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1. Introduction. Let G be a finite group and let P be one of its Sylow p-subgroups. If Q is another Sylow p-subgroup of G, the intersection $H = P \cap Q$ is said to be *tame* if both $N_P(H)$ and $N_Q(H)$ are Sylow p-subgroups of $N_G(H)$. The set of all nontrivial tame intersections of the form $H = P \cap Q$ will be denoted by $T(P)$.

Tame intersections are important because of a strong fusion result of J. Alperin [1] . We quote here only a consequence of Alperin's theorem, which characterizes locally the focal subgroup $P \cap G'$:

$$P \cap G' = \langle P \cap N_G(H)' \, , \, H \in T(P) \rangle \tag{1}$$

Let $\phi(G)$ be the Frattini subgroup of G and let $R(G)$ be the intersection of the maximal and normal subgroups of G. It is quite evident that $P \cap G' \leqslant P \cap R(G)$ and that $R(G)/G' = \phi(G/G')$. We call the subgroup $P \cap R(G)$ the *superfocal subgroup of* P (with respect to G). The reason of considering the superfocal subgroup is the following result of [2] :

$$G \text{ has a normal p-complement iff } P \cap R(G) = \phi(P) \tag{2}$$

Note that for every finite group G and each Sylow p-subgroup P of G, $\phi(P) \leqslant P \cap R(G)$. Thus the equality in the above inclusion assures the existence of a normal p-complement of G.

The aim of this note is to establish the following purely local characterization of the superfocal subgroup :

THEOREM. $$P \cap R(G) = \langle P \cap R(N_G(H)) \, , \, H \in T(P) \rangle$$

As an immediate consequence of this result we shall derive a strong form of a classical result of Frobenius which gives sufficient conditions for the existence of a normal p-complement. The following result was also obtained by Alperin [1] , by another proof :

COROLLARY. If $N_G(H)$ has a normal p-complement for every $H \in T(P)$, then G has also a normal p-complement.

2. The arguments. Before starting the proof of the theorem, let us make some notation conventions. If $H \leq G$, we shall denote by H_G° the intersection of those maximal subgroups of G which contain H. The following lemmas are elementary :

LEMMA 1. i) If $H \trianglelefteq G$, then $H_G^\circ / H = \phi(G/H)$.

ii) If $H \leq K \leq G$, then $H_G^\circ \leq K_G^\circ$.

LEMMA 2. If $H \leq G$, then $R(H) \leq R(G)$. In particular, if G is a p-group, then for every $H \leq G$ we have $\phi(H) \leq \phi(G)$.

The proof of the theorem uses at a key point the following characterization of the superfocal subgroup (see [2]) :

LEMMA 3. $$P \cap R(G) = (P \cap G')_P^\circ$$

The proof of the theorem. Set $P^* = \langle P \cap R(N_G(H)),\ H \in T(P) \rangle$. It is clear that $P^* \leq P \cap R(G)$ since by virtue of lemma 2, $P \cap R(N_G(H)) \leq P \cap R(G)$ for every $H \in T(P)$.

To prove the converse inclusion, observe first that by Alperin's result (1) $P \cap G' = \langle P \cap N_G(H)',\ H \in T(P) \rangle \leq \langle P \cap R(N_G(H)),\ H \in T(P) \rangle = P^*$ since $N_G(H)' \leq R(N_G(H))$ for every $H \in T(P)$. Thus $P \cap G' \leq P^*$ and by lemma 1 ii) we have that $(P \cap G')_P^\circ \leq (P^*)_P^\circ$. But by lemma 3 $(P \cap G')_P^\circ = P \cap R(G)$, so $P \cap R(G) \leq (P^*)_P^\circ$. Thus all we have now to prove is the equality $(P^*)_P^\circ = P^*$.

Note first that $P^* \trianglelefteq P$. Indeed, if $H \in T(P)$ and $x \in P$, then $H^x \in T(P)$ as one can easy verify. On the other hand, conjugation by $x \in P$ sends $P \cap R(N_G(H))$ into $P \cap R(N_G(H^x)) \leq P^*$.

Moreover, $\phi(P) \leq P^*$. Indeed, $\phi(P) = R(P) = R(P) \cap P \leq P \cap R(N_G(P))$. Since P itself is a tame intersection we obtain that $\phi(P) \leq P^*$.

Since $\phi(P) \leq P^{*} \trianglelefteq P$, the group P/P^{*} is elementary abelian. Therefore $\phi(P/P^{*}) = 1$. But by lemma 1 i), $\phi(P/P^{*}) = (P^{*})^{\circ}_{p}/P^{*}$. Then $(P^{*})^{\circ}_{p} = P^{*}$ and this completes the proof. //

Proof of the corollary. Suppose that $N_G(H)$ has a normal p-complement for every $H \in T(P)$. Then since H is a tame intersection, $N_P(H)$ is a Sylow p-subgroup of $N_G(H)$ and by virtue of (2) $N_P(H) \cap R(N_G(H)) = \phi(N_P(H))$. But $N_P(H) \cap R(N_G(H)) = P \cap R(N_G(H))$ and $\phi(N_P(H)) \leq \phi(P)$ by lemma 2. Thus $P \cap R(N_G(H)) \leq \phi(P)$ for every $H \in T(P)$ and by our theorem $P \cap R(G) \leq \phi(P)$. Since the inclusion $\phi(P) \leq P \cap R(G)$ is obvious, $P \cap R(G) = \phi(P)$ and G has a normal p-complement by (2).//

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