INSTITUTUL
DE
MATEMATICĂ

INSTITUTUL NAȚIONAL
PENTRU CREAȚIE
ȘTIINȚIFICĂ ȘI TEHNICĂ

ISSN 0250 3638

ISOMETRIC DILATIONS OF COMMUTING CONTRACTIONS. IV

by

Zoia CEAUSESCU and Ioan SUCIU

PREPRINT SERIES IN MATHEMATICS

No. 21/1985

BUCUREȘTI

7/83

ISOMETRIC DILATIONS OF COMMUTING

CONTRACTIONS.IV

by

Zoia CEAUSESCU and Ioan SUCIU*)

March 1985

*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd.Pacii 220, 79622 Bucharest, Romania

ISOMETRIC DILATIONS OF COMMUTING CONTRACTIONS. IV

Zoia CEAUSESCU and I.SUCIU

In this paper we continue the study developed in [4] of the set of all Ando dilations of a given pair of commuting contractions.

We show that any Ando dilation can be produced in a canonical way by an appropriate adequate isometry. Considering the labelling by choice sequences of the set of all adequate isometries we establish the obstructions in choice in order that the corresponding adequate isometry produces an Ando dilation (section 2). For a special class of Ando dilations we exhibit a system of free parameters which produces it (section 3). In section 1 we recall some known facts on adequate isometries (cf. [2], [3]). Considering only a particular case, of interest for our purpose, we give to the known general results a form which will be convenient in what follows.

1. Adequate isometries

For $j=1,2$ let T_j be a contraction on the Hilbert space H_j and let A be a contraction from H_1 to H_2 such that $AT_1=T_2A$. In [3] the concept of A-adequate isometry was introduced and used in the labelling of the set of all contractive intertwining dilations of A.

In this section we shall recall some results about A-adequate isometries in the particular case when $H_1=H_2=K_o$, $T_1=T_2=0_{K_o}$ - the null operator on K_o and $A=V_o$ - a contraction on K_o.

In this case a V_o-adequate isometry is an isometry V on a Hilbert space K containing K_o as a (closed) subspace such that

(1.1) $$K = \bigvee_{n\geq 0} V^n K_o ,$$

(1.2) $$V_o = P^K_{K_o} V | K_o .$$

Hence, a V_o-adequate isometry is in fact a minimal isometric dilation (not necessarly power dilation) of V_o. Since the term "minimal isometric dilation" is generally used for the minimal power dilation, we adopt the term "V_o-adequate isometry" in order to make this distinction. It is in concordance with the term introduced in [3] in the general context.

We say that two V_o-adequate isometries (K,V) and (K',V') coincide if there exists a unitary operator X from K on K' such that $X\, K_o = I_{K_o}$ - the identity operator on K_o, and $XV=V'X$.

The minimal isometric dilation $(\overset{\circ}{K}_o, \overset{\circ}{V}_o)$ of (K_o,V_o) is the unique (modulo the above coincidence) V_o-adequate isometry for which K_o is a semi-invariant subspace for $\overset{\circ}{V}$ i.e.

(1.3) $$V_o^n = P^{\overset{\circ}{K}_o}_{K_o} \overset{\circ}{V}{}_o^n | K_o .$$

In general, there are V_o-adequate isometries which do not coincide with $(\overset{\circ}{K}_o, \overset{\circ}{V}_o)$.

We shall recall here some parametrizations of the set of all V_o-adequate isometries which are particularizations to our situation of general results proved in [3], [2].

A sequence $\{(K_n,V_n)\}_{n\geq 1}$ will be called a generating sequence of V_o-adequate isometry if it is defined recurrently by the formulas

$$\mathcal{K}_n = \mathcal{K}_{n-1} \oplus \mathcal{D}_{V_{n-1}}$$

$(1.4)_n$
$$V_n = \begin{bmatrix} V_{n-1} & D_{V^*_{n-1}} C_{n-1} \\ D_{V_{n-1}} & -V^*_{n-1} C_{n-1} \end{bmatrix}$$

where, for each $n \geq 1$, C_{n-1} is a contraction from $\mathcal{D}_{V_{n-1}}$ into $\mathcal{D}_{V^*_{n-1}}$. Clearly the string $\{(\mathcal{K}_n, V_n)\}_{n=1}^{P}$ is well determined by the string $\{C_n\}_{n=0}^{P-1}$. We shall refer at the sequence $\{C_n\}_{n\geq 0}$ as a generating sequence of a V_o-adequate isometry with the meaning that C_n appears in the form $(1.4)_{n+1}$ of $(\mathcal{K}_{n+1}, V_{n+1})$.

Let $(\mathcal{K}_n, V_n)$ be a generating sequence of V_o-adequate isometry. Considering $\mathcal{K}_{n-1} \subset \mathcal{K}_n$ in a natural way it is easy to see that, for $n \geq 2$, $V_n | \mathcal{K}_{n-2} = V_{n-1} | \mathcal{K}_{n-2}$. It results that there exists the inductive limit $(\mathcal{K}\ , V\ ) = \varinjlim (\mathcal{K}_n, V_n)$ and $(\mathcal{K}\ , V\ )$ is a V_o-adequate isometry.

With a natural notion of coincidence for the generating sequences of V_o-adequate isometries we have

<u>Proposition 1.1. The map</u> $\{(\mathcal{K}_n, V_n)\}_{n\geq 1} \longrightarrow (\mathcal{K}\ , V\ ) = \varinjlim (\mathcal{K}_n, V_n)$ <u>is a bijective correspondence between the set of all generating sequences of V_o-adequate isometries and the set of all V_o-adequate isometries.</u>

If $(\mathcal{K}, V)$ is a V_o-adequate isometry than setting $\mathcal{K}_n = \bigvee_{j=0}^{n} V^j \mathcal{K}_o$ and $V_n = P^{\mathcal{K}}_{\mathcal{K}_n} V | \mathcal{K}_n$, it is easy to see that $\{(\mathcal{K}_n, V_n)\}_{n\geq 1}$ is the generating sequence of V_o-adequate isometry, corresponding to $(\mathcal{K}, V)$.

A sequence $\{R_n\}_{n\geq 0}$ will be called a V_o-<u>choice sequence</u> provided $R_o = V_o$ and for $n \geq 1$ R_n is a contraction from $\mathcal{D}_{R_{n-1}}$ into $\mathcal{D}_{R^*_{n-1}}$.

For a V_o-choice sequence $R=\{R_n\}_{n\geq 0}$ let us set

$$(1.5) \qquad K = K(R) = K_o \oplus \mathcal{D}_{R_o} \oplus \mathcal{D}_{R_1} \oplus \dots$$

and let V be the operator on K having, with respect to the decomposition (1.5) of K, the matrix

$$(1.6) \qquad V=V(R) = \begin{bmatrix} R_o & D_{R_o^*}R_1 & D_{R_o^*}D_{R_1^*}R_2 & \dots \\ D_{R_o} & -D_{R_o^*}R_1 & -R_o^*D_{R_1^*}R_2 & \dots \\ 0 & D_{R_1} & -R_1^* R_2 & \dots \\ \dots\dots\dots & & & \dots \end{bmatrix}$$

It is easy to see that (K,V) is a V_o-adequate isometry.

<u>Proposition 1.2. The map</u> $R \to (K(R), V(R))$ <u>is a bijective correspondence between the set of all</u> V_o<u>-choice sequences and the set of all</u> V_o<u>-adequate isometries.</u>

Let (K,V) be a V_o-adequate isometry, $\{(K_n,V_n)\}_{n\geq 1}$ be its generating sequence and $\{R_n\}_{n\geq 0}$ be the V_o-choice sequence of (K,V). We shall work freely with one of the three described above forms of (K_n,V_n):

I. The non specificated form

$$K_n = \bigvee_{j=0}^{n} V^n K_o$$

$$V_n = P^K_{K_n} V | K_o$$

II - The recursive matricial form

$$\mathcal{K}_n = \mathcal{K}_{n-1} \oplus \mathcal{D}_{V_{n-1}}$$

$$V_n = \begin{bmatrix} V_{n-1} & D_{V^*_{n-1}} C_{n-1} \\ D_{V_{n-1}} & -V^*_{n-1} C_{n-1} \end{bmatrix}$$

III - The matricial form

$$\mathcal{K}_n = \mathcal{K}_o \oplus \mathcal{D}_{R_o} \oplus \ldots \oplus \mathcal{D}_{R_{n-1}}$$

$$V_n = \begin{bmatrix} R_o & D_{R^*_o} R_1 & D_{R^*_o} D_{R^*_1} R_2 & \ldots & D_{R^*_o} \ldots D_{R^*_{n-2}} R_{n-1} \\ D_{R_o} & -R^*_o R_1 & -R^*_o D_{R^*_1} R_2 & \ldots & -R^*_o D_{R^*_1} \ldots D_{R^*_{n-2}} R_{n-1} \\ 0 & D_{R_1} & -R^*_1 R_2 & \ldots & -R^*_1 D_{R^*_2} \ldots D_{R^*_{n-2}} R_{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & -R^*_{n-2} R_{n-1} \end{bmatrix}$$

The identification between forms I and II is made having in mind that

$$\mathcal{K}_n = \mathcal{K}_{n-1} \vee V\mathcal{K}_{n-1} = \mathcal{K}_{n-1} \oplus \overline{[V - V_{n-1}]\, \mathcal{K}_{n-1}}$$

and $\|(V-V_{n-1})k_{n-1}\| = \|D_{V_{n-1}} k_{n-1}\|$. The identifications between forms II and III is made having in mind that

$$\left\| D_{V_{n-1}} \begin{bmatrix} k_{n-1} \\ D_{V_{n-2}} \begin{bmatrix} k_{n-2} \cdots \begin{bmatrix} 0 \\ D_{V_0} k_0 \end{bmatrix} \end{bmatrix} \end{bmatrix} \right\| = \| D_{R_{n-1}} \cdots D_{R_0} k_0 \| .$$

Clearly $(\mathcal{K}, V)$ coincides to the minimal isometric dilation of V_o if and only if $C_n = 0$ for any $n \geq 0$, or, equivalently, $R_n = 0$ for any $n \geq 1$.

