INSTITUTUL
DE
MATEMATICĂ

INSTITUTUL NAȚIONAL
PENTRU CREAȚIE
ȘTIINȚIFICĂ ȘI TEHNICĂ

ISSN 0250 3638

# A UNITARY INVARIANT FOR HYPONORMAL OPERATORS

by

Mircea MARTIN and Mihai PUTINAR

PREPRINT SERIES IN MATHEMATICS
No. 20/1985

BUCUREȘTI

# A UNITARY INVARIANT FOR HYPONORMAL OPERATORS

by

Mircea MARTIN and Mihai PUTINAR *)

March 1985

*) Department of Mathematics, The National Institute for Scientific and Technical Creation, Bdul Pacii 220, 79622 Bucharest, Romania.

# A UNITARY INVARIANT FOR HYPONORMAL OPERATORS

Mircea Martin and Mihai Putinar

In this paper is established a two-dimensional singular integral representation for an arbitrary hyponormal operator. The significant term in that model involves an operator valued distribution supported by the spectrum, which turns out to be a complete unitary invariant for the pure part of the operator. The technical tool in that representation is a family of contractions, indexed over the complex plane, which arises naturally from the hyponormality assumption. Other invariants, as for instance the principal function, are then recuperated from that distribution.

Let $H$ be a complex, separable Hilbert space and let $L(H)$ denote the algebra of all linear bounded operators on $H$. We recall that T in $L(H)$ is said to be a *hyponormal operator* if

$$TT^* \leq T^*T \text{ , or equivalently } \|T^*h\| \leq \|Th\| \text{ ,} h \in H \text{ .}$$

The hyponormal operator T is said to be *completely non-normal* or *pure* if there is no reducing subspace for T, on which its restriction would be a normal operator. The introductory texts [3] and [24] offer to the reader the basic principles and a comprehensive bibliography for the theory of hyponormal operators.

Let $D=[T^*,T]$ denote the self-commutator of a hyponormal operator T in $L(H)$. It is a non-negative semi-definite operator which satisfies the relations

$$D = [(T - \lambda)^*, (T - \lambda)] \leq ( T - \lambda)^*( T - \lambda)$$

for every complex number $\lambda$ in $\mathbb{C}$. Consequently there exists for any z in $\mathbb{C}$ a unique contraction $C(z)$ in $L(H)$ ,so that

$$( T - z )^* C(z) = D^{1/2} \quad \text{and} \quad C^*(z) | \mathrm{Ker}( T - z )^* = 0, \quad \text{(i)}$$

see for instance [9] . That family of contractions was introduced by Radjabalipour [20] in connection with Putnam's global resolvents phenomenon 19 ,and then it appeared persistently in Clancey's work [4,5,6] .The present paper is centered around that object,too.

Since the $L(H)$ -valued function C defined on $\mathbb{C}$ by the conditions (i) is locally integrable with respect to the planar Lebesgue measure, its complex derivative $\partial C = \partial C / \partial z$ in the sense of distributions satisfies on $\mathbb{C}$ the identity

$$( T - z )^* \partial C = 0 .$$

Let us introduce $X = ( \mathrm{Ran} D^{1/2} )^-$ ,the closure of the range of the operator $D^{1/2}$.Since $X$ is the actual domain of all operators C(z), it follows that every vector x in $X$ provides the globally defined eigendistribution $\partial C \cdot x$ of T . These $H$ -valued distributions span the pure part of the space $H$ ,with respect to the orthogonal decomposition of the operator T in pure and normal part,as it follows from the next

THEOREM A. *Let* T *be a hyponormal operator on the Hilbert space* $H$. *The linear span of the vectors* C(z)x ,*with* z *in* $\mathbb{C}$ *and* x *in* $X$ *is dense in* $H$ *if and only if the operator* T *is pure.*

That result was proved in the case rankD = 1 by Clancey [4].

Let us assume that T is a pure hyponormal operator.Then, because of Theorem A,the general scheme of producing functional representations for the hyponormal operator T described in [18] applies to the above generating family of eigendistributions.Thus the separate completion of the space $\mathcal{D}(\mathbb{C}) \hat{\otimes} H$ in the norm associated to the distribution kernel $\bar{\partial}_w \partial_z C^*(w)C(z)$ is unitary equivalent with the space $H$. In that identification the operator

T becomes

$$T\varphi = z\varphi + \pi^{-1}(D^{1/2}\,\partial C \cdot \varphi) * \frac{1}{\bar{\zeta}} \quad , \varphi \in \mathcal{D}(\mathbb{C}) \hat{\otimes} H \tag{ii}$$

while $T^*$ is represented by the multiplication operator with $\bar{z}$. We have denoted as usually by $\mathcal{D}$ the space of smooth, compactly supported functions, and by " $*$ " the convolution product of distributions; see the preliminaries below for more details.

Let us notice the close analogy between formula (ii) and Xia's one-dimensional singular integral model [22], where a similar expression appears for the real (or imaginary) part of the operator T, with the Hilbert transform instead of the Cauchy transform above. A dual representation to (ii) was established recently in the case rank$D$ = 1, in the papers [23], [16], see also [18].

The positive definite kernels like $C^*(w)C(z)$, which are related to a hyponormal operator by a singular integral representation as above, are characterized in the sequel by a first order linear partial differential equation, together with a boundedness condition. As a consequence of that computations we prove the following

THEOREM B. *The distribution* $\Gamma = -D^{1/2}\,\partial C$ *in* $\mathcal{D}'(\mathbb{C}) \hat{\otimes} L(X)$ *is a complete unitary invariant for a pure hyponormal operator* T *in* $L(H)$.

In this way one gets that the unitary equivalence class of a pure hyponormal operator T is decided on the space $X = (\operatorname{Ran} D^{1/2})^-$, a fact already known by the determining function method of Carey and Pincus [2]. However the relationship between their mosaic and the distribution $\Gamma = -D^{1/2}\partial C$ remains to be discussed elsewhere.

A recent result of Clancey [4] states, under the assumption that the operator $D^{1/2}$ is trace class, that the principal function of the operator T coincides, in our notations, with $-\text{Trace}(D^{1/2}\partial C)$. The principal function represents, up to now, the finest and well understood unitary invariant for that class of operators, see [11, 12], [15].

We prove that on the essential resolvent set of a pure hyponormal operator T, the complete unitary invariant $\Gamma$ coincides with a smooth, finite dimensional projection valued function, not necessarely self-adjoint. Moreover, the identity

$$\text{rank}\,\Gamma(z) = -\,\text{ind}(T - z), \quad z \in \rho_{ess}(T)$$

holds true. Thus, Clancey´s result mentioned above fits well with this picture. This behaviour of the invariant $\Gamma$ on the essential resolvent set is also close related to Cowen and Douglas theory.

All the statements below have elementary, and, as a matter of fact, independent of other references proofs.

The content is the following. In Section 1 we discuss a few facts concerning the function and distribution spaces which are used in the sequel.

In Section 2 one presents the basic properties of the contractive operator function associated to a hyponormal operator, including the proof of Theorem A.

Section 3 is devoted to introduce the operator valued distribution $\Gamma$ and to prove Theorem B. In particular we emphasize the behaviour of the restriction of $\Gamma$ on the essential resolvent set of the operator.

Section 4 contains the construction of a two-dimensional functional model for any pure hyponormal operator, by using the distribution $\Gamma$.

Section 5 concludes with a discussion concerning positive definite kernels and hyponormality.

## 1. PRELIMINARIES

The purpose of this section is to recall the notations and a few facts concerning the function and distribution spaces which are used in the paper. A complete reference for that subject is Schwartz´ paper [21] .

1.1. Let $H$ be a separable, complex Hilbert space and let $L(H)$ be the algebra of linear bounded operators on $H$ .
The trace-class ideal $C_1(H)$ of $L(H)$ coicides with the predual of the Banach space $L(H)$ , via the bilinear duality pairing

$$(T,S) \longmapsto \text{Trace}(TS) , \quad T \in L(H) , S \in C_1(H).$$

We denote as usually by z and $\bar{z}$ the complex coordinates on the complex plane $\mathbb{C}$ , and by $\partial = \partial/\partial z$ , $\bar{\partial} = \partial/\partial\bar{z}$ the corresponding vector fields.

