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Dan POLIŠEVSKI*)

Let $\sum^i$, $i\in\{1,2,\dots 6\}$, be the side faces of $Y=[0,1[^3$, and let Γ be a surface of class C^2 included in $\overline{Y}$, which cross the boundary of the cube following some regular curves which are reproduced identically on opposite faces. We assume that Γ separate Y into two domains, Y_S-the solid part and Y_f-the fluid part, with the property that repeating Y by periodicity, the reunion of all the fluid parts, respectively the solid parts, is connex in $\mathbb{R}^3$ and of class C^2. The origin of the coordinate system is set in a fluid ball; thus all the corners of $\overline{Y}$ are surrounded by fluid neighbourhoods (see Fig.1).

Let Ω be an open connected bounded set in $\mathbb{R}^3$, locally located on one side of the boundary $\partial\Omega$, a manifold of class C^2, composed of a finite number of connex components, and let $\varphi:\mathbb{R}\rightarrow[0,1[$ be the function which associates to any real number its fractional part; we say that a function $f:\mathbb{R}^3\rightarrow\mathbb{R}$ is Y-periodic if $f=f\circ\varphi$. Also, for any $\varepsilon\in(0,1)$ we define

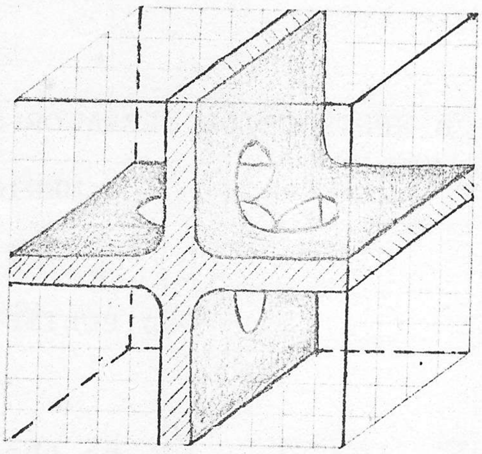
$$\varphi^{\varepsilon}(x)=((1/\varepsilon)x),$$

$$\Omega^{\varepsilon}_f=\{x\in\Omega \mid \varphi^{\varepsilon}(x)\in Y_f\},$$

$$\Omega^{\varepsilon}_S=\{x\in\Omega \mid \varphi^{\varepsilon}(x)\in Y_S\},$$

$$\Gamma^{\varepsilon}=\overline{\Omega}^{\varepsilon}_S\cap\overline{\Omega}^{\varepsilon}_f, \qquad (\partial\Omega)^{\varepsilon}_f=\overline{\Omega}^{\varepsilon}_f\cap\partial\Omega.$$

Fig. 1



Remark 1. If a fluid flow is considered through this εY--periodic media, that is in Ω_f^ε, and if the homogenization process is studied, one has to remove the fact that the velocity and the pressure are defined only in Ω_f^ε. While the velocity can be naturally continued with zero in Ω_S^ε, the continuation of the pressure to Ω is not so obvious, and it needs a special restriction operator from $\underset{\sim}{H}_o^1(\Omega)$ to $\underset{\sim}{H}_o^1(\Omega_f^\varepsilon)$. A construction of such an operator can be found in Sanchez-Palencia [1] Appendix, but from the physical point of view it is valid only for bidimensional flows, Y_S being strictly contained into Y; also any problem at the border there was avoided, defining Ω_f^ε as the domain obtained from Ω by picking out only the domains εY_S which do not intersect $\partial\Omega$.

Here, the above mentioned Tartar's construction is extended to the present case which is obviously tridimensional, with connex phases and biphasic boundary. □

<u>Lemma 1</u>. If $\underset{\sim}{u} \in \underset{\sim}{H}^1(Y)$ then there exists $f \in \mathscr{L}(\underset{\sim}{H}^1(Y), \underset{\sim}{H}^{1/2}(\partial Y_f))$ such that

(a) $f(\underset{\sim}{u}) = \underset{\sim}{0}$ on Γ

(b) If $\underset{\sim}{u} = \underset{\sim}{0}$ in Y_S, then $f(\underset{\sim}{u}) = \underset{\sim}{u}$.

