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ISSN 0250 3638

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PREPRINT SERIES IN MATHEMATICS
No. 26/1985

BUCURESTI

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April 1985

*) The National Institute for Scientific and Tehnical Creation

On the Spectrum of a Stable Rank 3 Reflexive Sheaf on $\mathbb{P}^3$

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0. Introduction

The spectrum of a stable rank 2 vector bundle on $\mathbb{P}^3$ was defined and studied by W.Barth and G.Elencwajg in [1]. Their work was extended by R.Hartshorne to stable rank 2 reflexive sheaves on $\mathbb{P}^3$ in [3] and [4]. Then C.Okonek and H.Spindler have defined in [6] the spectrum of a reflexive sheaf of arbitrary rank on $\mathbb{P}^3$ and even more, they have defined in [8] the spectrum of a torsion free sheaf.

The aim of this paper is to prove some finer properties of the spectrum of a stable rank 3 reflexive sheaf on $\mathbb{P}^3$ similar to the properties of the spectrum of a stable rank 2 reflexive sheaf proved by R.Hartshorne in [3] and [4]. Particularly, we show that in the cas $c_1=0$, the main properties of the spectrum do not depend on the generic splitting type. Our proofs are adaptations of the proofs given by R.Hartshorne for the rank 2 case in the two papers mentioned abov

As an application, we find further restrictions to be imposed to the Chern classes of a stable rank 3 vector bundle on $\mathbb{P}^3$. We will show, in a forthcoming paper, that these restrictions are sufficient to assure the existence of this kind of bundles.

1. Technical Results on $\mathbb{P}^2$

Throughout this paper we work over an algebraically closed field k of characteristic 0. Let E be a stable rank 3 vector bundle on $\mathbb{P}^2$ and let S be the graded k-algebra $\bigoplus_{\ell \in \mathbb{Z}} H^0(\mathcal{O}_{\mathbb{P}^2}(\ell))=k[x_0,x_1,x_2]$. Then $M=\bigoplus_{\ell \in \mathbb{Z}} H^1(E(\ell))$ has a natural structure of graded S-module.

Let N be a graded S-submodule of M. By the Theorem of Grauert-Mülich-Spindler, E has the generic splitting type $(0,0,0)$ or $(-1,0,1)$ if $c_1=0$, $(-1,0,0)$ if $c_1=-1$ and $(-1,-1,0)$ if $c_1=-2$.
Using the results of [6, Sect.1] one gets some information about N. The aim of this section is to show that, when E is stable, one can push one step further the results concerning N.

Definition. Let E be a rank 3 vector bundle on $\mathbb{P}^2$ with $c_1=0$, -1 or -2. A jumping line for E of order $r>0$ is a line $L\subset\mathbb{P}^2$ such that $H^0(E_L^*(-r))\neq 0$ and $H^0(E_L^*(-r-1))=0$.

Lemma 1.1. (Reduction step). Let E be a rank 3 vector bundle on $\mathbb{P}^2$ with $c_1=0$, -1 or -2 and L a jumping line for E of order $r>0$. We have $E_L=\mathcal{O}_L(-r)\oplus F_L$, where F_L is a locally free $\mathcal{O}_L$-module of rank 2.

(a) If $c_1=0$ or -1 then there are exact sequences:

$$0\to E'\to E\to\mathcal{O}_L(-r)\to 0 \qquad (1)$$

$$0\to E\to E'(1)\to F_L(1)\to 0 \qquad (2)$$

where E' is a rank 3 vector bundle on $\mathbb{P}^2$ with Chern classes $c_1'=c_1-1$ and $c_2'=c_2-r-c_1$.

(b) If $c_1=-2$ then there are exact sequences:

$$0\to E'(-1)\to E\to\mathcal{O}_L(-r)\to 0 \qquad (3)$$

$$0\to E\to E'\to F_L(1)\to 0 \qquad (4)$$

where E' is a rank 3 vector bundle on $\mathbb{P}^2$ with Chern classes $c_1'=0$ and $c_2'=c_2-r-1$.

Proof. (a) We get an epimorphism $E\to\mathcal{O}_L(-r)$ by composing the epimorphism $E_L\to\mathcal{O}_L(-r)$ with the restriction epimorphism $E\to E_L$.
Put $E'=\mathrm{Ker}(E\to\mathcal{O}_L(-r))$. The exact sequence:

$$0\to\mathcal{O}(-r-1)\to\mathcal{O}(-r)\to\mathcal{O}_L(-r)\to 0 \qquad (5)$$

shows us that $\mathcal{O}_L(-r)$ has homological dimension ≤ 1 at every point.
It follows that E' is a rank 3 vector bundle.

We have obtained the exact sequence (1). Using the exact sequence (5) we find that the Chern polynomial of $\mathcal{O}_L(-r)$ is :

$$c_t(\mathcal{O}_L(-r))=1+t+(r+1)t^2$$

Now, the Chern classes of E' can be computed using the exact sequence (1).

Applying the Snake lemma to the exact commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & E(-1) \arrow[d] & & \\
0 \arrow[r] & E' \arrow[r] \arrow[d, dashed] & E \arrow[r] \arrow[d] & \mathcal{O}_L(-r) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F_L \arrow[r] & E_L \arrow[r] \arrow[d] & \mathcal{O}_L(-r) \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

we get an exact sequence :

$$0 \longrightarrow E(-1) \longrightarrow E' \longrightarrow F_L \longrightarrow 0$$

which tensorized by $\mathcal{O}(1)$ gives us the exact sequence (2).

(b) Similarly :

Corollary 1.2. Under the hypothesis of Lemma 1.1 we have:

(a) If $c_1=0$ and E is semistable then the generic splitting type of E' is $(-1,0,0)$.

(a bis) If $c_1=0$ and E is stable then E' is stable.

(b) If $c_1=-1$ and E is stable then the generic splitting type of E' is $(-1,-1,0)$.

(c) If $c_1=-2$ and E is stable then E' is semistable.

Proof. (a) The exact sequence (1) and the dual of (2) show us that $H^0(E'(-1))=0$ and $H^0(E'^*(-2))=0$.

Case 1. E' is stable. In this case the assertion follows by the Theorem of Grauert-Mülich-Spindler.

Case 2. $H^0(E'^*(-1))\neq 0$. Then E' can be realized as an extension:

$$0 \longrightarrow F \longrightarrow E' \longrightarrow I_Z(-1) \longrightarrow 0$$

where Z is a closed subscheme of $\mathbb{P}^2$ of codimension ≥ 2 and F is a semistable rank 2 vector bundle on $\mathbb{P}^2$ with $c_1(F)=0$. The generic splitting type of F is $(0,0)$. It follows that if L is a generic line for F which does not intersect Z then $E'_L = \mathcal{O}_L \oplus \mathcal{O}_L \oplus \mathcal{O}_L(-1)$.

Case 3. $H^0(E')\neq 0$ and $H^0(E'^*(-1))=0$. Then E'^* can be realized as an extension:

$$0 \to F \to E'^* \to I_Z \to 0$$

where Z is a closed subscheme of $\mathbb{P}^2$ of codimension ≥ 2 and F is a stable rank 2 vector bundle on $\mathbb{P}^2$ with $c_1(F)=1$. The generic splitting type of F is $(0,1)$. It follows that if L is a generic line for F which does not intersect Z then $E'^*_L = \mathcal{O}_L(1) \oplus \mathcal{O}_L \oplus \mathcal{O}_L$.

(a bis), (b) and (c). Similarly.

Lemma 1.3. *Let E be a semistable rank 3 vector bundle on $\mathbb{P}^2$.*

(a) *If* $c_1=0$ *then:* $h^1(E(-1))=h^1(E(-2))=c_2$.

(a bis) *If* $c_1=0$ *and* E *is stable then:* $h^1(E)=h(E(-3))=c_2-3$.

(b) *If* $c_1=-1$ *then:* $h^1(E)=c_2-2$, $h^1(E(-1))=c_2$, $h^1(E(-2))=c_2-1$.

(c) *If* $c_1=-2$ *then:* $h^1(E)=c_2-2$, $h^1(E(-1))=c_2-1$, $h^1(E(-2))=$ $=c_2-3$.

Proof. One uses the Theorem of Riemann-Roch.

Lemma 1.4. *Let Z be a closed subscheme of $\mathbb{P}^2$ of codimension 2, N a graded S-submodule of* $\bigoplus_{\ell\in\mathbb{Z}} H^1(I_Z(\ell))$ *and* $n_\ell = \dim N_\ell$. *Put* $S_1=H^0(\mathcal{O}_{\mathbb{P}^2}(1))$. *Then:*

(i) $n_{-2}\leq n_{-1}$

(ii) *If* $0\neq n_{-2}= n_{-1}$ *then there is a nonzero element* $\psi\in N_{-2}$ *such that* $\dim(S_1\cdot\psi)=1$.

Proof. We have exact sequences:

$$H^0(\mathcal{O}(\ell)) \to H^0(\mathcal{O}_Z(\ell)) \xrightarrow{\delta} H^1(I_Z(\ell)) \to H^1(\mathcal{O}(\ell)) = 0.$$

Put $N'_\ell=\delta^{-1}(N_\ell)$ and $n'_\ell=\dim N'_\ell$. $H^0(\mathcal{O}(\ell))=0$ for $\ell\leq -1$ hence $N_\ell\cong N'_\ell$ for $\ell\leq -1$.

Let $\lambda_o, \lambda_1, \lambda_2$ be a basis of S_1 such that λ_o does not vanish at any point of Z. The morphisms $\mathcal{O}_Z(-2) \to \mathcal{O}_Z(-1)$ and $\mathcal{O}_Z(-1) \to \mathcal{O}_Z$ defined by the multiplication by λ_o are isomorphisms. Particularly, it follows that the multiplication by $\lambda_o : H^o(\mathcal{O}_Z(-2)) \to H^o(\mathcal{O}_Z(-1))$ is bijective, hence $n'_{-2} \leq n'_{-1}$.

Let U_o be the open set of $\mathbb{P}^2$ where λ_o does not vanish. We consider Z as a closed subscheme of $U_o = \operatorname{Spec} k[t_1, t_2]$, where $t_1 = \lambda_1/\lambda_o$ and $t_2 = \lambda_2/\lambda_o$. We identify $\mathcal{O}_Z(-2)|U_o$ and $\mathcal{O}_Z(-1)|U_o$ to $\mathcal{O}_Z$ using the isomorphisms considered above. Then N'_{-2} and N'_{-1} correspond to vector subspaces of $H^o(\mathcal{O}_Z)$. $H^o(\mathcal{O}_Z)$ is a quotient ring of $\mathcal{O}(U_o) = k[t_1, t_2]$.

