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The aim of this paper is to illustrate how one may obtain the rigourous derivation of constitutive restrictions, as an example for the Navier-Stokes-Fourier fluids, from the entropy inequality by using a new approach proposed by Suliciu in [12]. This procedure avoids the criticisms one may do to the previous methods of Coleman and Noll [1], Müller [5-7] and Liu [8].

Introduction

Coleman and Noll [1] (see also [2,3]) developed a general procedure for a systematical derivation of the constitutive restrictions imposed by entropy inequality.

The main fundamental objection against their method is the possibility to define thermodynamic processes by a suitable choice of body forces and radiation supplies in the equations of balance (which in general are regarded as assigned in advance).

Müller in [5-7] and also Liu [8,9] avoid this criticism constructing analytic thermodynamic processes from analytic initial data, by using the Cauchy-Kowalewsky theorem. But, in order to apply this theorem they need additionally *a priori* constitutive assumptions.

Thus, this procedure is not convenient because our purpose is to derive logical consequences from the entropy inequality which, eventually, should serve to prove existence theorems in mathematics and not viceversa.

Starting from this point of view, recently, Suliciu [12] has proposed an other approach for the rigourous derivation of constitutive restrictions from the entropy inequality.

This new procedure replaces the notion of thermodynamic admissible process with that of thermodynamic admissible rates.

One means by thermodynamic admissible rates (as in [12]) the time derivatives of thermodynamics fields at the initial time compatible with the general balance laws, with the constitutive assumptions under considerations and with the initial conditions of an initial value problem.

We then require that the entropy inequality in the form proposed by Müller [4-7] with the entropy and the entropy flux given by constitutive equations, holds at every state of the constitutive domain and for all thermodynamic admissible rates.

In the present paper, we apply this procedure for the simple case of Navier-Stokes-Fourier fluids and we obtain familiar restrictions (see for example Müller [11], Chap.1, and Coleman and Mizel [2]). What is new here is the modality to derive them.

Unlike Müller [5-7, 11], we do not use the notion of an ideal wall but, instead, we use the following unanimously accepted assertions: in thermostatics the Gibbs relation is valid and the absolute temperature is a positive-valued monotonically increasing function of the empirical temperature.

General balance equations and constitutive assumptions

The main objective of the thermodynamics of fluids is the determination of the five thermodynamics fields

$$\begin{aligned} &\text{density} && \rho(x_k,t), \\ &\text{velocity} && v_i(x_k,t), \\ &\text{(empirical) temperature} && \theta(x_k,t), \end{aligned} \tag{1}$$

in a body.

For this purpose it is customary to rely on the five equations of balance of mass, momentum and energy, which in a supply-free body and under sufficient smoothness assumptions have the forms

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + v_j \rho_{,j} + \rho v_{j,j} = 0 , \\ &\rho \frac{\partial v_i}{\partial t} + \rho v_j v_{i,j} - t_{ij,j} = 0 , \\ &\rho \frac{\partial \varepsilon}{\partial t} + \rho v_j \varepsilon_{,j} + q_{j,j} - t_{ij} v_{j,i} = 0 . \end{aligned} \tag{2}$$

These equations must be supplemented by constitutive equations which relate the stress t_{ij}, the heat flux q_i, and the specific internal energy ε to the fields ρ, v_i, θ for the material under consideration.

In the case of viscous, heat-conducting fluids it is assumed that the constitutive equations depend on

$$\rho , \ \theta , \ v_i , \ \theta_{,k} , v_{i,k}$$

and they obey the principle of material objectivity.

