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Horia I. ENE and Bogdan VERNESCU

1. INTRODUCTION

Homogenization deals with partial differential equations, in media with periodic structure, studying their behaviour as $\varepsilon \longrightarrow 0$, where ε usualy caracterizes the length of the period. Roughly speaking, we can distinguish two types of problems studied by means of the homogenization method: the first one consisting of problems that can be reduced to problems involving operators with periodic coefficients (functions of $\frac{x}{\varepsilon}$) defined in the whole domain, the second including the problems involving operators with constant coefficients, but in periodic structureddomains depending on ε ([1], [9]).

The problem that is studied in the present paper is in a way simmilar to the second type ones the difference consisting in the dependance on ε of the coefficients. If the domain hadn't had a periodic structure, the problem could have been regarded as a singular perturbation problem in the sense of [7].

In the second section of the paper we prove the existence and uniqueness of the solution of the problem for a given ε.

In order to obtain the homogenized problem in the third paragraph we use two-scale asymptotic expansions. The corresponding convergence theorem is proved in the next paragraph, following the general procedure of the homogenization method ([1], [9]).

The last paragraph deals with the mechanical model that has inspired the mathematical problem. Thus for a given ε, problem

(2.1) - (2.5) describes the Stokes flow in the presence of porous bodies. Therefore problem (3.14), (3.16) can be considered as modelling the flow in a porous fissured medium.

2. A SINGULAR PERTURBATION PROBLEM

Let Ω be a bounded open subset of R^N (N=2,3) of class C^2 and D_2 an open connected subset of Ω so that $\overline{D}_2 \subset \Omega$. We denote by $D_1 = \Omega - \overline{D}_2$, $\Gamma = \partial\Omega$ and $S = \partial D_2$. Let f be a given function of $L^2(\Omega)$.

We consider the following transmission problem:

(2.1) $-\varepsilon^2 \Delta v = -\text{grad } p + f$ in D_1

(2.2) $Kv = -\text{grad } p + f$ in D_2

(2.3) $\text{div } v = 0$, in Ω

(2.4) $v = 0$ on Γ

(2.5) $v \times n = 0$, $[v \cdot n] = 0$, $[p] = 0$ on S

where K is a symmetric and positive definite tensor with components $K_{ij} \in L^\infty(D_2)$, $[\,.\,]$ denotes the jump on the boundary S and n the unitary normal on S exterior to D_1. The first condition of (2.5) must be understood as the $v \times n$ component of the trace of $v/_{D_1}$.

In order to obtain the variational formulation of the problem (2.1) - (2.5) we have to prove the following lemma (for another proof see [6]).

LEMMA 2.1. The Hilbert space

$X = \{ v / v \in L^2(D_1),\ \text{div } v \in L^2(D_1),\ \text{curl } v \in L^2(D_1),$
$v \times n = 0 \text{ on } \partial D_1 \}$

is equal algebraicaly and topologicaly with:

$\{ v \in H^1(D_1) / v \times n = 0 \text{ on } \partial D_1 \}$

Proof.

Let $\varphi \in C^1(D_1)$ so that $\varphi \times n = 0$ on ∂D_1. We have:

(2.6) $\int_{D_1} (\text{grad } \varphi)^2 = \int_{D_1} ((\text{div } \varphi)^2 + (\text{curl } \varphi)^2) +$
$+ \int ((\frac{\partial \varphi}{\partial n} \cdot \varphi - \text{div } \varphi\ \varphi \cdot n)$

By extending $x \to n(x)$ in a neighbourhood of ∂D_1 [2] and observing that $\varphi = \lambda n$, on ∂D_1 we obtain the following estimation:

$$(2.7)\quad \left| \int_{\partial D_1} \left(\frac{\partial \varphi}{\partial n} \cdot \varphi \quad - \mathrm{div}\, \varphi \quad \varphi \cdot n \right) \right| \leq C \int_{\partial D_1} \varphi^2$$

and thus using the classical Lion's lemma (e.g. [8]):

$$\int_{\partial D_1} \varphi^2 \leq \frac{1}{2} \int_{D_1} (\mathrm{grad}\, \varphi)^2 + C \int_{D_1} \varphi^2$$

we get:

$$\int_{D_1} (\mathrm{grad}\, \varphi)^2 \leq C \int_{D_1} \varphi^2 + (\mathrm{div}\, \varphi)^2 + (\mathrm{curl}\, \varphi)^2$$

But the right hand side of the previous inequality represents the norm of X and consequently:

$$\| \varphi \|_{H^1} \leq C \quad \| \varphi \|_X$$

Using a density argument [2] we obtain the result.

