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0. INTRODUCTION

An algebraic singularity with linear resolution is a standard graded algebra $A=\bigoplus_{n\geq 0} A_n$ over a field $K=A_o$, such that between its Hilbert series $H_A(z)=\sum_{n\geq 0}(\dim_k A_n)z^n$ and its Poincaré series $P_A(z)=\sum_{m\geq 0}(\dim_k \mathrm{Tor}_m^A(K,K))z^m$ there is the connection: $P_A(-z)H_A(z)=1$.

We call such singularities "Fröberg rings" (after the Swedish mathematician R.Fröberg, who initiated their study).

In this note, we study certain such singularities, namely the ones defined by a particular class of standard graded abelian monoids.

Our main achievement is the revealing of the correct combinatorial interpretation of the relation $P_A(-z)H_A(z)=1$, for graded monoid algebras (Cor.5, par.3).

This is done by means of two ingredients: the characterization of the monoidal homology, due to A.O.Laudal and the monoidal version of the Möbius inversion formula, due to G.Lallement (whose remarkable proof to MacMahon's Master Theorem inspired this work).

The combinatorial restriction on a monoid M, imposed by the Fröberg relation for its monoid algebra $K[M]$, naturally leads to Cohen-Macaulay finite posets, whose study was done by Stanley, Reisner, Garcia, Backlawski, Björner e.a. Using their efficient results, we provide examples of such monoidal singularities, recovering, in particular, a class which was previously studied in a joint work of the author with N.Manolache.

1. HILBERT SERIES OF MONOIDAL GRADATIONS

Let $(M,+)$ be a cancellative, abelian monoid with unit element 0. An "ennumerative system" over M, is a pair (β,S) where $(S,\underline{m}_S)$ is a complete local noetherian ring and $\beta : M \to 1+\underline{m}_S$ is a monoid homomorphism, such that:

$$H_\beta = \sum_{x\in M}\beta(x)$$

exists in S ($1+\underline{m}_S$ being considered as a multiplicative submonoid in S). The element H_β is called "the Hilbert integral" of M with respect to the given ennumerative system.

The ennumerative systems of interest in Algebra are usually given by "monoidal gradations" on M, i.e. by those monoid homomorphisms $d:M \to Z_+^n$ of M in some free abelian monoid ($n\geq 1$), such that $d^{-1}(\nu)$ is finite for every $\nu \in Z_+^n$. Such a gradation is called "connected" when $d^{-1}(0)=\{0\}$.

When n=1, we call d a "simple" gradation (some authors call "multigradations" the ones corresponding to $n\geq 2$).

Suppose M carries a simple, connected, monoidal gradation $d:M \to Z_+$. To this gradation one associates the ennumerative pair:

$$(\beta_d, Z[[z]]),$$

where $\beta_d : M \to 1+zZ[[z]]$ is the monoid homomorphism:

$$\beta_d(x)=z^{d(x)} \quad , \quad (\forall)\ x\in M.$$

The Hilbert integral of this ennumerative system is usually called "the Hilbert series" of the graded monoid (M,d) and it is denoted by:

(3) $H_{M,d}(z)=\sum_{x\in M} z^{d(x)}$,

or simply by $H_M(z)$, when d is fixed and no confusion may arise. We adopt the following notations: a fixed gradation, as above, $M \to Z_+$ is written "$|\ |$" and, for any $m\geq 0$, we put $M_m=\{x\in M/|x|=m\}$. Thus: $M_o=\{0\}$, M_m is finite for $m>0$ and the family $\{M_m\}_{m\geq 0}$ is a partition of M, having the property: $M_m+M_{m'}\subseteq M_{m+m'}$, for any $m,m'\in Z_+$.

As an element of $Z[[z]]$, the Hilbert series of the graded monoid $(M,|\ |)$ may be written:

$$H_M(z)=\sum_{m\geq 0} h(m)z^m ,$$

where $h(m)=\# M_m$, for $m\geq 0$. The numerical function $m\mapsto h(m)$ is called "the Hilbert function" of the given gradation.