In this paper, we shall consider the special case of Navier-Stokes-Fourier fluids whose constitutive equations result by linearization of the most general constitutive equations of viscous, heat-conducting fluids and have the form (see for instance Müller [11], Chap.1, and Coleman and

Mizel [2])

$$t_{ij}=\hat{t}_{ij}(\rho,\theta,d_{mn},\theta_{,k})=-p_E(\rho,\theta)\delta_{ij}+\nu(\rho,\theta)d_{\ell\ell}\delta_{ij}+2\mu(\rho,\theta)d_{ij}$$

$$q_i=\hat{q}_i(\rho,\theta,d_{mn},\theta_{,k})=-\varkappa(\rho,\theta)\theta_{,i} \tag{3}$$

$$\varepsilon=\hat{\varepsilon}(\rho,\theta,d_{mn},\theta_{,k})=\varepsilon_E(\rho,\theta)+\lambda(\rho,\theta)d_{\ell\ell}$$

where $d_{ij}=v_{(i,j)}$ is the symmetric part of the velocity gradient $v_{i,j}$.

The constitutive functions $\hat{t}_{ij}$, $\hat{q}_i$ and $\hat{\varepsilon}$ are C^2 real-valued functions, defined on the constitutive domain $\mathcal{D}=(0,\infty)\times(\theta_\ell,\infty)\times R^6\times R^3$, where $\theta_\ell\in R$ is the lower bound for the fixed empirical temperature scale θ.

Here, since $-p_E\delta_{ij}=\hat{t}_{ij}(\rho,\theta,0,0)$, p_E is called the equilibrium pressure, $\varepsilon_E=\hat{\varepsilon}(\rho,\theta,0,0)$ the specific energy in equilibrium, μ the shear viscosity, $\nu+\frac{2}{3}\mu$ the bulk viscosity and $\varkappa$ the thermal conductivity.

We also assume that

$$\frac{\partial\hat{\varepsilon}(\rho,\theta,d_{mn},\theta_{,k})}{\partial\theta}\neq 0 \tag{4}$$

for any $(\rho,d_{mn},\theta_{,k})\in(0,\infty)\times R^6\times R^3$.

Thermodynamic admissible rates

After introducing the constitutive functions (3) in the equations of balance (2) we obtain a determinate set of differential equations for the thermodynamics fields $\rho(x_k,t)$, $v_i(x_k,t)$, $\theta(x_k,t)$, $(x_k,t)\in D\times I$, where D is an open and bounded set in R^3, $I=[t_o,t_1)$, $t_o\geq 0$.

Every solution of this system of equations is called usually a thermodynamic admissible process (see for example [1-9, 11]).

It is rational to suppose that every thermodynamic admissible process is generated by an initial and boundary value problem.

Let us consider for the system (2) and (3) an initia[l] value problem:

$$\rho(x_k, t_o) = \rho^o(x_k), \quad v_i(x_k, t_o) = v_i^o(x_k), \quad \theta(x_k, t_o) = \theta^o(x_k) \tag{5}$$

where $\rho^o(x_k) > 0$, $v_i^o(x_k) \in R$, $\theta^o(x_k) \in (\theta_\ell, \infty)$ for $x_k \in D$.

Following Suliciu [12], we give a precise meaning to the term <u>thermodynamic admissible rates</u> for a Navier-Stokes-Fourier fluid; these are defined to be the time derivatives of the density, velocity and (empirical) temperature at the initial time ($t = t_o$) compatible with the equations of balance (2), the constitutive equations (3) and with the initial data (5).

Using the notations

$$\left(\rho_{,t}\,,\ v_{i,t}\,,\ \theta_{,t}\right) = \frac{\partial}{\partial t}\left(\rho, v_i, \theta\right)(x_k, t_o), \tag{6}$$

we obtain the following relations for the thermodynamic admissible rates:

$$\rho_{,t} = -v_j^o \rho_{,j}^o - \rho^o v_{j,j}^o\ , \tag{7}$$

$$\begin{aligned} v_{i,t} &= \frac{1}{\rho^o} \frac{\partial \hat{t}_{ik}}{\partial x_k}\bigg|_{(x_\ell, t_o)} - v_k^o v_{i,k}^o = \\ &= \frac{1}{\rho^o}\left(\frac{\partial \hat{t}_{ik}}{\partial \rho}\bigg|_P \rho_{,k}^o + \frac{\partial \hat{t}_{ik}}{\partial \theta}\bigg|_P \theta_{,k}^o + \frac{\partial \hat{t}_{ik}}{\partial d_{mn}}\bigg|_P v_{(m,n)k}^o \right) - v_k^o v_{i,k}^o\,, \end{aligned} \tag{8}$$