Let now $\hat{V}_o$ be a contraction on the Hilbert space $\hat{\mathcal{K}}_o$ and let

$K_o \subset \hat{K}_o$ be invariant to $\hat{V}_o$. Denote $V_o = \hat{V}_o | K_o$. Let $(\hat{K},\hat{V})$ be a $\hat{V}_o$-adequate isometry. Denote

$$K = K_+(K_o) = \bigvee_{n \geq 0} \hat{V}^n K_o, \quad V = V_+(K_o) = V | K.$$

Then clearly (K,V) is a V_o-adequate isometry. Let $\{(\hat{K}_n,\hat{V}_n)\}_{n\geq 1}$ and $\{(K_n, V_n)\}_{n\geq 1}$ be the generating sequences of $(\hat{K},\hat{V})$ and (K,V) respectively. We have

$$K_n = \bigvee_{j=0}^{n} V^n K_o = \bigvee_{j=0}^{n} \hat{V}^n K_o \subset \bigvee_{j=0}^{n} \hat{V}^n \hat{K}_o = \hat{K}_n.$$

If we denote by Z_n the inclusion of K_n into $\hat{K}_n$ then, for $n \geq 1$, Z_n has the recursive matricial form

$$(1.7)_n \qquad Z_n = \begin{bmatrix} Z_{n-1} & a_{n-1} \\ 0 & b_{n-1} \end{bmatrix}$$

where a_{n-1} is the contraction from $\mathcal{D}_{V_{n-1}}$ into $\hat{K}_{n-1}$ defined by

$$(1.8)_n \qquad a_{n-1} D_{V_{n-k}} k_{n-1} = (\hat{V}_{n-1} Z_{n-1} - Z_{n-1} V_{n-1}) k_{n-1}, \; k_{n-1} \in K_{n-1},$$

and b_{n-1} is the contraction from $\mathcal{D}_{V_{n-1}}$ into $\mathcal{D}_{\hat{V}_{n-1}}$ defined by

$$(1.9)_n \qquad b_{n-1} D_{V_{n-1}} k_{n-1} = D_{\hat{V}_{n-1}} Z_{n-1} k_{n-1}, \quad k_{n-1} \in K_{n-1}.$$

Let us also note that

$$(1.10)_n \qquad V_n = P_K^{K_n} \hat{V}_n | K_n$$

or, in recursive matricial form

$$V_n = Z_n^* \hat{V}_n Z_n.$$

If we consider the matricial forms of (K_n, V_n) and $(\hat{K}_n, \hat{V}_n)$ then the matrix of Z_n has not a simple form. However we have the following

Remark 1.1. Let $k_o \in K_o$ and for $n \geq 1$ let $k_n = (k_o)_n$ be the element of K_n which in the matricial form of (K_n, V_n) is given by $0 \oplus \ldots \oplus 0 \oplus D_{R_{n-1}} \ldots D_{R_o} k_o$. Then in the recurssive matricial form of (K_n, V_n) we have

$$(k_o)_1 = \begin{bmatrix} 0 \\ D_{V_o} k_o \end{bmatrix}, \quad (k_o)_2 = \begin{bmatrix} 0 \\ D_{V_1} k_1 \end{bmatrix} \ldots \quad (k_o)_n = \begin{bmatrix} 0 \\ D_{V_{n-1}} k_{n-1} \end{bmatrix}.$$

It results

$$D_{V_n} Z_n k_n = D_{V_n} Z_n \begin{bmatrix} 0 \\ D_{V_{n-1}} k_{n-1} \end{bmatrix} = D_{\hat{V}_n} \begin{bmatrix} a_{n-1} D_{V_{n-1}} k_{n-1} \\ b_{n-1} D_{V_{n-1}} k_{n-1} \end{bmatrix} =$$

$$= D_{\hat{V}_n} \begin{bmatrix} 0 \\ b_{n-1} D_{V_{n-1}} k_{n-1} \end{bmatrix} = D_{\hat{V}_n} \begin{bmatrix} 0 \\ D_{\hat{V}_{n-1}} Z_{n-1} k_{n-1} \end{bmatrix}$$

This implies that in the matricial form of $(\hat{K}_{n+1}, \hat{V}_{n+1})$ the element $\begin{bmatrix} 0 \\ D_{\hat{V}_n} Z_n k_n \end{bmatrix}$ of $\hat{K}_{n+1}$ is given by $0 \oplus \ldots \oplus 0 \oplus D_{\hat{R}_{n-1}} \ldots D_{\hat{R}_o} k_o$.

Let us end this section with the following simple but usefull

Proposition 1.3. Let (K, V) be a V_o-adequate isometry and let $\{(K_n, V_n)\}_{n \geq 1}$ be its generating sequence. Let H be a closed subspace of K_o. Then the following are equivalent

(i) $V[K \ominus H] \subset K \ominus H$

(ii) $V_n[K_n \ominus H] \subset K_n \ominus H, \quad n \geq 0$

$$\text{(iii)} \begin{cases} V_o(K_o \ominus H) \subset K_o \ominus H \\ C_n \mathcal{D}_{V_n} \subset \mathcal{D}_{V_n^*} \ominus \overline{D_{V_n^*} H}, \ n \geq 0. \end{cases}$$

Proof. Since for $k_n \in \mathcal{K}_n$, k_n is in $K_n \ominus H$ if and only if $k_n \in K \ominus H$ an because

$$V_{n+1}\begin{bmatrix} k_n \\ 0 \end{bmatrix} = V\begin{bmatrix} k_n \\ 0 \end{bmatrix}$$

We clearly have (i)$\Leftrightarrow$(ii).

Since $0 \oplus D_{V_n} k_n \in K_{n+1} \ominus H$ from (ii) we obtain $V_{n+1}\begin{bmatrix} 0 \\ D_{V_n} k_n \end{bmatrix} \in K_{n+1} \ominus H$.

But

$$V_{n+1}\begin{bmatrix} 0 \\ D_{V_n} k_n \end{bmatrix} = \begin{bmatrix} D_{V_n^*} C_n D_{V_n} k_n \\ -V_n^* C_n D_{V_n} k_n \end{bmatrix}$$

and (iii) clearly results. A simple induction shows that (iii)$\Leftrightarrow$(ii).

2. Ando dilations as adequate isometries

In what follows we shall denote by $(H, [T,S])$ a pair of commuting contractions T,S on the Hilbert space H.

We say that $(K, [U,V])$ is an <u>Ando dilation</u> of $(H, [T,S])$ provided K is a Hilbert space containing H as a closed subspace, U,V are two commuting isometries on K and

$$(2.1) \qquad K = \bigvee_{n,m \geq 0} U^n V^m H$$

$$(2.2) \qquad T^n S^m = P_H^K U^n V^m | H, \ n,m \geq 0.$$

Two Ando dilations $(K[U,V])$ and $(K'[U',V'])$ of $(H, [T,S])$ <u>coincide</u> if there exists a unitary operator X from K on K' such that $X|H = I_H$ and $XU = U'X$, $XV = V'X$.

Let us denote

$$K_o=\bigvee_{n\geq 0} U^n H,\quad U_o=U|K_o,\quad V_o=P^K_{K_o} V|K_o.$$

It is clear that (K_o,U_o) is an identification of the minimal isometric dilation of T. It is easy to see that K_o reduces U and V_o is a contractive intertwining dilation of the commutant S of T. That is

$$U_oV_o=V_oU_o,\quad SP^{K_o}_H=P^{K_o}_H V_o.$$

We say that the Ando dilation $(K,[U,V])$ of $(H,[T,S])$ <u>crosses through</u> $(K_o,[U_o,V_o])$ if $(K_o,[U_o,V_o])$ is attached to $(K,[U,V])$ as above.

Since

$$K=\bigvee_{n,m\geq 0} U^nV^mH=\bigvee_{m\geq 0} V^mK_o$$

and $V_o=P^K_{K_o}V|K_o$ we conclude that (K,V) is a V_o-adequate isometr

For all what follows we shall fix $(K_o[U_o,V_o])$ consisting from an identification (K_o,U_o) of the minimal isometric dilation of T and a contractive intertwining dilation V_o of the commutant S of T.

We say that the V_o - adequate isometry (K,V) <u>produces</u> an Ando dilation of $(H,[T,S])$ if there exists an isometry U on K such that $(K,[U,V])$ is an Ando dilation of $(H,[T,S])$ which crosses through $(K_o,[U_o,V_o])$.

The above considerations show that any Ando dilation of $(H,[T,S])$ which crosses through $(K_o,[U_o,V_o])$ is produced by a V_o-adequate isometry. In [4] it was proved, in a slightly different terminology the following

Proposition 2.1. Let (K,V) be a V_o-adequate isometry. Then $(\mathcal{K},V)$ produces an Ando dilation of $(H,[T,S])$ if and only if the following conditions hold.

