \in \mathcal{G}(F)}$

Proof. We know that in $\mathcal{E}^*$ one has $\varprojlim \tilde{?}(\mathcal{F}) = (\tilde{F} \xrightarrow{\tilde{n}} \tilde{N})_{N \in \mathcal{G}(F)}$

It remains to show that $\tilde{D}_F$ is the initial c-uniformity J on $\tilde{F}$ induced by the family $\{\tilde{F} \xrightarrow{\tilde{n}} (\tilde{N};L)\}_{N \in \mathcal{G}(F)}$. Since for each N in $\mathcal{G}(F)$, $\tilde{n}:\tilde{F} \to \tilde{N}$ is uniform continuous in the uniformities $\tilde{D}_F$ and $\tilde{D}_N = L(N)$, it follows that $\tilde{D}_F \supseteq J$. On the other hand if $R \in \tilde{D}_F$ then R contains a $D_F(G)$ for a suitable G in $\mathcal{G}(F)$. The congruence $D_F(G)$ viewed as $\tilde{g}^{-1}(\Delta_{\tilde{G}})$ is an entourage of J. Hence $\tilde{D}_F$ has a base contained in J and consequently $\tilde{D}_F \subseteq J$.

Theorem 2.7.2 $T: C^{\wedge} \to c\text{-}u\text{-}\mathcal{E}^*$ is a fully faithful cofunctor.

Proof. It suffices to prove that T is a full cofunctor. Let $\Phi: T(F) \to T(H)$ be a morphism of $c\text{-}u\text{-}\mathcal{E}^*$. If H is in $\mathcal{G}$ then $T(H) = (\tilde{H};L)$ and the congruence $R = \Phi^{-1}(\Delta_{\tilde{H}})$ of $\Phi:\tilde{F} \to \tilde{H}$ contains a $D_F(N)$ for a suitable N in $\mathcal{G}(F)$. Consequently Φ factors through $\tilde{n}:\tilde{F} \to \tilde{N}$ to a morphism $\mu:\tilde{N} \to \tilde{H}$. Now $\mu = \tilde{\varphi}$ for a suitable $\varphi: H \to N$ so that we get $\Phi = \mu\tilde{n} = \tilde{\varphi}\tilde{n} = \widetilde{n\varphi} = T(n\varphi)$.

If H is not in $\mathcal{G}$ then let us consider H as the colimit of $\mathcal{G}(H)$ with the canonical morphisms $G \xrightarrow{g} H$, $G \in \mathcal{G}(H)$. By 2.7.1 $(T(H) \xrightarrow{\tilde{g}} T(G))_{G \in \mathcal{G}(H)}$ is the limit in $c\text{-}u\text{-}\mathcal{E}^*$ of $T(\mathcal{G}(H))$. For each (G,g) in $\mathcal{G}(H)$, $\tilde{g}\Phi = \tilde{\varphi}_g$ for a suitable $\varphi_g: G \to F$. For each $u:(G,g) \to (G',g')$ in $\mathcal{G}(H)$ one has

$$\widetilde{\varphi_{g'}u} = \tilde{u}\,\tilde{\varphi}_{g'} = \tilde{u}\tilde{g}'\Phi = \widetilde{g'u}\,\Phi = \tilde{g}\Phi = \tilde{\varphi}_g$$

and consequently $\varphi_{g'}u = \varphi_g$. Hence the family $\{\varphi_g\}_{g \in \mathcal{G}(H)}$ is a compatible family for the system $\mathcal{G}(H)$ and it defines a morphism

$\varphi: H \twoheadrightarrow F$ uniquely determined by $\varphi g = \varphi_g$ for all $(G,g)\in\mathcal{G}(H)$. Now one has $\tilde{g}\tilde{\varphi} = \widetilde{\varphi g} = \widetilde{\varphi}_g = \tilde{g}\tilde{\phi}$ for all $\tilde{g}$ with $(G,g)\in\mathcal{G}(H)$. Since $\{\tilde{g}\}_{(G,g)\in\mathcal{G}(H)}$ is a limit cone it follows that $\tilde{\varphi} = \tilde{\phi}$.

<u>Theorem 2.7.3</u> $T: C^\wedge \to \underline{c\text{-}u\text{-}\mathcal{E}^*}$ maps short exact sequences $\cdot \hookrightarrow \cdot \rightrightarrows \cdot$ of $C^\wedge$ to short exact sequences $\cdot \rightrightarrows \cdot \twoheadrightarrow \cdot$ of $\underline{c\text{-}u\text{-}\mathcal{E}^*}$.

<u>Proof.</u> Let $F \overset{\varphi}{\hookrightarrow} P \underset{u_2}{\overset{u_1}{\rightrightarrows}} H$ be a short exact sequence in $C^\wedge$. Hence φ is an equalizer of the pair $\{u_1, u_2\}$ and $\{u_1, u_2\}$ is a cokernel pair of φ. We already know that

$$\tilde{H} \underset{\tilde{u}_2}{\overset{\tilde{u}_1}{\rightrightarrows}} \tilde{P} \overset{\tilde{\varphi}}{\twoheadrightarrow} \tilde{F}$$

is a short exact sequence in $\mathcal{E}^*$. Hence it suffices to prove that if we consider the c-uniformity $\overline{D}_P$ on $\tilde{P}$ then $\overline{D}_F$ equals the final c-uniformity $\mathfrak{J}$ on $\tilde{F}$ induced by the regular epi $\tilde{\varphi}$ and $\overline{D}_H$ equals the initial c-uniformity $\mathfrak{I}$ on $\tilde{H}$ induced by the pair $\{\tilde{u}_1, \tilde{u}_2\}$.