$$\begin{aligned} \theta_{,t} = \frac{1}{\rho^o \dfrac{\partial \hat{\varepsilon}}{\partial \theta}\bigg|_P} \Bigg[& \hat{t}_{ij}\Big|_P v_{j,i}^o + \rho^{o2} \frac{\partial \hat{\varepsilon}}{\partial \rho}\bigg|_P v_{j,j}^o - \rho^o \frac{\partial \hat{\varepsilon}}{\partial d_{mn}}\bigg|_P d_{mn,t} - \\ & - \rho^o v_j^o \left(\frac{\partial \hat{\varepsilon}}{\partial \theta}\bigg|_P \theta_{,j}^o + \frac{\partial \hat{\varepsilon}}{\partial d_{mn}}\bigg|_P v_{(m,n)j}^o \right) - \end{aligned}$$

$$- \frac{\partial \hat{q}_j}{\partial \rho}\Big|_P \rho^\circ_{,j} - \frac{\partial \hat{q}_j}{\partial \theta}\Big|_P \theta^\circ_{,j} - \frac{\partial \hat{q}_j}{\partial \theta_{,k}}\Big|_P \theta^\circ_{,kj} \Big]$$

where $P=(\rho^\circ(x_k), \theta^\circ(x_k), v^\circ_{(m,n)}(x_k), \theta^\circ_{,\ell}(x_k)) \in \mathcal{D}$. (9)

The expresion

$$d_{mn,t} = \frac{\partial^2 v_{(m}}{\partial t \partial x_{n)}}(x_k, t_o), \tag{10}$$

of relation (9) can be calculated by means of (8) and we get

$$\begin{aligned}
d_{ij,t} &= -\frac{1}{\rho^{\circ 2}} \rho^\circ_{,(j} \frac{\partial \hat{t}_{i)k}}{\partial x_k}\Big|_{(x_\ell,t_o)} + \frac{1}{\rho^\circ} \frac{\partial^2 \hat{t}_{k(i}}{\partial x_{j)} \partial x_k}\Big|_{(x_\ell,t_o)} - v^\circ_{k,(j} v^\circ_{i),k} - v^\circ_k v^\circ_{(i,j)k} = \\
&= -\frac{1}{\rho^{\circ 2}} \rho^\circ_{,(j} \Big[\frac{\partial \hat{t}_{i)k}}{\partial \rho}\Big|_P \rho^\circ_{,k} + \frac{\partial \hat{t}_{i)k}}{\partial \theta}\Big|_P \theta^\circ_{,k} + \frac{\partial \hat{t}_{i)k}}{\partial d_{mn}}\Big|_P v^\circ_{(m,n)k} \Big] + \\
&\quad + \frac{1}{\rho^\circ} \Big[\frac{\partial \hat{t}_{k(i}}{\partial \rho}\Big|_P \rho^\circ_{,j)k} + \frac{\partial \hat{t}_{k(i}}{\partial \theta}\Big|_P \theta^\circ_{,j)k} + \frac{\partial \hat{t}_{k(i}}{\partial d_{mn}}\Big|_P v^\circ_{(m,n)j)k} + \\
&\quad + \rho^\circ_{,k} \Big(\frac{\partial^2 \hat{t}_{k(i}}{\partial \rho^2}\Big|_P \rho^\circ_{,j)} + \frac{\partial^2 \hat{t}_{k(i}}{\partial \theta \partial \rho}\Big|_P \theta^\circ_{,j)} + \frac{\partial^2 \hat{t}_{k(i}}{\partial d_{mn} \partial \rho}\Big|_P v^\circ_{(m,n)j)} \Big) + \\
&\quad + \theta^\circ_{,k} \Big(\frac{\partial^2 \hat{t}_{k(i}}{\partial \rho \partial \theta}\Big|_P \rho^\circ_{,j)} + \frac{\partial^2 \hat{t}_{k(i}}{\partial \theta^2}\Big|_P \theta^\circ_{,j)} + \frac{\partial^2 \hat{t}_{k(i}}{\partial d_{mn} \partial \theta}\Big|_P v^\circ_{(m,n)j)} \Big) + \\
&\quad + v^\circ_{(m,n)k} \Big(\frac{\partial^2 \hat{t}_{k(i}}{\partial \rho \partial d_{mn}}\Big|_P \rho^\circ_{,j)} + \frac{\partial^2 \hat{t}_{k(i}}{\partial \theta \partial d_{mn}}\Big|_P \theta^\circ_{,j)} + \frac{\partial^2 \hat{t}_{k(i}}{\partial d_{rs} \partial d_{mn}}\Big|_P v^\circ_{(r,s)j)} \Big) \Big] - \\
&\quad - v^\circ_{k,(j} v^\circ_{i),k} - v^\circ_k v^\circ_{(i,j)k} .
\end{aligned} \tag{11}$$