Let $\Omega$ be an open subset of $\mathbb{C}$ and let $E(\Omega,H) = E(\Omega)\hat{\otimes} H$ denote the Fréchet space of smooth, $H$-valued function on $\Omega$ . The $LF$ space of smooth, compactly supported functions on $\Omega$ is denoted as usually by $\mathcal{D}(\Omega, H)$ or $\mathcal{D}(\Omega)\hat{\otimes} H$.
The topological dual of $\mathcal{D}(\Omega,H)$ is the space $\mathcal{D}'(\Omega,H)$ of $H$-valued distributions on $\Omega$ . Since the space of scalar distributions

$\mathcal{D}'(\Omega)$ is nuclear, the topological isomorphism $\mathcal{D}'(\Omega, H) = \mathcal{D}'(\Omega) \hat{\otimes} H$ holds for every complete topological tensor product. Let

$$\langle \cdot , \cdot \rangle ; \mathcal{D}'(\Omega, H) \times \mathcal{D}(\Omega, H) \longrightarrow \mathbb{C}$$

denote the unique sesquilinear continuous map, which acts on simple tensor products by the formula

$$\langle u \otimes h, \varphi \otimes k \rangle = u(\bar{\varphi}) \langle h,k \rangle \text{ ,where } u \in \mathcal{D}'(\Omega), \varphi \in \mathcal{D}(\Omega), h,k \in H.$$

Let us notice that, in view of the natural embedding $\mathcal{D}(\Omega) \subsetneq \mathcal{D}'(\Omega)$, the equality

$$\langle \varphi, \varphi \rangle = \int_{\Omega} \| \varphi(z) \|^2 \, d\mu(z)$$

holds true for every $\varphi$ in $\mathcal{D}(\Omega, H)$, where $\mu$ stands for the Lebesque measure on $\mathbb{C}$. The completion of the space $\mathcal{D}(\Omega, H)$ with respect to the norm $\| \varphi \|^2_{2,\Omega} = \langle \varphi, \varphi \rangle$ is the Hilbert space $L^2(\Omega, H)$.

1.2. Similarely one denotes by $L^{\infty}(\Omega) \hat{\otimes}_{\varepsilon} L(H)$ the Banach space of $L(H)$-valued, measurable and essentially bounded functions on $\Omega$. We refer the reader to the monograph [10] for the integration theory of vector valued functions.
We have denoted by " $\hat{\otimes}_{\varepsilon}$ " the complete injective tensor product. Similarely " $\hat{\otimes}_{\pi}$ " stands for the complete projective tensor product, see [21].

For the convenience of the reader we sometimes denote bv $L(H)_{\sigma}$ or $H_{\sigma}$ the respective spaces endowed with the weak-star topology ( with respect to the predual space, $C_1(H)$, respectively $H$ ).

The Banach space $L^{\infty}(\Omega) \hat{\otimes}_{\varepsilon} L(H)$ is identified with the dual of $L^1(\Omega) \hat{\otimes}_{\pi} C_1(H)$, so that $L^{\infty}(\Omega) \hat{\otimes}_{\varepsilon} L(H)$, with the poinwise multiplication, is a von Neumann algebra.

1.3. The operator valued distribution space is denoted by $\mathcal{D}'(\Omega)\hat{\otimes}L(H)_\sigma$. It is the topological dual of the $LF$ space $\mathcal{D}(\Omega)\hat{\otimes}C_1(H)$, or equivalently, the set of linear continuous operators from $\mathcal{D}(\Omega)$ into $L(H)$.
We point out the natural embedding of $L^1_{loc}(\Omega, L(H))$ into $\mathcal{D}'(\Omega)\hat{\otimes}L(H)$.

We simply denote by $u\cdot\varphi = m(u,\varphi)$ the unique continuous bilinear map

$$m: (\mathcal{D}'(\Omega)\hat{\otimes}L(H)_\sigma) \times (E(\Omega)\hat{\otimes}H) \longrightarrow \mathcal{D}'(\Omega)\hat{\otimes}H_\sigma$$

which extends the application

$$m(u\otimes T, \varphi\otimes h) = (\varphi u)\otimes(Th)$$

where $u\in\mathcal{D}'(\Omega)$, $\varphi\in E(\Omega)$, $T\in L(H)$ and $h\in H$.
Let us point out that the evaluations $u\cdot\varphi$, with an arbitrary $\varphi$ as above, determine completely the distribution $u$.

1.4. The $\bar\partial$ derivative in the sense of distributions gives rise to a continuous operator

$$\bar\partial : L^1_{loc}(\mathbb{C})\hat{\otimes}_\pi E \longrightarrow \mathcal{D}'(\mathbb{C})\hat{\otimes}E,$$

where $E$ stands for a Banach space ( $E = H$ or $E = L(H)$ in this paper ).
The fundamental solution of the differential operator $\bar\partial$ is the function $-\pi^{-1}/z$ in $L^1_{loc}(\mathbb{C})$, that is, $\bar\partial(1/z) = -\pi\delta$.
Therefore the convolution identity

$$\bar\partial F * (-\pi^{-1}/z) = F \qquad (1.1)$$

holds true, whenever

$$F\in L^1_{loc}(\mathbb{C})\hat{\otimes}_\pi E,\quad \bar\partial F\in\mathcal{D}'(\mathbb{C})\hat{\otimes}E,\quad \operatorname{supp}(\bar\partial F)\ \text{compact},\quad \lim_{|z|\to\infty}\|F(z)\| = 0$$

Notice that $\operatorname{supp}(\bar\partial F)$ is compact, so that $F$ is a smooth function in the neighbourhood of infinity. We refer to (1.1) as to the (generalized) Cauchy-Pompeiu formula and to the convolution operator with $-\pi^{-1}/z$ as to the Cauchy transform.

The Cauchy transform $u = v * (-\pi^{-1}/z)$ of a compactly supported E-valued distribution v, is the unique solution in $\mathcal{D}'(\mathbb{C})\hat{\otimes}E$ of the equation $\bar{\partial}u = v$, vanishing at infinity.

Similar statements for the operator $\partial$ are obtained by complex conjugation.

We mention also the relation

$$(\partial F)\varphi = \partial(F\varphi) - F\partial\varphi \tag{1.2}$$

which will be used in the sequel for F in $L^{\infty}(\mathbb{C}) \hat{\otimes} L(H)$ and $\varphi$ in $\mathcal{D}(\mathbb{C}) \hat{\otimes} H$. If in addition $\mathrm{supp}(\partial F)$ is compact, then

$$(\partial F)\varphi * (-\pi^{-1}/z) = F\varphi + (F\partial\varphi)*(\pi^{-1}/z) \tag{1.3}$$

so that the right hand term contains only pointwise multiplications between operator and vector valued functions.

1.5. The identity (1.2) follows by an approximation by regularization argument.

Let us recall finally the notations concerning the regularization of distributions. Let $\rho$ be a non-negative element of $\mathcal{D}(\mathbb{C})$, so that

$$\int \rho(z)\, d\mu(z) = 1,$$

and let $\rho_\varepsilon$ be the function $\rho_\varepsilon(z) = \varepsilon^{-2}\rho(z/\varepsilon), \varepsilon > 0$.

We denote as usually the regularizations of a distribution u in $\mathcal{D}'(\mathbb{C}) \hat{\otimes} E$ by $u_\varepsilon = u * \rho_\varepsilon$, so that all $u_\varepsilon$ are smooth E-valued functions on $\mathbb{C}$ and $\lim_{\varepsilon\to 0} u_\varepsilon = u$ in the (weak) topology of the space $\mathcal{D}'(\mathbb{C})\hat{\otimes}E$.

## 2. THE CONTRACTIVE OPERATOR FUNCTION

In this section we give the basic properties of the contractive operator function associated to a hyponormal operator and we prove Theorem A .

2.1. Let us begin with a few conventions and notations. Let T denote throughout this section a hyponormal operator on the separable, complex Hilbert space $H$ , and let $D = [T^*, T]$ denote its self-commutator. For any complex number $\lambda$ , $T_\lambda$ and $T^*_\lambda$ stand for the expressions $T - \lambda$ and $(T - \lambda)^*$, respectively.

Since $D^{1/2}D^{1/2} \leq T^*_\lambda T_\lambda$ for every $\lambda$ in $\mathbb{C}$ , there exists by a well-known result of Douglas [9] a unique contraction $C(\lambda)$ in $L(H)$ , so that

$$T^*_\lambda C(\lambda) = D^{1/2} \quad , \quad C(\lambda) \mid \operatorname{Ker} T^*_\lambda = 0 \ .$$

It is straighforward to check that the function C is an element of the von Neumann algebra $L^\infty(\mathbb{C}) \hat{\otimes}_\varepsilon L(H)$ and, moreover it is an antianalytic function on the resolvent set $\rho(T)$ of T. Consequently the distribution $\partial C$ in $\mathcal{D}'(\mathbb{C}) \hat{\otimes} L(H)$ is supported by the spectrum $\sigma(T)$ of the operator T .

In fact we have the formula

$$C(\lambda) = \operatorname*{so-lim}_{\varepsilon \to 0} \ T_\lambda \int_\varepsilon^\infty t^{-1} E_\lambda(\, dt\,)\, D^{1/2} \tag{2.1}$$

where $E_\lambda$ stands for the spectral measure of the self-adjoint operator $T^*_\lambda T_\lambda$ .

2.2. We also denote by $T_z$ and $T^*_z$ the corresponding operator valued functions belonging to the space $E(\mathbb{C}) \hat{\otimes} L(H)$ .

Because of the natural, continuous, bilinear multiplication map

$$E(\mathbb{C}) \hat{\otimes} L(H) \quad \times \quad \mathcal{D}'(\mathbb{C}) \hat{\otimes} L(H)_{\sigma} \longrightarrow \mathcal{D}'(\mathbb{C}) \hat{\otimes} L(H)_{\sigma} \qquad (2.2)$$

the product $T_z^* u$, with $u$ in $\mathcal{D}'(\mathbb{C}) \hat{\otimes} L(H)$, makes a good sense. By taking the derivative in the relation of definition of $C$, we get

$$T_z^* \partial C = 0$$

in $\mathcal{D}'(\mathbb{C}) \hat{\otimes} L(H)$. However, we need in the sequel a slightly stronger result.