(i) If $\{C_n\}_{n\geq 0}$ is the generating sequence of (K,V) then

$$C_n\mathcal{D}_{V_n} \subset \mathcal{D}_{V_n^*} \ominus \overline{D_{V_n^*}H} \tag{2.3$_n$}$$

(ii) The formulas

$$\Gamma_n D_{V_n} = V_n U_n - U_n V_n, \quad n\geq 0 \tag{2.4$_n$}$$

$$Y_n D_{\Gamma_n} D_{V_n} = D_{V_n} U_n, \quad n\geq 0 \tag{2.5$_n$}$$

define a contraction Γ_n from $\mathcal{D}_{V_n}$ into $\ker U_n^*$ and an isometry Y_n from $\mathcal{D}_{\Gamma_n}$ into $\mathcal{D}_{V_n}$. The operator U_n, $n\geq 1$, which appears in $(2.4)_n$, $(2.5)_n$ is the isometry on K_n defined recursively, according to $(2.4)_{n-1}$, $(2.5)_{n-1}$, by the matrix

$$U_n = \begin{bmatrix} U_{n-1} & \Gamma_{n-1} \\ 0 & Y_{n-1}D_{\Gamma_{n-1}} \end{bmatrix} \tag{2.6$_n$}$$

Remarks. Two V_o-adequate isometries which produce coinciding Ando dilations of $(H,[T,S])$, coincide. If the V_o-adequate isometry (K,V) produces the Ando dilation $(K,[U,V])$ of $(H,[T,S])$ then U is uniquely determined by (K,V).

If $(\overset{\circ}{K}_o,\overset{\circ}{V}_o)$ is the minimal isometric dilation of V_o then it produces an Ando dilation $(\overset{\circ}{K}_o,[\overset{\circ}{U}_o,\overset{\circ}{V}_o])$ of $(H,[T,S])$. We shall call $(\overset{\circ}{K}_o,[\overset{\circ}{U}_o,\overset{\circ}{V}_o])$ the distinqueshed Ando dilation of $(H,[T,S])$ which crosses through $(K_o,[U_o,V_o])$.

For an Ando dilation $(K,[U,V])$ of $(H,[T,S])$ let us denote by $(\hat{K},[\hat{U},\hat{V}])$ its minimal-unitary-isometric extension. This means that $(\hat{K},\hat{U})$ is the minimal unitary extension of (K,U) and $\hat{V}$ is an isometry on $\hat{K}$ which commutes with $\hat{U}$ and extends V. Such an extension allways exists and it is unique. Let $\hat{K}_o=\bigvee_{n\geq 0}\hat{U}^{*n}K_o$ and $\hat{U}_o=\hat{U}|\hat{K}_o$. Clearly $(\hat{K}_o,\hat{U}_o)$ is the minimal unitary extension of (K_o,U_o) and consequentely the minimal unitary dilation of T. If $\hat{V}_o=P^{\hat{K}}_{\hat{K}_o}\hat{V}|\hat{K}_o$ then $\hat{V}_o$ is the unique extension of V_o to a commutant of $\hat{U}_o$.

Hence $(\hat{K}_o,[\hat{U}_o,\hat{V}_o])$ is uniquely determined by $(K_o,[U_o,V_o])$. Recall also that denoting $K_{*o}=\bigvee_{n\geq 0}\hat{U}_o^{*n}H$ then (cf. [6])

$$\hat{K}_o=\left[K_{*o}\ominus H\right]\oplus H\oplus\left[K_o\ominus H\right]. \tag{2.7}$$

Clearly $(\hat{K},\hat{V})$ is a $\hat{V}_o$-adequate isometry.

We say that a $\hat{V}_o$-adequate isometry $(\hat{K},\hat{V})$ <u>produces an Ando dilation</u> for $(H,[T,S])$ if there exists a unitary operator $\hat{U}$ on $\hat{K}$ which extends $\hat{U}_o$ and commutes with $\hat{V}$ such that

$$T^nS^m=P^{\hat{K}}_H\hat{U}^n\hat{V}^m|H,\ n,m\geq 0. \tag{2.8}$$

Setting $K=\bigvee_{n,m\geq 0}\hat{U}^n\hat{V}^mH$, $U=\hat{U}|K$, $V=\hat{V}|K$ then clearly $(K,[U,V])$ is an Ando dilation of $(H,[T,S])$ which crosses through $(K_o,[U_o,V_o])$ - the Ando dilation <u>produced</u> by $(\hat{K},\hat{V})$. Since $K=\bigvee_{n,m\geq 0}V^nU_o^mH=\bigvee_{n\geq 0}V^nK_o$, clearly the V_o-adequate isometry $(K,V)=$ $=(K_+(K_o),\ V_+(K_o^*))$ is attached to $(\hat{K},\hat{V})$ and K_o as in the preceding section and $(K,[U,V])$ is also produced by (K,V).

From

$$\bigvee_{n\geq 0}\hat{U}^{*n}K=\bigvee_{n\geq 0}\hat{U}^{*n}\bigvee_{m\geq 0}V^mK_o=\bigvee_{n\geq 0}V^n\bigvee_{m\geq 0}\hat{U}_o^{*m}K_o=\bigvee_{n\geq 0}V^n\hat{K}_o=\hat{K}$$

it results that $(\hat{K},\hat{U})$ is the minimal unitary extension of $(K,U$

Clearly then $(\hat{K},[\hat{U},\hat{V}])$ is the minimal unitary-isometric extension of $(K,[U,V])$.

We conclude that any Ando dilation of $(H,[T,S])$ which crosses through $(K_o[U_o,V_o])$ is produced as above by a (uniquely determined) $\hat{V}_o$-adequate isometry $(\hat{K},\hat{V})$.

Suppose now that the $\hat{V}_o$-adequate isometry $(\hat{K},\hat{V})$ produces the Ando dilation $(K,[U,V])$. Let $(\hat{K}_n,\hat{V}_n)$ and (K_n,V_n) be the generating sequences of $(\hat{K},\hat{V})$ and (K,V) respectively. Then clearly $\hat{K}_n$ is a reducing subspace for $\hat{U}$ and K_n is invariant for $\hat{U}$. Let $\hat{U}_n=\hat{U}|K_n$, $U_n=\hat{U}|K_n$. We shall call $(\hat{K}_n,[\hat{U}_n,\hat{V}_n])$ and $(K_n,[U_n,V_n])$ the <u>sequences of succesive dilations</u> of $(\hat{K},[\hat{U},\hat{V}])$ and $(K,[U,V])$ respectively. From

$$\bigvee_{j\geq 0}\hat{U}_n^{*j}K_n=\bigvee_{j\geq 0}\hat{U}^{*j}\bigvee_{p=0}^{n}\hat{V}^pK_o=\bigvee_{p=0}^{n}\hat{V}^p\bigvee_{j\geq 0}\hat{U}_o^{*}K_o=\bigvee_{p=0}^{n}\hat{V}^p\hat{K}_o=\hat{K}_n$$

it results that $(\hat{K}_n,\hat{U}_n)$ is the minimal unitary extension of (K_n,U_n). Since for any $k_n,g_n\in K_n$ we have

$$(U_n^*V_nU_nk_n,g_n) = (V_nU_n\,k_n,\,U_ng_n) = (\hat{V}\hat{U}k_n,\hat{U}g_n)=(\hat{V}k_n,g_n)=$$
$$=(V_n\,k_n,g_n)$$

it results

$$(2.9)_n \qquad U_n^*V_nU_n = V_n.$$

That means that V_n is a U_n - <u>Toeplitz operator</u>, and because $P_{K_n}^{K_n}\hat{V}_n|K_n=V_n$ we conclude that $\hat{V}_n$ is the <u>symbol</u> of the U_n-Toeplitz operator V_n. It is known then (cf. [5]) that we have

$$(2.10)_n \qquad V_n = \text{s-}\lim_{p\to\infty} \hat{U}_n^{*p} V_n P_{\hat{K}_n}^{\hat{K}n} \hat{U}_n^{p}.$$

Note also that for $j \leq n$, $\hat{K}_j$ reduces $\hat{U}_n$ and in the matricial from of $(\hat{K}_n, \hat{V}_n)$, $\hat{U}_n$ is given by the diagonal matrix

$$(2.11)_n \qquad \hat{U}_n = \begin{bmatrix} \hat{U}_o & 0 & \dots & 0 \\ 0 & \hat{Y}_o & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & \hat{Y}_{n-1} \end{bmatrix}$$

where for $0 \leq j \leq n-1$, $\hat{Y}_j = \hat{U}_o | \mathcal{D}_{\hat{R}_j}$, and $\hat{U}$ itself has the corresponding infinite diagonal matrix. From Proposition 2.1 it results that for the recursive matricial form of $(K_n, [U_n, V_n])$ the relations $(2.3)_n$-$(2.5)_n$ hold and U_n is given by the matrix $(2.6)_n$. We shall prove now the main result of this section

Theorem 1.1. *Let $(\hat{K}, \hat{V})$ be a $\hat{V}_o$-adequate isometry and $\{\hat{R}_n\}_{n \geq 0}$ be its $\hat{V}_o$-choice sequence. Let $(K,V) = (K_+(K_o), V_+(K_o))$ be the corresponding V_o-adequate isometry and $\{R_n\}_{n \geq 0}$ be its V_o-choice sequence. Then $(\hat{K}, \hat{V})$ (and consequentely (K,V)) produces an Ando dilation of $(H, [T,S])$ if and only if the following conditions hold.*