Entropy principle

The entropy principle is considered as a major restriction on the forms of constitutive functions which are compatible with thermodynamics.

We shall adopt here a modified form of the Müller's entropy principle (see for example [4-7]).

We assume that in every body the specific entropy η and the entropy flux ϕ_i are given by constitutive relations. The constitutive functions $\hat{\eta}$ and $\hat{\phi}_i$ are real functions defined on a constitutive domain $\mathcal{D}$ according to the principles of equipresence and of material objectivity.

The points of $\mathcal{D}$ will be called states.

In our case, for the Navier-Stokes-Fourier fluids

$$\mathcal{D} = (0, \infty) \times (\theta_\ell, \infty) \times R^6 \times R^3$$

and

$$\eta = \hat{\eta}(\rho, \theta, d_{mn}, \theta_{,k}) = \eta_E(\rho, \theta) + h(\rho, \theta) d_{\ell\ell}$$

$$\phi_i = \hat{\phi}_i(\rho, \theta, d_{mn}, \theta_{,k}) = -\varphi(\rho, \theta)\theta_{,i} \tag{12}$$

(see for example [11], Chap.1).

The specific production of entropy τ in a supply-free body, and under suitable smoothness assumptions is defined as

$$\tau = \rho \frac{\partial \eta}{\partial t} + \rho v_i \eta_{,i} + \phi_{i,i} \tag{13}$$

Using the method suggested by Suliciu in [12] we then require that the following postulate be accomplished.

POSTULATE: <u>The entropy inequality</u>

$$\tau \geq 0 \tag{14}$$

<u>must hold at every state of the constitutive domain $\mathcal{D}$ and for every thermodynamic admissible rates.</u>

The present procedure, for the rigourous derivation of constitutive restrictions from the entropy inequality is different for all previous methods.

Coleman and Noll in [1] introduce the notion of thermodynamic admissible processes and construct them by a suitable choice of body forces and radiation supplies in the equations of balance (although in general they are regarded as assigned a priori). This argument has been critisized by Woods in [10] (according to Müller [11], Section 5.4.2).

Müller in [5-7] and also Liu in [8,9] construct thermodynamic admissible processes from analytic initial data but they need additionally <u>a priori</u> constitutive assumptions (i.e. the analyticity of the constitutive functions) in order to apply the Cauchy-Kowalewsky theorem.

In contrast to this, we here consider (as in [12]), that there is not an a priori reason to suppose that an initial value problem has solution and therefore, it is more rational to replace the notion of thermodynamic admissible process with that of thermodynamic admissible rates and, eventually, use the restrictions of thermodynamics in order to prove existence theorems in mathematics and not viceversa.