LEMMA. *For any functions* $\varphi$ *in* $\mathcal{D}(\mathbb{C}) \hat{\otimes} H$ *and* $f$ *in* $\mathbb{L}^2(\mathbb{C}, H)$, *we have*

$$\lim_{\varepsilon \to 0} \left\langle T_z^* \partial C_{\varepsilon} \varphi \, , \, f \right\rangle_{2,\mathbb{T}} = 0 \qquad (2.3)$$

PROOF. We use the notations introduced at the end of Section 1. Recall that $C_{\varepsilon} = C * \rho_{\varepsilon}$.

Since

$$T_z^* C(w) = D^{1/2} - (\bar{z} - \bar{w}) \, C(w)$$

we get

$$T_z^* \partial C_{\varepsilon}(z) = D^{1/2} \int \partial\rho_{\varepsilon}(z-w) d\mu(w) - \int (\bar{z}-\bar{w}) C(w) \partial\rho_{\varepsilon}(z-w) d\mu(w).$$

But $\int \partial \rho_{\varepsilon}(z-w) d\mu(w) = 0$ and, by setting $\zeta = (z-w)/\varepsilon$, we obtain

$$T_z^* \partial C_{\varepsilon}(z) = - \int C(z - \varepsilon\zeta) \bar{\zeta} \, \partial\rho(\zeta) d\mu(\zeta) .$$

Thus, for a given function $\varphi$ in $\mathcal{D}(\mathbb{C}) \hat{\otimes} H$ the set

$$\{ T_z^* (\partial C_{\varepsilon}) \varphi \; : \varepsilon > 0 \}$$

is bounded in $\mathbb{L}^2(\mathbb{C}, H)$, hence its closure is weakly compact.

As $\lim_{\varepsilon\to 0} T_z^* \partial C_\varepsilon = 0$ in $\mathcal{D}'(\mathbb{C})\hat{\otimes}L(H)$, it follows that

$$\lim_{\varepsilon\to 0} T_z^* (\partial C_\varepsilon)\varphi = 0$$

in the weak topology of the Hilbert space $L^2(\mathbb{C},H)$, and the proof is complete.

2.3. Let $\lambda$ be a complex number. From the inequality $T_\lambda T_\lambda^* \leq T_\lambda^* T_\lambda$ it follows that there is a unique contraction $K(\lambda)$ which satisfies

$$T_\lambda^* = K(\lambda)\, T_\lambda \quad , \quad K(\lambda)\,|\,\mathrm{Ker}\, T_\lambda^* = 0$$

As in the case of the function $C$, the function $K$ is also an element of the von Neumann algebra $L^\infty(\mathbb{C})\hat{\otimes}_\varepsilon L(H)$. A link between the functions $C$ and $K$ appears in the next identity from [6]:

$$I - P(\lambda) - C(\lambda)C^*(\lambda) - K^*(\lambda)K(\lambda) = 0 \ , \ \lambda \in \mathbb{C} \qquad (2.4)$$

where $P(\lambda)$ denotes the orthogonal projection onto $\mathrm{Ker}\, T_\lambda^*$.

To verify (2.4), we note that the self-adjoint operators

$$I - P(\lambda) \ , \ C(\lambda)C^*(\lambda) \ , \ K^*(\lambda)K(\lambda)$$

vanish on $\mathrm{Ker}\, T_\lambda^* = (\mathrm{Ran}\, T_\lambda)^\perp$, hence (2.4) is equivalent with

$$T_\lambda^* \left( I - P(\lambda) - C(\lambda)C^*(\lambda) - K^*(\lambda)K(\lambda) \right) T_\lambda = 0 \qquad (2.5)$$

But

$$T_\lambda^* \left( I - P(\lambda) \right) T_\lambda = T_\lambda^* T_\lambda \ ,$$

$$T_\lambda^* \left( C(\lambda)C^*(\lambda) \right) T_\lambda = D^{1/2}D^{1/2} = T_\lambda^* T_\lambda - T_\lambda T_\lambda^* \ , \text{ and}$$

$$T_\lambda^* \left( K^*(\lambda)K(\lambda) \right) T_\lambda = T_\lambda T_\lambda^* \ ,$$

thus relation (2.5) is obvious.

As a consequence of (2.4) we have

$$T_\lambda = C(\lambda)\ D^{1/2} + K^*(\lambda)\ T^*_\lambda \qquad (2.6)$$

so that, in view of Lemma 2.2 it follows the next

LEMMA . *The identity*

$$T_z(\ \partial C)\cdot\varphi = \lim_{\varepsilon\ 0} CD^{1/2}\ \partial C_\varepsilon \varphi \qquad (2.7)$$

*holds in the weak topology of the space* $\mathcal{D}'(\mathbb{C})\hat{\otimes}H$ , *for any function* $\varphi$ *in* $\mathcal{D}(\mathbb{C})\hat{\otimes}H$ .

2.4. Next we restate and prove our first main result.

THEOREM A. *The hyponormal operator* T *in* $L(H)$ *is pure if and only if*

$$H = \text{span}\left\{ C(z)x : z\in\mathbb{C}\ ,\ x\in \text{Ran}\ D^{1/2} \right\}.$$

PROOF. It suffices to prove that the space

$$M = \text{span}\left\{ C(z)x : z\in\mathbb{C}\ ,\ x\in \text{Ran}\ D^{1/2} \right\}$$

reduces T. Indeed, the operators $C(\lambda)$, $\lambda$ in $\mathbb{C}$ ,take values,by their definition,only in the pure part of the space $H$ .

Let us remark that

$$\lim_{|z|\to\infty} zC(z) = -\ D^{1/2}$$

hence $\text{Ran}\ D^{1/2} \subset M$ . Because $\text{Ran}\ C(\lambda) = \text{Ran}\ C(\lambda)D^{1/2}$ for every $\lambda$ in $\mathbb{C}$ , it follows

$$M = \text{span}\left\{ C(\lambda)h : \lambda\in\mathbb{C}\ ,\ h\in H \right\}$$

The space $M$ is obviously invariant for $T^*$ .

By Lemma 2.3 we have in the space $\mathcal{D}'(\mathbb{C})\hat{\otimes}H$ the formula

$$T\,\partial(Ch) = z\,\partial(Ch) + \lim_{\varepsilon\to 0} CD^{1/2}\partial\,(C_\varepsilon h)$$

for every h in $H$ . Therefore, after a Cauchy transform we get

$$TC(\lambda)h = D^{1/2}h + \lambda C(\lambda)h + \pi^{-1}(Ch*\frac{1}{z})(\lambda) - \pi^{-1}\lim_{\varepsilon\to 0}(CD^{1/2}\partial(C_\varepsilon h)*\frac{1}{z})(\lambda)$$

for any $\lambda$ in $\mathbb{C}$ .

This shows that the space $M$ is invariant for the operator T, and the proof is complete.

2.5. We end this section with a technical result. Its meaning will become clear in Section 5.

PROPOSITION. *Let $\lambda$ be a complex number. For any function $\varphi$ in $\mathcal{D}(\mathbb{C})\hat{\otimes}H$ , the relation*

$$(z-\lambda)\,\partial(C^*(\lambda)C)\cdot\varphi = D^{1/2}\partial C\cdot\varphi - \lim_{\varepsilon\to 0} C^*(\lambda)CD^{1/2}\partial C_\varepsilon\cdot\varphi \qquad (2.8)$$

*holds true in the weak topology of $\mathcal{D}'(\mathbb{C})\hat{\otimes}H$ .*

PROOF. At the operator valued distribution level we have

$$(z-\lambda)\,\partial(C^*(\lambda)C) = C^*(\lambda)(\,T_\lambda - T_z\,)\partial C = D^{1/2}\partial C - C^*(\lambda)\,T_z\partial C$$

The proof is completed by substituting in this equation the relation (2.7).

## 3. COMPLETE UNITARY INVARIANTS

As indicated earlier, our main goal is to construct an appropriate complete unitary invariant for a pure hyponormal operator $T$ by using the associated contractive operator function $C$.

In fact we exhibit in this section two kinds of invariants, both of them defined on the closed subspace $X = (\operatorname{Ran}[T^*,T])^-$. Although these invariants exist in a general context, it is worthwile to remark that, in two relevant situations, they naturally arise by a geometrical approach.

Therefore, it is reasonable to begin our investigation by studying these two nice cases.

3.1. First, let us assume that the self-commutator $D = [T^*,T]$ of the pure hyponormal operator $T$ is of finite rank.

A well-known result of Clancey and Wadhwa [6] asserts that the restriction of the contractive operator function $C$ on the resolvent set $\rho(T)$ of $T$ is a complete unitary invariant for $T$.