Obviously one has $\overline{D}_F \sqsubseteq \mathfrak{J}$ and $\overline{D}_H \sqsupseteq \mathfrak{I}$. Let N be in $\mathcal{G}(H)$ with the inclusion $n: N \hookrightarrow H$. Let us consider the commutative diagram of presheaves

$$\begin{tikzcd}
P \arrow[r, hook, "\partial_i"] \arrow[rr, bend left, "u_i"] & P \amalg P \arrow[r, "{[u_1,u_2]}"] & H \\
M_i \arrow[u, hook, "m_i"] \arrow[r, hook, "d_i"] \arrow[rr, bend right, "v_i"'] & M \arrow[u, hook, "m"] \arrow[r, "v"] & N \arrow[u, hook, "n"']
\end{tikzcd}$$

where $\begin{cases} (1)\ nv=[u_1,u_2]m \text{ is a pullback} \\ (2)\ \partial_1,\partial_2 \text{ are the canonical morphisms of the coproduct } P^{(2)} \\ (3)\ \partial_i m_i = m d_i,\ i=1,2 \text{ , are pullbacks and } v_i = v d_i \end{cases}$

Since in $C^\sim$ coproducts are universal one gets that M is the coproduct of the pair $\{M_1,M_2\}$ with d_1,d_2 as canonical morphisms. Also $m=m_1 \amalg m_2$ and $v=[v_1,v_2]$. By (1) and (3) we get that $nv_i = u_i m_i$, $i=1,2$, are pullbacks. By 2.6.5 each u_i is a mono hence each M_i is in $\mathcal{G}$ as a subobject of N with the inclusion v_i. Consequently M is in $\mathcal{G}$. Also

$$\tilde{u}_1^{-1}(D_P(M_i))=D_H(u_i(M_i)) \qquad i=1,2$$

The inclusion of $u_i(M_i)$ in H is just nv_i . Hence $u_i(M_i)$ as a subobject of H is M_i with the inclusion nv_i or $u_i m_i$. Consequently

$$\bigcap_{i=1,2} \tilde{u}_i^{-1}(D_P(M_i))=\bigcap_{i=1,2} D_H(M_i)=D_H(\text{image of } M_1 \amalg M_2 \xrightarrow{[u_i m_i]_1^2} H)$$

Now $[u_1m_1,u_2m_2]=[u_1,u_2]\circ(m_1\amalg m_2)=nv$, and the image of $[u_1m_1,u_2m_2]$ can be taken as $N \xhookrightarrow{n} H$. Hence

$$\bigcap_{i=1,2} \tilde{u}_i^{-1}(D_P(M_i))=D_H(N) \qquad \text{is in } \mathcal{I}.$$

Consequently $\bar{D}_H$ has a base in I. It follows that $\bar{D}_H = I$.

Now we shall prove that $\bar{D}_F = J$. Since a base for $\bar{D}_P$ is D_P it follows that a base of J is

$$\mathfrak{S} = \{ R \in L(\tilde{F}) \mid \tilde{\varphi}^{-1}(R) \supseteq D_P(N) \text{ for a suitable } N \in \mathcal{G}(P) \}$$

Now if $N \xhookrightarrow{n} P$ is in $\mathcal{G}(P)$ and if $P \underset{v_2}{\overset{v_1}{\rightrightarrows}} Q$ is a co-kernel pair of n , then

$$N \xhookrightarrow{n} P \underset{v_2}{\overset{v_1}{\rightrightarrows}} Q$$

is a short exact sequence in $C^\sim$.

If

$$\begin{tikzcd}
F \amalg F \arrow[r, hook, "\varphi \amalg \varphi"] \arrow[dr, "{[w_1, w_2]}"'] & P \amalg P \arrow[r, "{[v_1, v_2]}"] & Q \\
& F' \arrow[ur, hook, "\varphi'"'] &
\end{tikzcd}$$

is a coequalizer decomposition of $[v_1, v_2] \circ (\varphi \amalg \varphi)$ then let $m : M \hookrightarrow F$ be the equalizer of the pair $\{w_1, w_2\}$. One has in $C^{\sim}$ the commutative diagram

(D)
$$\begin{tikzcd}
N \arrow[r, hook, "n"] & P \arrow[r, shift left, "v_1"] \arrow[r, shift right, "v_2"'] & Q \\
M \arrow[u, hook, "\psi"] \arrow[r, hook, "m"] & F \arrow[u, hook, "\varphi"] \arrow[r, shift left, "w_1"] \arrow[r, shift right, "w_2"'] & F' \arrow[u, hook, "\varphi'"']
\end{tikzcd}$$

where $\varphi' w_i = v_i \varphi$, $i=1,2$, by the above coequalizer decomposition, and ψ is canonically built so that $\varphi m = n \psi$. Hence M is in $\mathcal{G}$ as a subobject of N. Also (M, m) is in $\mathcal{G}(F)$. Now $\{w_1, w_2\}$ is a cokernel pair of m, by an application of <u>2.6.5</u> (firstly $[w_1, w_2]$ is a regular epi by construction, secondly: each w_i is a mono by the equality $\varphi' w_i = v_i \varphi$; finally if $w_1 x = w_2 y$ then $v_1 \varphi x = \varphi' w_1 x = \varphi' w_2 y = v_2 \varphi y$ and since $\{v_1, v_2\}$ is a cokernel pair in $C^{\sim}$ it follows $\varphi x = \varphi y$, hence $x = y$)

We get in $\tilde{C}^{*}$ the commutative diagram $\tilde{(D)}$:

$$\begin{tikzcd}
\tilde{Q} \arrow[r, shift left, "\tilde{v}_1"] \arrow[r, shift right, "\tilde{v}_2"'] \arrow[d, "\tilde{\varphi}'"'] & \tilde{P} \arrow[r, "\tilde{n}"] \arrow[d, "\tilde{\varphi}"] & \tilde{N} \arrow[d, "\tilde{\psi}"] \\
\tilde{F}' \arrow[r, shift left, "\tilde{w}_1"] \arrow[r, shift right, "\tilde{w}_2"'] & \tilde{F} \arrow[r, "\tilde{m}"] & \tilde{M}
\end{tikzcd}$$