Consequences of the entropy inequality

Let us consider the initial value problem (5) for the Navier-Stokes-Fourier fluid.

After introducing the constitutive functions $(12)_1$ and $(12)_2$, for η and ϕ_i, in the expression (13) of τ, and carrying out the indicated differentiations at the initial time $t=t_o$ and in an arbitrary point $x_k \in D$, we obtain the following explicit form of the entropy inequality

$$\tau\big|_{(x_k,t_o)} = \rho^\circ\left(\frac{\partial\hat{\eta}}{\partial\rho}\Big|_P \rho_{,t} + \frac{\partial\hat{\eta}}{\partial\theta}\Big|_P \theta_{,t} + \frac{\partial\hat{\eta}}{\partial d_{mn}}\Big|_P d_{mn,t}\right) +$$

$$+ \rho^\circ v_i^\circ\left(\frac{\partial\hat{\eta}}{\partial\rho}\Big|_P \rho^\circ_{,i} + \frac{\partial\hat{\eta}}{\partial\theta}\Big|_P \theta^\circ_{,i} + \frac{\partial\hat{\eta}}{\partial d_{mn}}\Big|_P v^\circ_{(m,n)i}\right) +$$

$$+\left(\frac{\partial\hat{\Phi}_i}{\partial\rho}\Big|_P \rho^o_{,i} + \frac{\partial\hat{\Phi}_i}{\partial\theta}\Big|_P \theta^o_{,i} + \frac{\partial\hat{\Phi}_i}{\partial\theta_{,k}}\Big|_P \theta^o_{,ki}\right) \geqslant 0 \tag{15}$$

where $P=(\rho^o(x_k), \theta^o(x_k), v^o_{(m,n)}(x_k), \theta^o_{,\ell}(x_k)) \in \mathcal{D}$.

This inequalitiy must hold at every state $P \in \mathcal{D}$ and for every thermodynamic admissible rates $\rho_{,t}$, $\theta_{,t}$ and $v_{i,t}$.

The inequalitiy (15) then becomes ofter substitutio of (7), (8) and (9)

$$\begin{aligned}\tau\Big|_{(x_k,t_o)} = &\left(\frac{\partial\hat{\Phi}_j}{\partial\rho}\Big|_P - \Lambda\Big|_P \frac{\partial\hat{q}_j}{\partial\rho}\Big|_P\right)\rho^o_{,j} + \\ &+\left(\frac{\partial\hat{\Phi}_j}{\partial\theta}\Big|_P - \Lambda\Big|_P \frac{\partial\hat{q}_j}{\partial\theta}\Big|_P\right)\theta^o_{,j} + \left(\frac{\partial\hat{\Phi}_{(j}}{\partial\theta_{,k)}}\Big|_P - \Lambda\Big|_P \frac{\partial\hat{q}_{(j}}{\partial\theta_{,k)}}\Big|_P\right)\theta^o_{,k} \\ &+\left(\frac{\partial\hat{\eta}}{\partial d_{mn}}\Big|_P - \Lambda\Big|_P \frac{\partial\hat{\varepsilon}}{\partial d_{mn}}\Big|_P\right)\rho^o\left(v^o_j v^o_{(m,n)j} + d_{mn,t}\right) + \\ &+\left[\Lambda\Big|_P \hat{t}_{ij}\Big|_P - \rho^{o2}\left(\frac{\partial\hat{\eta}}{\partial\rho}\Big|_P - \Lambda\Big|_P \frac{\partial\hat{\varepsilon}}{\partial\rho}\Big|_P\right)\delta_{ij}\right] v^o_{j,i} \geqslant 0\end{aligned} \tag{16}$$

where the quantity Λ has been introduced according to the definition:

$$\Lambda = \Lambda(\rho, \theta, d_{mn}, \theta_{,k}) = \frac{\dfrac{\partial\hat{\eta}}{\partial\theta}}{\dfrac{\partial\hat{\varepsilon}}{\partial\theta}} \tag{17}$$

and $d_{mn,t}$ is given by (11).