If $n'_{-2} = n'_{-1}$ then $N'_{-2} = N'_{-1}$, hence N'_{-2} is a vector subspace of $H^o(\mathcal{O}_Z)$ invariant with respect to the multiplication by t_1 and t_2. It follows that N'_{-2} is an ideal of $H^o(\mathcal{O}_Z)$. Let $z_1, \ldots, z_n$ be the points of Z. Then $H^o(\mathcal{O}_Z) = \mathcal{O}_{Z,z_1} \times \ldots \times \mathcal{O}_{Z,z_n}$, hence $N'_{-2} = I_1 \times \ldots \times I_n$, where I_j is an ideal of $\mathcal{O}_{Z,z_j}$. Suppose that $I_1 \neq 0$ and that $z_1 = (0,0)$. Then we can find $0 \neq \psi_1 \in I_1$ such that: $t_1 \cdot \psi_1 = 0$ and $t_2 \cdot \psi_1 = 0$. Put $\psi' = (\psi_1, 0, \ldots, 0) \in N'_{-2}$. Then $\dim (S_1 \cdot \psi') = 1$.

<u>Lemma</u> 1.5. <u>Let</u> E <u>be a vector bundle on</u> $\mathbb{P}^2$ <u>with</u> $H^o(E) = 0$. <u>If</u> $0 \neq \varphi \in H^1(E(-2))$ <u>then</u> $\dim (S_1 \cdot \varphi) \geq 2$.

<u>Proof</u>. See, for example, [5, Lemma 1.4, proof of the property (α2)]

<u>Proposition</u> 1.6. <u>Let</u> E <u>be a stable rank</u> 3 <u>vector bundle on</u> $\mathbb{P}^2$, $M = \bigoplus_{\ell \in \mathbb{Z}} H^1(E(\ell))$, N <u>a graded</u> S-<u>submodule of</u> M <u>and</u> $n_\ell = \dim N_\ell$.

(b) <u>If</u> $c_1 = -1$ <u>then</u>: $n_{-2} \neq 0$ <u>implies</u> $n_{-2} < n_{-1}$

(c) <u>If</u> $c_1 = -2$ <u>then</u>: $n_{-1} < c_2 - 1$ <u>implies</u> $n_{-1} \leq n_o$.

<u>Proof</u>. We shall use induction on $d = c_2$, the case $d = 1$ being vacuous. Now, suppose that (b) and (c) are true for all stable rank 3 vector bundles on $\mathbb{P}^2$ with $c_2 < d$.

<u>Step</u> 1. If $c_2 = d$ then (c) is true.

Put $F=E^*(-1)$. F is stable, $c_1(F)=-1$ and $c_2(F)=c_2-1<d$. N' is a graded quotient S-module of $M'=\bigoplus_{\ell\in\mathbb{Z}} H^1(E(\ell))' \cong \bigoplus_{\ell\in\mathbb{Z}} H^1(F(-\ell-2))$. Put $Q=\mathrm{Ker}\,(\bigoplus_{\ell\in\mathbb{Z}} H^1(F(-\ell-2)) \to N')$ and $q_\ell=\dim Q_\ell$. We have:

$$n_0=h^1(F(-2))-q_{-2}=c_2(F)-1-q_{-2}=c_2-2-q_{-2}$$

$$n_{-1}=h^1(F(-1))-q_{-1}=c_2(F)-q_{-1}=c_2-1-q_{-1}$$

If $n_{-1}<c_2-1$ then $q_{-1}\neq 0$. By the induction hypothesis, $q_{-2}<q_{-1}$ hence $n_{-1}\leq n_0$.

<u>Step</u> 2. If $c_2=d$ then (b) is true.

<u>Case</u> 1. $H^0(E^*)=0$. Then, according to the Bilinear map lemma [3, Lemma 5.1], we may suppose that there is a $\lambda\in H^0(\mathcal{O}(1))$, $\lambda\neq 0$, such that the multiplication by $\lambda: N_{-2}\to N_{-1}$ is not injective. Let L be the line of equation $\lambda=0$. Using the exact sequence:

$$H^0(E_L(-1)) \longrightarrow H^1(E(-2)) \xrightarrow{\lambda} H^1(E(-1))$$

one finds that $H^0(E_L(-1))\neq 0$, hence L is a jumping line for E of order $r>0$. Now, we apply the reduction step to E:

$$0\to E' \xrightarrow{\alpha} E \xrightarrow{\beta} \mathcal{O}_L(-r)\to 0$$

$$0\to E \xrightarrow{\gamma} E'(1) \to F_L(1)\to 0$$

The Chern classes of E' are $c_1'=-2$, $c_2'=c_2-r+1\leq d$ and E' is stable because $H^0(E^*)=0$. We have exact sequences:

$$H^0(E'(\ell+1)) \longrightarrow H^0(F_L(\ell+1)) \xrightarrow{\delta} H^1(E(\ell)) \xrightarrow{\gamma} H^1(E'(\ell+1))$$

Put $N_\ell'=\delta^{-1}(N_\ell)$ and $N_{\ell+1}''=\gamma(N_\ell)$. $H^0(E'(\ell+1))=0$ for $\ell\leq -1$ hence the sequence:

$$0\to N_\ell' \xrightarrow{\delta} N_\ell \xrightarrow{\gamma} N_{\ell+1}''\to 0$$

is exact for $\ell\leq -1$. Particularly:

$$n_{-1}=n_{-1}'+n_0''$$
$$n_{-2}=n_{-2}'+n_{-1}''$$

According to the proof of Lemma 1.1 the diagram:

$$\begin{tikzcd} & & N_{-2} \arrow[dl, "\gamma"'] \arrow[d, "\lambda"] \\ 0 = H^0(\mathcal{O}_L(-r-1)) \arrow[r] & H^1(E'(-1)) \arrow[r, "\alpha"] & H^1(E(-1)) \end{tikzcd} \qquad (6)$$

is commutative. The multiplication by $\lambda : N_{-2} \to N_{-1}$ is not injective, hence $\gamma : N_{-2} \to N''_{-1}$ is not injective, hence $N'_{-2} \neq 0$. It follows that $n'_{-2} < n'_{-1}$.

If $N''_{-1} \neq H^1(E'(-1))$ then by the induction hypothesis or by Step 1, $n''_{-1} \leq n''_0$ hence $n_{-2} < n_{-1}$.

Now, suppose that $N''_{-1} = H^1(E'(-1))$. We consider the exact sequences:

$$H^0(\mathcal{O}_L(\ell - r)) \to H^1(E'(\ell)) \xrightarrow{\alpha} H^1(E(\ell)) \xrightarrow{\beta} H^1(\mathcal{O}_L(\ell - r))$$

Put $N'''_\ell = \alpha^{-1}(N_\ell)$ and $N^{iv}_\ell = \beta(N_\ell)$. $H^0(\mathcal{O}_L(\ell - r)) = 0$ for $\ell \leq 0$ hence the sequence:

$$0 \to N'''_\ell \xrightarrow{\alpha} N_\ell \xrightarrow{\beta} N^{iv}_\ell \to 0$$

is exact for $\ell \leq 0$. Particularly:

$$n_{-1} = n'''_{-1} + n^{iv}_{-1}$$

$$n_{-2} = n'''_{-2} + n^{iv}_{-2}$$

Using the commutative diagram (6) and the fact that $\gamma(N_{-2}) = N''_{-1} = H^1(E'(-1))$ one finds that $N'''_{-1} = \alpha^{-1}(N_{-1}) = H^1(E'(-1))$. It follows that

$$n'''_{-2} \leq h^1(E'(-2)) = h^1(E'(-1)) - 2 = n'''_{-1} - 2$$

We also have $n^{iv}_{-2} \leq n^{iv}_{-1} + 1$, hence $n_{-2} < n_{-1}$.

Case 2. $H^0(E^*) \neq 0$. In this case E can be realized as an extension:

$$0 \to F \xrightarrow{\alpha} E \xrightarrow{\beta} I_Z \to 0$$

where Z is a closed subscheme of $\mathbb{P}^2$ of codimension ≥ 2 and F is a stable rank 2 vector bundle on $\mathbb{P}^2$ with $c_1(F) = -1$.

Case 2.1. Z is empty. Then we have exact sequences:

$$H^0(\mathcal{O}(\ell)) \longrightarrow H^1(F(\ell)) \xrightarrow{\alpha} H^1(E(\ell)) \longrightarrow H^1(\mathcal{O}(\ell))=0$$

Put $N'_\ell=\alpha^{-1}(N_\ell)$. $N'_\ell \cong N_\ell$ for $\ell \leq -1$. Using the results from [3, Sect.5] it follows that $n'_{-2}<n'_{-1}$, hence $n_{-2}<n_{-1}$.

Case 2.2. Z is not empty. Then we have exact sequences:

$$H^0(I_Z(\ell)) \longrightarrow H^1(F(\ell)) \xrightarrow{\alpha} H^1(E(\ell)) \xrightarrow{\beta} H^1(I_Z(\ell)).$$

Put $N'_\ell=\alpha^{-1}(N_\ell)$ and $N''_\ell=\beta(N_\ell)$. $H^0(I_Z(\ell))=0$ for $\ell \leq 0$ hence the sequence:

$$0 \longrightarrow N'_\ell \longrightarrow N_\ell \longrightarrow N''_\ell \longrightarrow 0$$

is exact for $\ell \leq 0$. Particularly:

$$n_{-1} = n'_{-1} + n''_{-1}$$

$$n_{-2} = n'_{-2} + n''_{-2}$$

We have $n''_{-2} \leq n''_{-1}$ (Lemma 1.4 (i)), and by [3, Sect.5], $n'_{-2}<n'_{-1}$ if $n'_{-2}\neq 0$. We have problems only if $n'_{-2}=0$ and $0\neq n''_{-2}= n''_{-1}$. If this is the case, then by Lemma 1.4 (ii), there is a nonzero element $\psi \in N''_{-2}$ such that $\dim (S_1\cdot\psi)=1$. Choose $\varphi\in N_{-2}$ such that $\beta(\varphi) =\psi$. According to Lemma 1.5 we have $\dim (S_1\cdot\varphi)\geq 2$. It follows that $N'_{-1} \neq 0$, hence $n'_{-2}<n'_{-1}$, hence $n_{-2}<n_{-1}$.

Corollary 1.7. Under the hypothesis of Proposition 1.6 we have:

(b) If $c_1= -1$ then: $n_{-3}=n_{-2}-1$ implies that there is a $\lambda\in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, $\lambda\neq 0$, such that $\lambda\cdot N_\ell = 0$ for all $\ell\leq -3$.