(i) *For any $n \geq 1$ we have*

$$(2.12)_n \qquad \hat{U}_o \hat{R}_n = \hat{R}_n \hat{U}_o | \mathcal{D}_{\hat{R}_{n-1}}.$$

(ii) *For any $n \geq 1$, $k_o \in K_o$, $k_{*o} \in K_{*o}$ we have*

$$(2.13)_n \qquad (\hat{R}_n D_{\hat{R}_{n-1}} \dots D_{\hat{R}_o} k_o, D_{\hat{R}_{n-1}^*} \dots D_{\hat{R}_o} k_{*o}) = (\hat{V}_{n-1}^2 S_{n-1} k_o, k_{*o})$$

where $S_o K_o = 0$ and for $n \geq 1$, S_n is the operator from K_o into $\hat{K}_n$ given by

$$(2.14)_n \qquad S_n k_o = Z_n (0 \oplus \dots \oplus 0 \oplus D_{\hat{R}_{n-1}} \dots D_{R_o} k_o).$$

Proof. Suppose that $(\hat{K},\hat{V})$ produces an Ando dilation $(K,[U,V])$ of $(H,[T,S])$. Writting $\hat{V}_n$ and $\hat{U}_n$ in the matricial form given by $(1.6)_n$ and $(2.11)_n$ respectively, and using $\hat{U}_n\hat{V}_n=\hat{V}_n\hat{U}_n$ we obtain $(2.12)_{n-1}$.

From $V_{n+1}=P^{\hat{K}_{n+1}}_{K_{n+1}}\hat{V}_{n+1}|K_{n+1}$ which in the recursive matricial form can be written as

$$V_{n+1} = Z^*_{n+1}\hat{V}_{n+1}Z_{n+1}$$

we obtain

$$(2.15)_n \qquad Z^*_n(\hat{V}_n a_n + D_{\hat{V}_n^*}\hat{C}_n b_n) = D_{V_n^*}C_n.$$

From Proposition 2.1 we have $D_{\hat{V}_n^*}C_n\mathcal{D}_{V_n}\subset K_n\oplus H$ and using $(2.15)_n$ we obtain

$$(2.16)_n (D_{\hat{V}_n^*}\hat{C}_n D_{\hat{V}_n} Z_n k_n, h) = -(\hat{V}_n a_n D_{V_n} k_n, h) = -(\hat{V}_n^2 Z_n k_n, h) + (V_n^2 k_n, h)$$

For n=0 it results

$$(2.16)_o \quad (D_{\hat{V}_o^*}\hat{C}_o D_{V_o} k_o, h) = -(\hat{V}_o^2 k_o, h) + (V_o^2, k_o, h) = 0$$

which in the matricial form gives

$$(D_{\hat{R}_o^*}\hat{R}_1 D_{\hat{R}_o} k_o, h) = 0$$

Since

$$(D_{\hat{R}_o^*}\hat{R}_1 D_{\hat{R}_o} k_o, \hat{U}_o^{*p} h) = (\hat{U}_o^p D_{\hat{R}_o^*}\hat{R}_1 D_{\hat{R}_o} k_o, h) = (D_{\hat{R}_o^*}\hat{R}_1 D_{\hat{R}_o} U_o^p k_o, h) = 0$$

for any $p\geq 0$, we obtain $(2.13)_1$.

Suppose now $n\geq 1$. Since $V_n(K_n\ominus H]\subset K_n\ominus H$ (see Proposition 1.3) from $(2.16)_n$ it results that for any $k_n\in K_n\ominus H$ we have

$$(D_{\hat{V}_n}^{*}\hat{C}_n D_{\hat{V}_n} Z_n k_n, h) = -(\hat{V}_n^2 Z_n k_n, h).$$

Since $U_n[K_n\ominus H]\subset K_n\ominus H$ we obtain

$$(D_{\hat{V}_n}^{*}\hat{C}_n D_{\hat{V}_n} Z_n k_n, \hat{U}_o^{*p}h) = (U_n^p D_{\hat{V}_n}^{*}\hat{C}_n D_{\hat{V}_n} Z_n k_n, h) =$$

$$= (D_{\hat{V}_n}^{*}\hat{C}_n \hat{U}_n^p Z_n k_n, h) = (D_{\hat{V}_n}^{*}\hat{C}_n Z_n U_n^p k_n, h) =$$

$$= (\hat{V}_n^2 Z_n U_n^p k_n, h) = (\hat{V}_n^2 \hat{U}_n^p Z_n k_n, h) = -(\hat{V}_n^2 Z_n k_n, \hat{U}_o^{*p}h).$$

Hence

$$(2.17)_n \qquad (D_{\hat{V}_n}^{*}\hat{C}_n D_{\hat{V}_n} Z_n k_n, k_{*o}) = -(\hat{V}_n^2 Z_n k_n, k_{*o}),\quad k_n\in K_n\ominus H,\ k_{*o}\in K_{*o}$$

Let now $k_o\in K_o$ and denote by k_n the element of $K_n\ominus H$ which in the matricial form of (K_n,V_n) is given by $k_n=0\oplus\ldots\ldots$ $\ldots\oplus 0\oplus D_{R_{n-1}}\ldots D_{R_o}k_o$. We have

$$(\hat{V}_{n+1}\begin{bmatrix}0\\ D_{V_n} Z_n k_n\end{bmatrix}, k_{*o}) = (D_{\hat{V}_n}^{*}\hat{C}_n D_{\hat{V}_n} Z_n k_n, k_{*o}) = -(\hat{V}_n^2 Z_n k_n, k_{*o}).$$

But accroding to the Remark 1.1, in the matricial form of $(\hat{K}_{n+1},\hat{V}_{n+1})$ the element $0\oplus D_{\hat{V}_n} Z_n k_n$ of $\hat{K}_{n+1}$ is given by $0\oplus\ldots\oplus 0\oplus D_{\hat{R}_n}\ldots D_{\hat{R}_o}k_o$.

It results

$$(D_{\hat{R}_n}^{*}\ldots D_{\hat{R}_o}^{*}\hat{R}_{n+1}D_{\hat{R}_n}\ldots D_{\hat{R}_o}k_o, k_{*o}) = (\hat{V}_{n+1}\begin{bmatrix}0\\ D_{V_n} Z_n k_n\end{bmatrix}, k_{*o}) =$$

$$= -(\hat{V}_n^2 Z_n k_n, k_{*o})$$

which is $(2.13)_{n+1}$.

We proved that $(2.12)_n$ and $(2.13)_n$, $n\geq 1$ are necessary conditions in order that $(\hat{K},\hat{V})$ produces an Ando dilations.

Suppose now that for any $n\geq 1$, $(2.12)_n$ and $(2.13)_n$ hold. From $(2.12)_n$ it results that $(2.11)_n$ defines a unitary operator $\hat{U}_n$ on $\hat{K}_n$ which extends $\hat{U}_o$ and commutes with $\hat{V}_n$ and consequently we can construct the unitary extension $\hat{U}$ of $\hat{U}_o$ to $\hat{K}$ which commutes with $\hat{V}$. If

$$(2.18) \qquad P^{K}_{H}V = P^{K}_{H}VP^{K}_{H}$$

holds then

$$\begin{aligned}P^{\hat{K}}_{H}\hat{V}^n\hat{U}^m h &= P^{K}_{H}V^nU^m_o h = P^{K}_{H}VP^{K}_{H}V^{n-1}U^m_o h = \\ &= P^{K_o}_{H}P^{K}_{K_o}VP^{K}_{H}V^{n-1}U^m_o h = P^{K_o}_{H}V_oP^{K}_{H}V^{n-1}U^m_o h = \ldots = \\ &= P^{K_o}_{H}V^n_oU^m_o h = S^nT^m h.\end{aligned}$$

So if (2.18) holds then $(\hat{K},\hat{V})$ produces an Ando dilation of $(H,[T,S])$.

But (2.18) is equivalent to $V[K\ominus H]\subset K\ominus H$ and, according to Proposition 1.3 this is equivalent to $V_n(K_n\ominus H]\subset K_n\ominus H$, for any $n\geq 0$ and also equivalent to $D_{V_n^*}C_nD_{V_n}\subset K_n\ominus H$ for any $n\geq 0$. For $n=0$ we have

$$\begin{aligned}(D_{V_o^*}C_oD_{V_o}k_o,h) &= (V_1\begin{bmatrix}0\\ D_{V_o}k_o\end{bmatrix},h) = (Z_1^*\hat{V}_1Z_1\begin{bmatrix}0\\ D_{V_o}k_o\end{bmatrix},h) = \\ &= (\hat{V}_1\begin{bmatrix}0\\ D_{\hat{V}_o}k_o\end{bmatrix},h) = (D_{\hat{R}_o^*}R_1D_{\hat{R}_o}k_o,h) = 0\end{aligned}$$

because of $(2.13)_1$.