By inspection of (16) one may see that $\tau\big|_{(x_k,t_o)}$ depends only on the values of the following quantities

ρ^o, θ^o, $\theta^o_{,i}$, $v^o_{i,j}$, $\rho^o_{,i}$, $\rho^o_{,ij}$, $\theta^o_{,ij}$, $v^o_{(i,j)k}$, $v^o_{(i,j)k\ell}$

which can be independently and arbitrarily specified.

The postulate (14) is equivalent to the assertion that $\tau\big|_{(x_k,t_o)}$ be $\geqslant 0$ at every arbitrarily fixed state $P=(\rho^o, \theta^o, v^o_{(m,n)}, \theta^o_{,k}) \in \mathcal{D}$ and for all choices of $\rho^o_{,i}$, $\rho^o_{,ij}$, $\theta^o_{,ij}$, $v^o_{(i,j)k}$, $v^o_{(i,j)k\ell}$ and $v^o_{[i,j]}$ *).

To derive the necessary conditions for the validity of the postulate, we first observe that (16) can be written,

---

*) $v_{[i,j]}$ is the antisymmetric part of the velocity gradient v

by means of (11), in the form

$$\tau\Big|_{(x_k,t_0)} = \left(\frac{\partial\hat{\eta}}{\partial d_{mn}}\Big|_P - \Lambda\Big|_P \frac{\partial\hat{\varepsilon}}{\partial d_{mn}}\Big|_P\right)\frac{1}{\rho^o}\,\rho^o_{,k(n}\,\frac{\partial\hat{t}_{m)k}}{\partial\rho}\Big|_P + $$
$$+ f(\rho^o, \theta^o, \theta^o_{,i}, v^o_{i,j}, \rho^o_{,i}, \theta^o_{,ij}, v^o_{(i,j)k}, v^o_{(i,j)kl}). \tag{18}$$

If we assign ρ^o, θ^o, $\theta^o_{,i}$, $v^o_{i,j}$, $\rho^o_{,i}$, $\theta^o_{,ij}$, $v^o_{(i,j)k}$, $v^o_{(i,j)kl}$ and $\rho^o_{,ij}$ as arbitrarily specified, then $\tau\big|_{(x_k,t_0)}$ is $\geqslant 0$ only if

$$\frac{\partial\hat{\eta}}{\partial d_{mn}} - \Lambda\frac{\partial\hat{\varepsilon}}{\partial d_{mn}} = 0\ . \tag{19}$$

Since the left hand side of the inequality (16) depends linearly on $\rho^o_{,i}$ and $\theta^o_{,ki}$ and since the inequality has to hold for any fixed $P = (\rho^o, \theta^o, v^o_{(i,j)}, \theta^o_{,k}) \in \mathcal{D}$ and for an arbitrary choice of $\rho^o_{,i}$ and $\theta^o_{,ki}$, the following conditions must hold

$$\frac{\partial\hat{\phi}_i}{\partial\rho} - \Lambda\frac{\partial\hat{q}_i}{\partial\rho} = 0\ , \tag{20}$$

$$\frac{\partial\hat{\phi}_{(i}}{\partial\theta_{,k)}} - \Lambda\frac{\partial\hat{q}_{(i}}{\partial\theta_{,k)}} = 0, \tag{21}$$

and

$$\left(\frac{\partial\hat{\phi}_i}{\partial\theta} - \Lambda\frac{\partial\hat{q}_i}{\partial\theta}\right)\theta^o_{,i} +$$
$$+\left[\Lambda\hat{t}_{ij} - \rho^{o2}\left(\frac{\partial\hat{\eta}}{\partial\rho} - \Lambda\frac{\partial\hat{\varepsilon}}{\partial\rho}\right)\delta_{ij}\right]v^o_{(i,j)} \geqslant 0. \tag{22}$$

It is obvious that (17), (19)-(22) is a necessary and sufficient set of restrictions on constitutive equations (3) and (12) for the inequality (14) to hold for every thermodynamic admissible process if this exists.