with short exact lines and $\tilde{\varphi}\tilde{v}_i=\tilde{w}_i\tilde{\varphi}'$, $i=1,2$ .By the exactness of the lines it follows that $\tilde{Q}$ resp. $\tilde{F}$ is canonically isomorphic to $D_P(N)$ resp. $D_F(M)$. (that the iso $\tilde{Q} \xrightarrow{\ \simeq\ } D_P(N)$ is canonical means that the morphisms $\tilde{v}_i\circ(\text{iso})$, $i=1,2$, are the canonical projections of $D_P(N)$ onto $\tilde{P}$). Actually in the commutative diagram

$$\begin{tikzcd}
D_P(N) \arrow[r, "\simeq"] \arrow[d] & \tilde{Q} \arrow[r, "\langle\tilde{v}_1,\tilde{v}_2\rangle"] \arrow[d, "\tilde{\varphi}'"] & \tilde{P}\times\tilde{P} \arrow[r, "\tilde{\varphi}\times\tilde{\varphi}"] & \tilde{F}\times\tilde{F} \\
D_F(M) \arrow[r, "\simeq"] & \tilde{F}' \arrow[urr, "\langle\tilde{w}_1,\tilde{w}_2\rangle"'] & &
\end{tikzcd}$$

the exterior contour can be viewed as a coequalizer decomposition of $(\tilde{\varphi}\times\tilde{\varphi})\langle\tilde{v}_1,\tilde{v}_2\rangle\circ(\text{iso})$. Consequently $D_F(M)=\tilde{\varphi}(D_P(N))$. Hence $\tilde{\varphi}(D_P(N))\in D_F$. Finally if $R\in\mathfrak{S}$ then $\tilde{\varphi}^{-1}(R)\supseteq D_P(N)$ and $\tilde{\varphi}(D_P(N))\subseteq\tilde{\varphi}\tilde{\varphi}^{-1}(R)=R$. Consequently $\bar{D}_F\supseteq J$ and $\bar{D}_F=J$.

2.7.4 Remark that if $\varphi: F \to H$ is a mono in $C^\wedge$ then for each N in $\mathcal{G}(H)$ the subalgebra $\tilde{\varphi}(D_H(N))$ of $\tilde{F} \times \tilde{F}$ is a congruence and $\tilde{\varphi}(D_H(N)) \in D_F$.

2.7.5 Let $\mathcal{D}$ be the category of all coherent objects of $\underline{\mathrm{Dis}\,\Gamma^*}$. If a separated and complete c-uniform $\mathcal{E}$-algebra $(A;\mathcal{A})$ has a base consisting of $\mathcal{D}$-special entourages then $(A;\mathcal{A}) \in \Gamma^*$ by the final remark of 1.6.10. Consequently any $\mathcal{D}$-strict c-uniform $\mathcal{E}$-algebra is in Γ^*.

Theorem (a) For each $F \in C^\wedge$, $T(F)$ is a $\mathcal{D}$-strict object in Γ^*

(b) T induces an equivalence between $(C^\wedge)^o$ and the full sub-category of $\underline{c\text{-}u\text{-}\mathcal{E}^*}$ of all $\mathcal{D}$-strict objects.

Proof (a) Let F be a presheaf. If $E \in \mathcal{E}$ and if $(\xi, \zeta) \in \bigcap_{N \in \mathcal{G}(F)} [D_F(N)](E)$ then $\xi|_N = \zeta|_N$ for all $N \in \mathcal{G}(F)$. Since $\bigcup \mathcal{G}(F) = F$, $\xi = \zeta$. Hence $\bigcap \tilde{D}_F = \Delta_{\tilde{F}}$ and T(F) is a separated c-u algebra. Since a base for $\tilde{D}_F$ is $\{D_F(N) \mid N \in \mathcal{G}(F)\}$ and since for each $N \xrightarrow{n} F$ in $\mathcal{G}(F)$ $T(F)/D_F(N) \xrightarrow{\sim} (\tilde{N};L) \in \mathcal{D}$ it follows that T(F) has a fundamental system of entourages consisting of $\mathcal{D}$-special entourages. By 2.7.1 $\left(T(F) \xrightarrow[\text{proj.}]{\text{can.}} T(F)/D_F(N)\right)_{N \in \mathcal{G}(F)} = \varprojlim_{\mathcal{G}(F)} T(\mathcal{F})$ in Γ^*. Consequently T(F) is a complete c-u algebra.

Now, let $R \overset{u}{\underset{v}{\rightrightarrows}} \tilde{F}$ be a $\mathcal{D}$-special congruence on T(F). Since the family of all $\mathcal{D}$-special entourages of T(F) is $\{D_F(N) \mid N \in \mathcal{G}(F)\}$ there exists a family $\{N_i \xrightarrow{n_i} F\}_{i \in I} \subseteq \mathcal{G}(F)$ so that $\bigcap_i D_F(N_i) = R$. Since $D_F(N_i) \cap D_F(N_j) = D_F(N_i \cup N_j)$ one can suppose that $\{N_i\}_{i \in I}$ is a filtered family of subobjects of F. Since R is a $\mathcal{D}$-special congruence on T(F), $\tilde{F}/R \in \mathcal{D}$. Hence there exists a $G \in \mathcal{G}$ so that $\tilde{F}/_R \overset{\Phi}{\underset{\sim}{\to}} \tilde{G}$. If $h: \tilde{F} \to \tilde{G}$ is the composition $\tilde{F} \xrightarrow{p} \tilde{F}/_R \overset{\Phi}{\underset{\sim}{\to}} \tilde{G}$ where p is the canonical regular epi then $R \overset{u}{\underset{v}{\rightrightarrows}} \tilde{F} \xrightarrow[h]{} \tilde{G}$ is a short exact sequence in $\mathcal{E}^*$. For each i, $\tilde{n}_i: \tilde{F} \to \tilde{N}_i$ factors through h to a regular epi $\mu_i: \tilde{G} \to \tilde{N}_i$. Since $G \in \mathcal{G}$, for each i there exists a mono $m_i: N_i \hookrightarrow G$ so that $\mu_i = \tilde{m}_i$. We shall show that $G \in \mathcal{G}(F)$ as the union of he family $\{(N_i;n_i)\}_i$ of subobjects of F. For that let us consider the monofiltered system $\mathcal{N}=\{N_i\}_i$ of subobjects of F with the transition morphisms the existing inclusions $\nu_{ij}:N_i \hookrightarrow N_j$. If $\varinjlim \mathcal{N} = (N_i \xrightarrow{\nu_i} N)_i$ then $N=\bigcup_i N_i$ where N is the subobject of F whose inclusion n in F is defined by $n\nu_i=n_i$ for all i. Also each N_i is cotained in N via ν_i. For each $\nu_{ij}:N_i \hookrightarrow N_j$ one has $\widetilde{\nu_{ij}}\mu_j h=\widetilde{\nu_{ij}}\tilde{n}_j=\widetilde{n_j\nu_{ij}}=\tilde{n}_i=\mu_i h$. Hence $m_j\nu_{ij}=m_i$ for each transition morphism of $\mathcal{N}$. Then there exists an $m:N\to G$ uniquely defined by $m\nu_i=m_i$ for all i. Since monofiltered colimits preserve monomorphisms , m is a mono and $\tilde{m}$ a regular epi.