Suppose now, by induction, that for some $n\geq 1$ we have $V_n(K_n\ominus H]\subset K_n\ominus H$. Then for any $k_n\in K_n\ominus H$ we have

$$(D_{V_n^*}C_nD_{V_n}k_n,h)=(V_{n+1}\begin{bmatrix}0\\ D_{V_n}k_n\end{bmatrix},h)=(Z^*_{n+1}\hat{V}_nZ_{n+1}\begin{bmatrix}0\\ D_{V_n}k_n\end{bmatrix},h)=$$

$$=(\hat{V}_{n+1}\begin{bmatrix}a_nD_{V_n}k_n\\ b_nD_{V_n}k_n\end{bmatrix},h)=(\hat{V}_na_nD_{V_n^*}k_n,h)+(\hat{V}_{n+1}\begin{bmatrix}0\\ D_{\hat{V}_n}Z_nk_n\end{bmatrix},h)=$$

$$=(\hat{V}_n^2Z_nk_n,h)-(V_n^2k_n,h)+(\hat{V}_{n+1}\begin{bmatrix}0\\ D_{\hat{V}_n}Z_nk_n\end{bmatrix},h)=$$

$$=(\hat{V}_n^2Z_nk_n,h)+(\hat{V}_{n+1}\begin{bmatrix}0\\ D_{V_n}Z_nk_n\end{bmatrix},h)$$

It k_n is the element of $K_n\ominus H$ which in the matricial form of (K_n,V_n) is given by $0\oplus\ldots\oplus 0\oplus D_{R_{n-1}}\ldots D_{R_o}k_o$ then using $(2.13)_n$ we obtain

$$(D_{V_n^*}C_nD_{V_n}k_n,h)=(\hat{V}_n^2Z_nk_n,h)+(D_{\hat{R}_o^*}\ldots D_{\hat{R}_{n-1}^*}\hat{R}_nD_{\hat{R}_{n-1}}\ldots D_{\hat{R}_o}k_o,h)=0.$$

Since for $n\geq 1$ $\overline{D_{V_n}K_n}=D_{V_n}(0\oplus\ldots\oplus 0\oplus\mathcal{D}_{R_{n-1}})$ we obtain

$$D_{V_n^*}C_nD_{V_n}K_n\subset K_n\ominus H$$

which clearly implies $V_{n+1}(K_{n+1}\ominus H)\subset K_{n+1}\ominus H$ and the induction is complete.

This finishes the proof of the Theorem.

The conditions $(2.12)_n$ and $(2.13)_n$, $n\geq 1$ describe all the obstructions on the $\hat{V}_o$-choice sequence $\{\hat{R}_n\}_{n\geq 0}$ in order that the corresponding $\hat{V}_o$-isometry $(\hat{K},\hat{V})$ produces an Ando dilation $(\tilde{K},[\tilde{U},\tilde{V}])$ of $(H,[T,S])$. They also sugest a recursive construction of $\{\hat{R}_n\}_{n\geq 0}$. Indeed if we denote

$$W_n=\hat{U}_o\Big|\overline{D_{\hat{R}_n}\ldots D_{\hat{R}_o}K_o}$$

$$W_{*n}=\hat{U}_o^*\Big|\overline{D_{\hat{R}_n^*}\ldots D_{\hat{R}_o^*}K_{*o}}$$

Mod 21.065

then W_n and W_{*n} are isometries. Let Q_n be the operator from $\overline{D_{\hat{R}_n}\dots D_{\hat{R}_o}K_o}$ into $\overline{D_{\hat{R}_n^*}\dots D_{\hat{R}_o^*}K_{*o}}$ defined by

$$(2.19)_n \quad (Q_nD_{\hat{R}_n}\dots D_{\hat{R}_o}k_o, D_{\hat{R}_n^*}\dots D_{\hat{R}_o^*}k_{*o})=(\hat{V}_n^2S_nk_o,k_{*o})$$

According to $(2.13)_{n+1}$ Q_n is a contraction. We also have

$$(2.20)_n \qquad Q_nW_n=W_{*n}^*Q_n .$$

So $\hat{R}_{n+1}$ is a contractive intertwining dilation of the contraction Q_n which intertwines W_n and W_{*n}^* and can be constructed following the methods from intertwining dilation theory (cf. [2]). Unfortunately the choice of $\hat{R}_{n+1}$ as a contractive intertwining dilation of Q_n is not free. It has to be chosen such that Q_{n+1} defined by $(2.19)_{n+1}$ be a contraction. It can happen that $(2.13)_j$ holds for any $1\leq j\leq n$ but the choice of $\hat{R}_{n+1}$ which satisfies $(2.13)_{n+1}$ be impossible. An example in this sense is given in [4], III .

In [4],I we indirectly proved that if we start with $\hat{R}_1$ verifying $(2.13)_1$ and having sufficiently small norm we can produce a sequence $\{R_n\}_{n\geq 1}$ which verifies $(2.13)_n$ for any $n\geq 1$.

In the next section we shall study a class of Ando dilation of $(H,[T,S])$ which cross through $(K_o,[U_o,V_o])$ for which, we can exhibit a system of free parameters which describe it.

3. U-diagonal Ando dilation

Let $(K,[U,V])$ be an Ando dilation of $(H,[T,S])$ and let $(K_n[U_n,V_n])_{n\geq 0}$ be its sequence of succesive dilations. We say that $(K,[U,V])$ is U-diagonal provided $U_nV_n=V_nU_n$ for any $n\geq 0$.

Since $\Gamma_nD_{V_n}=V_nU_n-U_nV_n$ it results from Proposition 2.1

that $(K,[U,V])$ is U-diagonal if and only if for each $n\geq 1$ U_n has the diagonal form

$$U_n=\begin{bmatrix} U_{n-1} & 0 \\ 0 & Y_{n-1} \end{bmatrix}$$

where $Y_{n-1}D_{V_{n-1}}=D_{V_{n-1}}U_{n-1}$. Clearly then U itself has a diagonal matricial form.

<u>Proposition 3.1. Let</u> $(K,[U,V])$ <u>be an Ando dilation of</u> $(H,[T,S])$, $(\hat{K},[\hat{U},\hat{V}])$ <u>be the minimal unitary - isometric extension of</u> $(K,[U,V])$ <u>and</u> $\{\hat{K}_n,[\hat{U}_n,\hat{V}_n]\}_{n\geq 0}$ $\{K_n,[U_n,V_n]\}_{n\geq 0}$ <u>be the corresponding sequences of succesive dilations. The following assertions are equivalent.</u>

(i) $(K,[U,V])$ <u>is</u> U-<u>diagonal</u>.

(ii) For each $n\geq 0$, $\hat{V}_n|K_n=V_n$.

(iii) <u>The recursive matricial form of the embedding</u> Z_n <u>of</u> K_n <u>into</u> $\hat{K}_n$ <u>is diagonal</u> i.e.

$$Z_n=\begin{bmatrix} Z_{n-1} & 0 \\ 0 & b_{n-1} \end{bmatrix}$$

<u>with</u> $b_{n-1}D_{V_{n-1}}=D_{\hat{V}_{n-1}}Z_{n-1}$.

<u>Proof</u>. Since according to $(2.10)_n$, for any $n\geq 0$ and $k_n\in K_n$ we have

$$\hat{V}_nk_n=\lim_{p\to\infty}\hat{U}_n^{*p}V_nU_n^pk_n$$

clearly (i) $\Rightarrow$ (ii). If (ii) holds then for any $k_n\in K_n$ we have

$$(V_nU_n-U_nV_n)k_n=\hat{V}_n\hat{U}_nk_n-\hat{U}_n\hat{V}_nk_n=0$$

hence (ii) $\Longrightarrow$ (i).

Since in general

$$Z_n = \begin{bmatrix} Z_{n-1} & a_{n-1} \\ 0 & b_{n-1} \end{bmatrix}$$

with $a_{n-1} D_{V_{n-1}} k_{n-1} = (\hat{V}_{n-1} Z_{n-1} - Z_{n-1} V_{n-1}) k_{n-1}$ clearly (i) $\Longleftrightarrow$ (ii).

Let us fix again $(K_o, [U_o, V_o])$ and consequently $(\hat{K}_o, [\hat{U}_o, \hat{V}_o])$ with the semnification given in the preceding section.

Theorem 3.1. *Let $(\hat{K}, \hat{V})$ be a $\hat{V}_o$-adequate isometry and let $\{R_n\}_{n\geq 0}$ be its $\hat{V}_o$-choice sequence. Then $(\hat{K}, \hat{V})$ produces a U-diagonal Ando dilation of $(H, [T, S])$ if and only if for any $n \geq 1$ we have*

$(3.1)_n$ $$\hat{U}_o \hat{R}_n = \hat{R}_n \hat{U}_o | \mathcal{D}_{R_{n-1}}$$

$(3.2)_{n,0}$ $$D_{\hat{R}^*_{n-1}} \ldots D_{\hat{R}^*_o} \hat{R}_n D_{\hat{R}_{n-1}} \ldots D_{\hat{R}_o} K_o \subset K_o \ominus H$$

$(3.2)_{n,j}$ $$\hat{R}^*_{j-1} D_{\hat{R}^*_j} \ldots D_{\hat{R}^*_{n-1}} \hat{R}_n D_{\hat{R}_{n-1}} \ldots D_{\hat{R}_o} K_o \subset \overline{D_{\hat{R}_{n-1}} \ldots D_{\hat{R}_o} K_o}, \quad 1 \leq j \leq n-1,$$

$(3.2)_{n,n}$ $$\hat{R}^*_{n-1} \hat{R}_n D_{\hat{R}_{n-1}} \ldots D_{\hat{R}_o} K_o \subset \overline{D_{\hat{R}_{n-1}} \ldots D_{\hat{R}_o} K_o}.$$