Now, we are able to derive the same constitutive restrictions as those obtained by Müller [11], Chap.1, using the method of Lagrange multipliers (see also [2]).

If we introduce the constitutive functions $(3)_2$ and $(12)_2$ in (21) we get

$$\varphi(\rho,\theta)+\Lambda\varkappa(\rho,\theta)=0. \tag{23}$$

Assuming that $\varkappa(\rho,\theta)\neq 0$ for any $(\rho,\theta)\in(0,\infty)\mathrm{x}R$, it follows that

$$\Lambda=\Lambda(\rho,\theta) \quad \text{and} \quad \hat{\Phi}_i=\Lambda(\rho,\theta)\hat{q}_i. \tag{24}$$

From (24) and (20) we obtain that $\frac{\partial\Lambda}{\partial\rho}=0$.

Therefore,

$$\Lambda=\Lambda(\theta) \quad \text{and} \quad \hat{\Phi}_i=\Lambda(\theta)\hat{q}_i. \tag{25}$$

Since (19) and (25) implies that $\hat{\eta}-\Lambda\hat{\varepsilon}$ is independent of d_{mn} , and assuming that $\frac{d\Lambda}{d\theta}\neq 0$ for any $\theta\in R$, one obtains from relation (17), written in the form

$$\frac{\partial(\hat{\eta}-\Lambda\hat{\varepsilon})}{\partial\theta}=-\hat{\varepsilon}\frac{d\Lambda}{d\theta}$$

that $\hat{\varepsilon}$ is also independent of d_{mn}

Thus, $\hat{\eta}$ and $\hat{\varepsilon}$ are both functions of ρ and θ only, which means that the coefficients λ and h from $(3)_3$ and $(12)_1$ both vanish.

We can therefore summarize conditions (17) and (19)-(21) for the Navier-Stokes-Fourier fluid in the form:

$$\begin{aligned}
\hat{\varepsilon} &= \varepsilon_E(\rho,\theta),\\
\hat{\eta} &= \eta_E(\rho,\theta),\\
\hat{q}_i &= -\varkappa(\rho,\theta)\theta_{,i}\,,\\
\hat{\Phi}_i &= -\Lambda(\theta)\varkappa(\rho,\theta)\theta_{,i}\,,\\
\Lambda(\theta) &= \frac{\frac{\partial\eta_E}{\partial\theta}}{\frac{\partial\varepsilon_E}{\partial\theta}}.
\end{aligned} \tag{26}$$

Introducing the constitutive relations $(3)_1$ and $(26)_{1-4}$, in the left hand side of the residual entropy inequality (22), we obtain the specific entropy production for a Navier-Stokes-Fourier fluid

$$\tau = \tau(\varrho, \theta, d_{ij}, \theta_{,k}) = -\frac{d\Lambda}{d\theta} \varkappa \theta_{,i} \theta_{,i} + \\ + \left[\Lambda(\theta)(2\mu d_{ij} + \nu d_{kk} \delta_{ij}) - \left(p_E \Lambda(\theta) + \varrho^2 \frac{\partial(\eta_E - \Lambda(\theta)\varepsilon_E)}{\partial\varrho}\right)\delta_{ij}\right] d_{ij} \tag{27}$$

We see that τ assumes its minimum value, namely zero, for $d_{ij}=0$ and $\theta_{,i}=0$.