$$
\begin{tikzcd}
R \arrow[rr, shift left] \arrow[rr, shift right] & & \tilde{F} \arrow[rr, "h"] \arrow[dr, "\tilde{n}_i"] \arrow[ddr, "\tilde{n}"'] & & \tilde{G} \arrow[dl, "\mu_i=\tilde{m}_i"'] \arrow[ddl, bend left, "\tilde{m}"] \\
& D_F(N_i) \arrow[ul, hook] \arrow[ur, shift left] \arrow[ur, shift right] & & \tilde{N}_i & \\
& & & \tilde{N} \arrow[u, "\tilde{\nu}_i"'] &
\end{tikzcd}
$$

If R_i is the congruence of μ_i then $\bigcap_i R_i$=(the congruence of $\tilde{m}$). Indeed $\mu_i f=\mu_i g$ for all i iff $\tilde{\nu}_i\tilde{m}f=\tilde{\nu}_i\tilde{m}g$ for all i iff $\tilde{m}f=\tilde{m}g$ (since $\tilde{?}:C^\wedge\to\mathcal{E}^*$ preserves colimits). Also $h^{-1}(R_i)=h^{-1}(\mu_i^{-1}(\Delta_{\tilde{N}_i}))=\tilde{n}_i^{-1}(\Delta_{\tilde{N}_i})=$ $=D_F(N_i)$ for each i. Since h is a regular epi $R_i=h(D_F(N_i))$ for each i. Consequently (the congruence of $\tilde{m}h$)=h^{-1}(the congruence of $\tilde{m}$)= $=h^{-1}(\bigcap_i R_i)=h^{-1}(\bigcap_i h(D_F(N_i)))=\bigcap_i h^{-1}h(D_F(N_i))=\bigcap_i D_F(N_i)=R$ by the fact that h is a regular epi and each $D_F(N_i)$ contains the congruence of h. Consequently $\tilde{m}$ is a mono , hence an iso; it follows that m is an iso too. Then G is a subobject of F with the inclusion $g=nm^{-1}$;moreover G equals N as subobjects of F.

$$
\begin{tikzcd}
N_i \arrow[rrd, hook, bend left, "n_i"] \arrow[rd, hook, "\nu_i"] \arrow[rdd, bend right, "m_i"'] & & \\
& N \arrow[r, hook, "n"] \arrow[d, "m"] & F \\
& G \arrow[ur, dashed, "g"'] &
\end{tikzcd}
$$

Now by the equalities $\tilde{m}_i h=\tilde{n}_i=\tilde{m}_i\tilde{g}$, $i\in I$, and by the fact that $\{\tilde{m}_i\}_i$ is a monomorphic family it follows that $h=\tilde{g}$. Consequently $R=D_F(G)$ is a $\mathcal{D}$-special entourage of T(F).

(b) It suffices to prove that for each $\mathcal{D}$-strict object

$(A;\mathcal{A})$ of Γ^* there exists an iso $(A;\mathcal{A}) \longrightarrow T(F)$ for a suitable F in $C^\wedge$. Let $(A;\mathcal{A})$ be a $\mathcal{D}$-strict object of Γ^*. $\mathfrak{S}=\{R\in\mathcal{A} \mid A/_R \in \mathcal{D}\}$ is a base for $\mathcal{A}$; moreover if for each pair $R\subseteq R'$ of $\mathfrak{S}$ $p_{RR'}: A/_R \twoheadrightarrow A/_{R'}$ and $p_R: A \twoheadrightarrow A/R$ are the canonically existing regular epimorphisms then $\{(A;\mathcal{A}) \xrightarrow{p_R} (A/R;L)\}_{R\in\mathfrak{S}} = \varprojlim_{\mathfrak{S}} ((A/R;L);p_{RR'})$ in Γ^*. Each A/R is isomorphic to a suitable $\widetilde{G}_R$ with $G_R\in\mathcal{G}$. Let $\Phi_R: A/_R \xrightarrow{\sim} \widetilde{G}_R$ be the iso. For each pair $R\subseteq S$ of $\mathfrak{S}$ there exists $\varphi_{SR}: G_S \hookrightarrow G_R$ so that the diagram

$$\begin{tikzcd}
A/_R \arrow[r, "p_{RS}"] \arrow[d, "\Phi_R"', "\wr"] & A/S \arrow[d, "\Phi_S", "\wr"'] \\
\widetilde{G}_R \arrow[r, "\widetilde{\varphi_{SR}}"] & \widetilde{G}_S
\end{tikzcd}$$