We are interested in special gradations of the kind just described. Namely, such a gradation is called "standard" if its first-degree component M_1 generates the whole monoid M.

Obviously, a standard gradation is a surjective monoid homomorphism from M to Z_+. If M carries a standard gradation, then it is finitely generated and, moreover, has finite decompositions (i.e. every element $x\in M$ defines only a finite number of sequences $(y_1,\ldots,y_p)$, with $y_j\in M\setminus\{0\}$ for $j=1,2,\ldots,p$ and $x=\sum_{j=1}^{p} y_j$).

2. MÖBIUS TRANSFORMS OF HILBERT SERIES

Returning to the general setting, let (M,+) be a cancellative, abelian monoid with unit element 0. Suppose, further, that M has finite decompositions.

On M, we consider the natural poset structure, given by the partial order:

(1) for x,y M: $x \leq y$ iff $(\exists)$ $z \in M$ and $x+z=y$.

The Möbius function μ_M of the poset $(M, \leq)$ naturally defines by restriction to the intervals $[0,x]$, a function $\mu : M \rightarrow Z$, which we call, abusively, "<u>the Möbius function</u>" of M. Therefore (by the well-known connection between the Möbius function and the "chain" function on a poset, cf. [1]) one may define:

(2) $(\forall)$ $x \in M \setminus \{0\}$, $\mu(x) = \delta_o(x) - \delta_1(x) = \mu_M(0,x)$

where $\delta_o(x)$ $(\delta_1(x))$ is the number of decompositions of x with an even (odd) number of terms. We complete (2) with $\mu(0)=1$.

Now, let (β, S) be an ennumerative system for M, defining the Hilbert integral H_β.

<u>1. Definition</u>

The "<u>Möbius transform</u>" $\overline{H}_\beta$ of H_β is the elment of S:

(3) $\overline{H}_\beta = \sum_{x \in M} \mu(x) \beta(x)$.

(where every product $\mu(x)\beta(x)$ is taken in the natural Z-algebra structure on the commutative ring S).

<u>1. Proposition</u>

<u>Let M be a cancellative, abelian monoid and let (β, S) be an ennumerative system for M.</u>

<u>If M has finite decomposition, then:</u>

(4) $H_\beta \cdot \overline{H}_\beta = 1$

Proof

$$H_\beta \cdot \overline{H}_\beta = (\sum_{x\in M} \beta(x))(\sum_{y\in M} \mu(y)\beta(y)) = \sum_{x,y\in M} \mu(y)\beta(x)\beta(y) = "$$

$$= \sum_{x,y\in M} \mu(y)\beta(xy) = \sum_{z\in M}(\sum_{x+y=z} \mu(y))\beta(z) .$$

But, since M has finite decompositions, the Möbius inversion formulas (cf. [1]) immediately give: $\sum_{x+y=z} \mu(y) = \begin{cases} 1, & \text{if } z=0 \\ 0, & \text{if } z\neq 0 \end{cases}$, ending the proof of the proposition.

2. Corollary

Let (M,+) be a cancellative, abelian monoid, such that there is a standard gradation $|\ | : M \to Z_+$, with Hilbert series $H_M \in Z[[z]]$. Then $H_M(z) = \sum_{x\in M} z^{|x|}$ is invertible in $Z[[z]]$ and its inverse is the Möbius transform: $\overline{H}_M(z) = \sum_{x\in M} \mu(x) z^{|x|}$.

Proof

Apply Proposition 1 to the ennumerative system associated to the given gradation (cf. §1).

In the conditions of Corollary 2, we write: $H_M(z) = \sum_{m\geq 0}(\sum_{|x|=m} 1) z^m$, respectively: $\overline{H}_M(z) = \sum_{m\geq 0}(\sum_{|x|=m} \mu(x)) z^m$.

3. MONOIDAL HOMOLOGY

We fix, now, a base field K and consider the monoid K-algebra $K[M]:=R$ of a cancellative, abelian monoid (M,+).