Proof. Suppose that $(\hat{K}, \hat{V})$ produces the U-diagonal Ando dilation $(K, [U, V])$ of $(H, [T, S])$ and let $(\hat{K}, [\hat{U}, \hat{V}])$ be the minimal unitary - isometric extension of $(K, [U, V])$. From Proposition 3.1 it results that $\hat{V}_n | K_n = V_n$ and the embedding Z_n of K_n in $\hat{K}_n$ has a diagonal recurssive matricial form. It results that if we writte $\hat{K}_n$ as

$$\hat{K}_n = \hat{K}_o \oplus \mathcal{D}_{\hat{R}_o} \oplus \ldots \oplus \mathcal{D}_{\hat{R}_{n-1}}$$

then the subspace K_n of $\hat{K}_n$ is

$$K_n = K_o \oplus \overline{D_{\hat{R}_o} K_o} \oplus \overline{D_{\hat{R}_1} D_{\hat{R}_o} K_o} + \ldots + \overline{D_{\hat{R}_{n-1}} \ldots D_{\hat{R}_o} K_o}.$$

From

$$\hat{V}_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ D_{R_{n-1}} \ldots D_{R_o} k_o \end{bmatrix} = \begin{bmatrix} D_{\hat{R}_o}^* \ldots D_{\hat{R}_{n-1}}^* \hat{R}_n D_{\hat{R}_{n-1}} \ldots D_{\hat{R}_o} k_o \\ -\hat{R}_o^* D_{\hat{R}_1}^* \ldots D_{\hat{R}_{n-1}}^* \hat{R}_n D_{\hat{R}_{n-1}} \ldots D_{\hat{R}_o} k_o \\ \vdots \\ -\hat{R}_{n-2}^* D_{\hat{R}_{n-1}}^* \hat{R}_n D_{\hat{R}_{n-1}} \ldots D_{\hat{R}_o} k_o \\ \hat{R}_{n-1}^* \hat{R}_n D_{\hat{R}_{n-1}} \ldots D_{\hat{R}_o} k_o \end{bmatrix}$$

and

$$\hat{V}_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ D_{\hat{R}_{n-1}} \ldots D_{\hat{R}_o} k_o \end{bmatrix} = V_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ D_{\hat{R}_{n-1}} \ldots D_{\hat{R}_o} k_o \end{bmatrix} \in K_n \ominus H$$

we obtain $(3.2)_{n,j}$ for $0 \leq j \leq n$. The conditions $(3.1)_n$ result as in the proof of Theorem 2.1.

Suppose now that for any $n \geq 1$, $(3.1)_n$ and $(3.2)_{n,j}$, $0 \leq j \leq n$, hold. Using $(3.1)_n$ we shall construct as in the proof of Theorem 2.1 the unitary operator $\hat{U}$ on $\hat{K}$ which extends $\hat{U}_o$ and commutes with $\hat{V}$. Let $(K,V) = (K_+(K_o), V_+(K_o))$ be the V_o-adequate isometry attached to $(\hat{K},\hat{V})$ and $\hat{K}_o$ as in the first section and (K_n, V_n) be its generating sequence.

We shall prove first that

$$(3.3)_n \quad \mathcal{K}_n = K_o \oplus \overline{D_{\hat{R}_o} K_o} \oplus \ldots \oplus \overline{D_{\hat{R}_{n-1}} \ldots D_{\hat{R}_o} K_o}$$

Since $K_{n+1} = K_n \vee VK_n$ proceeding by induction, it is sufficient to prove that for any $k_o \in K_o$ we have

$$V\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ D_{\hat{R}_{n-1}} \dots D_{\hat{R}_o} k_o \end{bmatrix} \in K_o \oplus \overline{D_{\hat{R}_o} K_o} \oplus \dots \oplus \overline{D_{\hat{R}_n} \dots D_{\hat{R}_o} K_o} .$$

But from $(3.2)_{n,j}$, $0 \le j \le n$ we have

$$V\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ D_{\hat{R}_{n-1}} \dots D_{\hat{R}_o} k_o \end{bmatrix} = \hat{V}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ D_{\hat{R}_{n-1}} \dots D_{\hat{R}_o} k_o \end{bmatrix} =$$

$$= \begin{bmatrix} D^*_{\hat{R}_o} \dots D^*_{\hat{R}_{n-1}} \hat{R}_n D_{\hat{R}_{n-1}} \dots D_{\hat{R}_o} k_o \\ -\hat{R}^*_o D^*_{\hat{R}_1} \dots D^*_{\hat{R}_{n-1}} \hat{R}_n D_{\hat{R}_{n-1}} \dots D_{\hat{R}_o} k_o \\ \vdots \\ -\hat{R}^*_{n-1} \hat{R}_n D_{\hat{R}_{n-1}} \dots D_{\hat{R}_o} k_o \\ D_{\hat{R}_n} D_{\hat{R}_{n-1}} \dots D_{\hat{R}_o} k_o \end{bmatrix} \in K_o \oplus \overline{D_{\hat{R}_o} K_o} \oplus \dots \oplus \overline{D_{\hat{R}_n} \dots D_{\hat{R}_o} K_o}$$

and $(3.3)_n$ is proved for any $n \ge 1$.

It results

$$(3.4) \qquad K = K_o \oplus \overline{D_{\hat{R}_o} K_o} \oplus \overline{D_{\hat{R}_1} D_{\hat{R}_o} K_o} \oplus \dots$$

From (3.4) and $(3.2)_{n,0}$ we obtain $V[K \ominus H] \subset K \ominus H$ which together with the existence of $\hat{U}$ shows that $(\hat{K}, \hat{V})$ produces an Ando dilation $(K, [U,V])$ of $(H, [T,S])$. Using again $(3.3)_n$ and $(3.2)_{n,j}$, $0 \le j \le n$ we obtain $\hat{V}_n | K_n = V_n$ and by Proposition 3.1 we conclude that $(K, [U,V])$ is U-diagonal.

The proof of the Theorem is complete.

Based on Theorem 3.1 we shall give now a labelling by a system of free parameters of the set of all U-diagonal Ando dilations of $(H, [T,S])$ which cross through a fixed $(K_o, [U_o, V_o])$.

A sequence $\Theta = \{[L_{n-1}, F_{n-1}, \Theta_n(\lambda)]\}_{n \ge 1}$ will be called

$(\mathcal{K}_0, [U_0, V_0])$-<u>adequate sequence of analytic functions</u> if its terms $[\mathcal{L}_{n-1}, \mathcal{F}_{n-1}, \Theta_n(\lambda)]$ are analytic functions defined in the open unit disc D of the complex plane, taking values contractions from the Hilbert space $\mathcal{L}_{n-1}$ into the Hilbert space $\mathcal{F}_{n-1}$ which are recurrently constructed following the procedure presented below:

Let us denote

$$(3.5)_0 \qquad \hat{R}_0 = \hat{V}_0$$

and

$$(3.6) \qquad \begin{aligned} \mathcal{L}_0 &= \overline{D_{\hat{R}_0} \mathcal{K}_0} \ominus \hat{U}_0 \overline{D_{\hat{R}_0} \mathcal{K}_0}; \quad \mathcal{R}_0 = \bigcap_{p\geq 0} \hat{U}_0^p \overline{D_{\hat{R}_0} \mathcal{K}_0} \\ \mathcal{L}_{*0} &= \overline{D_{\hat{R}_0^*} \mathcal{K}_{*0}} \ominus U_0^* \overline{D_{\hat{R}_0^*} \mathcal{K}_{*0}}; \quad \mathcal{R}_{*0} = \bigcap_{p\geq 0} \hat{U}_0^{*p} \overline{D_{\hat{R}_0^*} \mathcal{K}_{*0}} \end{aligned}$$

Then $\mathcal{L}_0$ and $\mathcal{L}_{*0}$ are wandering subspaces for $\hat{U}_0$ and

$$(3.8)_0 \qquad \begin{aligned} \overline{D_{\hat{R}_0} \mathcal{K}_0} &= M_+(\mathcal{L}_0) \oplus \mathcal{R}_0 \\ \overline{D_{R_0^*} K_{*0}} &= M_-(\mathcal{L}_{*0}) \oplus \mathcal{R}_{*0} \end{aligned}$$

when, for the wandering subspace L of the unitary operator U we denoted

$$M(\mathcal{L}) = \bigoplus_{n\in\mathbb{Z}} U^n \mathcal{L}, \quad M_+(\mathcal{L}) = \bigoplus_{n\geq 0} U^n \mathcal{L}, \quad M_-(\mathcal{L}) = \bigoplus_{n\geq 0} U^{*n} \mathcal{L}.$$

Let us denote

$$(3.9)_0 \qquad \begin{aligned} \mathcal{D}_0 &= \bigoplus_{p\geq 1} \hat{U}_0^{*p} L_0 = M(\mathcal{L}_0) \ominus M_+(\mathcal{L}_0) = \mathcal{D}_{\hat{R}_0} \ominus \overline{D_{\hat{R}_0} K_0} \\ \mathcal{D}_{*0} &= \bigoplus_{p\geq 1} \hat{U}_0^p \mathcal{L}_{*0} = M(\mathcal{L}_{*0}) \ominus M_-(\mathcal{L}_{*0}) = \mathcal{D}_{\hat{R}_0^*} \ominus \overline{D_{\hat{R}_0^*} \mathcal{K}_{*0}} \end{aligned}$$

and let $H_{0,0}$ be the contraction from $\mathcal{D}_{*o}$ into $\mathcal{D}_o$ defined by

$$(3.10)_{0,0} \qquad H_{0,0} = P_{\mathcal{D}_o}\hat{R}_o^*|\mathcal{D}_{*0}.$$

Since we have

$$(3.11)_{0,0} \qquad H_{0,0}\hat{U}_o|\mathcal{D}_{*o} = P_{\mathcal{D}_o}\hat{U}_o H_{0,0}$$

it results that the subspace $M_o = \ker H_{0,0}$ of $\mathcal{D}_{*o}$ is invariant to $\hat{U}_o$. Since $\hat{U}_o|\mathcal{D}_{*o}$ is a unilateral shift, $\hat{U}_o|M_o$ will be a unilateral shift too. Hence we have

$$(3.12)_o \qquad M_o = \operatorname{Ker} H_{0,0} = M_+(F_o)$$

with

$$(3.13) \qquad F_o = M_o \ominus \hat{U}_o M_o.$$

We shall choose as the first term is our sequence <u>an arbitrary constractive analytic function</u> $[L_o, F_o, \Theta_1(\lambda)]$.