commutes. Obviously $\varphi_{SR}\circ\varphi_{US}=\varphi_{UR}$ for any three R,S,U in $\mathfrak{S}$ with $R\subseteq S\subseteq U$. Hence $\{G_R;\varphi_{SR}\}_{\mathfrak{S}^\circ}$ is a monofiltered system in $\mathcal{G}$. If $\varinjlim_{\mathfrak{S}^\circ}(G_R;\varphi_{SR})=(G_R \xhookrightarrow{\varphi_R} F)_{R\in\mathfrak{S}}$ then $\varprojlim_{\mathfrak{S}}(\widetilde{G}_R;\widetilde{\varphi}_{SR})=(\widetilde{F} \xrightarrow{\widetilde{\varphi}_R} \widetilde{G}_R)_{R\in\mathfrak{S}}$ and $\widetilde{F}$ is canonically isomorphic to A in $\mathcal{E}^*$. Let $\Phi: A \xrightarrow{\sim} \widetilde{F}$ be this canonical isomorphism. In what follows we shall identify A and $\widetilde{F}$ by the aid of Φ. Consequently $\mathcal{A}$ becomes the initial c-uniformity (which we shall denote also by $\mathcal{A}$) on $\widetilde{F}$ induced by the family $\{\widetilde{\varphi}_R:\widetilde{F}\longrightarrow(\widetilde{G}_R;L)\}_{R\in\mathfrak{S}}$. Obviously by this identification $(\widetilde{F};\mathcal{A})$ is a $\mathcal{D}$-strict object of Γ^* and $D_F(G_R)=R$ for each $R\in\mathfrak{S}$. We must show that $\mathcal{A}=\overline{D_F}$. Since each $(\widetilde{G}_R;L)=T(G_R)$ it follows that each $\widetilde{\varphi}_R$ is uniform continuous in $\overline{D_F}$ and $L_{\mathcal{E}^*}(\widetilde{G}_R)=\overline{D_{G_R}}$. Hence $\overline{D_F}\supseteq\mathcal{A}$. Now let $N \xhookrightarrow{n} F$ be in $\mathcal{G}(F)$. Remark that each $N_R=N\cap G_R$ is in $\mathcal{G}(F)$ too. Then $\bigcup\{N_R \mid R\in\mathfrak{S}\}=N\cap(\bigcup\{G_R \mid R\in\mathfrak{S}\})$ since $\{G_R \mid R\in\mathfrak{S}\}$ is a filtered family of subobjects of F. Hence $\bigcup\{N_R \mid R\in\mathfrak{S}\}=N\cap F=N$ as subobjects of F. Consequently $\bigcap\{D_F(N_R) \mid R\in\mathfrak{S}\}=D_F(N)$. Also since for each $R\in\mathfrak{S}$ N_R is contained in G_R it follows that $D_F(N_R)\supseteq D_F(G_R)=R$ for each $R\in\mathfrak{S}$. Hence each $D_F(N_R)\in\mathcal{A}$. Since $(F,\mathcal{A})/D_F(N_R)$ is canonically isomorphic to $(\widetilde{N}_R,L)$ which is in $\mathcal{D}$ it follows that each $D_F(N_R)$ is a $\mathcal{D}$-special entourage of $(F;\mathcal{A})$. Actually $D_F(N)$ is a

$\mathcal{D}$-special congruence on $(\tilde{F};\mathcal{A})$. Since $(\tilde{F};\mathcal{A})$ is $\mathcal{D}$-strict, $D_F(N)\in\mathcal{A}$. Consequently $\bar{D}_F$ has a base in $\mathcal{A}$. Hence $\bar{D}_F=\mathcal{A}$.

2.8 Applications and remarks

2.8.1 If for a presheaves category $C^\wedge$ we choose the pair $(\mathcal{G},\mathcal{E})$ so that $\mathcal{E}$ is a finitary algebraic theory (i.e. $\mathcal{E}=\{1,E,E^2,\dots\}$) then $\mathcal{E}^*$ is an algebraic category (Lawvere) and $\underline{c\text{-}u\text{-}\mathcal{E}^*}$ is a category of p-u $\mathcal{E}$-algebras in the usual sense since if $(A;\mathcal{A})\in$ $\in\underline{c\text{-}u\text{-}\mathcal{E}^*}$ and if $\alpha:E^k\longrightarrow E$ is a k-ary operation then for each $R\in\mathcal{A}$ $\alpha^{-1}(R)\supseteq\bigcap_{i=1}^{k}\pi_i^{-1}(R)$.

2.8.2 If there exists an $n\in\mathbb{N}$ so that each $G\in\mathcal{G}$ is a subobject of an E^k for a suitable $k\leq n$ then we can replace $\mathcal{E}$ by the truncate algebraic theory Σ whose primitive operations are all the k-ary operations of $\mathcal{E}$ with $k\leq n$. Indeed, the cofunctor $\tilde{?}:C^\wedge\longrightarrow\Sigma^*$, $F\mapsto\tilde{F}=$ $=\mathrm{Hom}_{C^\wedge}(F;E)$ is a faithful cofunctor which preserves colimits and short exact sequences $\cdot\rightarrowtail\cdot\twoheadrightarrow\cdot$ of $C^\wedge$. Also $\widetilde{E^k}$ is the free Σ-algebra on k generators for each $k\leq n$. 2.3.6 becomes " If F is a subpresheaf of an E^k with $k\leq n$ then the map $\mathrm{Hom}_{C^\wedge}(H;F)\longrightarrow\mathrm{Hom}_{\Sigma^*}(\tilde{F};\tilde{H})$ induced by $\tilde{?}$ is a bijection for any $H\in C^\wedge$". By the aid of $\mathcal{G}(F)$ we produce a c-uniformity on $\tilde{F}$ just like in 2.5. The c-structure Γ on Σ/Σ^* which we consider is $\Gamma(L_k)=(L_k;\mathcal{X}_k)$ where

$$\mathcal{X}_k=\begin{cases}\bar{D}_{E^k}, & k\leq n\\ \text{the final c-uniformity for } \bigcup_{i\leq n}\left\{(\widetilde{E^i};\bar{D}_{E^i})\xrightarrow{\varphi}L_k\,\middle|\,\varphi\in\Sigma^*\right\} & \text{if } k>n.\end{cases}$$