(c) If $c_1= -2$ then:

(ii) $n_{-2}\neq 0$ implies $n_{-2}<n_{-1}$

(iii) $n_{-3}=n_{-2}-1$ implies that there is a $\lambda\in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, $\lambda\neq 0$, such that $\lambda\cdot N_\ell = 0$ for all $\ell\leq -3$.

Proof. One uses the same kind of argument as in the proof of [6, Theorem 1.1 (iii) and (iv)] observing, for (c)(ii), that if $\lambda\in H^0(\mathcal{O}(1))$ then the multiplication by $\lambda : N_{-2} \longrightarrow H^1(E(-1))$ can't be surjective because $n_{-2}\leq h^1(E(-2))< h^1(E(-1))$.

Proposition 1.8. Let E be a stable rank 3 vector bundle on $\mathbb{P}^2$ with $c_1=0$, $M=\bigoplus_{\ell\in\mathbb{Z}} H^1(E(\ell))$, N a graded S-submodule of M and $n_\ell = \dim N_\ell$.

Then: $0<n_{-2}<c_2$ implies $n_{-2}<n_{-1}$.

Proof. According to the Bilinear map lemma we may suppose that there is a $\lambda\in H^0(\mathcal{O}(1))$, $\lambda\neq 0$, such that the multiplication by $\lambda: N_{-2}\to N_{-1}$ is not injective. Let L be the line of equation $\lambda=0$. Using the exact sequence :

$$H^0(E_L(-1))\to H^1(E(-2))\xrightarrow{\lambda} H^1(E(-1))$$

we find that $H^0(E_L(-1))\neq 0$, hence there is an $r>0$ such that $H^0(E_L(-r))\neq 0$ and $H^0(E_L(-r-1))=0$. It follows that $E_L=F_L\oplus\mathcal{O}_L(r)$ where F_L is a locally free $\mathcal{O}_L$-module of rank 2.

Now, we apply the reduction step to E using an epimorphism $E_L\to F_L$:

$$0\longrightarrow E'\xrightarrow{\alpha} E\xrightarrow{\beta} F_L\longrightarrow 0$$

$$0\longrightarrow E\xrightarrow{\gamma} E'(1)\longrightarrow \mathcal{O}_L(r+1)\longrightarrow 0$$

E' is a stable rank 3 vector bundle on $\mathbb{P}^2$ with $c_1(E')=-2$. We have exact sequences:

$$H^0(E'(\ell+1))\longrightarrow H^0(\mathcal{O}_L(\ell+r+1))\xrightarrow{\delta} H^1(E(\ell))\xrightarrow{\gamma} H^1(E'(\ell+1))$$

Put $N'_\ell=\delta^{-1}(N_\ell)$ and $N''_{\ell+1}=\gamma(N_\ell)$. $H^0(E'(\ell+1))=0$ for $\ell\leq -1$ hence the sequence:

$$0\longrightarrow N'_\ell\xrightarrow{\delta} N_\ell\xrightarrow{\gamma} N''_{\ell+1}\longrightarrow 0$$

is exact for $\ell\leq -1$. Particularly:

$$n_{-1}=n'_{-1}+n''_0$$

$$n_{-2}=n'_{-2}+n''_{-1}$$

We also consider the exact sequences :

$$H^0(E(\ell))\to H^0(F_L(\ell))\to H^1(E'(\ell))\xrightarrow{\alpha} H^1(E(\ell))\xrightarrow{\beta} H^1(F_L(\ell)).$$

Put $N'''_\ell=\alpha^{-1}(N_\ell)$ and $N^{iv}_\ell=\beta(N_\ell)$. $H^0(E(\ell))=0$ for $\ell<0$, hence the se-

$$0 \longrightarrow H^0(F_L(\ell)) \longrightarrow N_\ell''' \xrightarrow{\alpha} N_\ell \xrightarrow{\beta} N_\ell^{iv} \longrightarrow 0$$

is exact for $\ell \leq 0$.

We have an exact commutative diagram:

$$\begin{tikzcd}
 & & & N_{-2} \arrow[dl, "\gamma"'] \arrow[d, "\lambda"] \\
0 \arrow[r] & H^0(F_L(-1)) \arrow[r] & H^1(E'(-1)) \arrow[r, "\alpha"] & H^1(E(-1))
\end{tikzcd} \tag{7}$$

<u>Case</u> 1. $F_L = \mathcal{O}_L(a) \oplus \mathcal{O}_L(b)$ with $a<0$ and $b \leq 0$. In this case $H^0(F_L(-1))=0$. The multiplication by $\lambda : N_{-2} \rightarrow N_{-1}$ is not injective, hence using the diagram (7), one findsthat $\gamma : N_{-2} \rightarrow N''_{-1}$ is not injective. It follows that $N'_{-2} \neq 0$, hence $n'_{-2} < n'_{-1}$.

If $N''_{-1} \neq H^1(E'(-1))$ then by Proposition 1.6 (c), it follows that $n''_{-1} \leq n''_0$, hence $n_{-2} < n_{-1}$.

Now, suppose that $N''_{-1} = H^1(E'(-1))$. Using the commutative diagram (7) it follows that $N'''_{-1} = \alpha^{-1}(N_{-1}) = H^1(E'(-1))$. $H^0(F_L(\ell))=0$ for $\ell \leq -1$ hence the sequence:

$$0 \longrightarrow N_\ell''' \xrightarrow{\alpha} N_\ell \xrightarrow{\beta} N_\ell^{iv} \longrightarrow 0$$

is exact for $\ell \leq -1$. Particularly:

$$n_{-1} = n'''_{-1} + n^{iv}_{-1}$$

$$n_{-2} = n'''_{-2} + n^{iv}_{-2}$$

We have: $n'''_{-2} \leq h^1(E'(-2)) = h^1(E'(-1)) - 2 = n'''_{-1} - 2$.

If $N^{iv}_{-1} \neq H^1(F_L(-1))$ then $n^{iv}_{-2} \leq n^{iv}_{-1} + 1$, hence $n_{-2} < n_{-1}$.

If $N^{iv}_{-1} = H^1(F_L(-1))$ then $N_{-1} = H^1(E(-1))$, hence $n_{-2} < c_2 =$

$= h^1(E(-1)) = n_{-1}$.

<u>Case</u> 2. $F_L = \mathcal{O}_L(a) \oplus \mathcal{O}_L(b)$ with $a<0$ and $b>0$.

Firstly, suppose $n'_{-2} \neq 0$. If follows that $n'_{-2} < n'_{-1}$. If $N''_{-1} \neq H^1(E'(-1))$, then by Proposition 1.6 (c), $n''_{-1} \leq n''_0$ hence $n_{-2} < n_{-1}$.

If $N''_{-1}=H^1(E'(-1))$ then $N'''_{-1}=H^1(E'(-1))$. We have:

$$n_{-1}=(n'''_{-1}-h^0(F_L(-1)))+n^{iv}_{-1}$$

$$n_{-2}=(n'''_{-2}-h^0(F_L(-2)))+n^{iv}_{-2}$$

Now, $n'''_{-2}\leq h^1(E'(-2))=h^1(E'(-1))-2=n'''_{-1}-2$, hence $n'''_{-2}-h^0(F_L(-2))\leq$ $\leq n'''_{-1}-h^0(F_L(-1))-1$.

If $N^{iv}_{-1}\neq H^1(F_L(-1))$ then $n^{iv}_{-2}\leq n^{iv}_{-1}$, because $H^1(F_L(-1))\cong$ $\cong H^1(\mathcal{O}_L(a-1))$ and $H^1(F_L(-2))\cong H^1(\mathcal{O}_L(a-2))$. It follows that $n_{-2}<n_{-1}$.

If $N^{iv}_{-1}=H^1(F_L(-1))$ then $N_{-1}=H^1(E(-1))$ hence $n_{-2}<c_2=$ $=h^1(E(-1))=n_{-1}$.

Now, suppose $n'_{-2}=0$. It follows that the morphism $\gamma: N_{-2}\to$ $\to N''_{-1}$ is injective. Let K be the kernel of the morphism $N_{-2}\to N_{-1}$ defined by the multiplication by λ and C the cokernel of this morphism. Using the commutative diagram (7) one finds that $\dim K\leq h^0(F_L(-1))$. Put $R_{-\ell-3}=H^1(E(\ell))/N_\ell$ and $R=\bigoplus_{\ell\in\mathbb{Z}} R_{-\ell-3}$. R' is a graded S-submodule of $\bigoplus_{\ell\in\mathbb{Z}} H^1(E^*(\ell))$. Consider the exact commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[d] & H^0(E_L(-1)) \arrow[d] & & \\
0 \arrow[r] & N_{-2} \arrow[r] \arrow[d, "\lambda"] & H^1(E(-2)) \arrow[r] \arrow[d, "\lambda"] & R_{-1} \arrow[r] \arrow[d, "\lambda"] & 0 \\
0 \arrow[r] & N_{-1} \arrow[r] \arrow[d] & H^1(E(-1)) \arrow[r] \arrow[d] & R_{-2} \arrow[r] & 0 \\
 & C \arrow[d] & H^1(E_L(-1)) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

If the multiplication by $\lambda: R_{-1}\to R_{-2}$ is surjective, then by the Snake lemma, $\dim C\geq h^1(E_L(-1))$. But $\dim C=n_{-1}-n_{-2}+\dim K$ and $h^1(E_L(-1))=h^1(E(-1))-h^1(E(-2))+h^0(E_L(-1))=h^0(E_L(-1))$. We have $\dim K\leq h^0(F_L(-1))<h^0(E_L(-1))$ hence $n_{-1}-n_{-2}>0$.

If the multiplication by $\lambda : R_{-1} \to R_{-2}$ is not surjective, then the multiplication by $\lambda : R'_{-2} \to R'_{-1}$ is not injective, hence the S-submodule R' of $\bigoplus_{\ell \in \mathbb{Z}} H^1(E^*(\ell))$ satisfies the hypothesis of the Case 1. It follows that $r_{-2} < r_{-1}$, hence $n_{-2} < n_{-1}$.

<u>Corollary 1.9.</u> <u>Under the hypothesis of Proposition</u> 1.8 <u>we have:</u> $n_{-3} = n_{-2} - 1$ <u>implies that there is a</u> $\lambda \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, $\lambda \neq 0$, <u>such that</u> $\lambda \cdot N_\ell = 0$ <u>for all</u> $\ell \leq -3$.

<u>Proof.</u> One uses the same kind of argument as in the proof of [6, Theorem 1.1 (iv)] observing that if $\lambda \in H^0(\mathcal{O}(1))$ then the multiplication by $\lambda : N_{-3} \to H^1(E(-2))$ can't be surjective because $n_{-3} \leq h^1(E(-3)) < h^1(E(-2))$.