(c) For every $i \in \{1,2,\dots 6\}$ it holds

$$\int_{\Sigma_f^i} f(\underset{\sim}{u}) \cdot \underset{\sim}{\nu}\, d\sigma = \int_{\Sigma^i} \underset{\sim}{u} \cdot \underset{\sim}{\nu}\, d\sigma \tag{1}$$

where $\underset{\sim}{\nu}$ denotes the unit outward normal on Σ^i, and $\Sigma_f^i = \Sigma^i \cap \overline{Y}_f$.

<u>Proof</u>. First, for every $\underset{\sim}{v} \in \underset{\sim}{C}(\overline{Y})$ we will define an element $f(\underset{\sim}{v}) \in \underset{\sim}{C}(\partial Y_f)$ satisfying the properties (a), (b) and (c).

Let $\Gamma_1 = \Gamma \cap \Sigma^1$, where Σ^1 is the face $x_1 = 0$, and let Γ_1^α be a convex component of Γ^1. Let γ_1^α be the curve obtained by a uniform "dilatation" of Γ_1^α on the normal, of thickness $t_\alpha > 0$ sufficiently small, such that, if we denote with Σ_M^α the part of Σ_f^1 contained between γ_1^α and Γ_1^α and with $\Sigma_M = \bigcup_\alpha \Sigma_M^\alpha$, every connex component of Σ_M is, for some α, Σ_M^α (see Fig.2). Let

$$\begin{cases} x_2 = g(s) \\ x_3 = h(s) \end{cases}$$

be the parametric equations of the regular curve Γ_1^α, that is $g, h \in C^2$ and $(g')^2 + (h')^2 = 1$, where s is the arc length on Γ_1^α. Denoting with l_α the length of Γ_1^α, we assume that t_α is also such small

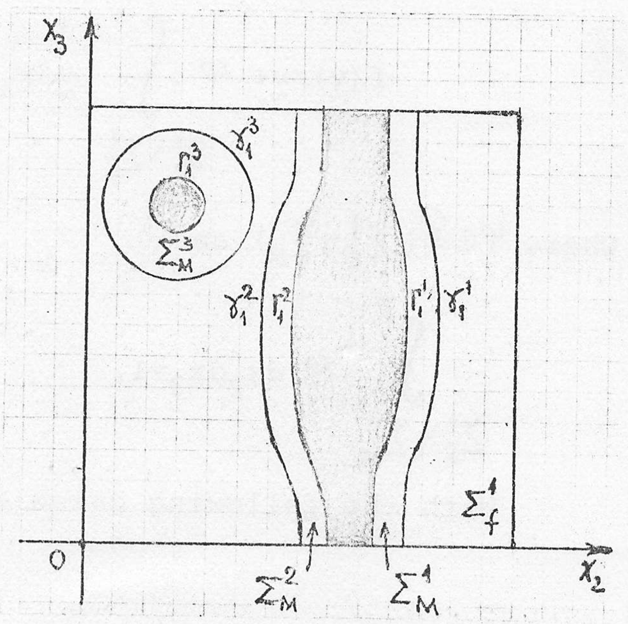


Fig. 2

as the transformation of $(x_2,x_3)\in\Sigma_M^\alpha$ into $(s,n)\in(0,l_\alpha)\times(0,t_\alpha)$, given by

$$x_2=g(s)-nh'(s)$$
$$x_3=h(s)+ng'(s)$$

has a strictly positive Jacobian determinant:

$$\frac{\partial(x_2,x_3)}{\partial(s,n)} = 1+n(g''h'-g'h'')>0$$

Here, we have to remark that if Γ_1^α attains an edge of $\bar{Y}$, then the normal of Γ_1^α in that point is the edge itself, because of the assumption that repeating Y by periodicity the reunion of all the fluid parts is of class C^2.