Obviously Γ^* consists of all c-u Σ-algebras $(A;\mathcal{A})$ so that each Σ^*-morphism $E^k\longrightarrow A$, $k\leq n$, is uniform continuous in $\bar{D}_{E^k}$ and $\mathcal{A}$. Also each $\tilde{G}$ with $G\in\mathcal{G}$ is a finitely generated Σ-algebra with less than n generators. 2.6.4 becomes " $A\in\underline{\mathrm{Dis}\,\Gamma^*}$ is isomorphic to a $\tilde{G}$ with $G\in\mathcal{G}$ iff there exists a sequence in Σ^*, $\tilde{G}_1\twoheadrightarrow R\underset{v}{\overset{u}{\rightrightarrows}}\tilde{G}_2\xrightarrow{\Phi_2}A$ where $\{u,v\}$ is a kernel pair of Φ_2 and $G_i\in\mathcal{G}$, $i=1,2$.". Now we take $\mathcal{D}$ the full subcategory of $\underline{\mathrm{Dis}\,\Gamma^*}$ of all finitely generated objects A so that A has less than n generators and so that for each finitely generated Σ-algebra B of $\underline{\mathrm{Dis}\Gamma^*}$ with less than n generators the congruence of any algebra morphism $B \twoheadrightarrow A$ is a finitely generated Σ-algebra with less than n generators. It is easy to prove that $T(\mathcal{G})$ is equivalent to $\mathcal{D}$. Finally $\underline{2.7.2}$ - $\underline{2.7.5}$ can be proved literally.

$\underline{2.8.1.3}$ If E is any injective cogenerator of $C^\wedge$ then each object of $C^\wedge$ is a subobject of a suitable power of E. If we take $\mathcal{E}$ to be the infinitary algebraic theory of all powers of E then for $\mathcal{G}=C^\wedge$, each T(F) is a discrete object of $\underline{c\text{-}u\text{-}\mathcal{E}^*}$ and consequently $(C^\wedge)^o$ is equivalent to $\mathcal{E}^*$. We have obtained thus particular cases of Linton's contravariant representation theorem as well as Paré's theorem on the dual of an elementary topos ([15],[11])

<u>2.8.2. A topological (linear in Leftschetz sense) version of Linton's theorem</u> ([15]). Let $\mathcal{C}$ be a category as in Linton's contravariant representation theorem ([15]), i.e. an algebraic category so that $\mathcal{C}$ has an injective cogenerator E and each mono in $\mathcal{C}$ is regular). Let $\underline{a}$ be the rank ([15]) of $\mathcal{C}$. If there is no such rank then $\underline{a}=\infty$. Let $\underline{b}$ a regular cardinal and let $(E;\underline{b})^*$ be the algebraic category (product preserving functors) over the truncated theory $(E;\underline{b})$ of all powers of E in $\mathcal{C}$ whose exponent is smaller than $\underline{b}$. Also let $\underline{c} \geq \aleph_0$ be a regular cardinal so that any $\underline{c}$-generated object of $\mathcal{C}$ is a subobject of a suitable E^k with $k \leq \underline{b}$. Finally let $\mathcal{G}$ be the set of all $\underline{c}$-generated objects of $\mathcal{C}$. For each $X \in \mathrm{ob}\,\mathcal{C}$, $\mathcal{G}(X)$ denotes the set of all subobjects of X which are in $\mathcal{G}$. The cofunctor $\tilde{?}:\mathcal{C} \longrightarrow (E;\underline{b})^*$, $\tilde{X}=\mathrm{Hom}_{\mathcal{C}}(X;E)$, $\tilde{f}(\xi)=\xi f$, commutes with colimits and preserves all short exact sequences $\rightarrowtail \rightrightarrows$ of $\mathcal{C}$. For each $Y \xrightarrow{y} X$ in $\mathcal{G}(X)$ the congruence of $\tilde{y}:\tilde{X} \longrightarrow \tilde{Y}$ is $D_X(Y)=\{(\xi;\zeta)\in \tilde{X}\times\tilde{X} \mid \xi|_Y=\zeta|_Y\}$

Since $\bigcap_i D_X(Y_i)=D_X(\bigcup_i Y_i)$, the family $D_X=\{D_X(Y) \mid Y\in \mathcal{G}(X)\}$ is closed in $L_{(E;b)^*}(X)$ under $\underline{c}$-meets. In particular D_X is a filter base in $L_{(E;b)^*}(X)$ and the c-uniformity $\bar{D}_X$ on X generated by D_X is stable at $\underline{c}$-meets. Let us call such a uniformity a $(\underline{c})$-c-uniformity. If $\underline{\mathcal{U}=(c)\text{-}c\text{-}u\text{-}(E;b)^*}$ is the category of all $(\underline{c})$-c-uniform objects in

$(E;\underline{b})^*$ then the cofunctor $\widetilde{?}$ induces a cofunctor $T:\mathcal{C}\to\mathcal{H}$, $T(X)=(\tilde{X};\overline{\overline{D}}_X)$. As in <u>2.3.6</u> we get that $\mathrm{Hom}_{\mathcal{C}}(X;Y)\xrightarrow{T}\mathrm{Hom}_{(E;\underline{b})^*}(\tilde{Y};\tilde{X})$ is a bijection provided Y is a subobject of an E^k with $k<\underline{b}$. Since $\underline{c}$ is at least $\aleph_0$ it follows that $(\tilde{X}\xrightarrow{\tilde{y}}\tilde{Y})_{(Y;y)\in\mathcal{G}(X)}=\varprojlim_{Y\in\mathcal{G}(X)}(\tilde{X}/_{D_X(Y)},\widetilde{\nu_{YY'}})$ where $\nu_{YY'}$ are the inclusions. Similarly to <u>2.7.2</u> it results that $T:\mathcal{C}\to\mathcal{H}$ is a full cofunctor. Finally <u>2.7.5</u> becomes

<u>Theorem 2.8.2.1</u> T induces an equivalence between $\mathcal{C}^0$ and the category of all $T(\mathcal{G})$-strict objects of $\mathcal{H}$.