<u>Proposition 1.10.</u> <u>Let</u> E <u>be a semistable rank</u> 3 <u>vector bundle on</u> $\mathbb{P}^2$ <u>with</u> $c_1 = 0$, $M = \bigoplus_{\ell \in \mathbb{Z}} H^1(E(\ell))$, N <u>a graded</u> S-<u>submodule of</u> M <u>and</u> $n_\ell = \dim N_\ell$. <u>Then:</u>

(i) $n_{-2} \leq n_{-1}$

(ii), $n_{-3} \neq 0$ <u>implies</u> $n_{-3} < n_{-2}$

(iii) $n_{-4} = n_{-3} - 1$ <u>implies that there is a</u> $\lambda \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, $\lambda \neq 0$, <u>such that</u> $\lambda \cdot N_\ell = 0$ <u>for all</u> $\ell \leq -4$.

<u>Proof</u> (i). If E is stable then the assertion follows from the Proposition 1.8. If not, then E (or E^*) can be realized as an extension:

$$0 \longrightarrow F \longrightarrow E \longrightarrow I_Z \longrightarrow 0$$

where Z is a closed subscheme of $\mathbb{P}^2$ of codimension ≥ 2 and F is a semistable rank 2 vector bundle on $\mathbb{P}^2$ with $c_1(F) = 0$. It follows that the generic splitting type of E is (0,0,0). Now, our assertion follows from [6, Theorem 1.1 (ii)].

(ii) and (iii). One uses the same argument as in the proof of [6, Theorem 1.1 (iii) and (iv)].

<u>Proposition 1.11.</u> <u>Let</u> E <u>be a semistable rank</u> 3 <u>vector bundle on</u> $\mathbb{P}^2$,

$M = \bigoplus_{\ell \in \mathbb{Z}} H^1(E(\ell))$, R <u>a graded quotient</u> S<u>-module of</u> M <u>and</u> $r_\ell = \dim R_\ell$.

(a) <u>If</u> $c_1=0$ <u>then:</u>

(i) $r_{-2} \geqq r_{-1}$

(ii) $r_0 \neq 0$ <u>implies</u> $r_{-1} > r_0$

(iii) $r_0 = r_1 + 1$ <u>implies that there is a</u> $\lambda \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, $\lambda \neq 0$, <u>such that</u> $\lambda \cdot R_\ell = 0$ <u>for all</u> $\ell \geqq 0$

(a bis) <u>If</u> $c_1=0$ <u>and</u> E <u>is stable then:</u>

(ii) $0 < r_{-1} < c_2$ <u>implies</u> $r_{-2} > r_{-1}$

(iii) $r_{-1} = r_0 + 1$ <u>implies that there is a</u> $\lambda \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, $\lambda \neq 0$, <u>such that</u> $\lambda \cdot R_\ell = 0$ <u>for all</u> $\ell \geqq -1$.

(b) <u>If</u> $c_1 = -1$ <u>then:</u>

(i) $r_{-1} < c_2$ <u>implies</u> $r_{-2} \geqq r_{-1}$

(ii) $r_0 \neq 0$ <u>implies</u> $r_{-1} > r_0$

(iii) $r_0 = r_1 + 1$ <u>implies that there is a</u> $\lambda \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, $\lambda \neq 0$, <u>such that</u> $\lambda \cdot R_\ell = 0$ <u>for all</u> $\ell \geqq 0$.

(c) <u>If</u> $c_1 = -2$ <u>then:</u>

(ii) $r_0 \neq 0$ <u>implies</u> $r_{-1} > r_0$

(iii) $r_0 = r_1 + 1$ <u>implies that there is a</u> $\lambda \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, $\lambda \neq 0$, <u>such that</u> $\lambda \cdot R_\ell = 0$ <u>for all</u> $\ell \geqq 0$.

<u>Proof</u>. All these assertions rollow from the previous results by duality.

2. <u>Results Concerning the Spectrum</u>

Let $\mathcal{E}$ be a semistable rank 3 reflexive sheaf on $\mathbb{P}^3$ with $c_1 = 0, -1$ or -2. By Schneider's restriction theorems [2, Theorem 1.6 and Theorem 3.4] the restriction $\mathcal{E}_H$ to a general plane $H \subset \mathbb{P}^3$ is semistable unless $c_1 = -1$ and $\mathcal{E} = \Omega_{\mathbb{P}^3}(1)$ or $c_1 = -2$ and $\mathcal{E} = T_{\mathbb{P}^3}(-2)$.

Let's suppose, for the moment, that the restriction $\mathcal{E}_H$ to a general plane is semistable. Let $H \subset \mathbb{P}^3$ be a plane which does not contain any singular point of $\mathcal{E}$ and such that $\mathcal{E}_H$ is semistable on $H\cong\mathbb{P}^2$. Let $h=0$ an equation of H. We have exact sequences:

$$H^1(\mathcal{E}(\ell-1)) \xrightarrow{h} H^1(\mathcal{E}(\ell)) \longrightarrow H^1(\mathcal{E}_H(\ell)) \longrightarrow H^2(\mathcal{E}(\ell-1)) \xrightarrow{h} H^2(\mathcal{E}(\ell)).$$

Put:

$$N_\ell=\mathrm{Im}(H^1(\mathcal{E}(\ell)) \longrightarrow H^1(\mathcal{E}_H(\ell))), \quad n_\ell = \dim N_\ell$$

$$R_\ell=\mathrm{Ker}(H^2(\mathcal{E}(\ell-1)) \longrightarrow H^2(\mathcal{E}(\ell))), \; r_\ell = \dim R_\ell$$

The spectrum of $\mathcal{E}$ is an increasing sequence of integers $k_{\mathcal{E}} = (k_1,\dots,k_m)$ determined by the relations:

$$n_\ell - n_{\ell-1} = \mathrm{card}\{i \mid k_i = -\ell-1\} \qquad \text{for } \ell \leqq -1 \tag{1}$$

$$r_{\ell+1} - r_{\ell+2} = \mathrm{card}\{i \mid k_i = -\ell-3\} \text{ for } \ell \geqq -3 \text{ if } c_1=0 \tag{2}$$

and for $\ell \geqq -2$ if $c_1= -1$ or -2.

In the case $c_1=0$ we have to verify that the definition is correct, that is, we have to show that $n_{-1}-n_{-2} = r_{-2}-r_{-1}$. But this follows from the relations:

$$n_{-1}+r_{-1}=h^1(\mathcal{E}_H(-1))=h^1(\mathcal{E}_H(-2))=n_{-2}+r_{-2}$$

(1) and (2) implies that:

$$h^1(\mathcal{E}(\ell))=h^0(\bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^1}(k_i+\ell+1)) \qquad \text{for } \ell \leqq -1 \tag{3}$$

$$h^2(\mathcal{E}(\ell))=h^1(\bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^1}(k_i+\ell+1)) \qquad \text{for } \ell \geqq -3 \text{ if } c_1=0 \tag{4}$$

and for $\ell \geqq -2$ if $c_1= -1$ or -2.

If $\mathcal{E}=\Omega(1)$ then Bott's formulae imply that (3) and (4) are satisfied if we put $k_{\mathcal{E}}=(0)$. Similarly, if $\mathcal{E}=T(-2)$ we may put $k_{\mathcal{E}}= (-1)$.

We have the following relations:

$$m=c_2 \text{ if } c_1=0 \text{ or } -1 \text{ and } m=c_2-1 \text{ if } c_1= -2 \tag{5}$$

$$-2\sum k_i=c_3\,, \qquad \text{if } c_1=0 \tag{6a}$$

$$-2\sum k_i = c_3 + c_2\,, \qquad \text{if } c_1 = -1 \tag{6b}$$

$$-2\sum k_i = c_3 + 2c_2 - 2\,, \qquad \text{if } c_1 = -2. \tag{6c}$$

Only the case $c_1=0$ needs an argument (for $c_1= -1$ or -2 see [6, Lemma 3.4 and Lemma 3.5]). So in the case $c_1=0$ (1) and (2) imply that

$$m = n_{-1} + r_{-1} = h^1(\mathcal{E}_H(-1)) = c_2$$

(3) and (4) imply that:

$$-\chi(\mathcal{E}(-1)) = h^1(\mathcal{E}(-1)) - h^2(\mathcal{E}(-1)) =$$

$$= \sum_{i \mid k_i \geq 0} (k_i+1) - \sum_{i \mid k_i \leq -2} (-k_i-1) = \sum_{i=1}^{m} (k_i+1) = \sum k_i + m$$

and, by the Theorem of Riemann-Roch, $\chi(\mathcal{E}(-1)) = \frac{1}{2} c_3 - c_2$.

<u>Proposition</u> 2.1. <u>Let</u> $\mathcal{E}$ <u>be a semistable rank</u> 3 <u>reflexive sheaf on</u> $\mathbb{P}^3$ <u>and</u> $k_{\mathcal{E}} = (k_1,\dots,k_m)$ <u>its spectrum.</u>

(a) <u>Suppose</u> $c_1=0$. <u>If there is an</u> i <u>such that</u> $k_i \leq -1$ (<u>resp.</u> $k_i \geq 1$) <u>then</u> $k_i, k_i+1,\dots,-1$ (resp. $1,2,\dots,k_i$) <u>occur in the spectrum.</u>

(b) <u>Suppose</u> $c_1= -1$ <u>or</u> -2. <u>If there is an</u> i <u>such that</u> $k_i \leq -1$ (<u>resp.</u> $k_i \geq 0$) <u>then</u> $k_i, k_i+1,\dots,-1$ (<u>resp.</u> $0,1,\dots,k_i$) <u>occur in the spectrum.</u>

<u>Proof.</u> One uses the same argument as in the proof of [6, Proposition 3.3] using convenient results from Section 1.

<u>Proposition</u> 2.2. <u>Let</u> $\mathcal{E}$ <u>be a rank</u> 3 <u>reflexive sheaf on</u> $\mathbb{P}^3$ <u>such that the restriction</u> $\mathcal{E}_H$ <u>to a general plane</u> $H \subset \mathbb{P}^3$ <u>is stable and let</u> $k_{\mathcal{E}} = (k_1,\dots,k_m)$ <u>be its spectrum.</u>

(a) <u>Suppose</u> $c_1=0$. <u>If</u> 0 <u>does not occur in the spectrum then</u> $k_{m-2}=k_{m-1}=k_m= -1$ or $k_1=k_2=k_3=1$.