Then we put in Σ_M^α

$$f(\underset{\sim}{v})(s,n)=\underset{\sim}{v}(s,n)-P(n/t_\alpha)\underset{\sim}{v}(s,0)$$

where $P(x)=-10x^3+18^2-9x+1$.

In $\Sigma_f^1\setminus\Sigma_M$ we put

$$f(\underset{\sim}{v})=\underset{\sim}{v}+\varphi\int_{\Sigma^1\setminus\Sigma_f^1}\underset{\sim}{v}dx_2dx_3$$

where $\varphi\in\mathcal{D}(\Sigma_f^1\setminus\Sigma_M)$ and

$$\int_{\Sigma_f^1\setminus\Sigma_M}\varphi\, dx_2dx_3=1.$$

With the following calculation

$$\int_{\Sigma_f^1} f(\underset{\sim}{v})dx_2dx_3 = \int_{\Sigma_f^1\setminus\Sigma_M} (\underset{\sim}{v}+\varphi \int_{\Sigma^1\setminus\Sigma_f^1} \underset{\sim}{v}dx_2dx_3)dx_2dx_3 + \sum_\alpha \int_{\Sigma_M^\alpha} f(\underset{\sim}{v})dx_2dx_3 =$$

$$= \int_{\Sigma_f^1\setminus\Sigma_M} \underset{\sim}{v}dx_2dx_3 + \left(\int_{\Sigma_f^1\setminus\Sigma_M} \varphi\, dx_2dx_3\right)\left(\int_{\Sigma^1\setminus\Sigma_f^1} \underset{\sim}{v}dx_2dx_3\right) + \sum_\alpha \int_0^{t_\alpha}\int_0^{l_\alpha} (v(s,n)-P(n/t_\alpha)\underset{\sim}{v}(s,0))\cdot$$

$$\cdot\frac{\partial(x_2,x_3)}{\partial(s,n)}dsdn = \int_{\Sigma^1\setminus\Sigma_M} \underset{\sim}{v}dx_2dx_3 + \sum_\alpha \int_0^{t_\alpha}\int_0^{l_\alpha} \underset{\sim}{v}(s,n)\frac{\partial(x_2,x_3)}{\partial(s,n)}dsdn -$$

$$-\sum_\alpha\left(\int_0^{l_\alpha}\underset{\sim}{v}(s,0)ds\right)\left(\int_0^{t_\alpha}P(n/t_\alpha)dn\right) - \sum_\alpha\left(\int_0^{l_\alpha}\underset{\sim}{v}(s,0)(g''h'-g'h'')ds\right)\left(\int_0^{t_\alpha}nP(n/t_\alpha)dn\right)$$

and because

$$\int_0^{t_\alpha}P(n/t_\alpha)dn = \int_0^{t_\alpha}nP(n/t_\alpha)dn = 0$$

the property (c) is checked for i=1. Using the same method we define $f(\underset{\sim}{v})$ on the other side faces. As the property (a) require $f(\underset{\sim}{v})=\underset{\sim}{0}$ on Γ, we have the whole construction of $f(\underset{\sim}{v})$ on ∂Y_f and it is easy to verify that $f(\underset{\sim}{v})\in C(\partial Y_f)$ (we remark here that $P(0)=1$ and $P(1)=0$). We see also that if $\underset{\sim}{v}=\underset{\sim}{0}$ in Y_S, then $\underset{\sim}{v}=\underset{\sim}{0}$ on $(\Sigma^1\setminus\Sigma_f^1)$ and hence $f(\underset{\sim}{v})=\underset{\sim}{v}$ from the definition.