The main algebraic information about the singularity R, is contained into the (homologically graded) Tor-algebra:

(5) $\mathrm{Tor}^R_{\bullet}(K,K)=\bigoplus_{m\geq 0}\mathrm{Tor}^R_m(K,K)$.

The numerical object associated to this algebra is the corresponding "Poincaré series", defined by:

(6) $P_M(z)=\sum_{m\geq 0}(\dim_K \mathrm{Tor}^R_m(K,K))z^m \in 1+Z[[z]]\cdot z$.

The coefficients of $P_M(z)$ are the main topological invariants of the singularity R. They are called "the Betti numbers" of R, being usually denoted by: $(b_m(R))_{m\geq 0}$.

It is not very easy to compute P_M , in general.

However, a general description of $\mathrm{Tor}^R_{\bullet}(K,K)$, for monoid algebras, was done by A.O.Laudal in [3], and is the following:

Theorem

Let (M,+) be a cancellative, abelian monoid and let K be any field. Put R=K[M], the corresponding monoid algebra. For any element $x\in M\setminus\{0\}$, let $\Delta(x)$ be the chain complex associated to the subposet $(\hat{X}\setminus\{0,X\},\leq)$ of $(M,\leq)$, where $\hat{X}$ denotes the principal order-ideal $\{y\in M/\ y\leq x\}$.

Then:

(7) $$\mathrm{Tor}^R_m(K,K)=\begin{cases} K, & \text{if } m=0 \\ K^s, & \text{if } m=1 \\ \bigoplus_{x\in M\setminus\{0\}} H_{m-2}(\Delta(x);K), & \text{if } m\geq 2 \end{cases}$$

(where $H_p(\Delta(x);K)$ denotes the p-th homology group of $\Delta(x)$, with coefficients in K), s being the number of minimal elements in $M\setminus\{0\}$. This result enables us to compute the components of $\mathrm{Tor}^R_{\bullet}(K,K)$ and to derive valuable information about the Betti numbers, in some interesting special cases.

We begin by reminding some useful notions about finite posets.

Let $(P,\leq)$ be a finite (non-void) graded poset, of rank $r>0$. Thus, P has $\hat{0}$ (absolute minimum) and $\hat{1}$ (absolute maximum) and every maximal chain in P, has lenght r. $\overline{P}=P\setminus\{\hat{0},\hat{1}\}$ is the "proper part" of P. The degree on P will be denoted by "deg". The simplicial complex $\Delta(P)$, whose i-simplices are precisely the i-chains of P: $y_0<y_1<\dots<y_i$, for $i\geq -1$ ($\emptyset$ being also an (-1)--chain), defines over any field K a canonical "chain complex" whose homology modules are denoted by:

$$\tilde{H}_j(P,K),\quad j=0,1,\dots,r$$

(since $\tilde{H}_j(P,K)=0$, for $j\leq -1$ or $j>r$).

The "reduced Euler characteristic" of P is the alternating sum:

$$\chi(P)=\sum_{i=0}^{r}(-1)^i \dim_K \tilde{H}_i(P,K)\ .$$

A remarkable result is the following:

Theorem (Ph. Hall)

Let $(P,\leq)$ be a finite (non-empty) graded poset. Then:

(8) $\chi(P)=\mu(P)$,

where $\mu(P)=\mu(\hat{0},\hat{1})$ is the total Möbius function on P.

An important class of posets is the following one (cf. [2]).

2. Definition

Let $(P,\leq)$ be a graded (non-necessarily finite) poset.

Then P is "Cohen-Macaulay" over the field K, if for each open interval (x,y) in P: $\tilde{H}_i(x,y)=0$, if $i\neq r(x,y)-2$ (where $r(x,y)=$ $=\deg(y)-\deg(x)$ is the dimension of the simplicial complex $\Delta((x,y))$, when $\{(x,y)=z\in P/x<y<z\}$ is ordered with the induced order form P).