By the "rigidity theorem" of Cowen and Douglas [7] it follows that $T$ is determined up to unitary equivalence by the hermitian anti-holomorphic vector bundle defined over $\rho(T)$ by the correspondence

$$\rho(T) \ni \lambda \longrightarrow \operatorname{Ran} C(\lambda). \qquad (3.1)$$

Moreover, the results of Cowen and Douglas prompts one to compute the curvature operator of this vector bundle. By a direct computation one finds, as a more or less expected conclusion, that the $L(X)$ -valued real-analytic function $\Theta$ defined by

$$\Theta : \rho(T) \longrightarrow L(X) \quad , \quad \Theta(\lambda) = C^*(\lambda)C(\lambda) \qquad (3.2)$$

is a complete unitary invariant for $T$.

For example, the unitary orbit of a pure hyponormal operator $T$ having the self-commutator $D$ of one dimensional range, is determined by a scalar function. More precisely, if one denotes $D = h \otimes h$ , with $h$ in $\mathcal{H}$ , then the complex valued real-analytic function $\vartheta$ defined by

$$\vartheta : \rho(T) \longrightarrow \mathbb{C} \quad , \quad \vartheta(\lambda) = \| C(\lambda)h \| \tag{3.3}$$

is a complete unitary invariant of $T$, see [4] ,[6] .

3.2. Our next example is based on a different assumption. Let $\rho_{ess}(T)$ denote the essential resolvent set of $T$. Since $T$ is a pure hyponormal operator, for any $\lambda$ in $\rho_{ess}(T)$ the operator $T^*_\lambda T_\lambda$ is invertible, hence we find the formula (see for instance [1] )

$$C(\lambda) = T_\lambda (T^*_\lambda T_\lambda)^{-1} D^{1/2} . \tag{3.4}$$

Thus the function $C$ is smooth on $\rho_{ess}(T)$.

We suppose further on that

$$\text{span}\left\{ C(\lambda)x \ : \ \lambda \in \rho_{ess}(T),\ x \in X \right\} = \mathcal{H} . \tag{3.5}$$

Let $\Omega$ denote the union of all bounded connected components of $\rho_{ess}(T)$, so that $\Omega$ is a subset of the spectrum $\sigma(T)$ of $T$. A simple Cauchy transform argument gives that (3.5) is equivalent with

$$\text{span}\left\{ \partial C(\omega)x \ : \ \omega \in \Omega \ ,\ x \in X \right\} = \mathcal{H} \tag{3.6}$$

The results proved in Section 2 show that for any $\omega$ in $\Omega$ we have

$$\text{span}\left\{ \partial C(\omega)x \ : \ x \in X \right\} = \text{Ker } T^*_\omega \tag{3.7}$$

and the eigenspaces $\text{Ker } T^*_\omega$ caracterise the unitary orbit of $T$.

More precisely, from the already mentioned work of Cowen and Douglas [7] it follows that the generalized hermitian anti-holomorphic vector bundle $\xi_T$ defined over $\Omega$ by the correspondence

$$\Omega \ni \omega \longrightarrow \operatorname{Ker} T_\omega^* = \operatorname{Ran} \partial C(\omega) \subset H \qquad (3.8)$$

is a complete unitary invariant for T.

Let v be a vector in Ker $T_\omega^*$ ,for a fixed point $\omega$ in $\Omega$ . Since $T_\omega$ is injective,the vector v is uniquely determined by its image $T_\omega v$ .But we have

$$T_\omega = C(\omega)D^{1/2} + K^*(\omega)T_\omega^* \qquad (3.9)$$

and consequently $T_\omega v = C(\omega)D^{1/2}v$ ,so that the vector v is uniquely determined by its image $D^{1/2}v$ in $X$ .

Now,let $\Gamma$ denote the $L(X)$-valued function on $\Omega$ defined by

$$\Gamma : \Omega \longrightarrow L(X) \quad , \Gamma(\omega) = - D^{1/2}\partial C(\omega) . \qquad (3.10)$$

From the last remarks,we obtain that the vector bundle $\xi_T$ defined by (3.8) is equivalent with the hermitian anti-holomorphic vector bundle $\eta_T$ defined over $\Omega$ by the assignement

$$\Omega \ni \omega \longrightarrow \operatorname{Ran} \Gamma(\omega) \subset X . \qquad (3.11)$$

We may resume by asserting that if T satiafies the condition (3.5), then the function $\Gamma$ is a complete unitary invariant for T.

It is relevant to point out two important properties of the invariant $\Gamma$ . First,we note that the function $\Gamma$ is projection-valued. Indeed,from (3.7) and (3.9) we find the equation

$$T_\omega \partial C(\omega) = C(\omega)D^{1/2}\partial C(\omega) .$$

By differetiating it,we obtain

$$- \partial C(\omega) + T_\omega \partial^2 C(\omega) = \partial C(\omega)D^{1/2}\partial C(\omega) + C(\omega)D^{1/2} \partial^2 C(\omega) .$$

From $T_\omega^* \partial C(\omega) = 0$ we also have $T_\omega^* \partial^2 C(\omega) = 0$, and by using again (3.9) we get

$$T_\omega \partial^2 C(\omega) = C(\omega) D^{1/2} \partial^2 C(\omega) .$$

Thus

$$- \partial C(\omega) = \partial C(\omega) D^{1/2} C(\omega)$$

and consequently

$$\Gamma(\omega) = \Gamma(\omega)^2 \quad , \omega \in \Omega . \tag{3.12}$$

Second, in connection with our earlier comments about the vector bundles $\xi_T$ and $\eta_T$, we point out the equality

$$\text{rank } \Gamma(\omega) = \dim \text{Ker } T_\omega^*$$

which implies

$$\text{Trace } \Gamma(\omega) = - \text{ind } ( T - \omega ) . \tag{3.13}$$

Let us remark that both properties (3.12) and (3.13) do not depend on the assumption (3.5). This assumption reflects only the fact that $\Gamma$ is a *complete* unitary invariant for T.

Moreover, let us mention in passing another important consequence of the condition (3.5). For detailes the reader is advised to consult [7] or [13]. As we already noticed, the condition (3.5) implies that the hermitian anti-holomorphic vector bundle $\eta_T$ defined by (3.11) caracterises the unitary orbit of T. The equivalence class of the vector bundle $\eta_T$ can be described in terms of a finite number of derivatives of its canonical curvature operator. But a direct computation shows that the derivatives of the canonical curvature operator can be expressed by using appropriate derivatives of the function $\Gamma$ .

It turns out that two pure hyponormal operators T and T′ which posses the same essential resolvent set and which satisfy the assumption

$$\text{span} \bigcup_{\omega \in \Omega} \text{Ker } T^*_\omega = \text{span} \bigcup_{\omega \in \Omega} \text{Ker } T'^*_\omega = \mathcal{H} \qquad (3.14)'$$

are unitarily equivalent if and only if for any point $\omega$ in $\Omega$ there exists an unitary operator

$$U_\omega : (\text{Ran} [T^*,T])^- \longrightarrow (\text{Ran} [T'^*,T'])^-$$

such that

$$U_\omega \partial^p \bar{\partial}^q \Gamma(\omega) U^*_\omega = \partial^p \bar{\partial}^q \Gamma'(\omega) \qquad (3.15)$$

for all $0 \leq p,q \leq \dim \text{Ker } T^*_\omega = \dim \text{Ker } T'^*_\omega$.

3.3. The remaing part of this section is devoted to establish the existence, in the general case, of the invariants introduced above under aditional assumptions.

We use the same notations as in Section 2.

PROPOSITION. *The germ at the infinity of the* $L(X)$*-valued function* $\Theta = C^* C$ *is a complete unitary invariant for a pure operator* T.

PROOF. Let T and T′ be two pure hyponormal operators on $\mathcal{H}$ and let D, C, and D′, C′ denote the self-commutators and the corresponding contractive operator functions of T and T′.

Let $\mathcal{X} = (\text{Ran } D)^-$ and $\mathcal{X}' = (\text{Ran } D')^-$ and let us suppose that $U : \mathcal{X} \longrightarrow \mathcal{X}'$ is a unitary operator such that

$$U C^*(z) C(z) U^* = C'^*(z) C'(z) \qquad (3.16)$$

for $z$ in a neighborhood of infinity. By identifying the power expansions, from (3.16) it follows easily that

$$UD^{1/2}T^nT^{*m}D^{1/2}U^* = D'^{1/2}T'^nT'^{*m}D'^{1/2} \tag{3.17}$$

for all non-negative integers $n$ and $m$.