The necessary conditions for this local minimum are the following

$$\left.\frac{\partial\tau}{\partial\theta_{,i}}\right|_{(\varrho,\theta,0,0)} = 0, \qquad \left.\frac{\partial\tau}{\partial d_{ij}}\right|_{(\varrho,\theta,0,0)} = 0 \tag{28}$$

and

$$\left.\frac{\partial^2\tau}{\partial\theta_{,i}\,\partial d_{mn}}\right|_{(\varrho,\theta,0,0)} \text{ - is positive semi-definite.} \tag{29}$$

More explicitly, the relations (28) read

$$\frac{\partial\eta_E}{\partial\varrho} = \Lambda\left(\frac{\partial\varepsilon_E}{\partial\varrho} - \frac{1}{\varrho^2} p_E\right). \tag{30}$$

Relations (29), and the previous assumptions that $\varkappa \neq 0$ and $\frac{d\Lambda}{d\theta} \neq 0$, also imply

$$\varkappa \frac{d\Lambda}{d\theta} < 0, \quad \mu\Lambda(\theta) \geqslant 0, \quad (3\nu + 2\mu)\Lambda(\theta) \geqslant 0. \tag{31}$$

From (17) and (30) we obtain the following integrability condition

$$\frac{1}{\Lambda(\theta)}\cdot\frac{d\Lambda(\theta)}{d\theta}=\frac{\frac{\partial p_E}{\partial\theta}}{\rho^2\frac{\partial\varepsilon_E}{\partial\rho}-p_E} \tag{32}$$

for the total differential of the function η_E

$$d\eta_E=\Lambda(\theta)\left[\frac{\partial\varepsilon_E}{\partial\theta}\,d\theta+\left(\frac{\partial\varepsilon_E}{\partial\rho}-\frac{1}{\rho^2}p_E\right)d\rho\right] \tag{33}$$

which we can rewrite in a more familiar form

$$d\eta_E=\Lambda(\theta)\left[d\varepsilon_E+p_E\,d\left(\frac{1}{\rho}\right)\right]. \tag{34}$$

We <u>assume</u> that at equilibrium situations, i.e. when $d_{ij}=0$ and $\theta_{,i}=0$, the Gibbs relation of thermostatistics should hold

$$d\eta_E=\frac{1}{T(\theta)}\left[d\varepsilon_E+p_E\,d\left(\frac{1}{\rho}\right)\right] \tag{35}$$

where $T=T(\theta)$ is the absolute temperature of thermostatics. At the same time, we know from thermostatics that the absolute temperature is a positive - valued, monotonically increasing function of the empirical temperature θ, i.e.

$$T(\theta)>0 \quad\text{and}\quad \frac{dT(\theta)}{d\theta}>0 \quad\text{for any}\quad \theta\in(\theta_e,\infty). \tag{36}$$

Therefore, we obtain from (34) and (35) that $\Lambda=\Lambda(\theta)$ is the reciprocal of the absolute temperature

$$\Lambda(\theta)=\frac{1}{T(\theta)}\,, \tag{37}$$

and from here it results that $\Lambda=\Lambda(\theta)$ is a universal function of θ, i.e. it is the same function for all Navier-Stokes-Fourier fluids for a fixed empirical temperature scale.

Using relations (31) and (36) we obtain the following familiar restrictions for a Navier-Stokes-Fourier fluid

$$\varkappa > 0, \qquad \mu \geqslant 0, \qquad 3\nu + 2\mu \geqslant 0. \tag{38}$$

Also the relations (25) and (37) imply

$$\phi_i = \frac{q_i}{T} \tag{39}$$

which means that the entropy flux is the heat flux divided by the absolute temperature and thus the entropy principle implies the Clausius -Duhem inequality in a supply - free body.

Remark: Unlike Müller in [5, 11] we do not use the notion of an ideal wall to derive the universal character of function $\Lambda=\Lambda(\theta)$.

We also do not need to apply the integrability condition (32) for the ideal gas (as Müller did in [5], p.15, and [11], Section 1.3.2.4, although he has excluded it by supposing that $\varkappa \neq 0$ to derive (32)) in order to obtain that $\Lambda(\theta)$ is the reciprocal of the absolute temperature.

R E F E R E N C E S

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