It follows that, for each open interval (x,y) of a Cohen--Macaulay poset P, only "the top homology" is non-vanishing and moreover, if $\delta(x,y)=r(x,y)-2$, then:

$$\text{(9)}\qquad \dim_K H_{\delta(x,y)}((x,y);K)=(-1)^{r(x,y)}\cdot\mu_P(x,y)\quad,$$

μ_P being the Möbius function on P.

Now, we are in position to formulate our

3. Proposition

Let (M,+) be a cancellative, abelian monoid, such that there is a standard gradation $|\;|:M\longrightarrow Z_+$.

Suppose M has the following property:

(*) $(\forall)\ x\in M,\ \hat{X}=\{y\in M/y\leq x\}$ is a Cohen-Macaulay poset.

Then, for any field K:

$$\text{(10)}\qquad \mathrm{Tor}_m^{K[M]}(K,K)=\begin{cases} K, & \text{if } m=0\\ K^{h_1(M)}, & \text{if } m=1\\ \bigoplus_{|x|=m} H_{m-2}(\Delta(x);K), & \text{if } m\geq 2\end{cases}$$

where $h_1(M)=\#\{x\in M\ /\ |x|=1\}$.

Proof

Using (7), the only assertion to be proved is the one about the homological degrees $m\geq 2$ (the elements $\{x\in M/|x|=1\}$ being minimal in $M\setminus\{0\}$).

However, by (*) and by Definition 2, $H_j(\Delta(x),K)=0$ for $j\neq r(x,0)-2$ (since $\hat{x}=[0,x]$ in $(M,\leq)$), implying that

$$\bigoplus_{x\in M\setminus\{0\}} H_{m-2}(\Delta(x);K) \text{ reduces to } \bigoplus_{x:r(x,0)-2=m-2} H_{m-2}(\Delta(x);K)=$$

$$=\bigoplus_{x:r(x,0)=m} H_{m-2}(\Delta(x);K)=(\text{since the degree of the poset } (M,\leq)$$

is given by the fixed standard gradation) $=\bigoplus_{|x|=m} H_{m-2}(\Delta(x);K)$.

4. Corollary

Let $(M,+)$ be as in the enounce of Proposition 3. The Poincaré series of $K[M]$ is:

$$(11)\quad P_M(z)=\sum_{m\geq 0}(-1)^m\Big(\sum_{|x|=m}\mu(x)\Big)z^m ,$$

μ being the Möbius function of the monoid M (cf. (2), § 2).

Proof

By (9) and (10), we succesively get, for $m\geq 2$:

$$\dim_K Tor_m^{K[M]}(K,K)=\sum_{|x|=m}\dim_K H_{m-2}(\Delta(x),K)=$$

$$=\sum_{|x|=m}(-1)^{r(x,0)}\mu_M(0,x)=(\text{since the degree on the poset } M$$

is given by the gradation $|\ |$) $=\sum_{|x|=m}(-1)^m\mu_M(0,x)=$ (by (2), § 2) $=(-1)^m\sum_{|x|=m}\mu(x)$.

For $m=0$, the unique element of degree zero in M is 0, therefore $\mu(0)=1$. For $m=1$, the elements of $M_1=\{x\in M/\ |x|=1\}$ are the ones of degree 1 and each of them dominates 0, hence:

$\mu(x)=\mu(0,x)=-1$, if $|x|=1$.

This ends the proof of the assertion.

5. Corollary

Let $(M,+)$ be as in the enounce of Proposition 3. Let $H_M(z)$ be the Hilbert series of the standard gradation on M and let $P_M(z)$ be the Poincaré series of $K[M]$ (K any field).

Then the following holds:

(12) $P_M(-z)H_M(z)=1$ in $Z[[z]]$.

Proof

The relation follows immediately from (11), Cor.4 together with Corollary 2, § 2.

We call a monoid algebra verifying (12) a "Fröberg ring" , such that our last result states that, if M has Cohen-Macaulay principal order-ideals (cond. (*) of Prop.3), then $K[M]$ is a Fröberg ring (which means that K has a linear minimal free resolution over $K[M]$).

We are left with the problem of deciding what (*) of Proposition 3 really means. We try to do this into the next section.