(b) <u>Suppose</u> $c_1= -1$. <u>If</u> 0 <u>does not occur in the spectrum then</u> -1 <u>occurs at least twice.</u>

(c) <u>Suppose</u> $c_1= -2$. <u>If</u> -1 <u>does not occur in the spectrum then</u> 0 <u>occurs at least twice.</u>

Proof. Let $H\subset\mathbb{P}^3$ be a plane which does not contain any singular point of $\mathcal{E}$ and such that $\mathcal{E}_H$ is stable. Put $E=\mathcal{E}_H$.

(a) (1) implies that $n_{-1}=n_{-2}$. Then by Proposition 1.8, $n_{-2}=0$ or $n_{-2}=h^1(E(-2))$. If $n_{-2}=0$ then $n_\ell=0$ for all $\ell\leqq -2$ and $r_{-1}=h^1(E(-1))$ hence $r_{-1}-r_0\geqq h^1(E(-1))-h^1(E)=3$. It follows that $k_{m-2}=k_{m-1}=k_m=-1$. Similarly, if $n_{-2}=h^1(E(-2))$ then $k_1=k_2=k_3=1$.

(b) (1) implies that $n_{-1}=n_{-2}$. Then by Proposition 1.6 (b), $n_{-1}=0$ hence $r_{-1}=h^1(E(-1))$. It follows that $r_{-1}-r_0\geqq h^1(E(-1))-h^1(E)=2$, that is, -1 occurs at least twice in the spectrum.

(c) (2) implies that $r_{-1}=r_0$. Then by Proposition 1.11 (c)(ii), $r_{-1}=0$ hence $n_{-1}=h^1(E(-1))$. It follows that $n_{-1}-n_{-2}\geqq h^1(E(-1))-$ $-h^1(E(-2))=2$, that is, 0 occurs at least twice in the spectrum.

Definition. Let $\mathcal{E}$ be a semistable rank 3 reflexive sheaf on $\mathbb{P}^3$ with $c_1=0$, -1 or -2. An unstable plane for $\mathcal{E}$ of order $r>0$ is a plane $H\subset\mathbb{P}^3$ such that:

$$H^0(\mathcal{E}_H^*(-r))\neq 0 \quad\text{and}\quad H^0(\mathcal{E}_H^*(-r-1))=0 \qquad (7)$$

Here $\mathcal{E}_H^*$ means the dual of $\mathcal{E}_H$ as an $\mathcal{O}_H$-module.

Remark 2.3. By Serre duality on $H\cong\mathbb{P}^2$, condition (7) above is equivalent to:

$$H^2(\mathcal{E}_H(r-3))\neq 0 \quad\text{and}\quad H^2(\mathcal{E}_H(r-2))=0 \qquad (8)$$

Let $h=0$ be an equation of a plane $H\subset\mathbb{P}^3$. Using the exact sequences

$$H^2(\mathcal{E}(r-2))\longrightarrow H^2(\mathcal{E}_H(r-2))\longrightarrow H^3(\mathcal{E}(r-3))=0$$

$$H^2(\mathcal{E}_H(r-3))'\longrightarrow H^2(\mathcal{E}(r-3))'\xrightarrow{h} H^2(\mathcal{E}(r-4))'$$

one finds that condition (8) is satisfied if $H^2(\mathcal{E}(r-2))=0$ and if the multiplication by $h:H^2(\mathcal{E}(r-3))'\longrightarrow H^2(\mathcal{E}(r-4))'$ is not injective.

Particularly, suppose that $h^2(\mathcal{E}(r-2))=0$ and $h^2(\mathcal{E}(r-4))\leqq$ $\leqq h^2(\mathcal{E}(r-3))+2$. Then by the Bilinear map lemma applied to :

$$H^2(\mathcal{E}(r-3))'\times H^0(\mathcal{O}_{\mathbb{P}^3}(1))\longrightarrow H^2(\mathcal{E}(r-4))'$$

there is an $h\in H^0(\mathcal{O}(1))$, $h\neq 0$, such that the multiplication by $h: H^2(\mathcal{E}(r-3))' \to H^2(\mathcal{E}(r-4))'$ is not injective, hence there is an unstable plane for $\mathcal{E}$ of order r.

Example 2.3.1. Let N be a nullcorrelation bundle on $\mathbb{P}^3$ and $\mathcal{E}=S^2N$. Then $k_{\mathcal{E}}=(0,0,0,0)$.

Indeed, we have $c_1(\mathcal{E})=0$, $c_2(\mathcal{E})=4$, and $c_3(\mathcal{E})=0$. The possible spectra of $\mathcal{E}$ are $(0,0,0,0)$, $(-1,0,0,1)$, $(-1,-1,1,1)$ and $(-2,-1,1,2)$. If the spectrum of $\mathcal{E}$ is not $(0,0,0,0)$ then by Remark 2.3, there is an unstable plane for $\mathcal{E}$, which is not the case.

Example 2.3.2. Let $\mathcal{E}$ be a stable rank 3 reflexive sheaf on $\mathbb{P}^3$ with $c_1=0$ which verifies the equivalent conditions (i), (ii), (iii) of [2, Proposition 5.1]. Then $k_{\mathcal{E}}=(-c_2+1,\dots,-1,0)$.

Indeed, let $k_{\mathcal{E}}=(k_1,\dots,k_m)$. Using the exact sequence:

$$0 \to \Omega(1) \to \mathcal{E} \to \mathcal{O}_{H_0}(-c_2+1) \to 0$$

one finds that $h^1(\mathcal{E}(-1))=1$, hence $k_m=0$. Now, the relation $-2\sum k_i = c_3=c_2^2-c_2$ implies that $k_{\mathcal{E}}=(-c_2+1,\dots,-1,0)$.

Example 2.3.3. Let $\mathcal{E}$ be a stable rank 3 reflexive sheaf on $\mathbb{P}^3$ with $c_1=0$ and $c_2=2$. Then the spectrum of $\mathcal{E}$ is connected. (See also [7]).

Indeed, we have to eliminate the spectrum $(-1,1)$. If $k_{\mathcal{E}}=(-1,1)$ then by Remark 2.3, there is an unstable plane for $\mathcal{E}$ of order 1 hence $\mathcal{E}$ satisfies the condition (iii) of [2, Proposition 5.1]. By Example 2.3.2 it follows that $k_{\mathcal{E}}=(-1,0)$, which is a contradiction.

Example 2.3.4. Let $\mathcal{E}$ be a stable rank 3 reflexive sheaf on $\mathbb{P}^3$ with $c_1=0$ and $c_2=3$. Then the spectrum of $\mathcal{E}$ is connected.

Indeed, if the restriction $\mathcal{E}_H$ to a general plane $H\subset\mathbb{P}^3$ is stable, then by Proposition 2.1 (a) and Proposition 2.2 (a), the spectrum of $\mathcal{E}$ is connected.

Now, suppose that the restriction of $\mathcal{E}$ to a general plane is not stable.

Case 1. $h^0(\mathcal{E}_H)=0$ for a general plane $H \subset \mathbb{P}^3$. In this case, for a general plane $H \subset \mathbb{P}^3$, $\mathcal{E}_H$ can be realized as an extension:

$$0 \longrightarrow F \longrightarrow \mathcal{E}_H \longrightarrow I_{Z,H} \longrightarrow 0$$

where Z is a closed subscheme of H of codimension ≥ 2 and F is a stable rank 2 vector bundle on $H \cong \mathbb{P}^2$ with $c_1(F) = 0$. We have $3=c_2(\mathcal{E}_H)=c_2(F)+h^0(\mathcal{O}_Z)$ and $c_2(F) \geq 2$, hence Z is empty or Z is one simple point. If 0 occurs in the spectrum then the spectrum is connected. If 0 does not occur in the spectrum then by (1), $n_{-1}=n_{-2}$. It follows that $N_{-1}=0$, or $N_{-1}=H^1(\mathcal{E}_H(-1))$, or $N_{-1}=\mathrm{Ker}(H^1(\mathcal{E}_H(-1)) \longrightarrow H^1(I_{Z,H}(-1)))$.

In the first two cases we must have $k_{\mathcal{E}}=(-1,-1,-1)$ and $k_{\mathcal{E}} = (1,1,1)$, respectively. In the third case we apply the reduction step to $\mathcal{E}$:

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow I_{Z,H} \longrightarrow 0$$

$\mathcal{E}'$ is a stable rank 3 reflexive sheaf on $\mathbb{P}^3$ with $c_1(\mathcal{E}')= -1$ and $c_2(\mathcal{E}')=3$. Using the exact sequence:

$$H^0(I_{Z,H}(\ell)) \longrightarrow H^1(\mathcal{E}'(\ell)) \longrightarrow H^1(\mathcal{E}(\ell)) \longrightarrow H^1(I_{Z,H}(\ell))$$

one finds that $h^1(\mathcal{E}(\ell))=h^1(\mathcal{E}'(\ell))$ for all $\ell \leq -1$, hence the nonnegative part of $k_{\mathcal{E}}$ is identical to the nonnegative part of $k_{\mathcal{E}'}$. By Proposition 2.1 (b) the nonnegative part of $k_{\mathcal{E}'}$ is empty, hence we must have $k_{\mathcal{E}} = (-1,-1,-1)$.

Case 2. $h^0(\mathcal{E}_H) \neq 0$ for a general plane $H \subset \mathbb{P}^3$. Then let $k_{\mathcal{E}}=(k_1,k_2,k_3)$. If $k_1 \geq 0$ then the possible spectra of $\mathcal{E}$ are $(0,0,0)$, $(0,0,1)$, $(0,1,1)$, $(1,1,1)$ and $(0,1,2)$ which are all connected. If $k_1 \leq -1$, then $k_{\mathcal{E}}=(-1,-1,-1)$ or by Remark 2.3, there is an unstable plane for $\mathcal{E}$, hence $\mathcal{E}$ satisfies the condition (i) of [2, Proposition 5.1]. By Example 2.3.2, $k_{\mathcal{E}} = (-2,-1,0)$.

The following two results improve [8, Theorem 3.3 and Teorem 4.1] in the case of the stable rank 3 reflexive sheaves and are similar to [4, Proposition 3.1 and Proposition 5.1]. The proof is not substantially different from the proof of the mentioned results, except that now one has further information on $\mathbb{P}^2$. However, we give all the details, for the convenience of the reader.

Lemma 2.4. Let $\mathcal{E}$ be a semistable rank 3 reflexive sheaf on $\mathbb{P}^3$ with $c_1=0,-1$ or -2. Let n be an integer such that $n\geq -1$ if $c_1=0$ or -1 and $n\geq 0$ if $c_1=-2$, and let $b\in H^2(\mathcal{E}(n))'$. For any plane $H\subset\mathbb{P}^3$ let $b_H\in H^1(\mathcal{E}_H(n+1))'$ be the image of b by the morphism $\delta': H^2(\mathcal{E}(n))'\longrightarrow$ $\longrightarrow H^1(\mathcal{E}_H(n+1))'$.

