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by

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0. INTRODUCTION

The classical theory of distributed control systems is mostly concerned with linear models in which control appears in linear fashion either in the inhomogeneous term of the equation or in the boundary data (see [7], [13], [17]). More recently work has begun on nonlinear control systems described by variational inèqualities motivated by free boundary problems models in fluid dynamics, heat conduction, diffusion processes and linear elasticity (see [8], [9], [11] for basic results and applications of general theory of variational inequalities). In [14], F.Mignot obtains a quite complete set of necessary conditions for optimality in quadratic control problems governed by the "obstacle problem" and "Signorini problem". As a matter of fact, the theory developed in [14] can be put in a more general context and in particular in the case of control problems governed by parabolic variational inèqualities [15]. The works of Ch.Saguez surveyed in [16] are on these lines with main emphasis on approximation results for optimal control of free boundary systems of parabolic type. (In this context we must als mention the pioneering work of Yvon [19].) A general theory of necessary conditions for optimal control problems governed by variational inequalities can be found in the author's book [5].

Here we briefly survey some of these results and use or adapt them in order to get explicit description of the optimal controls for some nonlinear models recently considered in literature ([6], [10], [12], [18].)

1. OPTIMAL CONTROL OF THE OBSTACLE PROBLEM

Throughout this section Ω is a bounded and open subset of R^N having a sufficiently smooth boundary Γ. Let A_o be the second order ellyptic differential operator

$$(1.1)\quad A_o y = -\sum_{i,j=1}^{n} (a_{ij}(x) y_{x_i})_{x_j} + a_o(x) y$$

where $a_{ij} \in C^1(\bar{\Omega})$, $a_o \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$ for all i,j, $a_o \geq o$ in Ω and for some $\omega > o$

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \omega |\xi|_n^2 \quad \forall \xi \in R^n,\ x \in \Omega.$$

Finally, let $a: H^1(\Omega) \times H^1(\Omega) \longrightarrow R$ be the Dirichlet form associated with A_o,

$$(1.2)\quad a(y,z) = \sum_{i,j=1}^{n} \int_\Omega (a_{ij}(x) y_{x_i} z_{x_j} + a_o(x) yz)\,dx \ ,\quad y,z \in H^1(\Omega).$$

(Here and throughout in sequel, $H^k(\Omega)$ and $H^k_o(\Omega)$ are usual Sobolev spaces on Ω.)

Let U be a real Hilbert space and let B be a linear continuous operator from U to $L^2(\Omega)$. For a fixed $f \in L^2(\Omega)$ and some input $u \in U$ consider the obstacle problem associated with operator (1.1)

$$(A_o y - Bu - f)(y - \psi) = o \quad \text{a.e. in } \Omega$$

(1.3) $A_o y - Bu - f \geqslant o, \; y \geqslant \psi$ a.e. in Ω

$$y = o \quad \text{in } \Gamma$$

where $\psi \in H^2(\Omega)$ is a given function such that $\psi \leq o$ in Γ. Under our assumptions, the free boundary problem (1.3) has a unique solution $y \in H^1_o(\Omega) \cap H^2(\Omega)$.

More precisely, y is the solution to variational inequality

(1.4) $a(y, y-z) \leq (y-z, f+Bu) \quad \forall z \in K$

where $K = \{ y \in H^1_o(\Omega); y \geqslant \psi \text{ a.e. in } \Omega \}$ and (.,.) is the usual scalar product in $L^2(\Omega)$.

We associate with state system (1.3) the following optimal control problem

<u>Minimize</u>

(1.5) $g(y) + h(u)$

<u>on all</u> $y \in H^1_o(\Omega) \times H^2(\Omega)$ <u>and</u> $u \in U$ <u>satisfying</u> Eqs.(1.3).

Here $g: L^2(\Omega) \longrightarrow R$ is locally Lipschitz, $h: L^2(\Omega) \longrightarrow \overline{R} =]-\infty, +\infty]$ is convex, lower semicontinuous and

(1.6) $g(y^u) + h(u) \longrightarrow +\infty \quad$ for $\|u\|_U \longrightarrow +\infty$

where y^u is the solution to Eq.(1.3)((1.4)).

Then by standard arguments we infer that problem (1.5) admits at least one optimal pair (y^*, u^*) (see for instance [5] p.62). As regards the characterization of optimal arcs (the maximum principle) we have the following result ([5], p.82).

THEOREM 1. <u>Let</u> (y^*, u^*) <u>be an arbitrary optimal pair in control problem</u> (1.5). <u>Then there is</u> $p \in H^1_o(\Omega)$ <u>with</u> $A_o p \in (L^\infty(\Omega))^*$ <u>and</u> $\xi \in L^2(\Omega)$ <u>such that</u> $\xi \in \partial g(y^*)$ <u>and</u>

(1.7) $(A_o p)_a + \xi = o$ a.e. in $\{x; y^*(x) > \psi(x)\}$

(1.8) $p(A_o y^* - Bu^* - f) = o$ a.e. in Ω

(1.9) $B^* p \in \partial h(u^*)$.