Finally, let $\underset{\sim}{u}\in \underset{\sim}{H}^1(Y)$; because

$$\left|f(\underset{\sim}{v})\right|_{\underset{\sim}{H}^{1/2}(\partial Y_f)} \leqslant C_1\left|\underset{\sim}{v}\right|_{\underset{\sim}{H}^1(\partial Y)} \leqslant C_2\left|\underset{\sim}{v}\right|_{\underset{\sim}{H}^1(Y)}$$

for any $\underset{\sim}{v}\in \underset{\sim}{C}(\bar{Y})$, and as $\underset{\sim}{C}(\bar{Y})$ is dense in $\underset{\sim}{H}^1(Y)$ we define $f(\underset{\sim}{u})$ as the limit of $f(\underset{\sim}{v}_k)$ for $\underset{\sim}{v}_k\in \underset{\sim}{C}(\bar{Y})$ and $\underset{\sim}{v}_k \longrightarrow \underset{\sim}{u}$ strongly in $\underset{\sim}{H}^1(Y)$.

Obviously $f(\underset{\sim}{u})$ is well-defined and everything holds at the limit also. □

Lemma 2. If $\underset{\sim}{u}\in \underset{\sim}{H}^1(Y)$, then there exist a unique $\underset{\sim}{v}\in \underset{\sim}{H}^1(Y_f)$ and a unique $q\in L^2(Y_f)/\mathbb{R}$, solution of the problem

(2) $-\Delta \underset{\sim}{v}+\nabla q=-\Delta \underset{\sim}{u}$ in Y_f

(3) $\operatorname{div} \underset{\sim}{v}=\operatorname{div} \underset{\sim}{u} + 1/(|Y_f|)\int_{Y_S} \operatorname{div} \underset{\sim}{u} dy$ in Y_f

(4) $\underset{\sim}{v}=f(\underset{\sim}{u})$ on ∂Y_f (f given by Lemma 1)

Moreover, there exists a constant $C(Y_f)$ such that

(5) $$\left|\underset{\sim}{v}\right|_{\underset{\sim}{H}^1(Y_f)} \leqslant C(Y_f)\left|\underset{\sim}{u}\right|_{\underset{\sim}{H}^1(Y)}$$

($|Y_f|$ is the measure of Y_f).

Proof. Using the result proved in Cattabriga [2] we have only to check the compatibility condition

(6) $$\int_{\partial Y_f} \underset{\sim}{v}\cdot\underset{\sim}{\nu}\, d\sigma = \int_{Y_f} \operatorname{div} \underset{\sim}{v}\, dy$$

which follows from

$$\int_{\partial Y_f} f(\underset{\sim}{u})\cdot\underset{\sim}{\nu}\, d\sigma = \sum_{i=1}^{6}\int_{\Sigma_f^i} f(\underset{\sim}{u})\cdot\underset{\sim}{\nu}\, d\sigma = \sum_{i=1}^{6}\int_{\Sigma^i} \underset{\sim}{u}\cdot\underset{\sim}{\nu}\, d\sigma = \int_{\partial Y} \underset{\sim}{u}\cdot\underset{\sim}{\nu}\, d\sigma =$$

$$=\int_Y \operatorname{div} \underset{\sim}{u}\, dy=\int_{Y_f} \left(\operatorname{div} \underset{\sim}{u}+1/(|Y_f|)\int_{Y_S} \operatorname{div} \underset{\sim}{u}\, dy_s\right)dy_f$$

where we have used property (c) of f. □

With these two lemmas we can prove our main result.

Theorem. For any $\varepsilon > 0$ sufficiently small there exists a restriction operator $R_\varepsilon \in \mathcal{L}(\underset{\sim}{H}^1_o(\Omega), \underset{\sim}{H}^1_o(\Omega^\varepsilon_f))$ such that

(A) If $\underset{\sim}{u} \in \underset{\sim}{H}^1_o(\Omega^\varepsilon_f)$ is continued with zero in $\Omega \setminus \Omega^\varepsilon_f$, then $R_\varepsilon \underset{\sim}{u} = \underset{\sim}{u}$.

(B) If $\underset{\sim}{u} \in \underset{\sim}{H}^1_o(\Omega)$ and div $\underset{\sim}{u}=0$, then $\mathrm{div}(R_\varepsilon \underset{\sim}{u})=0$.