On the other hand, we have the following bracket relation

$$[T^{*p},T^q] = \sum_{r=0}^{p-1}\sum_{s=0}^{q-1} T^{*r}T^s[T^*,T]\,T^{q-1-s}T^{*p-1-r} \tag{3.18}$$

By using an obvious induction argument, from (3.17) and (3.18) we obtain

$$UD^{1/2}T^nT^{*p}T^qT^{*m}D^{1/2}U^* = D'^{1/2}T'^nT'^{*p}T'^qT'^{*m}D'^{1/2}. \tag{3.19}$$

Since $T$ and $T'$ are pure, we have

$$\text{span}\,\{T^qT^{*m}D^{1/2}x : q,m \geq 0,\ x\in \mathcal{K}\} = \mathcal{H}$$

and

$$\text{span}\,\{T'^qT'^{m}D'^{1/2}x' : q,m \geq 0,\ x'\in \mathcal{K}'\} = \mathcal{H}.$$

Now we can define the unitary operator $V$ on $\mathcal{H}$ by setting

$$V(T^qT^{*m}D^{1/2}x) = T'^qT'^{*m}D'^{1/2}Ux \tag{3.20}$$

By (3.19) the equations (3.20) really define a unitary operator and, moreover, $VT = T'V$. Thus $T$ and $T'$ are unitarily equivalent.

3.4. Our next task is to prove the second main result of the paper.

THEOREM B. *The $L(X)$-valued compactly supported distribution $\Gamma = -D^{1/2}\partial C$ is a complete unitary invariant for a pure hyponormal operator* $T$.

PROOF. Let $T$ and $T'$ be two pure hyponormal operators on $\mathcal{H}$. We define $D$, $C$, $\mathcal{K}$, and $D'$, $C'$, $\mathcal{K}'$, as in the proof of Proposition 3.4.

Let $\Gamma = -D^{1/2}\partial C$ and $\Gamma' = -D'^{1/2}\partial C'$ and let us suppose that $U : \mathcal{K} \longrightarrow \mathcal{K}'$ is a unitary operator such that

$$U \Gamma U^* = \Gamma' \tag{3.21}$$

By applying the Cauchy transform one finds

$$U D^{1/2} C(z) U^* = D'^{1/2} C'(z) . \tag{3.22}$$

But by Lemma 2.3 we know that

$$T (\partial C)\cdot \varphi = z\partial C\cdot \varphi + \lim_{\varepsilon\to 0} C D^{1/2}\partial C_\varepsilon \cdot \varphi$$

for any function $\varphi$ in $\mathcal{D}(\mathbb{C})\hat{\otimes}\mathcal{K}$. Then it follows that

$$D^{1/2}T^{n+1}(\partial C)\cdot\varphi = z D^{1/2}T^n\partial C\cdot\varphi + \lim_{\varepsilon\to 0} D^{1/2}T^n C D^{1/2}\partial C_\varepsilon\cdot\varphi .$$

Now, by an induction argument, from (3.22) and the last equations we obtain

$$UD^{1/2}T^n C(z)U^* = D'^{1/2}T'^n C'(z) . \tag{3.23}$$

But for any $z$ in a neighborhood of infinity we can write the power expansion

$$C^*(z) = D^{1/2}T_z^{-1} = -D^{1/2}\sum_{n\geq 0} T^n/z^{n+1}$$

and a similar expresion for $C'^*(z)$.

By (3.24) we find

$$U\, C^{*}(z) C(z)\, U^{*} = C'^{*}(z) C'(z) \qquad (3.25)$$

for large values of $|z|$, thus $T$ and $T'$ are unitarily equivalent by Proposition 3.3 .

## 4. A FUNCTIONAL MODEL

Daoxing Xia was the first who remarked , and since then his paper [22] is a basic reference of the theory , that every pure hyponormal operator can be realized on a direct integral over the real line,as a combination between multiplication operators with bounded measureable functions and the Hilbert transform. That approach was intensively and succesfully used in the study of this class of operators , cf. [2], [15], [24].

On the other hand,most of the concrete hyponormal and especially subnormal operators,act naturally on function spaces defined on a domain of the complex plane.However,only the last years brought some light in the two-dimensional representation problem for arbitrary hyponormal operators ( with rank-one self-commutator ),cf. [4], [16], [18], [23] .

In a previous paper [18] it was developed a general framework for producing two-dimensional functional models for arbitrary hyponormal operators,by starting with a generating subspace of eigendistributions of the adjoint operator. Theorem A above provides a canonical and,in a generic sense,minimal subspace of such eigendistributions.This section is devoted to its corresponding two-dimensional singular integral model.

4.1. We recall a few of the notations introduced in the preceding sections. Let T be a hyponormal operator on the Hilbert space $\mathcal{H}$ ,and let $D = [T^*,T]$ stand for its self-commutator. C is the associated contractive operator function,with the initial space $X = (\operatorname{Ran} D)^-$ and $\Gamma = -D^{1/2}\,\partial C$ .

4.2. In virtue of Theorem A above,the closed linear span in the sapce $W_T^{-2}(\mathcal{H})$ of the elements $\partial C\cdot x$,with $x$ in $X$ ,is a generating subspace G of eigendistributions, for the pure part of the operator T, in the terminology and with the notations of [18] . Consequently, the compresion $K_T^G$ of the distribution kernel $K_T$ to this space is

$$K_T^G(w,z) = \bar{\partial}_w\,\partial_z\,C^*(w)C(z).$$

The remaining part of this section is devoted to restate in a more precise form Theorem 4.6 from [18] ,in the case of this generating subspace. For the sake of completeness we avoid the references to that paper.

4.3. Let us assume in addition that T is a pure hyponormal operator.

Let $\mathbb{H}$ be the separate completion of the space $\mathcal{D}(\mathbb{C})\otimes X$ with respect to the seminorm

$$\|\varphi\|^2_{\mathbb{H}} = \left\|\int C(z)\,\partial\varphi(z)\,d\mu(z)\right\|^2_{\mathcal{H}}, \quad \varphi\in\mathcal{D}(\mathbb{C})\otimes X \qquad (4.1)$$

It is plain to check that the operator $U:\mathbb{H}\longrightarrow\mathcal{H}$ defined by

$$U(\varphi) = \int C(z)\,\partial\varphi(z)\,d\mu(z), \quad \varphi\in\mathcal{D}(\mathbb{C})\otimes X \qquad (4.2)$$

is an isometry. Moreover $U$ is onto because of Theorem A and of the assumption on complete non-normality ( compare with the proof of Corollary 5.2 from [18] ). It is also straightforward to verify that the multiplication operators with $z$ and $\bar{z}$ on $\mathcal{D}(\mathbb{C})\otimes X$ induce two well defined bounded linear operators on $\mathcal{H}$, which will be denoted by the same symbols, respectively.

The main result is the following.

4.4. THEOREM . *Let $T$ be a pure hyponormal operator on the Hilbert space $H$, and let $U : \mathcal{H} \longrightarrow H$ be the corresponding unitary operator, defined on the function space $\mathcal{H}$.*

*The following identities*

a) $$U^*T^*U(\varphi) = \bar{z}\varphi$$

b) $$U^*T\,U(\varphi) = z\varphi - \pi^{-1}(\Gamma\cdot\varphi) * 1/\bar{z}$$

c) $$U^*[T^*,T]\,U(\varphi) = \pi^{-1}\int D^{1/2}C(z)\,\partial\varphi(z)\,d\mu(z)$$

*hold true for every function $\varphi$ in $\mathcal{D}(\mathbb{C})\otimes X$.*

PROOF. Let $\varphi$ denote throughout this section an arbitrary element of $\mathcal{D}(\mathbb{C})\otimes X$.

a) We have successively by Stokes theorem

$$T^*U(\varphi) = \int T^*C(z)\,\partial\varphi(z)\,d\mu(z) =$$

$$= \int D^{1/2}\,\partial\varphi(z)\,d\mu(z) + \int C(z)\,\partial(\bar{z}\varphi)(z)\,d\mu(z) =$$

$$= U(\bar{z}\varphi).$$

b) In order to prove the identity b) we recall the relation (2.7), which combined with the observation (1.2) from the preliminaries yields

$$TU(\varphi) = \int TC(z)\,\partial\varphi(z)\,d\mu(z) =$$

$$= \int C(z)\,\partial(z\varphi)(z)\,d\mu(z) - (T_z \partial C\cdot\varphi)(\mathbb{1}) =$$

$$= U(z\varphi) + \lim_{\varepsilon\to 0} \int C(z)\Gamma_\varepsilon(z)\,\varphi(z)\,d\mu(z) =$$

$$= U(z\varphi) - \pi^{-1} U(\lim_{\varepsilon\to 0} (\Gamma_\varepsilon\varphi) * 1/\bar{z}) =$$

$$= U(z\varphi - \pi^{-1}(\Gamma\cdot\varphi) * 1/\bar{z}).$$

Because $U(\varphi) = 0$ whenever $\operatorname{supp}(\varphi)$ is disjoint of $\sigma(T)$, we put by definition

$$U(\psi) = U(\chi\psi), \qquad \psi \in \mathcal{E}(\mathbb{C}) \otimes X \tag{4.3}$$

for any function $\chi$ in $\mathcal{D}(\mathbb{C})$ with $\chi \equiv 1$ in a neighborhood of $\sigma(T)$. This is the meaning of the expression $U(\Gamma_\varepsilon\varphi * 1/\bar{z})$, while the last equality is a convention of notation.