4. MONOIDS WITH COHEN-MACAULAY PRINCIPAL ORDER-IDEALS

We begin by reminding some combinatorics.

To any finite graded poset $(P,\leq)$ one associates its Stanley-Reisner ring $\mathcal{R}_P$, defined as follows: let $\overline{P}=\{x_1,\dots,x_n\}$ be the proper part of P and let $K[X_1,\dots,X_n]$ be the polynomial ring in $\#\overline{P}$ indeterminates (over our base field K). Let I be the ideal of $K[X_1,\dots,X_n]$, generated by all the monomials X_iX_j , such that x_i and x_j are incomparable in $\overline{P}$. Then: $\mathcal{R}_P=K[X_1,\dots,X_n]/I$. The importance of this ring lies in the following

Theorem (Reisner)

P is Cohen-Macaulay (as a poset) if and only if the ring $\mathcal{R}_P$ is Cohen-Macaulay.

There is no really simple proof to this result (the quickest one is due to Hochster). Subsequent work of Stanley, Garsia, Baclawski, Björner, e.a. provided testing criteria for the C.-M property of large classes of finite posets.

We use here one of these criteria, suited to our situation.

Namely, following [2], we consider a labeling of each edge $x \to y$ in a finite graded poset P, say $\lambda(x \to y) \in Z$ (or $\lambda(x \to y)$ in any totally ordered finite set, which is the same).

Then an arbitrary chain of P: $c=(x_o \leq x_1 \leq \ldots \leq x_m)$ gets a natural labeling, namely: $\lambda(c)=(\lambda(x_o \to x_1), \lambda(x_1 \to x_2), \ldots$ $\ldots, \lambda(x_{m-1} \to x_m))$, with $\lambda(c) \in Z^m$. Therefore, the chains of P may be totally ordered, according to the lexicographic order of their labels (written $<_L$ here).

An "edge-wise lexicographic labeling" (called EL-labeling, for short) is a labeling of the edges of P, with the following properties:

(a) for every interval $[x,y]$ of P, there is an unique chain $a_{x,y}$ $: x=x_o \leq x_1 \ldots \leq x_m=y$, such that: $\lambda(x_o \to x_1) \leq \lambda(x_1 \to x_2) \leq \ldots \leq \lambda(x_{m-1} \to x_m)$ in Z.

(b) for every other m-chain b in $[x,y]$: $\lambda(b) >_L \lambda(a_{x,y})$.

2. Definition

A finite graded poset $(P, \leq)$ is called "lexicographically shellable" if it has an EL-labeling.

The importance to us, of this notion, lies in the following result.

6. Proposition

(i) Any lexicographically shellable poset is Cohen-Macaulay

(ii) A distributive lattice is lexicographically shellable

(The proof of this Proposition may be found in [2]. We only remark that (ii) is immediate, in virtue of the structure of any distributive lattice, as the lattice of all order ideals over its join-irreducibles. (i) follows from the above quoted Theorem of Reisner, using a testing criterium for Cohen-Macaulayness, due to Garsia).

Having, thus, plenty of examples for Cohen-Macaulay posets, we state two more properties of these objects, used in the sequel.

To formulate them, we remind that, for any graded poset $(P,\leq)$ of rank $r>0$ and for any non-void subset $S\subseteq\{0,1,\dots,r\}$, the "rank-selected subposet" P_S is defined as $\{x\in P/\deg x\in S\}$, with the restricted order from P.

Also, if $(P,\leq)$ and $(Q,\leq)$ are two graded posets of the same rank $r>0$, we define the "ordinal sum" $P\circ Q$ as the disjoint union $\dot{\bigcup}_{m=0}^{r}(P_m \times Q_m)$ with the induced product structure of the poset $P\times Q$ (here P_m, Q_m denote the sets of all degree m elements in P, Q respectively).

7. Proposition

(i) If $(P,\leq)$ is a Cohen-Macaulay poset, then P_S is Cohen-Macaulay for any $S\subseteq\{0,1,\dots,r\}$, $r=\mathrm{rk}P$.