(C) For any $\underset{\sim}{u} \in \underset{\sim}{H}^1_o(\Omega)$ the following estimations hold:

$$\left| R_\varepsilon \underset{\sim}{u} \right|_{\underset{\sim}{L}^2(\Omega^\varepsilon_f)} \leqslant C(Y_f)\left(\left| \underset{\sim}{u} \right|_{\underset{\sim}{L}^2(\Omega)} + \varepsilon \left| \nabla \underset{\sim}{u} \right|_{\underset{\sim}{L}^2(\Omega)} \right) \tag{7}$$

$$\left| \nabla R_\varepsilon \underset{\sim}{u} \right|_{\underset{\sim}{L}^2(\Omega^\varepsilon_f)} \leqslant C(Y_f)\left((1/\varepsilon) \left| \underset{\sim}{u} \right|_{\underset{\sim}{L}^2(\Omega)} + \left| \nabla u \right|_{\underset{\sim}{L}^2(\Omega)} \right) \tag{8}$$

Proof. Noticing that every (εY)-cube is of the form $\varepsilon Y^{\underset{\sim}{n}} = \prod_{i=1}^{3} [\varepsilon n_i, \varepsilon n_i + \varepsilon[$ with $\underset{\sim}{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3$, then the (εY)-cubes which intersect Ω can be indexed following

$$\underset{\sim}{\mathbb{Z}}_\varepsilon = \left\{ \underset{\sim}{n} \in \mathbb{Z}^3 \;\middle|\; \varepsilon Y^{\underset{\sim}{n}} \cap \Omega \neq \emptyset \right\}.$$

For the (εY)-cubes which are cutted by $\partial\Omega$, we shall denote by $\widetilde{(\cdot)}$ all the parts of them which are still contained in Ω.

Reminding that $\partial\Omega$ is of class C^2 and that all the corners of Y are surrounded by fluid neighbourhoods, we choose $\varepsilon > 0$ as small as if $\partial\Omega$ has an intersection with an $(\varepsilon \widetilde{\Sigma}^i)$-side face such that $\left| \varepsilon \widetilde{\Sigma}^i_f \right| = 0$ then the adjacent $\varepsilon\widetilde{Y}$-cube, outward to Ω, has $\left| \varepsilon \widetilde{Y}_f \right| = 0$. That is always possible, because the tangent plane to $(\varepsilon\Gamma)$ on $(\varepsilon\Gamma) \cap (\varepsilon\Sigma^i)$ is orthogonal to $(\varepsilon\Sigma^i)$. (see Fig.3).

Let $\underset{\sim}{u} \in \underset{\sim}{H}_0^1(\Omega)$; in Ω_S^ε we define

(9) $R_\varepsilon \underset{\sim}{u} = \underset{\sim}{0}$

In Ω_f^ε we shall construct R_ε in every $\varepsilon Y_f^{\underset{\sim}{n}}$, $\underset{\sim}{n} \in \underset{\sim}{\mathbb{Z}}_\varepsilon$. That is why we have to consider two situations.

First, if $\varepsilon Y^{\underset{\sim}{n}} \subset \Omega$, then we define

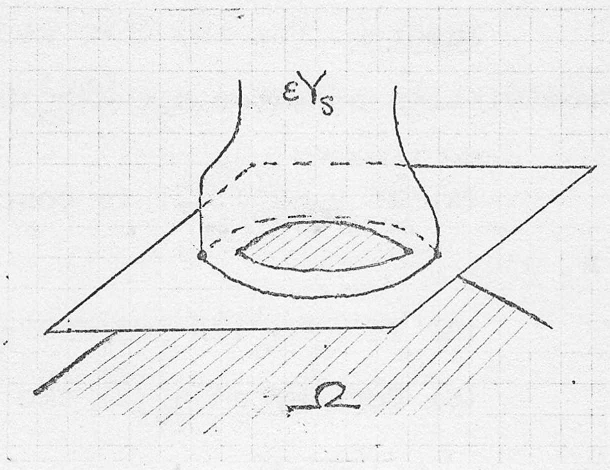


Fig. 3

(10) $R_\varepsilon \underset{\sim}{u} = \underset{\sim}{v} \circ \varphi^\varepsilon$ in $\varepsilon Y_f^{\underset{\sim}{n}}$