*c)* Before to compute the self-commutator of the operator $U\,TU^*$, let us remark that the identity *b)* can be reformulated in the following non-distributional form:

$$b)' \quad U^*TU(\varphi) = z\varphi - D^{1/2}C\varphi - \pi^{-1}\int \frac{D^{1/2}C(\zeta)\,\partial\varphi(\zeta)}{\bar{\zeta} - \bar{z}}\,d\mu(\zeta)$$

Therefore we get

$$U^*[T^*,T]U(\varphi) = [U^*T^*U, U^*TU](\varphi) =$$

$$= \pi^{-1}\int D^{1/2}C(\zeta)\,\partial\varphi(\zeta)\,d\mu(\zeta)$$

and the proof is complete.

4.5. We remark finally that the range of the operator $U^*[T^*,T]U$ consists of classes of constant functions like $\chi \otimes x$, with $x$ in $X$. Moreover, a short computation which will appear in the next section shows that

$$U(\chi \otimes x) = \pi D^{1/2}x \quad , \qquad x \in X \quad .$$

## 5. POSITIVE DEFINITE KERNELS AND HYPONORMALITY

The functional model of a pure hyponormal operator $T$ in $L(H)$ ,which was described in the preceding section,relies on two operator valued objects, namely on the kernel

$$K(w,z) = C^*(w)C(z) \in L^{\infty}(\mathbb{C}^2) \hat{\otimes}_{\varepsilon} L(X)$$

and on the unitary invariant distribution

$$\Gamma = -D^{1/2}\partial C \in \mathcal{D}'(\mathbb{C}) \hat{\otimes} L(X) .$$

They were related by the equation (2.8) . The purpose of the first part of this section is to characterize those positive definite kernels $K \in L^{\infty}(\mathbb{C}^2) \hat{\otimes} L(X)$ which produce as in Theorem 4.4 at the level of an associated first order Sobolev space with respect to the operator $\partial$ ,pure hyponormal operators.

Roughly speaking,the kernel $I - K$ turns out to be , in a weak sense, a generalized analytic function ( in the terminology of Vekua ) off the diagonal of the space $\mathbb{C}^2$ , subject to a uniform boundedness condition. The equation of definition for K is precisely (2.8),with the distribution $\Gamma$ as a given data. This equation was studied for the first time by Clancey [4],in the scalar case ( of hyponormal operators with rank-one self-commutator ).
We should mention that the kernel associated in a similar way to a normal operator is,off the diagonal of $\mathbb{C}^2$,an analytic function in the

first variable and anti-analytic in the second variable, cf. [18].

In the last part of this section is described the procedure of recuperating the operator T, and implicitely the Hilbert space $\mathcal{H}$, from the kernel $C^*(w)C(z)$, with $|w|, |z| >> 0$. It turns out that the central object in this construction is also a positive definite kernel, which depends on a pair of discrete variables. It is characterized by a linear equation together with a boundedness condition.

5.1. THEOREM. *Let* $X$ *be a separable Hilbert space and let* $K : \mathbb{C}^2 \longrightarrow L(X)$ *be a measureable, positive definite function with the following properties*

*a) There is a constant* $M$, *so that*

$$\|K(w,z)\| = M(1+|wz|)^{-1} \qquad (5.1)$$

*for every pair* $(w,z) \in \mathbb{C}^2$, *and*

*b) There is a compactly supported distribution* $\Gamma$ *in* $\mathcal{D}(\mathbb{C}) \hat{\otimes} L(X)$, *such that for every function* $\varphi$ *in* $\mathcal{D}(\mathbb{C}) \hat{\otimes} X$, *the relation*

$$(w-z)[\partial_z K(w,z)]\varphi(z) = \Gamma(z)\varphi(z) - \lim_{\varepsilon \to 0} K(w,z)\Gamma_\varepsilon(z)\varphi(z) \qquad (5.2)$$

*holds true in the weak topology of* $\mathcal{D}(\mathbb{C}^2) \hat{\otimes} X$.

*Then there exist a Hilbert space* $H$ *and a pure hyponormal operator* T *in* $L(H)$ *and a canonical isometric embedding of the space* $(\mathrm{Ran}\, D^{1/2})^-$ *into* $X$, *so that*

$$K(w,z) = C^*(w)C(z) \quad \textit{almost everywhere in } \mathbb{C}^2 \qquad (5.3)$$

*and*

$$\Gamma = -D^{1/2}\partial C. \qquad (5.4)$$

PROOF. We divide the proof of the theorem in many steps.

*i) The Kolmogorov factorization of the kernel K* .

By a general well-known result due to Kolmogorov, see for instance [14] for a proof in the scalar case, there exist a Hilbert space $\mathcal{M}$ and linear bounded operators

$$C_1(z) : \mathcal{X} \to \mathcal{M} \qquad z \in \mathbb{C}$$

so that

$$K(w,z) = C_1^*(w) C_1(z) \quad , (w,z) \in \mathbb{C}^2 . \tag{5.5}$$

Moreover, the minimal space $\mathcal{M}$ with this property, which is generated by the vectors $C_1(z)x$ , $z \in \mathbb{C}$ , $x \in \mathcal{X}$ , is unique up to a unitary isomorphism.

Then the properties of the kernel K are transmitted to the function $C_1$. Namely, one gets from (5.1) the estimate

$$\| C_1(z) \|^2 = M( 1+ |z|^2 )^{-1}, \quad z \in \mathbb{C} \tag{5.6}$$

and since K is measureable and the space $\mathcal{M}$ is minimal it turns out that $C_1$ is also an operator valued measureable function.

The equation (5.2) implies that $C_1$ is an anti-analytic function off the support of the distribution $\Gamma$ . Indeed, a Fourier transform argument, or an evaluation of $\partial_z K(w,z)$ in an appropriate Sobolev space, show that the distribution $\partial_z K(w,z) \cdot \varphi(z)$ can not be supported by the diagonal, for every $\varphi$ in $\mathcal{D}(\mathbb{C}) \hat{\otimes} \mathcal{X}$ .

Because of (5.6), the function $C_1$ has a power series expansion at infinity of the form

$$C_1(z) = - R/\bar{z} + \sum_{n \geq 1} R_n/\bar{z}^{n+1} \tag{5.7}$$

for large values of $|z|$ .

Let us write the equation (5.2) for $|w|$ large:

$$(w-z)C_1^*(w)\,\partial C_1(z)\varphi(z) = \Gamma(z)\,\varphi(z) - C_1^*(w)\lim_{\varepsilon\to 0} C_1(z)\Gamma_\varepsilon(z)\varphi(z)$$

By identifying the first term of the two analytic series in $w$, with coefficients in the space $\mathcal{D}'(\mathbb{C})\hat{\otimes}\mathcal{X}$, one gets

$$\Gamma = -R^*\partial C_1. \qquad (5.8)$$

We put $F = -R^*C_1$, so that $F \in L^\infty(\mathbb{C})\hat{\otimes}_\varepsilon \mathcal{L}(\mathcal{X})$ and

$$\Gamma = \partial F. \qquad (5.9($$

By (5.5) we have

$$\lim_{|z|,|w|\to\infty} w\bar{z}K(w,z) = \lim_{|z|\to\infty} \bar{z}F(z) = R^*R. \qquad (5.10)$$

Let us denote finally by $A$ the square root $(R^*R)^{1/2}$ in $L(\mathcal{H})$.

*ii) The functional space associated to* $K$.

Let $H$ denote the separate completion of the space $\mathcal{D}(\mathbb{C})\hat{\otimes}\mathcal{X}$ in the following seminorm

$$\|\varphi\|_H^2 = \int \langle K(w,z)\partial\varphi(z), \partial\varphi(w)\rangle \, d\mu(w)d\mu(z)$$

Since the kernel $K$ is antianalytic in $z$ off the support of the distribution $\Gamma$, we obtain $\|\varphi\|_H = 0$ whenever the supports $\text{supp}(\varphi)$ and $\text{supp}(\Gamma)$ are disjoint.Consequently,the multiplication operator with $\bar{z}$ is bounded in the above seminorm.We denote by $S^*$ is extension to $H$, and we write formally

$$S^*\varphi = \bar{z}\varphi.$$

Unless otherwise stated, $\varphi$ and $\psi$ denote throughout this section elements of the space $\mathcal{D}(\mathbb{C})\hat{\otimes}\mathcal{X}$.

We assert that the adjoint S of $S^*$ in $\mathcal{L}(\mathbb{H})$ acts by the formula:

$$S\varphi = z\varphi - \pi^{-1}(\Gamma\cdot\varphi) * 1/\bar{z}. \qquad (5.11)$$

Indeed, at the level of distributions, (5.11) is equivalent with

$$\partial(S\varphi) = z\partial\varphi + \varphi + \Gamma\cdot\varphi = z\partial\varphi + \varphi + \lim_{\varepsilon\to 0}\Gamma_\varepsilon\varphi$$

so that

$$\langle S\varphi,\psi\rangle_{\mathbb{H}} = \int\langle K(w,z)\partial(S\varphi)(z),\partial\psi(w)\rangle\, d\mu(z)d\mu(w) =$$

$$= \int\langle K(w,z)\; z\,\partial\varphi(z),\partial\psi(w)\rangle\, d\mu(z)d\mu(w) \; +$$

$$\int\langle K(w,z)\varphi(z),\partial\psi(w)\rangle\, d\mu(z)d\mu(w) \; +$$

$$\lim_{\varepsilon\to 0}\int\langle K(w,z)\Gamma_\varepsilon(z)\varphi(z),\partial\psi(w)\rangle\, d\mu(z)d\mu(w).$$

Here is implicetely used the convention, together the existence of the limit :

$$K(w,z)\Gamma\varphi = \lim_{\varepsilon\to 0} K(w,z)\,\Gamma_\varepsilon(z)\,\varphi(z)$$

which is asserted in the statement.