Assume that for all sufficiently general planes $H\subset\mathbb{P}^3$, $b_H\neq 0$ and is annihilated by a nonzero linear form $\lambda_H\in H^0(\mathcal{O}_H(1))$. Then b is annihilated by a unique (up to scalar) nonzero linear form $f\in H^0(\mathcal{O}_{\mathbb{P}^3}(1))$.

Proof. We intend to apply [4, Proposition 2.1].

If $\mathcal{E}=\Omega(1)$ then $H^2(\mathcal{E}(n))=0$ for all $n\in\mathbb{Z}$ and if $\mathcal{E}=T(-2)$ then $H^2(\mathcal{E}(n))=0$ for all $n\geq -1$. It follows that in these cases our assertion is vacuous. Hence we may assume that the restriction of $\mathcal{E}$ to a general plane is semistable.

Let $S=\bigoplus_{\ell\in\mathbb{Z}} H^0(\mathcal{O}_{\mathbb{P}^3}(\ell))=k[x_0,x_1,x_2,x_3]$ and let M be the graded S-module $\bigoplus_{\ell\in\mathbb{Z}} H^2(\mathcal{E}(\ell))'$, where we define the grading of M by taking $H^2(\mathcal{E}(\ell))'$ to have degree $-\ell+n$. Let $H_0\subset\mathbb{P}^3$ be a plane such that:

(i) H_0 does not contain any singular point of $\mathcal{E}$

(ii) $\mathcal{E}_{H_0}$ is semistable

(iii) $b_{H_0}\neq 0$ and is annihilated by a nonzero linear form $\lambda_{H_0}\in H^0(\mathcal{O}_{H_0}(1))$.

Let $L\subset H_0$ be a generic line of $\mathcal{E}_{H_0}$ the position of which will be specified later. Let $V=\{h\in H^0(\mathcal{O}_{\mathbb{P}^3}(1))\mid h$ vanishes on $L\}$. Now, we show that S,V,M verify the hypothesis of [4, Proposition 2.1].

Firstly, we show that V is in general position for M in degrees ≤ 1. Let $h\in V$, $h\neq 0$, and let $H\subset\mathbb{P}^3$ be the plane of equation $h=0$. Then $H\supset L$. We have exact sequences:

$$H^2(\mathcal{E}_H(\ell+1))' \longrightarrow H^2(\mathcal{E}(\ell+1))' \xrightarrow{h} H^2(\mathcal{E}(\ell))'$$

$H^2(\mathcal{E}_H(\ell+1))' \cong H^0(\mathcal{E}_H^*(-\ell-4))=0$ if $-\ell\leq 2$ (because $H^0(\mathcal{E}_L^*(\ell))=0$ if $\ell\leq -2$) hence the multiplication by $h: H^2(\mathcal{E}(\ell+1))' \longrightarrow H^2(\mathcal{E}(\ell))'$ is injective if $-\ell+n\leq 1$.

Let h, h' be a basis of V. Using the exact sequences:

$$H^2(\mathcal{E}(\ell+1))' \xrightarrow{h} H^2(\mathcal{E}(\ell))' \xrightarrow{\delta'} H^1(\mathcal{E}_H(\ell+1))'$$

one finds that M/hM is a graded S/hS-submodule of $\bigoplus_{\ell\in\mathbb{Z}} H^1(\mathcal{E}_H(\ell+1))'$ where we define the grading of $\bigoplus_{\ell\in\mathbb{Z}} H^1(\mathcal{E}_H(\ell+1))'$ by taking $H^1(\mathcal{E}_H(\ell+1))'$ to have degree $-\ell+n$. Let $\lambda\in H^0(\mathcal{O}_H(1))$ be the image of h'.

$\lambda=0$ is an equation of L on H. We have exact sequences:

$$H^1(\mathcal{E}_L(\ell+2))' \longrightarrow H^1(\mathcal{E}_H(\ell+2))' \xrightarrow{\lambda} H^1(\mathcal{E}_H(\ell+1))'$$

$H^1(\mathcal{E}_L(\ell+2))=0$ if $-\ell\leq 2$ hence the multiplication by λ: $H^1(\mathcal{E}_H(\ell+2))' \longrightarrow H^1(\mathcal{E}_H(\ell+1))'$ is injective if $-\ell+n\leq 1$.

We have shown that V is in general position for M in degrees ≤ 1.

Now, we verify the second condition of [4, Proposition 2.1]. As we observed before, if $h\in V$, $h\neq 0$, then M/hM is a graded S/hS-submodule of $\bigoplus_{\ell\in\mathbb{Z}} H^1(\mathcal{E}_H(\ell+1))'$. By the property (iii) of H_0 and by our hypothesis, b_H is annihilated by a nonzero linear form $\lambda_H\in H^0(\mathcal{O}_H(1))$ for all sufficiently general $h\in V$. Hence the second condition of [4, Proposition 2.1] holds.

It remains to verify the third condition of [4, Proposition 2.1]. Let $h_o \in V$ be such that $h_o=0$ is an equation of H_o. Put $E=\mathcal{E}_{H_o}$. M/h_oM is a graded S/h_oS-submodule of $N = \bigoplus_{\ell\in\mathbb{Z}} H^1(E(\ell+1))'$. Let $L_o \subset H_o$ be the line of equation $\lambda_{H_o}=0$. Let $h\in V$ be such that h_o, h is a basis of V and let $\lambda\in H^0(\mathcal{O}_{H_o}(1))$ be the image of h. $\lambda=0$ is an equation of L on H_o. $(S/VS)_1 = H^0(\mathcal{O}_L(1))$.

Let $\bar{b}\in M/h_oM$ and $\bar{\bar{b}}\in M/VM$ be the images of b. By the identification of M/h_oM with a submodule of N, $\bar{b}$ corresponds to b_{H_o}. b_{H_o} is annihilated by λ_{H_o} hence $\bar{\bar{b}}$ is annihilated by $\lambda_{H_o}|L$.

If we choose L such that $L\neq L_o$ then $\lambda_{H_o}|L\neq 0$.

We have $\dim H^0(\mathcal{O}_L(1))=2$. It follows that, in order to verify the third condition of [4, Proposition 2.1] it suffices to show that there is a $u\in H^0(\mathcal{O}_L(1))$ such that $u\cdot\bar{\bar{b}}\neq 0$.

There is a morphism (not necessarily injective) of graded S/VS-modules $M/VM \to N/\lambda N$. The image of $\bar{\bar{b}}$ in $N/\lambda N$ is identical to the image $\bar{b}_{H_o}$ of b_{H_o} in $N/\lambda N$. Using the exact sequences:

$$H^1(E(\ell+2))' \xrightarrow{\lambda} H^1(E\,(\ell+1))' \xrightarrow{\delta'} H^0(E_L(\ell+2))'$$

one finds that $N/\lambda N$ is a graded S/VS-submodule of $\bigoplus_{\ell\in\mathbb{Z}} H^0(E_L(\ell+2))'$. $\bar{b}_{H_o}\in N/\lambda N$ corresponds to the image $b_L\in H^0(E_L(n+2))'$ of $b_{H_o}\in H^1(E(n+1))'$ by the morphism $\delta' : H^1(E(n+1))' \to H^0(E_L(n+2))'$.

It follows that it suffices to show that one can choose L such that there is a $u\in H^0(\mathcal{O}_L(1))$ such that $u\cdot b_L\neq 0$.

<u>Case</u> 1. $n\geq 0$ or $n=-1$, $c_1=0$ and $E_L=\mathcal{O}_L^3$. In this case, it suffices to show that one can choose L such that $b_L\neq 0$. Indeed, $E_L(n+1)=\mathcal{O}_L(a_1)\oplus\mathcal{O}_L(a_2)\oplus\mathcal{O}_L(a_3)$ with $a_i\geq 0$, hence the morphism:

$$H^0(E_L(n+1))\otimes_k H^0(\mathcal{O}_L(1)) \longrightarrow H^0(E_L(n+2))$$

is surjective. It follows that if $b_L \neq 0$ then there is a $u \in H^0(\mathcal{O}_L(1))$ such that $b_L : H^0(E_L(n+2)) \to k$ composed with the multiplication by $u : H^0(E_L(n+1)) \to H^0(E_L(n+2))$ is nonzero, that is, such that $u \cdot b_L \neq 0$.

We have an exact sequence:

$$H^1(E_{L_0}(n+1))' \to H^1(E(n+1))' \xrightarrow{\lambda_{H_0}} H^1(E(n))'$$

$\lambda_{H_0} \cdot b_{H_0} = 0$ implies that b_{H_0} is the image of a nonzero element $b_0 \in H^1(E_{L_0}(n+1))' \cong \mathrm{Hom}(E_{L_0}(n+1), \mathcal{O}_{L_0}(-2))$. There is an $r \geqq 2$ and a nonzero $\varphi \in H^0(\mathcal{O}_{L_0}(r-2))$ such that $b_0 : E_{L_0}(n+1) \to \mathcal{O}_{L_0}(-2)$ is the composition of an epimorphism $E_{L_0}(n+1) \to \mathcal{O}_{L_0}(-r)$ and of the multiplication by $\varphi : \mathcal{O}_{L_0}(-r) \to \mathcal{O}_{L_0}(-2)$.

Let $p : E(n+1) \to \mathcal{O}_{L_0}(-r)$ be the composed map $E(n+1) \to E_{L_0}(n+1) \to \mathcal{O}_{L_0}(-r)$. Using the exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & E(n+1) \arrow[r, "\lambda"] \arrow[d, "p"] & E(n+2) \arrow[r] \arrow[d, "p(1)"] & E_L(n+2) \arrow[r] \arrow[d, "p(1)|L"] & 0 \\
0 \arrow[r] & \mathcal{O}_{L_0}(-r) \arrow[r, "\lambda"] & \mathcal{O}_{L_0}(-r+1) \arrow[r] & \mathcal{O}_{L \cap L_0}(-r+1) \arrow[r] & 0
\end{tikzcd}$$

one finds a commutative diagram:

$$\begin{tikzcd}
H^0(E_L(n+2)) \arrow[r, "\delta"] \arrow[d, "H^0(p(1)|L)"'] & H^1(E(n+1)) \arrow[d, "H^1(p)"] \\
H^0(\mathcal{O}_{L \cap L_0}(-r+1)) \arrow[r, "\delta"] \arrow[d, "\varphi"'] & H^1(\mathcal{O}_{L_0}(-r)) \arrow[d, "\varphi^0"] \\
H^0(\mathcal{O}_{L \cap L_0}(-1)) \arrow[r, "\delta"] & H^1(\mathcal{O}_{L_0}(-2))
\end{tikzcd}$$

The multiplication by $\varphi : H^1(\mathcal{O}_{L_0}(-r)) \to H^1(\mathcal{O}_{L_0}(-2))$ composed with $H^1(p)$ is b_{H_0} and b_{H_0} composed with δ is b_L. The morphism $\delta : H^0(\mathcal{O}_{L \cap L_0}(-1)) \to H^1(\mathcal{O}_{L_0}(-2))$ is bijective.