where $\underset{\sim}{v}$ is given by Lemma 2 for $\underset{\sim}{u}(\varepsilon \underset{\sim}{n} + \varepsilon(.)) \in \underset{\sim}{H}^1(Y)$.

Second, if $\varepsilon Y^{\underset{\sim}{n}}$ is cutted by $\partial\Omega$, then we can repeat the construction of f on $\tilde{\Sigma}_f^i$ as in Lemma 1, but only if $|\tilde{\Sigma}_f^i| \neq 0$. Let $\underset{\sim}{u}^* \in \underset{\sim}{H}^1(Y)$ be the continuation of $\underset{\sim}{u}(\varepsilon \underset{\sim}{n} + \varepsilon(.))$ with zero in $(Y \setminus \tilde{Y})$; we remark here that it make sense since $\underset{\sim}{u} \in \underset{\sim}{H}_0^1(\Omega)$. Thus we can consider a system analogous to (2)-(4):

(11) $-\Delta \underset{\sim}{v} + \nabla q = -\Delta \underset{\sim}{u}^*$ in Y_f

(12) $\operatorname{div} \underset{\sim}{v} = \operatorname{div} \underset{\sim}{u}^* + (1/|Y_f|) \int_{Y_S} \operatorname{div} \underset{\sim}{u}^* dy - (1/|Y_f|) \sum_{i(*)} \int_{\tilde{\Sigma}_S^i} \underset{\sim}{u}^* \cdot \underset{\sim}{\gamma}\, d\sigma$ in Y_f

(13) $\begin{cases} \underset{\sim}{v} = f(\underset{\sim}{u}^*) & \text{on } \tilde{\Sigma}_f \\ \underset{\sim}{v} = \underset{\sim}{0} & \text{on } \partial Y_f \setminus \tilde{\Sigma}_f \end{cases}$

where $\sum_{i(*)}$ denotes the sum extended over all those indices i for which $|\tilde{\Sigma}_f^i| = 0$. Again we have to check the compatibility relation (6):

$$\int_{Y_f} \operatorname{div} \underset{\sim}{v}\, dy = \int_{\tilde{Y}} \operatorname{div} \underset{\sim}{u}^* - \sum_{i(*)} \int_{\tilde{\Sigma}_S^i} \underset{\sim}{u}^* \cdot \underset{\sim}{\nu}\, d\sigma =$$

$$= \sum_{i=1}^{6} \int_{\tilde{\Sigma}^i} \underset{\sim}{u}^* \cdot \underset{\sim}{\nu}\, d\sigma - \sum_{i(*)} \int_{\tilde{\Sigma}_S^i} \underset{\sim}{u}^* \cdot \underset{\sim}{\nu}\, d\sigma = \sum_{i(**)} \int_{\tilde{\Sigma}^i} \underset{\sim}{u}^* \cdot \underset{\sim}{\nu}\, d\sigma =$$

$$= \sum_{i(**)} \int_{\tilde{\Sigma}_f^i} f(\underset{\sim}{u}^*) \cdot \underset{\sim}{\nu}\, d\sigma = \int_{\tilde{\Sigma}^f} f(\underset{\sim}{u}^*) \cdot \underset{\sim}{\nu}\, d\sigma$$

where $\sum_{i(**)}$ denotes the sum extended over all those indices i for which $|\tilde{\Sigma}_f^i| \neq 0$. Hence, there exists a unique solution of (11)-(13), $\underset{\sim}{v} \in \underset{\sim}{H}^1(Y_f)$ and $q \in L^2(Y_f)/\mathbb{R}$. If div $\underset{\sim}{u}=0$ then, reminding the property with which ε was choosen, for a face with $|\varepsilon\tilde{\Sigma}_f^i| = 0$, we have in the outward adjacent $\varepsilon\tilde{Y}$-cube