In view of the equation (5.2) we obtain

$$\langle S\varphi,\psi\rangle = \int\langle wK(w,z)\partial\varphi(z),\partial\psi(w)\rangle + \lim_{\varepsilon\to 0}\int\langle(\Gamma_\varepsilon\varphi)(z),\partial\psi(w)\rangle =$$

$$= \langle\varphi, S^*\psi\rangle,$$

because $\int\partial\psi(w)d\mu(w) = 0$.

In fact (5.9) leads to a functional expression for the operator S , namely

$$S\varphi = (z+F)\varphi + \pi^{-1}\int\frac{F(\zeta)\partial\varphi(\zeta)}{\bar{\zeta}-\bar{z}}\, d\mu(\zeta).$$

Thus the range of the operator $[S^*, S]$ consists of classes in the space $\mathcal{H}$ of identically constant functions in a neighbourhood of supp($\Gamma$). Let $\chi \otimes x$ be the representative of such a function, with $\chi$ in $\mathcal{D}(\mathbb{C})$, $\chi \equiv 1$ on supp($\Gamma$), and x in $\mathcal{X}$. The relations (5.5) and (5.7) imply

$$\int K(w,z)\,\bar\partial\chi(z)\,d\mu(z) = \pi\, C_1^*(w)\, R\,. \tag{5.12}$$

Consequently we have

$$\langle \chi \otimes x\,,\ \varphi\rangle_{\mathcal{H}} = \langle x\,,\ \pi\int R^* C_1(w)\bar\partial\varphi(w)\,d\mu(w)\rangle$$

so that

$$\langle [S^*, S]\varphi, \varphi\rangle = \|\int F(z)\,\bar\partial\varphi(z)\,d\mu(z)\|^2 .$$

In conclusion we have proved that the operator S on $\mathcal{H}$ is hyponormal. We shall prove that this is the expected operator. It remains to compare its kernel $C^*(w)C(z)$ with $K(w,z)$.

*iii) The isometric embedding of $\mathcal{H}$ into M.*

The definition of the scalar product on the space $\mathcal{H}$, shows that the operator

$$V\varphi = \int C_1(z)\,\bar\partial\varphi(z)\,d\mu(z)$$

extends up to a isometry $V : \mathcal{H} \to \mathcal{M}$. Let us denote

$$T = VSV^* ,$$

so that T is a hyponormal operator on the Hilbert space $\mathcal{M}$. Moreover, T as well its adjoint $T^*$ vanish on the orthogonal complement of the space

$$\text{span} \left\{ \int C_1(z)\bar{\partial}\varphi(z)\,d\mu(z) \;:\; \varphi \in \mathcal{D}(\mathbb{C})\hat{\otimes}\mathcal{X} \right\}.$$

The formulae given in part ii) of the proof imply

$$T^* \int C_1\bar{\partial}\varphi\, d\mu = \int C_1\bar{z}\,\bar{\partial}\varphi\, d\mu$$

and

$$[T^*, T]\int C_1\bar{\partial}\varphi\, d\mu = RR^* \int C_1\bar{\partial}\varphi\, d\mu = A^2 \int C_1\bar{\partial}\varphi\, d\mu .$$

Hence $\bar{\partial}( T_z^* C_1 ) = 0$ in the sense of distributions, and $[T^*, T]^{1/2} = A$. By using (5.5) one gets

$$T_z^* C_1(z) = R \;, \quad z \in \mathbb{C} \;, \tag{5.13}$$

Let $R^* = U(RR^*)^{1/2} = UA$ be the polar decomposition of the operator $R^*$. The partial isometry $U$ identifies the space $(\operatorname{Ran} A)^-$ with a subspace of $\mathcal{X}$.

On the other hand, there is the contractive operator function $C$ in $L^\infty(\mathbb{C})\hat{\otimes} \mathcal{L}((\operatorname{Ran} A)^-)$ attached to the hyponormal operator $T$, so that $T_z^* C(z) = A$. Therefore we obtain from (5.13) the identity

$$T_z^* ( C_1(z) - C(z)U^* ) = 0 \;, z \in \mathbb{C} \tag{5.14}$$

*iv) The comparison of the two kernels.*

Let $\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_n$ be the orthogonal decomposition of the Hilbert space $\mathcal{H}$ into pure and normal part, with respect to the hyponormal operator $T$. The funtion $C$ takes values only in the pure subspace $\mathcal{H}_p$.

Let us define the operator valued function $C' : \mathbb{C} \to \mathcal{L}(\mathcal{X}, \mathcal{H})$ as follows

$$C'(z) = \begin{cases} C_1(z) & \text{if } \operatorname{Ker} T_z^* \cap \operatorname{Ker} T_z = (0) \\ C(z)\,U^* & \text{otherwise ,} \end{cases}$$

Since the Hilbert space $\mathcal{H}$ is separable , the condition

$$\operatorname{Ker} T_z^* \cap \operatorname{Ker} T_z \neq (0)$$

holds only for a countable set of points $z$ in $\mathbb{C}$ ( on these subspaces the operator $T$ is normal ). Therfore the function $C'$ is still measureable, and, moreover, its class in $L^\infty(\mathbb{C}) \hat{\otimes}_\varepsilon L(\mathcal{X}, \mathcal{M})$ coincides with that of $C_1$ .

Then we infer from (5.14) that

$$C'(z) - C(z)\,U^* \in L(\mathcal{X}, \mathcal{M}_p), \; z \in \mathbb{C}$$

But $\mathcal{M}_p \subset V\mathcal{H}$ , and consequently , the space $V\mathcal{H}$ is generated by the vectors of the form $\int (C' - CU^*)\bar{\partial}\varphi \, d\mu$ and $\int CU^* \bar{\partial}\varphi \, d\mu$ , so that we conclude $\mathcal{M}_p = V\mathcal{H}$ .

In other words, the hyponormal operator $S$ ,as well as the restriction of $T$ to the subspace $V\mathcal{H}$ , are pure.

The equation (5.2) together with the Kolmogorov factorisation (5.5) , imply the relation

$$T_z \bar{\partial} C' \varphi = - \lim_{\varepsilon \to 0} C' \Gamma_\varepsilon \varphi \qquad , \; \varphi \in \mathcal{D}(\mathbb{C}) \hat{\otimes} \mathcal{X} \tag{5.15}$$

which holds at the level of distributions. On the other hand we have a similar expression for the function $CU^*$ , namely

$$T_z \bar{\partial} CU^* \varphi = \lim_{\varepsilon \to 0} CU^* U D^{1/2} \bar{\partial} C_\varepsilon U^* \varphi . \tag{5.16}$$

Because the invariant $\Gamma$ is completely determined by the values of the operator function $\Theta(z) = C^*(z)C(z)$ for $|z| \gg 0$ ( see 5.2 bellow ), we infer from (5.14) that

$$\Gamma = - UD^{1/2}\partial CU^* .$$

Then by comparing (5.15) with (5.16) one gets

$$T_z \partial ( C' - CU^* ) \varphi = 0 \quad , \quad \varphi \in \mathcal{D}(\mathbb{C}) \hat{\otimes} \mathcal{X} \quad .$$

Thus, by respecting the inductive proof of Theorem B we obtain

$$A T^n ( C' - CU^* ) = 0$$

for every non-negative integer n .

Finally we get in virtue of relation (5.13) the identities

$$A T^n T^{*m} ( C' - CU^* ) = 0 \quad ; \quad n,m \geq 0$$

which prove, together with the observation that the operator T is pure on the space $V\mathcal{H}$ , that $C' = CU^*$ , and the proof of Theorem 5.1 is complete.

5.2. Our next result gives a constructive way of recuperating the space $\mathcal{H}$ and the pure hyponormal operator T from the values of the function $\Theta(z) = C^*(z)C(z)$ , for $|z| \gg 0$ .
This is the precise meaning of the assertion that the function $\Theta$ determines completely the invariant $\Gamma$ ( see the last part of the proof of Theorem 5.1 ) .

In order to state the result, let $(\mathbb{Z}^+)^2$ denote the semigroup of all paires of non-negative integers, with the generators $\iota = (1,0)$ , $\varkappa = (0,1)$ and the zero element $\theta = (0,0)$ .

PROPOSITION. *Let X be a separable, complex Hilbert space, and let*

$$N : (\mathbb{Z}^+)^2 \times (\mathbb{Z}^+)^2 \longrightarrow L(X)$$

*be a positive definite function with the following properties:*

*a) There is a constant* $M$ *, so that*

$$\sum_{\alpha,\beta} \langle N(\alpha+\iota,\beta+\iota)\, x_\beta\, ,\, x_\alpha \rangle \leq M \sum_{\alpha,\beta} \langle N(\alpha,\beta)\, x_\beta\, ,\, x_\alpha \rangle \tag{5.17}$$

*for any finite subset* $\{x_\alpha\}$ *of* $X$ .