Let us remark that what is imposed in this choice are the subspaces L_o, F_o (in fact only their dimensions) the choice of the parameter $\Theta_1(\lambda)$ as a contractive analytic function from $\mathcal{D}$ into $L(L_o, F_o)$ being totally free.

Let us denote by $\Phi_{U,L} = \Phi_L$ the Fourier representation of the bilateral shift U on $M(L)$ as multiplication by coordinate function e^{it} on $L^2(L)$, i.e.

$$\Phi_L \sum_{n\in Z} U^n \ell_n = \sum_{n\in Z} e^{int}\ell_n,\ \ell_n \in L,\ \sum_{n\in Z}\|\ell_n\|^2 < \infty .$$

Using $(3.8)_o$ we shall define the contraction $\hat{R}_1$ from $\mathcal{D}_{\hat{R}_o}$ into $\mathcal{D}_{\hat{R}_o^*}$ by

$$(3.5)_1 \qquad \hat{R}_1 = \Phi^*_{F_o^*} \hat{\Theta}_1 \Phi_{L_o} \oplus O_{R_o}$$

where by Θ_1 we denoted the contraction from $L^2(L_o)$ into $L^2(F_o)$ given by the pointwise multiplication with the boundary values $\Theta_1(e^{it})$ of $\Theta_1(\lambda)$.

Let us mention the following properties of $\hat{R}_1$. From the definition $(3.5)_1$ it is clear that

$$(3.14)_1 \qquad \hat{R}_1\hat{U}_o|\mathcal{D}_{\hat{R}_o} = \hat{U}_o\hat{R}_1 .$$

Using

$$\hat{R}_1\overline{D_{\hat{R}_o}K_o} \subset M_+(F_o) \subset \mathcal{D}_{*o} = \mathcal{D}_{\hat{R}_o^*} \ominus \overline{D_{\hat{R}_o^*}K_{*o}}$$

we obtain

$$(3.15)_{1,0} \qquad (\hat{R}_1 D_{\hat{R}_o}k_o, D_{\hat{R}_o^*}k_{*o}) = 0, \quad k_o \in K_o,\ k_{*o} \in K_{*o}.$$

Note also that from $H_{0,0}\hat{R}_1\mathcal{D}_{\hat{R}_o}K_o = \{0\}$ it results

$$(3.15)_{1,1} \qquad \hat{R}_o^*\hat{R}_1 D_{\hat{R}_o}K_o \subset \overline{D_{\hat{R}_o}K_o}$$

Let now $n \geq 1$. Suppose that, after the n^{th} step of our construction for any p, $0 \leq p \leq n-1$ we produced L_p, L_{*p} by $(3.6)_p$, $\mathcal{D}_p, \mathcal{D}_{*p}$ by $(3.9)_p$, $H_{p,j}$, $0 \leq j \leq p$ by $(3.10)_{p,j}$, M_p by $(3.12)_p$ and F_p by $(3.13)_p$. Suppose that $[L_p, F_p, \Theta_{p+1}]$ was chosen as an arbitrary contractive analytic function on $\mathcal{D}$ and $\hat{R}_{p+1}$ was constructed by $(3.5)_{p+1}$. Suppose also the relations $(3.14)_{p+1}$ and $(3.15)_{p+1,j}$ $0\leq j\leq p+1$ hold.

We shall describe now the $(n+1)^{th}$-step of our construction.

Define

$$(3.6)_n \quad \begin{aligned} &\mathcal{L}_n=\overline{D_{\hat{R}_n}\ldots D_{\hat{R}_o}\mathcal{K}_o}\ominus \hat{U}_o\overline{D_{\hat{R}_n}\ldots D_{\hat{R}_o}\mathcal{K}_o};\ \mathcal{R}_n=\bigcap_{p\geq 0}\hat{U}_o^{p}\overline{D_{\hat{R}_n}\ldots D_{\hat{R}_o}\mathcal{K}_o}\\ &\mathcal{L}_{*n}=\overline{D_{\hat{R}_n^*}\ldots\ldots\ldots D_{\hat{R}_o^*}\mathcal{K}_{*o}}\ominus U_o^*\overline{D_{\hat{R}_n}\ldots D_{\hat{R}_o}\mathcal{K}_{*o}};\ \mathcal{R}_{*n}=\bigcap_{p\geq 0}U_o^*\overline{D_{\hat{R}_n^*}\ldots D_{\hat{R}_o}\mathcal{K}_{*o}}. \end{aligned}$$

We have

$$(3.7)_n \begin{cases} \overline{D_{\hat{R}_n}\ldots D_{\hat{R}_o}\mathcal{K}_o}=M_+(\mathcal{L}_n)\oplus\mathcal{R}_n\\ \overline{D_{\hat{R}_n^*}\ldots D_{\hat{R}_o^*}\mathcal{K}_{*o}}=M_-(\mathcal{L}_{*n})\oplus\mathcal{R}_{*n} \end{cases}$$

and

$$(3.8)_n \begin{cases} \mathcal{D}_{R_n}=M(\mathcal{L}_n)\oplus\mathcal{R}_n\\ \mathcal{D}_{R_n^*}=M(\mathcal{L}_{*n})\oplus\mathcal{R}_{*n} \end{cases}$$

Denote

$$(3.9)_n \quad \begin{aligned} &\mathcal{D}_n=\bigoplus_{p\geq 1}U_o^{*p}\mathcal{L}_n=M(\mathcal{L}_n)\ominus M_+(\mathcal{L}_n)=\mathcal{D}_{\hat{R}_n}\ominus\overline{D_{\hat{R}_n}\ldots D_{\hat{R}_o}\mathcal{K}_o}\\ &\mathcal{D}_{*n}=\bigoplus_{p\geq 1}U_o^{p}\mathcal{L}_{*n}=M(\mathcal{L}_{*n})\ominus M_-(\mathcal{L}_{*n})=\mathcal{D}_{\hat{R}_n^*}\ominus\overline{D_{\hat{R}_n^*}\ldots D_{\hat{R}_o^*}\mathcal{K}_{*o}} \end{aligned}$$

and let $H_{n,j}$, $0\leq j\leq n$ be the contraction from $\mathcal{D}_{*n}$ to $\mathcal{D}_j$ defined by

$$(3.10)_{n,j} \qquad H_{n,j}=\begin{cases} P_{\mathcal{D}_j}\hat{R}_j^{*}D_{\hat{R}_{j+1}^{*}}\cdots D_{\hat{R}_n^{*}}\big|\mathcal{D}_{*n}, & 0\leq j\leq n-1 \\ P_{\mathcal{D}_n}\hat{R}_n^{*}\big|\mathcal{D}_{*n}, & j=n. \end{cases}$$

Since for each j, $1\leq j\leq n$, R_j is a contraction from $\mathcal{D}_{\hat{R}_{j-1}}$ into $\mathcal{D}_{\hat{R}_{j-1}^{*}}$ we have $\mathcal{D}_{\hat{R}_j}\subset\mathcal{D}_{\hat{R}_{j-1}}$, $\mathcal{D}_{\hat{R}_j^{*}}\subset\mathcal{D}_{\hat{R}_{j-1}^{*}}$. Also $\mathcal{D}_j\subset\mathcal{D}_{\hat{R}_j}$, $\mathcal{D}_{*j}\subset\mathcal{D}_{\hat{R}_j^{*}}$. Hence $(3.10)_{n,j}$ has sens for any j, $0\leq j\leq n$, and defines a contraction $H_{n,j}$ from $\mathcal{D}_{*n}$ to $\mathcal{D}_j$. Moreover it is easy to see that

$$(3.11)_{n,j} \qquad H_{n,j}\hat{U}_o\big|\mathcal{D}_{*n} = P_{\mathcal{D}_j}\hat{U}_o H_{n,j}, \quad 0\leq j\leq n.$$

It results that the subspace $\mathcal{M}_n=\bigcap_{j=0}^{n}\operatorname{Ker} H_{n,j}$ of $\mathcal{D}_{*n}$ is invariant to $\hat{U}_o$ and $\hat{U}_o\big|\mathcal{D}_{*n}$ being a unilateral shift, $\hat{U}_o\big|\mathcal{M}_n$ will be a unilateral shift too. Hence we have

$$(3.12)_n \qquad \mathcal{M}_n=:\bigcap_{j=0}^{n}\operatorname{Ker} H_{n,j}=M_+(\mathcal{F}_n)$$

with

$$(3.13)_n \qquad \mathcal{F}_n=\mathcal{M}_n\ominus\hat{U}_o\mathcal{M}_n$$

We shall choose as the $(n+1)^{\text{th}}$-term of our sequence an <u>arbitrary</u> contractive analytic function $\{\mathcal{L}_n,\mathcal{F}_n,\Theta_{n+1}(\lambda)\}$.