If we denote by:

$H = H(\Omega, D_1) = \{ \; v \in L^2(\Omega) / \;\; v \in H^1(D_1)$, div $v = 0$ in Ω , $v = 0$ on Γ , $v \times n = 0$ on $S \}$

and we consider the equations (2.1) - (2.5) in the distribution sense then the variational formulation is:

$$(2.8) \begin{cases} \text{find } v \in H \text{ such that:} \\ a(v,w) = (f,w), \quad \text{for all } w \in H \end{cases}$$

where $a: H \times H \to R$ is given by :

$$a(v,w) = \varepsilon^2 \int_{D_1} \mathrm{curl}\, v \quad \mathrm{curl}\, w + \int_{D_2} Kvw$$

and $(.,.)$ denotes the scalar product of $L^2(\Omega)$.

The equivalence between problem (2.1) - (2.5) and problem (2.8) may be obtained as follows.

First we have to remark that:

(2.9) $-\Delta v = \mathrm{curl}\, (\mathrm{curl}\, v) - \mathrm{grad}\, (\mathrm{div}\, v)$

(2.10) $(\mathrm{curl}\, v, w) = (v, \mathrm{curl}\, w) - \langle v, w \times n \rangle$

for all $v, w \in H$, where $\langle \, , \, \rangle$ denotes the duality product in $H^{-1/2}(S)$. Thus (2.8) is obtained.

For proving the converse implication we have to get, in the classical manner for the Stokes equations (e.g. [11]), the existence of p, by writing by means of (2.9), (2.10) the problem (2.8) in the form:

(2.11) $\langle -\varepsilon^2 \Delta v - f, w \rangle + (Kv-f,w)=0$

and by choosing divergence free vectors w in $\mathcal{D}(D_1)$, $\mathcal{D}(D_2)$ respectively. We get in fact the existence and uniqueness, up to constants, of $p_1 \in L^2(D_1)$ and $p_2 \in H^1(D_2)$. The only boundary condition that is not a consequence of the definition of H is $(2.5)_3$; this can be obtained in the following sense:

$$\langle \nabla p_1, w \rangle = \int_S p_2 \, w.n \quad \text{for all } w \in H$$

and assuming that the constants have been chosen in a suitable way.

THEOREM 2.1. There exists a unique solution of problem (2.8).

Proof.

The existence and uniqueness of the solution yields from the Lax-Milgram theorem. For proving the coercivity of $a(v,w)$ we have to prove the next lemma.

LEMMA 2.2. There exists $C>0$ so that:

(2.12) $\|v\|_{L^2(D_1)} \leq C \|\text{curl } v\|_{L^2(D_1)}$

for all $v \in \tilde{X} = \{ v \in X \mid \text{div } v=0, \int_S v \cdot n=0 \}$

Proof.

By contradiction there exists a sequence $(v_n) \subset X$ so that:

$$\|\text{curl } v_n\|_{L^2(D_1)} \leq \frac{1}{n}, \quad \|v_n\|_{L^2(D_1)} = 1$$

Thus (v_n) is bounded in $\tilde{X}$ and hence weakly convergent to an element $v \in \tilde{X}$ that satisfies:

(2.13) $\text{div } v=0 \quad , \quad \int_S v \cdot n=0$

(2.14) $\text{curl } v=0 \quad , \quad v \times n=0 \quad \text{on } \partial D_1$

(2.15) $\| v \|_{L^2(D_1)} = 1$

Next (2.13) and (2.14) yield respectively [4] :

$v \in \text{curl}\, H^1(D_1)$, $v \in (\text{curl}\, H^1(D_1))^{\perp}$

and thus $v=0$; but this contradicts (2.15).

<u>REMARK 2.1.</u> The set $D_1 = \Omega - \bar{D}_2$ may be also supposed multi connected, without any change in the proofs.