*b) There is a function* $G : (\mathbb{Z}^+)^2 \longrightarrow L(X)$ *, so that*

$$N(\alpha+\iota,\beta) - N(\alpha,\beta+\varkappa) = \sum_{r=0}^{p-1} N(\alpha, r\iota)\, G(\beta-(r+1)\iota) \tag{5.18}$$

*and*

$$N(\theta,\alpha) = N(m\varkappa, n\varkappa) = G(\alpha) \tag{5.19}$$

*for any* $\alpha=(m,n)$ *and* $\beta=(p,q)$ *in* $(\mathbb{Z}^+)^2$ .

*Then there exist a Hilbert space* $H$ *, a pure hyponormal operator* $T$ *in* $L(H)$ *and a canonical isometric embedding of the space* $(\operatorname{Ran} D^{1/2})^-$ *into* $X$ *, such that*

$$N(\alpha,\beta) = D^{1/2} T^n T^{*m} T^p T^{*q} D^{1/2} \quad , \alpha=(m,n) \quad , \beta=(p,q) . \tag{5.20}$$

REMARK. Before to begin the proof of Proposition 5.2 , we have to remark that, via the equations (5.20) , the arguments involved in the proof of Proposition 3.3 lead easily to a connection between the positive definite kernel N with the properties (5.17) , (5.18) , (5.19) and the function Ⓗ .

PROOF. By the already mentioned result of Kolmogorov, there exist a Hilbert space $\mathcal{H}$ ,unique up to a unitary isomorphism, and linear bounded operators

$$R(\alpha) : X \longrightarrow \mathcal{H} \quad , \alpha \in (\mathbb{Z}^+)^2$$

so that

$$N(\alpha,\beta) = R^*(\alpha)R(\beta) \quad , \quad \alpha , \beta \in (\mathbb{Z}^+)^2 \qquad (5.21)$$

and

$$\mathcal{H} = \text{span} \left\{ R(\beta)\ x \ : \beta \in (\mathbb{Z}^+)^2 , x \in X \right\}. \qquad (5.22)$$

By condition (5.17) , we can define a linear bounded operator $T$ on $\mathcal{H}$ , as follows

$$T\,R(\beta)x = R(\beta+\iota)x \quad , \beta \in (\mathbb{Z}^+)^2 \quad , x \in X . \qquad (5.23)$$

From (5.18) , (5.21) and (5.22) we derive , by a direct computation

$$T^*R(\beta)x = R(\beta+\varkappa)x + \sum_{r=0}^{p-1} R(r\iota)\ G(\beta-(r+1)\iota\ )\ , \ \beta=(p,q), x \in X . \qquad (5.24)$$

From (5.23) , (5.24) and (5.19) it follows that

$$[T^*,T]\,R(\beta)x = R(\theta)G(\beta)x = R(\theta)N(\theta,\beta)x = R(\theta)R^*(\theta)R(\beta)x$$

for every $\beta$ in $(\mathbb{Z}^+)^2$ and $x$ in $X$ .

Thus $D = [T^*,T] = R(\theta)R^*(\theta)$ hence $T$ is a hyponormal operator . It is pure, because of the condition (5.22) and the equations (5.23) , (5.24) .

Let $U$ be the partial isometry which appeares in the polar decomposition

$$R^*(\theta) = U\,D^{1/2}$$

The operator $U$ identifies the space $( \mathrm{Ran}\, D^{1/2} )^-$ with a subspace of $\mathcal{X}$.

Let us denote

$$\tilde{N}(\alpha,\beta) = D^{1/2}T^nT^{*m}T^pT^{*q}D^{1/2}, \quad \alpha = (m,n), \ \beta = (p,q)$$

and

$$\tilde{G}(\alpha) = D^{1/2}T^nT^{*m}D^{1/2}, \quad \alpha = (m,n).$$

We have to prove that

$$N(\alpha,\beta) = U\,\tilde{N}(\alpha,\beta)\,U^*, \qquad \alpha,\beta \in (\mathbb{Z}^+)^2. \tag{5.25}$$

A simple bracket identity shows that $\tilde{N}$ is uniquely determined by $\tilde{G}$, and, moreover that $\tilde{N}$ and $\tilde{G}$ are related by the equations (5.18), (5.19). Since $N$ is also uniquely determined by $G$, it turns out that (5.25) will be a consequence of the next relation

$$G(\alpha) = U\,\tilde{G}(\alpha)\,U^*, \qquad \alpha \in (\mathbb{Z}^+)^2. \tag{5.26}$$

In order to prove it, we remark that we have from (5.24)

$$T^{*m} R(\theta) = R(m\varkappa), \qquad m \in \mathbb{Z}^+.$$

Therefore, for any $\alpha = (m,n)$ we obtain

$$U\,\tilde{G}(\alpha)\,U^* = UD^{1/2}T^nT^{*m}D^{1/2}U^* =$$

$$= R^*(\theta)\, T^n\, T^{*m}\, R(\theta) =$$

$$= R^*(n\varkappa)\, R(m\varkappa) =$$

$$= N(n\varkappa, m\varkappa) = G(\alpha),$$

which ends the proof.

# REFERENCES


1. APOSTOL, G.; CLANCEY, K.: Generalized inverses and spectral theory, Trans.Amer.Math.Soc. 215(1976), 293-300.
2. CAREY, R.W.; PINCUS,J.D.: Mosaics, principal functions, and mean motion in von Neumann alebras, Acta Math. 138(1977), 153-218.
3. CLANCEY,K.: Seminormal operators, Lecture Notes in Math. No.742, Springer, Berlin-Heidelberg-New York, 1979.
4. CLANCEY,K.: Hilbert space operators with one-dimensional self-commutator, J.Operator Theory 13(1985), 265-289.
5. CLANCEY,K.: The Cauchy transform of the principal function associated with a non-normal operator, preprint 1983.
6. CLANCEY,K.; WADHWA,B.L.: Local spectra of seminormal operators, TRans.Amer.Math.Soc. 280(1983), 415-428.
7. COWEN,M.J.;DOUGLAS,R.G.: Complex geometry and operator theory, Acta.Math. 141(1978), 187-261.
8. CURTO,R.E.; SALINAS,N.: Generalized Bergman kernels and the Cowen-Douglas theory, Amer.J.Math. 106 : 2(1984), 447-488.
9. DOUGLAS,R.G.: On majorization, factorization and range inclusion of operators on Hilbert space, Proc. Amer.Math.Soc. 17(1966), 413-415.
10. DUNFORD,N.; SCHWARTZ.J.: Linear Operators, Part I, Interscience Publishers, New-York, 1958.
11. HELTON,J.W.; HOWE,R.: Integral operators, commutator traces, index and homology, Lecture Notes in Math.No.345, Springer, 1973.
12. HELTON,J.W.; HOWE,R.: Trace of commutators of integral operators, Acta Math.135(1975), 271-305.
13. MARTIN,M.: Hermitian geometry and involutive algebras, Math.Z. 188(1985), 359-382.
14. PARTHASARATHY,K.R.; SCHMIDT,K.: Positive Definite Kernels, Continuous Tensor Products, and Central Limit Theorems of Probability Theory, Lecture Notes in Math. No.272, Springer, Berlin-Heidelberg-New York 1972.
15. PINCUS,J.D.: Commutators and systems of singular integral equations.I, Acta Math. 121(1968), 219-249.
16. PINCUS,J.D.; XIA,D.; XIA,J.: The analytic. model of a hyponormal operator with rank one self-commutator, Integral Equations and Operator Theory 7(1984), 516-535.
17. PUTINAR,M.: Hypormal operators are subscalar, J. Operator Theory

12(1984), 385-395.

18. PUTINAR,M.: Hyponormal operators and eigendistributions, Operator Theory: Advances and Applications No. , Birkhauser Verlag, Basel-Boston-Stuttgart, 1985.

19. PUTNAM,C.R.: Resolvent vectors, invariant subspaces and sets of zero capacity, Math.Ann. 205(1973), 165-171.

20. RADJABALIPOUR,M.: On majorization and normality of operators, Proc.Amer.Math.Soc. 62 : 1(1977), 105-110.

21. SCHWARTZ,L.: Theorie des distributions a valeurs vectorielles. I., Ann.Inst.Fourier 7 (1957), 1-141; II, ibidem 8 (1958), 1-209.

22. XIA,D.: On non-normal operators, Chinese Math. 3(1963), 232-246.

23. XIA,D.: On the kermels associated with a class of hyponormal operators, Integral Equations and Operator Theory 6(1983), 444-452.

24. XIA,D.: Spectral Theory of Hyponormal Operators, Birkhauser Verlag, Basel-Boston-Stuttgart, 1983.

Department of Mathematics, INCREST
Bdul Pacii 220, 79622 Bucharest
Romania.