<u>Proof</u> Firstly let us show that any $T(X)$ is a $T(\mathcal{G})$-strict object. Since D_X is a base for $\overline{\overline{D}}_X$, the family of all $T(\mathcal{G})$-special entourages of $T(X)$ is a base for $\overline{\overline{D}}_X$. Since $\varprojlim_{\mathcal{G}(X)}(\tilde{X}/_{D_X(Y)})=\tilde{X}$, $T(X)$ is separated and complete. Now let $R\in\mathcal{L}_{(E;\underline{b})^*}(\tilde{X})$ be so that $\tilde{X}/_R\simeq\tilde{G}$ with $G\in\mathcal{G}$ and $R=\bigcap_{i\in I}D_X(Y_i)$ with $\{(Y_i,y_i)\}_{i\in I}\subseteq\mathcal{G}(X)$. Then $R=$ $=D_X(\bigcup_{i\in I}Y_i)=\{(\xi;\zeta)\in\tilde{X}\times\tilde{X}\mid \xi|_{\bigcup Y_i}=\zeta|_{\bigcup Y_i}\}$. We must show that $Y=\bigcup_i Y_i$ is $\underline{c}$-generated. Let $y:Y\hookrightarrow X$ be the inclusion. If $u,v:X\rightrightarrows X'$ is a cokernel pair of y in $\mathcal{C}$ then $\widetilde{X'}\overset{\tilde{u}}{\underset{\tilde{v}}{\rightrightarrows}}\tilde{X}\xrightarrow{\tilde{y}}\tilde{Y}$ is a short exact sequence in $(E;\underline{b})^*$. Since the congruence of $\tilde{y}$ is R the canonical isomorphisms in $(E;\underline{b})^*$ results :

$$\begin{array}{ccccccc}
\widetilde{X'} & \overset{\tilde{u}}{\underset{\tilde{v}}{\rightrightarrows}} & \tilde{X} & \xrightarrow{\tilde{y}} & \tilde{Y} & & \\
\wr\| & & \| & & \wr\| & & \\
R & \rightrightarrows & \tilde{X} & \xrightarrow{\text{can}} & \tilde{X}/_R & \simeq & \tilde{G}
\end{array}$$

Hence there exists a morphism $\varphi:Y\to G$ with $G\in\mathcal{G}$ so that $\tilde{\varphi}$ is an iso in $\underline{(E;b)^*}$. Since E is a cogenerator in $\mathcal{C}$, φ is an epi. Since $\widetilde{?}$ is faithful φ is a mono. By the assumptions made on $\mathcal{C}$ we get that φ is an iso in $\mathcal{C}$. Hence $Y\in\mathcal{G}$ and $R\in D_X$.

Suppose now that $(A;\mathcal{A})\in\mathcal{H}$ is a $T(\mathcal{G})$-strict object. Let $\mathfrak{G}$ be a base for $\mathcal{A}$, so that $A/_R\simeq\widetilde{G_R}$ with $G_R\in\mathcal{G}$ for each $R\in\mathfrak{G}$. Since $A\simeq\varprojlim_{R\in\mathfrak{G}}(A/_R\simeq\widetilde{G_R})$ it follows that for $X=\varinjlim_{R\in\mathfrak{G}}G_R$ -which is a mono-filtered colimit- we get an iso $\Phi:A\to\tilde{X}$. We shall identify A to $\tilde{X}$ by Φ. By this identification $\mathcal{A}\subseteq\overline{\overline{D}}_X$ since $\mathfrak{G}\subseteq D_X$. Now if $Y\in\mathcal{G}(X)$

then $Y=\bigcup\{Y_R=Y\cap G_R \mid R\in\mathfrak{S}\}$. Hence $D_X(Y)=\bigcap\{D_X(Y_R) \mid R\in\mathfrak{S}\}$. Since $Y_R\subseteq G_R$, $D_X(G_R)\subseteq D_X(Y_R)$ for each $R\in\mathfrak{S}$. Hence all $D_X(Y_R)$ are elements of $\mathcal{A}$. If for each $R\in\mathfrak{S}$ $y_R:Y_R\longrightarrow G_R$ denotes the inclusion then $\widetilde{y_R}:\widetilde{G_R}\longrightarrow\widetilde{Y_R}$ is uniform continuous in $\overline{D_{G_R}}$ and $\overline{D_{Y_R}}$. Hence

$$\widetilde{X}\xrightarrow{\text{can}}\widetilde{G_R}\xrightarrow{\widetilde{y_R}}\widetilde{Y_R}$$ is uniform continuous in $\mathcal{A}$ and $\overline{D_{Y_R}}$.

Consequently if $N\in\mathcal{G}(Y_R)$ then $D_X(N)\in\mathcal{A}$ and $\widetilde{X}/_{D_X(N)}\simeq\widetilde{N}$ with $N\in\mathcal{G}$ .Actually we can suppose that $\mathfrak{S}=\{R\in\mathcal{A} \mid A/_R\simeq\widetilde{G}$ with $G\in\mathcal{G}\}$. Hence $D_X(N)\in\mathfrak{S}$ for all $N\in\mathcal{G}(Y_R)$ and any $R\in\mathfrak{S}$ .Since $Y_R=\bigcup\mathcal{G}(Y_R)$ it follows that $D_X(Y)=\bigcap\limits_R D_X(Y_R)=\bigcap\limits_R \bigcap\limits_{N\in\mathcal{G}(Y_R)} D_X(N)$ and each $D_X(N)$ is in $\mathfrak{S}$. Finally since $(\widetilde{X};\mathcal{A})$ is $T(\mathcal{G})$-strict and since $D_X(Y)$ is a $T(\mathcal{G})$-special congruence on $(\widetilde{X};\mathcal{A})$ we conclude that $D_X(Y)\in\mathcal{A}$ and $(\widetilde{X};\mathcal{A})$ is isomorphic to $T(X)$.