3. TWO - SCALE ASYMPTOTIC EXPANSIONS

In the general framework of the homogenization theory we consider a domain Ω with a periodic structure. We denote by Y the parallelipipedic cell formed by two parts Y_1 and Y_2 separated by a smooth boundary S. We also denote by Y_1, Y_2, the union of all Y_1 and Y_2 parts. If $\Omega = \Omega_{\varepsilon 1} \cup \Omega_{\varepsilon 2}$ where $\Omega_{\varepsilon i} = \{x \in \Omega \;/ x \in \varepsilon Y_i\}$ $i=1,2$, we consider the problem:

$$(3.1)\begin{cases} \text{find } v^{\varepsilon} \in H(\Omega, \Omega_{\varepsilon 1}) \\ a(v^{\varepsilon}, w) = (f, w) \quad \text{for all } w \in H(\Omega, \Omega_{\varepsilon 1}) \end{cases}$$

where $a: H(\Omega, \Omega_{\varepsilon 1}) \times H(\Omega, \Omega_{\varepsilon 1}) \to R$ is given by:

$$(3.2)\; a(v,w) = \varepsilon^2 \int_{\Omega_{\varepsilon 1}} \text{curl}\, v \;\; \text{curl}\, w + \int_{\Omega_{\varepsilon 2}} K\, v\, w$$

As we have proved in the previous section the above problem is equivalent to a transmission problem of the type (2.1) - (2.5).

In order to study the asymptotic behaviour of the solution when $\varepsilon \to 0$ we introduce the two-scale expansions:

$$(3.3)\; v^{\varepsilon}(x) = v^0(x,y) + \varepsilon\, v^1(x,y) + \ldots$$

$$(3.4)\; p^{\varepsilon}(x) = p^0(x) + \varepsilon\, p^1(x,y) + \ldots$$

By a formal matching of the powers of ε we obtain the local problem :

$$(3.5)\; -\Delta_y v^0 = -\,\text{grad}_x p^0 + f - \text{grad}_y p^1 \quad \text{in } Y_1$$

$$(3.6)\; K v^0 = -\text{grad}_x p^0 + f - \text{grad}_y p^1 \;\; \text{in } Y_2$$

(3.7) $\operatorname{div}_y v^0=0$ in Y

(3.8) v^0, p^1 Y-periodic functions

(3.9) $v^0 \times n=0$, $[v^0 \cdot n]=0$, $[p^1]=0$ on S.

Introducing the Hilbert space:

$$H\,per(Y, Y_1) = \left\{ v \in L^2(Y)/v \in H^1(Y_1),\ \operatorname{div} v=0 \text{ in } Y, \right.$$
$$\left. v \times n=0 \text{ on } S,\ v \text{ Y-periodic} \right\}$$

the variational formulation of the local problem is:

(3.10) $$\int_{Y_1} \operatorname{curl} v^0 \cdot \operatorname{curl} w + \int_{Y_2} K v^0 w = (f - \operatorname{grad}_x p^0) \int_Y w$$

The left hand side of (3.10) defines a bilinear, symmetric and coercive functional and hence the local problem has a unique solution (the coercivity can be proved by contradiction, as in lemma 2.2, using the periodicity condition and the fact that curl v=0 implies $v=\operatorname{grad}\psi$).

If we introduce $\varphi^i \in H_{per}(Y,Y_1)$ the solution of

(3.11) $$\int_{Y_1} \operatorname{curl} \varphi^i \operatorname{curl} w + \int_{Y_2} K \varphi^i w = \int_Y w_i, \quad (\forall)\ w \in H_{per}(Y,Y_1)$$

then the solution of (3.10) may be written in the form:

(3.12) $$v^0=(f_i - \frac{\partial p^0}{\partial x^i})\varphi^i$$

By applying the mean value operator:

(3.13) $$\tilde{\cdot} = \frac{1}{|Y|}\int_Y \cdot$$

to (3.12) we get:

(3.14) $$\tilde{v}^0_j = L_{ij}(f_i - \frac{\partial p^0}{\partial x_i})$$

where

(3.15) $$L_{ij} = \tilde{\varphi}^i_j$$

On the other hand from equation (2.3), written for v^ε, using (3.3) and (3.13) we deduce:

(3.16) $\operatorname{div}_x \tilde{v}^0 = 0$

and thus from (3.14) and (3.16) we have an elliptic problem for p^0.