If L does not contain any point where φ vanishes then the multiplication by φ: $H^0(\mathcal{O}_{L\cap L_0}(-r+1)) \to H^0(\mathcal{O}_{L\cap L_0}(-1))$ is bijective.

$p(1)|L$ is the composition $E_L(n+2) \to E_{L\cap L_0}(n+2) \longrightarrow$ $\longrightarrow \mathcal{O}_{L\cap L_0}(-r+1)$. The map $H^0(E_{L\cap L_0}(n+2)) \to H^0(\mathcal{O}_{L\cap L_0}(-r+1))$ is surjective. Using the exact sequence:

$$0 \longrightarrow E_L(n+1) \xrightarrow{\lambda_{H_0}|L} E_L(n+2) \longrightarrow E_{L\cap L_0}(n+2) \longrightarrow 0$$

one obtains an exact sequence:

$$H^0(E_L(n+2)) \longrightarrow H^0(E_{L\cap L_0}(n+2)) \longrightarrow H^1(E_L(n+1)) = 0.$$

Hence $H^0(E_L(n+2)) \to H^0(E_{L\cap L_0}(n+2))$ is surjective. Consequently, $H^0(p(1)|L)$ is surjective. It follows that b_L is surjective, hence $b_L \neq 0$.

<u>Case</u> 2. $n=-1$, $c_1=0$ and $E_L=\mathcal{O}_L(-1) \oplus \mathcal{O}_L \oplus \mathcal{O}_L(1)$. Then an argument used in the proof of the Case 1 shows us that there is a unique (up to scalar) nonzero element $\beta\in H^0(E_L(1))'$ such that $u\cdot\beta=0$ for all $u\in H^0(\mathcal{O}_L(1))$, namely the one whose kernel is $H^0(\mathcal{O}_L(1) \oplus \mathcal{O}_L(2))$. Preserving the notations used in the Case 1 we apply the reduction step to E and L_0 :

$$0 \to E' \to E \xrightarrow{p} \mathcal{O}_{L_0}(-r) \to 0$$

The above sequence twisted by $\mathcal{O}_{H_0}(1)$ and restricted to L gives an exact sequence:

$$0 \to E'_L(1) \to E_L(1) \xrightarrow{p(1)|L} \mathcal{O}_{L\cap L_0}(-r+1) \longrightarrow 0$$

It follows that we have an exact sequence:

$$0 \to H^0(E'_L(1)) \to H^0(E_L(1)) \xrightarrow{H^0(p(1)|L)} H^0(\mathcal{O}_{L\cap L_0}(-r+1))$$

If L is a generic line for E' then by Corollary 1.2(a),

$E'_L(1)=\mathcal{O}_L \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L(1)$ hence: Ker $b_L = \mathrm{Im}(H^0(E'_L(1)) \longrightarrow H^0(E_L(1))) \neq$ $\neq H^0(\mathcal{O}_L(1) \oplus \mathcal{O}_L(2))$. It follows that there is a $u \in H^0(\mathcal{O}_L(1))$ such that $u \cdot b_L \neq 0$.

Case 3. $n=-1$, $c_1=-1$. In this case one uses the same argument as in the Case 2.

Proposition 2.5. Let $\mathcal{E}$ be a rank 3 reflexive sheaf on $\mathbb{P}^3$ with $c_1=0,-1$ or -2 and such that the restriction of $\mathcal{E}$ to a general plane is stable. Let $k_{\mathcal{E}}=(k_1,\ldots,k_m)$ be the spectrum of $\mathcal{E}$. Assume that there is an i with $2 \leq i \leq m-1$ such that $k_{i-1} < k_i < k_{i+1} \leqq 0$, if $c_1=0$ and such that $k_{i-1} < k_i < k_{i+1} \leqq -1$ if $c_1=-1$ or -2.

Then there is an unstable plane for $\mathcal{E}$ of order $-k_1$ and $k_1 < k_2 < \ldots < k_i$.

Proof. Firstly, we show that there is an unstable plane for $\mathcal{E}$ of order $-k_1$.

By (3) and (4), $H^2(\mathcal{E}(-k_1-2))=0$ and $H^2(\mathcal{E}(-k_1-3)) \neq 0$. We choose a nonzero $b \in H^2(\mathcal{E}(-k_1-3))'$ and we show that b verifies the hypothesis of Lemma 2.4. Indeed, $-k_1-3 \geqq -1$ if $c_1=0$ and $-k_1-3 \geqq 0$ if $c_1=-1$ or -2. Let $H \subset \mathbb{P}^3$ be a plane which does not contain any singular point of $\mathcal{E}$ and such that $\mathcal{E}_H$ is stable. Let $h=0$ be an equation of H. From the exact sequence:

$$0=H^2(\mathcal{E}(-k_1-2))' \longrightarrow H^2(\mathcal{E}(-k_1-3))' \xrightarrow{\delta'} H^1(\mathcal{E}_H(-k_1-2))'$$

it follows that $b_H \neq 0$.

Let R_ℓ be the kernel of the multiplication by $h: H^2(\mathcal{E}(\ell-1)) \longrightarrow$ $\longrightarrow H^2(\mathcal{E}(\ell))$ and let $r_\ell = \dim R_\ell$. We have $-k_i-3 \geqq -2$. By (2) one has:

$$r_{-k_i-2} - r_{-k_i-1} = \mathrm{card}\{j \mid k_j = k_i\} = 1$$

We have $-k_i-2 \geqq -1$ if $c_1=0$ and $-k_i-2 \geqq 0$ if $c_1=-1$ or -2. By Proposition 1.11 or by [6, Theorem 1.1.iv'] it follows that there is a nonzero $\lambda_H \in H^0(\mathcal{O}_H(1))$ such that $\lambda_H \cdot R_\ell = 0$ for all $\ell \geqq -k_i-2$, hence such that $\lambda_H \cdot R'_\ell = 0$ for all $\ell \geq -k_i - 1$. For $\ell = -k_1 - 2$ we get $\lambda_H \cdot R'_{-k_1-2} = 0$ and particularly, $\lambda_H \cdot b_H = 0$.

By Lemma 2.4 there is a nonzero linear form $h_o \in H^0(\mathcal{O}_{\mathbb{P}^3}(1))$ which annihilates b. By Remark 2.3, the plane H_o of equation $h_o = 0$ is an unstable plane for $\mathcal{E}$ of order $-k_1$.

Now, we will show that $k_1 < k_2 < \ldots < k_i$.

We have $H^0(\mathcal{E}^*_{H_o}(k_1)) \neq 0$ and $H^0(\mathcal{E}^*_{H_o}(k_1 - 1)) = 0$. Let $\varphi : \mathcal{E}_{H_o} \longrightarrow \mathcal{O}_{H_o}(k_1)$ be a nonzero morphism. From the fact that $\mathcal{E}_{H_o}$ is torsion free on $H_o \cong \mathbb{P}^2$ it follows that φ is surjective, with the exception of a finite number of points.

Let $H \subset \mathbb{P}^3$ be a plane which does not contain any singular point of $\mathcal{E}$, such that $\mathcal{E}_H$ is stable and such that the line $L = H \cap H_o$ does not contain any point where φ is not surjective. By restricting φ to L we get an epimorphism $\varphi_L : \mathcal{E}_L \rightarrow \mathcal{O}_L(k_1)$. Now, we apply the reduction step to $\mathcal{E}_H$ and L :

$$0 \longrightarrow E'(t) \longrightarrow \mathcal{E}_H \longrightarrow \mathcal{O}_L(k_1) \longrightarrow 0$$

where $t = 0$ if $c_1 = 0$ or -1 and $t = -1$ if $c_1 = -2$. Consider the exact commutative diagram:

$$\begin{tikzcd}
 & H^1(\mathcal{E}(\ell)) \arrow[r] \arrow[dd] & H^1(\mathcal{E}_{H_o}(\ell)) \arrow[d, "H^1(\varphi(\ell))"] & \\
 & & H^1(\mathcal{O}_{H_o}(\ell + k_1)) = 0 \arrow[d] & \\
H^1(E'(\ell + t)) \arrow[r] & H^1(\mathcal{E}_H(\ell)) \arrow[r] \arrow[d] & H^1(\mathcal{O}_L(\ell + k_1)) \arrow[r] & H^2(E'(\ell + t)) \\
 & R_\ell \arrow[d] \arrow[ru, dashed] & & \\
 & 0 & &
\end{tikzcd}$$

It follows that the composition $H^1(\mathcal{E}(\ell)) \longrightarrow H^1(\mathcal{E}_H(\ell)) \longrightarrow$

$\longrightarrow H^1(\mathcal{O}_L(\ell+k_1)$ is 0 hence we obtain an exact sequence:

$$H^1(E'(\ell+t)) \longrightarrow R_\ell \longrightarrow H^1(\mathcal{O}_L(\ell+k_1)) \longrightarrow H^2(E'(\ell+t))$$

Put $Q_{\ell+t} = \mathrm{Im}(H^1(E'(\ell+t)) \longrightarrow R_\ell)$ and $q_\ell = \dim Q_\ell$.

$H^2(E'(\ell+t)) \cong H^0(E'^*(-\ell-t-3))'$ and $H^0(E'^*(-\ell-t-3))$ can be embeded in $H^0(\mathcal{E}_H^*(-\ell-2))$ which is 0 for $\ell \geqq -1$. It follows that the sequence:

$$0 \longrightarrow Q_{\ell+t} \longrightarrow R_\ell \longrightarrow H^1(\mathcal{O}_L(\ell+k_1)) \longrightarrow 0$$

is exact for $\ell \geqq -1$. We have $r_{-k_i-2} - r_{-k_i-1} = 1$ and $-k_i-2 \geqq -1$, hence $q_{-k_i-2+t} = q_{-k_i-1+t}$.

Next we will show that $q_{-k_i-2+t} = 0$.

<u>Case</u> 1. $c_1=0$. In this case $c_1(E') = -1$ and E' is stable, We have $q_{-k_i-2} = q_{-k_i-1}$ and $-k_i-2 \geqq -1$ hence by Proposition 1.11 (b) (ii) or by [6, Theorem 1.1 iii'], $q_{-k_i-2} = 0$.