$$0 = \int_{\varepsilon\tilde{Y}} \operatorname{div} \underset{\sim}{u}\, dy = \int_{(\partial\Omega)\cap\,\varepsilon\tilde{Y}} \underset{\sim}{u} \cdot \underset{\sim}{\nu}\, d\sigma + \int_{\varepsilon\tilde{\Sigma}_S^i} \underset{\sim}{u} \cdot \underset{\sim}{\nu}\, d\sigma$$

and as $\underset{\sim}{u} \in \underset{\sim}{H}_0^1(\Omega)$ it follows

$$\int_{\tilde{\Sigma}_S^i} \underset{\sim}{u}^* \cdot \underset{\sim}{\nu}\, d\sigma = 0 .$$

Then from (12) we get

$$\mathrm{div}\, \underset{\sim}{v} = -(1/|Y_f|) \sum_{i(*)} \int_{\tilde{\Sigma}_S^i} \underset{\sim}{u}^* \cdot \underset{\sim}{\nu}\, d\sigma = 0 .$$

Moreover an estimation analogous to (5) hold:

$$(14) \qquad |\underset{\sim}{v}|_{\underset{\sim}{H}^1(Y_f)} \leqslant C(Y_f) |\underset{\sim}{u}^*|_{\underset{\sim}{H}^1(Y)} .$$

But $\underset{\sim}{v}=\underset{\sim}{0}$ in $Y_f \setminus \tilde{Y}_f$ and thus (14) becomes

$$(15) \qquad |\underset{\sim}{v}|_{\underset{\sim}{H}^1(\tilde{Y}_f)} \leqslant C(Y_f) |\underset{\sim}{u}(\varepsilon \underset{\sim}{n} + \varepsilon (.))|_{\underset{\sim}{H}^1(\tilde{Y})}$$

The definition of the restriction operator in $\varepsilon Y_f^{\underset{\sim}{n}}$ follows now naturally:

$$(16) \qquad R_\varepsilon \underset{\sim}{u} = \underset{\sim}{v} \circ \varphi^\varepsilon$$

where $\underset{\sim}{v}$ is given by (11)-(13).

It is clear that the properties (A) and (B) result straightly from the definition of R_ε . The property (C) can be proved by evaluating the integral

$$(17) \qquad I_\varepsilon = \int_{\Omega_f^\varepsilon} ((R_\varepsilon \underset{\sim}{u})^2 + \varepsilon^2 (\nabla R_\varepsilon \underset{\sim}{u})^2) dx$$

We decompose I_ε on every fluid part of the $\varepsilon Y^{\underset{\sim}{n}}$-cubes, making the following change of variables

$$(18) \qquad \underset{\sim}{v}(y) = R_\varepsilon\, \underset{\sim}{u}(\varepsilon \underset{\sim}{n} + \varepsilon y)$$

Then using in $\varepsilon Y^{\underset{\sim}{n}}$ the corresponding estimation, (5) or (15), and recomposing the right hand side of the global estimation we finally get

$$I_\varepsilon \leqslant c(Y_f)\left(|\underset{\sim}{u}|_{\underset{\sim}{L}^2(\Omega)} + \varepsilon^2 |\nabla \underset{\sim}{u}|_{\underset{\sim}{L}^2(\Omega)}\right) \tag{19}$$

which obviously completes the proof. □

REFERENCES

[1] E.SANCHEZ-PALENCIA, Topics in Non-Homogeneous Media and Vibration Theory, Lecture Notes in Physics, 127, Springer--Verlag, Berlin, 1980.

[2]. L.CATTABRIGA, Rend. Mat. Sem. Univ. Padova, 31, 1961, p.308-340.

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