REMARK 3.1. L is a symmetric and positive definite tensor. For proving this we have to use the symmetry and coercivity of K and a (2.12) type inequality for $H_{per}(Y,Y_1)$ functions.

4. A CONVERGENCE THEOREM

If $\bar{Y}_1 \subset Y$ we have the following:

THEOREM 4.1. If v^ε, p^ε is the solution of the transmission problem (2.1) - (2.5) in $\Omega_{\varepsilon 1}$ and v^0, p^0 the solutions of (3.14), (3.16) then :

$$v^\varepsilon \longrightarrow \tilde{v}^0 \quad \text{weakly in } L^2(\Omega)$$

$$p^\varepsilon \longrightarrow p^0 \quad \text{strongly in } L^2(\Omega)/_R$$

when ε tends to zero.

Proof.

By using lemma 2.2 we deduce from (3.1) that:

(4.1) $\|v^\varepsilon\|_{L^2(\Omega)} \leq C$

(4.2) $\|\operatorname{curl} v^\varepsilon\|_{L^2(\Omega_{\varepsilon 1})} \leq C\, \varepsilon^{-1}$

Considering test functions $w \in H_0^1(\Omega)$ in the transmission problem equivalent to (3.1) we get:

$$|\langle \operatorname{grad} p^\varepsilon, w\rangle| = \left|\varepsilon^2 \int_{\Omega_{\varepsilon 1}} \operatorname{curl} v^\varepsilon \operatorname{curl} w + \right.$$

$$\left. + \int_{\Omega_{\varepsilon 2}} K v^\varepsilon w - \int_\Omega f w \right| \leq C \|w\|_{L^2(\Omega)} + C\varepsilon \|\operatorname{curl} w\|_{L^2(\Omega_{\varepsilon 1})}$$

and therefore, for ε sufficiently small:

(4.4) $\|\operatorname{grad} p^\varepsilon\|_{H^{-1}(\Omega)} \leq C$

(4.5) $\|p^\varepsilon\|_{L^2(\Omega)/_R} \leq C$

Out of (4.1) and (4.5) there exists $v^* \in L^2(\Omega)$ and

$p^* \in L^2(\Omega)/_R$ so that $v^\varepsilon \rightharpoonup v^*$ weakly in $L^2(\Omega)$ and $p^\varepsilon \rightharpoonup p^*$ weakly in $L^2(\Omega)/_R$. respectively grad $p^\varepsilon \to$ grad p^* weakly in $H^{-1}(\Omega)$.

Choosing w^ε a weakly convergent sequence in $\overset{\circ}{H}{}^1_0(\Omega)$ to w^* and using (4.3) we get:

$$|\langle \text{grad } p^\varepsilon, w^\varepsilon\rangle - \langle \text{grad } p^*, w^*\rangle| \leq$$

$$\leq |\langle \text{grad } p^\varepsilon, w^\varepsilon - w^*\rangle| + |\langle \text{grad } p^\varepsilon - \text{grad } p^*, w^*\rangle| \leq$$

$$\leq C\|w^\varepsilon - w^*\|_{L^2(\Omega)} + C\varepsilon\|\text{curl } (w^\varepsilon - w^*)\|_{L^2(\Omega_{\varepsilon 1})} +$$

$$+ |\langle \text{grad } p^\varepsilon - \text{grad } p^*, w^*\rangle|$$

Because the right hand side terms of the previous inequality tend to zero we have the strong convergence of p^ε to p^* in $L^2(\Omega)/_R$.