(ii) If $(P,\leq)$ and $(Q,\leq)$ are lexicographically shellable, then $P\circ Q$ is lexicographically shellable.

((i) was proved in [2]. (ii) follows by considering the labeling: $\lambda_{P\circ Q}((x_1,y_1)\to(x_2,y_2))=(\lambda_P(x_1\to x_2),\mathrm{rk}P+\lambda_Q(y_1\to y_2))$, where we consider the usual lexicographic order on the pairs (a,b) $(a,b\in Z)$ and where λ_P, λ_Q are EL-labelings of P, Q respectively).

Now, we come back to graded monoids.

Let $(M, |\ |)$ be a standard graded monoid (cf. § 1) and let $s > 0$ be an integer. The "s-th Veronese selection" of $(M, |\ |)$ is the submonoid of M, generated by its s-th component M_s (in the given gradation). This submonoid is denoted by M(s). It is also standard graded by $(1/s)|\ |$, the m-th component of M being M_{ms}, for every $m \geq 0$.

More, M(s) is normally embedded in M (i.e. the inner poset structure of M(s) coincides with the restriction of the poset structure on M).

If $(M, |\ |)$ and $(M', |\ |')$ are standard graded monoids, then their "Segre product" is the monoid $(M \circ M', \|\ \|)$ defined as the submonoid $\dot{\bigcup}_{m\geq 0} M_m \times M'_m$ of $M \times M'$. $M \circ M'$ is also standard graded by $\|(x,y)\| = m$, if $|x| = |y|' = m$ and it is normally embedded in $M \times M'$.

We prove the following:

8. Proposition

(i) Let M be a standard graded monoid. If $(M, \leq)$ has Cohen--Macaulay principal order ideals, the same is true for any Veronese selection M(s), $s \geq 1$.

(ii) Let M, M′ be standard graded monoids. If $(M, \leq)$ and $(M', \leq)$ have lexicographically shellable principal order-ideals, the same is true for their Segre product $M \circ M'$.

Proof

(i) follows from Prop.7, (i), because any principal order ideal in M(s) is a rank-selection into a principal order ideal of M.

(ii) follows from Prop.7, (ii) because any principal order ideal in $M \circ M'$ is an ordinal sum of principal order ideals in

M, M′ respectively.

9. Proposition

Let $n > 0$ be an integer and consider the free monoid Z_+^n, graded by the usual total degree gradation (with respect to its canonical basis). Then Z_+^n has Cohen-Macaulay principal order ideals.

Proof

The assertion follows from Proposition 6, because any principal order ideal in Z_+^n is a distributive lattice.

Remark

It may be proved that any Veronese selection into a free abelian monoid, is lexicographically shellable.

Putting together the above results, we derive the following

10. Proposition

Let $n_1, \ldots, n_t > 0$ and $s_1, \ldots, s_t > 0$ be integers. Consider the monoid $M(n_1, \ldots, n_t; s_1, \ldots, s_t) = Z_+^{n_1}(s_1) \circ \ldots \circ Z_+^{n_t}(s_t)$. Then, for any field K, the monoid algebra $K[M(n_1, \ldots, n_t; s_1, \ldots, s_t)]$ is a Fröberg ring.

Proof

The assertion follows from Propositions 9, 8 and Cor. 5 of §3. (It also follows from the total semimodularity(*) of the principal order ideals of $M(n_1, \ldots, n_t; s_1, \ldots, s_t)$, a fact implying their shellability (cf. [2]), hence their CM-ness)

Of course, using the above developed technique, wider classes of Fröberg monoidal rings may be derived. In particular this may be

(*) This semimodularity is a consequence of the fact that $M(n_1, \ldots, n_t; s_1, \ldots, s_t)$ is defined by certain quadratic relations.

successfully applied to algebraic singularities arising from (finite) abelian group actions on polynomials. Also, for standard graded monoids with quadratic defining relations, our technique eventually leads to characterizations of these relations, which assure the Fröberg property of the corresponding monoid algebras.

March, 21 1985

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