Using $(3.8)_n$ we shall define the contraction $\hat{R}_{n+1}$ from $\mathcal{D}_{R_n}$ into $\mathcal{D}_{R_n^{*}}$ by

$$(3.5)_{n+1} \qquad \hat{R}_{n+1}=\Phi_{\mathcal{F}_n}^{*}\hat{\Theta}_{n+1}\Phi_{\mathcal{L}_n}\oplus O_{R_n},$$

Clearly $\hat{R}_{n+1}$ verifies

$$(3.14)_{n+1} \qquad \hat{R}_{n+1}\hat{U}_o\big|\mathcal{D}_{R_n}=\hat{U}_o R_{n+1}$$

Using

$$\hat{R}_{n+1}\overline{D_{\hat{R}_n}\cdots D_{\hat{R}_o}K_o}\subset M_+(F_n)\subset \mathcal{D}_{*n}=\mathcal{D}_{\hat{R}^*_n}\ominus \overline{D_{\hat{R}^*_n}\cdots D_{\hat{R}^*_o}K_{*o}}$$

we obtain

$$(3.15)_{n+1,0}\quad (\hat{R}_{n+1}D_{\hat{R}_n}\cdots D_{\hat{R}_o}k_o, D_{\hat{R}^*_n}\cdots D_{\hat{R}^*_o}k_{*o})=0,\ k_o\in K_o,\ k_{*o}\in K_{*o}.$$

From $H_{n,j}\hat{R}_{n+1}D_{\hat{R}_n}\cdots D_{\hat{R}_o}k_o=0$, $k_o\in K_o$, $\quad 0\leq j\leq n$

we obtain

$$(3.15)_{n+1,j+1}\quad \begin{cases} \hat{R}^*_j D_{\hat{R}^*_{j+1}}\cdots D_{\hat{R}^*_n}\hat{R}_{n+1}D_{\hat{R}_n}\cdots D_{\hat{R}_o}K_o\subset \overline{D_{\hat{R}_j}\cdots D_{\hat{R}_o}K_o}, & 0\leq j\leq n-1\\ \hat{R}^*_n\hat{R}_{n+1}\overline{D_{\hat{R}_n}\cdots D_{\hat{R}_o}K_o}\subset \overline{D_{\hat{R}_n}\cdots D_{\hat{R}_o}K_o}, & j=n \end{cases}$$

In this way we described completely the recurrsive construction of a $(K_o,[U_o,V_o])$-choice sequence of analytic functions $\theta=\{[L_{n-1},F_{n-1},\theta_n(\lambda)]\}_{n\geq 1}$. This construction also produces the V_o-choice sequence $\{\hat{R}_n\}_{n\geq 0}=\{\hat{R}_n(\theta)\}_{n\geq 0}$. We shall call $\{\hat{R}_n(\theta)\}_{n\geq 0}$ the V_o-choice sequence <u>canonically attached to the</u> $(K_o,[U_o,V_o])$ - choice sequence of analytic functions $=\{[L_{n-1},F_{n-j},\theta_n(\lambda)]\}_{n\geq 1}$.

We can state now the main result of this section.

<u>Theorem 3.2. Let</u> $\theta=\{[L_{n-1},F_{n-1},\theta_n(\lambda)]\}_{n\geq 1}$ <u>be a</u> $(K_o,[U_o,V_o])$ - <u>choice sequence of analytic function and let</u> $\{\hat{R}_n\}_{n\geq 0}=\{R_n(\theta)\}_{n\geq 0}$ <u>be the</u> V_o-<u>choice sequence cannonically attached to</u> θ. <u>Then the</u> $\hat{V}_o$-<u>adequate isometry</u> $(\hat{K},\hat{V})=(\hat{K}(\theta),\hat{V}(\theta))$ <u>corresponding to</u> $\{\hat{R}_n\}_{n\geq 0}=\{\hat{R}_n(\theta)\}_{n\geq 0}$ <u>produces a U-diagonal Ando dilation</u> $(K,[U,V])=(K(\theta),[U(\theta),V(\theta)])$ of $(H,[T,S])$ <u>which crosses through</u> $(K_o,[U_o,V_o])$. <u>Moreover, the map</u> $(K(\theta),[U(\theta),V(\theta)])$ <u>is an one-to-one correspondence between the set of all</u>

$(K_o, [U_o, V_o])$ - adequate sequences of analytic functions and the set of all U-diagonal Ando dilations of $(H, [T,S])$ which cross through $(K_o, [U_o, V_o])$.

Proof. Let $\Theta = \{[L_{n-1}, F_{n-1}, \Theta_n(\lambda)]\}_{n\geq 1}$ be a $(K_o, [U_o, V_o])$-choice sequence of analytic functions and $\{\hat{R}_n\}_{n\geq 0} = \{\hat{R}_n(\Theta)\}_{n\geq 0}$. Then the conditions $(3.14)_n$ and $(3.15)_{n,j}$, $0\leq j\leq n$ hold for any $n\geq 1$. But $(3.14)_n$ is the same as $(3.1)_n$ and $(3.15)_{n,j}$, is the same as $(3.2)_{n,j}$, $0\leq j\leq n$. From Theorem 3.1 it results that $(\hat{K}(\Theta), \hat{V}(\Theta))$ produces a U-diagonal Ando dilation.

So the map $\Theta \to (K(\Theta), [U(\Theta), V(\Theta)])$ is well defined and we can show easily that it is an injective map.

In order to prove that it is also a surjective map let $(K, [U,V])$ be a U-diagonal Ando dilation of $(H, [T,S])$ which crosses through $(K_o [U_o, V_o])$ and let $(\hat{K}, [\hat{U}, \hat{V}])$ be its unitary-isometric extension. If $\{\hat{R}_n\}_{n\geq 0}$ is the $\hat{V}_o$-choice sequence of $(\hat{K}, \hat{V})$ then from Theorem 3.1 it results that for any $n\geq 1$ the relations $(3.1)_n$ and $(3.2)_{n,j}$ $0\leq j\leq n$, hold.

As in the recursive construction of $(K_o, [U_o, V_o])$-choice sequences of analytic functions we shall produce all th elements defined by the formulas $(3.6)_n \div (3.13)_n$. The relations $(3.2)_{n,j}$, $0\leq j\leq n$ imply

$$(3.16)_n \qquad \hat{R}_n \overline{D_{\hat{R}_{n-1}} \ldots D_{\hat{R}_o} K_o} \subset M_+(F_{n-1}) .$$

Since $\hat{R}_{n-1}$ is reducing for $\hat{U}_o$ so will be $\hat{R}_n R_{n-1}$ which together with $(3.16)_n$ implies $\hat{R}_n | R_{n-1} = 0$. It results that there exists a contractive anaytic function $[L_{n-1}, F_{n-1} \Theta_n(\lambda)]$ such that

$$\hat{R}_n = \Phi^*_{F_{n-1}} \hat{\Theta}_n \Phi_{L_{n-1}} \oplus O_{R_{n-1}} .$$

The resulting $\Theta = \{[L_{n-1}, F_{n-1} \Theta_n(\lambda)]\}_{n\geq 1}$ is a $(K_o, [U_o, V_o])$-

choice sequence of analytic functions and $\hat{R}_n = \hat{R}_n(\theta)$. Hence $(K,[U,V]) = (K(\theta),[U(\theta),V(\theta)])$.

The proof of the Theorem is complete.

Corollary 3.1. The distingueshed Ando dilation $(\overset{\circ}{K}_o,[\overset{\circ}{U}_o, \overset{\circ}{V}_o])$ of $(H,[T,S])$ is the only U-diagonal Ando dilation of $(H,[T,S])$ which crosses through $(K_o,[U_o,V_o])$ if and only if either $L_o=\{0\}$ or Ker $H_{o,o}=\{0\}$.

R E F E R E N C E S

1 Ando, T.: On a pair of commutative contractions, Acta Sci.Math.(Szeged) 24(1963), 88-90.

2 Ceauşescu, Zoia: Operatorial extrapolations, Thesis, Bucharest, 1980.

3 Ceauşescu, Zoia; Foiaş, C.: On intertwining dilations.VI, Rev.Roumaine Math.Pures Appl.23(1978), 1471-1482.

4 Ceauşescu, Zoia; Suciu, I.: Isometric dilations of commuting contractions I, J.Operator Theory 12(1984), 65-88, II in Dilation theory, Toeplitz operators and other topics, Birkhäuser Verlag, 1983, pp.55-80, III in Spectral theory of linear operators and related topics, Birkhäuser Verlag, 1984, pp.47-59.

5 Douglas, R.G.: On the operator equations $S^{*}X = X$ and related topics, Acta Sci.Math.(Szeged) 30(1969), 19-32.

6 Sz.-Nagy, B.; Foiaş, C.: Harmonic analysis of operators on Hilbert space, North-Holland-Akademiai Kiado, Amsterdam-Budapest, 1970.