2.8.2.2 Remark that for arbitrary $\underline{c}$ (with respect to $\underline{b}$) an object $(A;\mathcal{A})$ of $\mathcal{H}$ is not a uniform algebra in the usual sense since only the operations of finite arity are uniform continuous in the product uniformity. On the other side $\mathcal{H}$ has small products. More generally the initial $(\underline{c})$-c-uniformity induced by the family $\{f_i:A\longrightarrow(A_i;\mathcal{A}_i)\}_{i\in I}$ with $\{(A_i;\mathcal{A}_i)\}_{i\in I}\subseteq \mathrm{ob}\,\mathcal{H}$, is the uniformity generated by $\{\bigcap\limits_{j\in J} f_j^{-1}(R_j) \mid J\subseteq I$ with $\mathrm{card}(J)<\underline{c}$ and $R_j\in\mathcal{A}_j$ for each $j\in J\}$

From this point of view for an $(A;\mathcal{A})\in \mathrm{ob}\,\mathcal{H}$ only operations of arity k smaller than $\underline{c}$ are uniform continuous in the product $(\underline{c})$-c--uniformity on A^k. In particular if $\underline{c}\geq\underline{b}$ then all objects of $\mathcal{H}$ are $(\underline{c})$-uniform algebras in the usual sense. Moreover if $\underline{b}\leq\aleph_0$ then also all objects of $\mathcal{H}$ are $(\underline{c})$-uniform $(E;\underline{b})$-algebras in the usual sense.

2.8.2.3 If $\underline{b}=\infty$ or $\underline{b}$=(the rank of $\mathcal{E}$) where $\mathcal{E}$ is the theory of all powers of E in $\mathcal{C}$, then we can choose $\underline{c}=\infty$. In this case each $T(X)$ is discrete and the dual $\mathcal{D}$ of $\mathcal{C}$ consists of all $(A;\mathcal{A})\in \mathrm{ob}\underline{(\infty)\text{-c-u-}\mathcal{E}^*}$ which are separated, complete and $T(\mathcal{C})$-strict objects By separation we get that any such $(A;\mathcal{A})$ must be discrete too. Hence $\mathcal{D}$ is a full subcategory of $\mathcal{E}$-algebras. Since for discrete c-u objects the $T(\mathscr{C})$-strict condition is obviously equivalent to the fact that A is a subdirect product of $\overline{\mathscr{L}(\{*\})}$ ($\mathscr{L}$ denotes a left adjoint of $\mathscr{C} \xrightarrow{\text{forgetful}}$ SET) one gets the original Linton's theor

2.8.2.4 If R is a ring , if M is an R-module and if $\mathscr{M}$ is the theory of all finite powers of M in R-mod then $\mathscr{M}^*$ is equivalent to the category of all $End_R(M)$-modules , It follows from 2.8.2.1 , 2.6.4 and 2.6.5 (which is true in R-mod) a duality theorem for R-mod. This remark and 2.8.3 below prove the first half of Oberst duality theorem for abelian toposes.

2.8.3 Since 2.6.5 is true in a Grothendieck-Giraud topos all results from 2.2 to 2.75 are true in such a topos. Hence 2.7.5 is in fact a duality theorem for Grothendieck-Giraud toposes.

2.8.4 Let us compute the dual of SET. SET is the category of all presheaves on C where C is the category with one object and one morphism. We can take $\mathscr{G}$ to be the full subcategory of all finite sets. We can take $E=\{0;1\}$ and $\mathscr{E}$ the full subcategory of all finite powers of E. $\mathscr{E}^*$ is equivalent to the category of all Boolean rings. Since each $\tilde{G}$ with $G\in\mathscr{G}$ is a finite set it follows that all T(F) with F∈SET are compact p-u spaces and hence (by 1.4.8) Stone spaces. Since each $T(E^k)$ with $k\in\mathbb{N}$ is discrete Γ^* is the category of all c-uniform Boolean rings and DisΓ^* is the category of all Boolean rings. Also $T(\mathscr{G})$ is the category of all finite Boolean rings which in turn is the category of all coherent objects of DisΓ^*. Since compact p-u spaces are exactly the SET_f-stict p-u spaces it follows that SET^o is equivalent to the category of all compact c-uniform Boolean rings (compact is a quality of the associated p-u space).

Since by [12].VI.2.9 , any Stone Boolean ring is a profinite Boolean ring, all Stone Boolean rings are c-uniform Boolean rings. Hence SET^o is equivalent to the category of all topological Stone Boolean rings and all continuous ring morphisms between them.

2.8.5 Let us compute the dual of the category of all Boolean algebras.i.e. Boolean rings. Standard arguments based on prime (maximal) ideals shows that $\mathbb{Z}_2=\{o;1\}$ is an injective cogenerator in the category $\underline{BR}$ of all Boolean rings. Hence we shall take $E=\mathbb{Z}_2$ and $\mathcal{E}$ the full subcategory of $\underline{BR}$ of all finite powers of E. $\mathcal{E}$ is the trivial theory. Indeed let $w:E^n \to E$ be a ring morphism. Since w preserves the unit element w is a surjection. Hence its kernel is a maximal ideal of E^n, say $\underline{a}$. At least one of the elements $e_i=(o,..,o,\underset{i}{1},o,..,o), i=1,..,n$, is not contained in $\underline{a}$, say e_1. Then $e_i \in \underline{a}$, $i=2,..,n$, since $e_1 e_i=o$ for all $i \geq 2$. Hence $\underline{a}$ is the kernel of the projection of E^n onto its first factor. Hence w is a trivial operation of $\mathcal{E}$. Actually $\mathcal{E}^*$ is equivalent to $\underline{SET}$. In order to get the dual of $\underline{BR}$ we shall apply Theorem $\underline{2.8.2.1}$.(see for the regularity of monomorphisms). $\mathcal{G}$ is the category of all finite Boolean rings. Since each $\tilde{G}$ with $G \in \mathcal{G}$ is a finite set it follows that $(\underline{BR})^o$ is a full subcategory of $\underline{SET}_f$-strict p-u spaces. By $\underline{2.8.4}$ $T(\mathcal{G})$ is equivalent to $\underline{SET}_f$. Hence $(\underline{BR})^o$ is equivalent to the category of all $\underline{SET}_f$-strict p-u spaces which are exactly the compact p-u spaces. By $\underline{1.4.8}$ we get that $(\underline{BR})^o$ is equivalent to the category of all Stone spaces.

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