<u>Case</u> 2. $c_1 = -1$. In this case $c_1(E') = -2$ and the generic splitting type of E' is $(-1,-1,0)$. We have $q_{-k_i-2} = q_{-k_i-1}$ and $-k_i-2 \geqq 0$ hence by [6, Theorem 1.1 iii'], $q_{-k_i-2} = 0$.

<u>Case</u> 3. $c_1 = -2$. In this case $c_1(E')=0$ and E' is semistable. We have $q_{-k_i-3} = q_{-k_i-2}$ and $-k_i-3 \geqq -1$ hence by Proposition 1.11 (a) (ii) or by [6, Theorem 1.1 iii'], $q_{-k_i-3} = 0$.

It follows that $q_\ell = 0$ for all $\ell \geqq -k_i-2+t$, that is, $q_{\ell+t} = 0$ for all $\ell \geqq -k_i-2$. We have $h^1(\mathcal{O}_L(\ell+1)) = h^1(\mathcal{O}_L(\ell))-1$ for all $\ell \leqq -2$. It follows that $r_{\ell+1} - r_{\ell+2} = 1$ for $-k_i-3 \leqq \ell \leqq -k_1-3$, hence $k_1 < k_2 < \ldots < k_i$.

<u>Corollary</u> 2.6. <u>Let</u> $\mathcal{E}$ <u>be a rank</u> 3 <u>vector bundle on</u> $\mathbb{P}^3$ <u>with</u> $c_1=0,-1$ <u>or</u> -2 <u>and such that the restriction of</u> $\mathcal{E}$ <u>to a general plane is stable. Let</u> $k_\mathcal{E} = (k_1,\ldots,k_m)$ <u>be the spectrum of</u> $\mathcal{E}$. <u>Put</u> $\mathcal{F} = \mathcal{E}^*$ <u>if</u> $c_1=0$ <u>and</u> $\mathcal{F} = \mathcal{E}^*(-1)$ <u>if</u> $c_1 = -1$ <u>or</u> -2. <u>Assume that there is an</u> i <u>with</u> $2 \leqq i \leqq m-1$ <u>such that</u> $0 \leqq k_{i-1} < k_i < k_{i+1}$.

Then there is an unstable plane for $\mathcal{F}$ *of order* k_m *if* $c_1=0$ *and of order* k_m+1 *if* $c_1=-1$ *or* -2 *and* $k_i<k_{i+1}<\dots<k_m$.

Proof. One applies Proposition 2.5 to $\mathcal{F}$ taking into account that by [6, Lemma 3.2], $k_{\mathcal{F}}=(-k_m,\dots,-k_1)$ if $c_1=0$ and $k_{\mathcal{F}}=(-k_m-1,\dots$ $\dots,-k_1-1)$ if $c_1=-1$ or -2.

Corollary 2.7. *Let* $\mathcal{E}$ *be a stable rank 3 vector bundle on* $\mathbb{P}^3$ *with* $c_1=0$. *Suppose that* $c_3>\frac{1}{2}\cdot c_2^2+2$. *Then there is an unstable plane for* $\mathcal{E}$ *of order* c_2-q *for some* q *with* $1\leqq q<\frac{1}{2}\cdot c_2$, *and in that case:*

$$c_3=c_2^2-c_2\qquad,\quad \text{if}\quad q=1$$

$$c_2^2-(2q-1)c_2\leqq c_3\leqq c_2^2-(2q-1)c_2+2(q^2-q+1),\quad \text{if}\quad q\geqq 2$$

Furthermore, assume, for a fixed $q\geqq 2$, *that* $c_3<c_2^2-$ $-(2q-1)c_2+\frac{1}{2}(q^2-2q)$. *Then there is an unstable plane for* $\mathcal{E}^*$ *of order* $q-d-1$ *for some* d *with* $0\leqq d<\frac{1}{2}\cdot q-1$, *and in that case:*

$$c_2^2-(2q-1)c_2+2dq-2d(d+1)\leqq c_3\leqq c_2^2-(2q-1)c_2+2dq$$

Proof. The relation $c_3>\frac{1}{2}\cdot c_2^2+2$ is possible only if $c_2\geqq 4$. $\mathcal{E}$ can't be the second symmetric power of a nullcorrelation bundle because in that case $c_3=0$. If $c_3=c_2^2-c_2$ then by [2, Proposition 5.1], there is an unstable plane for $\mathcal{E}$ of order c_2-1. If $c_3<c_2^2-c_2$ then by [2, Theorem 0.1] the restriction of $\mathcal{E}$ to a general plane is stable.

Consider the spectrum of $\mathcal{E}$, $k_{\mathcal{E}}=(k_1,\dots,k_m)$. If $\mathcal{E}$ does not satisfy the hypothesis of Proposition 2.5 then every k_i with $k_1<k_i\leqq -1$ must occur at least twice in the spectrum. Recalling that $c_3=-2\sum k_i$, the largest value of c_3 will correspond to the spectrum:

$$-\tfrac{1}{2}\cdot c_2,\ -\tfrac{1}{2}\cdot c_2+1,\ -\tfrac{1}{2}\cdot c_2+1,\dots,-2,-2,-1,-1,-1\quad \text{if}\quad c_2 \text{ is even}$$

or $-\frac{1}{2}\cdot(c_2-1),\ -\frac{1}{2}\cdot(c_2-1),\dots,-2,-2,-1,-1,-1$ if c_2 is odd

(one takes into account the Proposition 2.2 (a)).

These spectra give $c_3=\frac{1}{2}\cdot c_2^2+2$ and $c_3=\frac{1}{2}(c_2^2+3)$, respectively.

Thus if $c_3 > \frac{1}{2} \cdot c_2^2 + 2$, $\mathcal{E}$ must satisfy the hypothesis of Proposition 2.5. In this case $k_1 = -c_2 + q$ for some q with $2 \leq q < \frac{1}{2} \cdot c_2$ (recall that $c_3 < c_2^2 - c_2$), and by Proposition 2.5, there is an unstable plane for $\mathcal{E}$ of order $c_2 - q$. The minimum and the maximum value of c_3 for the given q correspond to the spectra:

$$-c_2+q, \ldots, -2, -1, 0, 1, 2, \ldots, q-1$$

and

$$-c_2+q, \ldots, -q, -q+1, -q+1, \ldots, -2, -2, -1, -1, -1$$

which give $c_3 = c_2^2 - (2q-1)c_2$ and $c_3 = c_2^2 - (2q-1)c_2 + 2(q^2 - q + 1)$, respectively.

Now, suppose that $q \geq 2$ is fixed . If $\mathcal{E}$ does not satisfy the hypothesis of Corollary 2.6 then every k_i with $1 \leq k_i < k_m$ must occur at least twice in the spectrum. The minimum value of c_3 will correspond to the spectrum:

$$-c_2+q, \ldots, -1, 0, 1, 1, \ldots, \tfrac{1}{2} \cdot q-1, \tfrac{1}{2} \cdot q-1, \tfrac{1}{2} \cdot q \quad \text{if } q \text{ is even}$$

or

$$-c_2+q, \ldots, -1, 0, 1, 1, \ldots, \tfrac{1}{2} \cdot (q-1), \tfrac{1}{2} \cdot (q-1) \quad \text{if } q \text{ is odd.}$$

These spectra give $c_3 = c_2^2 - (2q-1)c_2 + \frac{1}{2} \cdot (q^2 - 2q)$ and $c_3 = c_2^2 - (2q-1)c_2 + \frac{1}{2} \cdot (q-1)^2$, respectively. Thus if $c_3 < c_2^2 - (2q-1)c_2 + \frac{1}{2}(q^2 - 2q)$, $\mathcal{E}$ must satisfy the hypothesis of Corollary 2.6. In this case $k_m = q-d-1$ for some d with $0 \leq d < \frac{1}{2} \cdot q - 1$, and by Corollary 2.6, there is an unstable plane for $\mathcal{E}^*$ of order $q-d-1$. The minimum and the maximum value of c_3 for the given d correspond to the spectra:

$$-c_2+q, \ldots, -1, 0, 1, 1, \ldots, d, d, d+1, \ldots, q-d-1$$

and

$$-c_2+q, \ldots, -d-1, -d, -d, \ldots, -1, -1, 0, 1, 2, \ldots, q-d-1$$

which give $c_3 = c_2^2 - (2q-1)c_2 + 2dq - 2d(d+1)$ and $c_3 = c_2^2 - (2q-1)c_2 + 2dq$, respectively.

The proof of the following two corollaries is similar to that of Corollary 2.7 and it will be omitted.

Corollary 2.8. Let $\mathcal{E}$ be a stable rank 3 reflexive shear on $\mathbb{P}^3$ with $c_1 = -1$. Suppose that $c_3 > \frac{1}{2}\cdot(c_2^2+1)$. Then there is an unstable plane for $\mathcal{E}$ of order c_2-q for some q with $1 \leqq q < \frac{1}{2}\cdot(c_2-1)$, and in that case:

$$c_2^2-2qc_2 \leqq c_3 \leqq c_2^2-2qc_2+2q^2$$

Furthermore, assume $\mathcal{E}$ locally free and that $c_3 < c_2^2-2qc_2 + \frac{1}{2}\cdot(q^2-2q)$. Then there is an unstable plane for $\mathcal{E}^*(-1)$ of order $q-d$ for some d with $0 \leqq d < \frac{1}{2}\cdot q-1$, and in that case:

$$c_2^2-2qc_2+2dq-2d(d+1) \leqq c_3 \leqq c_2^2-2qc_2+2dq+2d$$

Corollary 2.9. Let $\mathcal{E}$ be a stable rank 3 reflexive sheaf on $\mathbb{P}^3$ with $c_1 = -2$. Suppose that $c_3 > \frac{1}{2}(c_2-1)^2$. Then there is an unstable plane for $\mathcal{E}$ of order c_2-q for some q with $1 \leqq q < \frac{1}{2}\cdot(c_2-1)$, and in that case:

$$c_2^2-(2q+1)c_2+2q \leqq c_3 \leqq c_2^2-(2q+1)c_2+2q^2$$

Furtherore, assume $\mathcal{E}$ locally free and that $c_3 < c_2^2-(2q+1)c_2 + \frac{1}{2}\cdot(q^2+3)$. Then there is an unstable plane for $\mathcal{E}^*(-1)$ of order $q-d-1$ for some d with $0 \leqq d < \frac{1}{2}\cdot(q-3)$, and in that case:

$$c_2^2-(2q+1)c_2+2(d+1)q-2d(d+2) \leqq c_3 \leqq c_2^2-(2q+1)c_2+2(d+1)q \ .$$

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