We have to deduce next that $v^* = v^0$ and $p^* = p^0$. For this we consider the equivalent system of (3.11):

$$(4.6)\begin{cases} -\Delta_y \varphi^i = -\text{grad}_y \psi^i + e_i & \text{in } Y_1 \\ K\varphi^i = -\text{grad}_y \psi^i + e_i & \text{in } Y_2 \\ \text{div}_y \varphi^i = 0 & \text{in } y \\ \varphi^i, \psi^i & \text{Y-periodic} \\ \varphi^i \times n = 0, \quad [\varphi^i \cdot n] = 0, \quad [\psi^i] = 0 & \text{on } S \end{cases}$$

By denoting $x = \varepsilon y$ in (4.6) we get a system for the functions

$$\varphi^i_\varepsilon(x) \equiv \varphi^i\left(\frac{x}{\varepsilon}\right), \quad \psi^i_\varepsilon(x) \equiv \psi^i\left(\frac{x}{\varepsilon}\right)$$

and because φ^i, ψ^i are independent of ε

$$\|\varphi^i_\varepsilon\|_{L^2(\Omega)} \leq C, \quad \|\psi^i_\varepsilon\|_{L^2(\Omega)} \leq C, \quad \|\text{curl } \varphi^i_\varepsilon\|_{L^2(\Omega_{\varepsilon 1})} \leq C\varepsilon^{-1}$$

As it is usual in the proof of the homogenization process [10], we take ϕv^ε, with $\phi \in \mathcal{D}(\Omega)$, as test function in the variational formulation of (4.6) and hence as $\varepsilon \to 0$:

(4.7) $$\varepsilon^2 \int_{\Omega_{\varepsilon 1}} \text{curl}\ \varphi_\varepsilon^{\ i}\ \ \text{curl}\ (\emptyset\ v^\varepsilon) + \int_{\Omega_{\varepsilon 2}} K\ \ \varphi_\varepsilon^{\ i} \emptyset v^\varepsilon \to \int_\Omega \emptyset v_i^*$$

On the other hand from (3.1):

(4.8) $$\varepsilon^2 \int_{\Omega_{\varepsilon 1}} \text{curl}\ v^\varepsilon\ \ \text{curl}\ (\emptyset\ \ \varphi_\varepsilon^{\ i}) + \int_{\Omega_{\varepsilon 2}} Kv^\varepsilon \emptyset\ \varphi_\varepsilon^{\ i} \to$$

$$\to L_{ij} \int_\Omega (f_j \emptyset + p^* \frac{\partial \emptyset}{\partial x_j})$$

By subtracting (4.7) from (4.8) we observe that the right hand side term vanishes; this yields:

$$v_i^* = L_{ij}(f_j - \frac{\partial p^*}{\partial x_j}) \qquad \text{in } \Omega$$

$$\text{div } v^* = 0 \qquad \text{in } \Omega$$

which are in fact (3.14) and (3.16).

5. A RELATED MECHANICAL PROBLEM

The problem studied in the previous sections may be considered as the mathematical model of the motion of a viscous fluid through a porous fissured medium.

Thus by taking $\varepsilon^2 v = u$ in the problem (2.1) - (2.5), we obtain the equations describing the flow in the presence of a porous body. In this case (2.1) is the Stokes equation, (2.2) the Darcy's law, (2.3) the continuity equation and (2.4), (2.5) are the usual boundary conditions (e.g. [5]). Then the theorem 2.1 gives us the existence and uniqueness result for this mechanical problem. The same result for the case of motion in the presence of several bodies can be proved analogously. The small parameter ε is related to the characteristic dimensions of the pores. In fact K of equation (2.2) is the inverse of the permeability tensor, and it is known [3] that the permeability tensor is proportional to the square of the characteristic lenght of the pores.

Problem (3.1) can be thus considered as modelling the flow of a fluid in a fissured porous medium. In the periodic model considered Y_1 and Y_2 denote the fluid and respectively the porous part of the period. The model implies two hypotheses:

i) the dimension of the porous cells are the same as the dimension of the fissures

ii) the ratio between the characteristic length of the pores and the characteristic length of the cells is the same as the ratio between the dimension of the fissures and the dimension of the porous body.

If we denote by l, l_f the characteristic lengths of the pores and of the fissures respectively and by L the dimension of the porous body, then the small parameter involved in the problem is:

$$\varepsilon = \frac{l}{l_f} = \frac{l_f}{L}$$

The study of the local problem yields a Darcy's law (3.14) for the flow in such a medium, but with a permeability tensor (3.15) which differs from the classical one. This new tensor is a genuine permeability tensor, because by considering dimensional quantities in the definition of L_{ij} we obtain that it is proportional to l_f^2 and μ^{-1} (μ being the viscosity coefficient, that in the mathematical formulation of the problem was considered 1 for simplicity).

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