If $1 \leq n \leq 3$ then Eq.(1.7) can be strengthen to

(1.10) $(A_o p + \xi)(y^* - \psi) = o$ a.e. in Ω.

We have denoted by $(A_o p)_a$ the absolutely continuous part of the measure $A_o p \in (L^\infty(\Omega))^*$ and by $(A_o p + \xi)(y^* - \psi)$ the product of the measure $A_o p + \xi$ with the function $y^* - \psi$ which belongs to $C(\bar{\Omega})$ if $1 \leq n \leq 3$. By ∂g we have denoted the Clarke generalized gradient of g and by ∂h the subdifferential of h.

Theorem 1 has been proved along with other related results in [5] (p.84) (see also [2]) via an approximating process.

In few words the idea is to approximate problem (1.5) by the following one:

Minimize

(1.11) $g^\varepsilon(y) + h(u) + \frac{1}{2} \|u - u^*\|_U^2$

on all (y,u) subject to

(1.12) $A_o y + \beta^\varepsilon(y - \psi) = f + Bu$ in Ω

$y = o$ in Γ.

Here $\varepsilon > o$, g^ε and β^ε are smooth approximations of g and β_ε. In particular β^ε can be chosen as a mollifier of $\beta_\varepsilon(r) = -\varepsilon^{-1} r^-$, i.e.,

(1.13) $\beta^\varepsilon(r) = -\varepsilon^{-1} \int_{-\infty}^{\infty} ((r - \varepsilon^2\theta)^- - \varepsilon^2\theta^+) \rho(\theta) d\theta$.

Let $(y_\varepsilon, u_\varepsilon)$ be any approximating pair in problem (1.11). Then there is $p_\varepsilon \in H_0^1(\Omega) \cap H^2(\Omega)$ such that

$$A_o y_\varepsilon + \beta^\varepsilon(y_\varepsilon - \psi) = f + Bu_\varepsilon \quad \text{in } \Omega$$

$$-A_o p_\varepsilon - \dot\beta^\varepsilon(y_\varepsilon - \psi) p_\varepsilon = \nabla g^\varepsilon(y_\varepsilon) \quad \text{in } \Omega$$

$$B^* p_\varepsilon \in \partial h(u_\varepsilon) + u_\varepsilon - u^* .$$

It turns out that for $\varepsilon \longrightarrow o$,

$$u_\varepsilon \longrightarrow u^* \quad \text{strongly in } U$$

$$y_\varepsilon \longrightarrow y^* \quad \text{weakly in } H^2(\Omega) \text{ and strongly in } H^1_o(\Omega)$$

$$\beta^\varepsilon(y_\varepsilon - \psi) \longrightarrow f + Bu^* - A_o y^* \quad \text{weakly in } L^2(\Omega)$$

and on a subsequence (again denoted ε)

$$p_\varepsilon \longrightarrow p \quad \text{weakly in } H^1_o(\Omega)$$

$$\nabla g^\varepsilon(y_\varepsilon) \longrightarrow \xi \in \partial g(y^*) \quad \text{weakly in } L^2(\Omega)$$

$$p_\varepsilon \beta^\varepsilon(y_\varepsilon - \psi) \longrightarrow p\,(f + Bu^* - A_o y^*) = o \quad \text{strongly in } L^1(\Omega)$$

$$\dot\beta^\varepsilon(y_\varepsilon - \psi) p_\varepsilon \longrightarrow \mu \quad \text{weak star in } (L^\infty(\Omega))^*$$

$$(y_\varepsilon - \psi)\dot\beta^\varepsilon(y_\varepsilon - \psi) \longrightarrow o \quad \text{strongly in } L^1(\Omega).$$

All these relations taken together imply (1.7) $\sim$ (1.10) as claimed.

At first glance Eqs.(1.7) $\sim$ (1.10) seem too intricate to provide a precise description of optimal controllers. However, we will see below that they contain sufficient information for the explicit solution of some typical problems (admittedly of limited generality).

The Control of elastic string. Consider the following problem ([18]): Maximize

(1.14) $\displaystyle\int_o^1 y(x)dx$

on all $y \in H^2(o,1)$, $u \in U$ subject to

(1.15) $y_{xx} + u \leq o,\ y \geq o,\ (y_{xx}+u)y = o$ a.e. in $(o,1)$

$y(o) = y(1) = 1$

where

(1.16) $U = \left\{u \in L^{\infty}(o,1); -N \leq u(x) \leq o,\ \int_o^1 u(x)dx = -M\right\}, (N \geq M).$

Physically, (1.15) represent in the plane oXy the equations of an elastic string clamped at the points (0,1), (1.1) and pressured by a vertical force $u \in U$ and limited from below by a rigid obstacle $y=o$. This is a problem of the form (1.5) where

$$g(y) = -\int_o^1 y(x)dx \quad , \quad y \in L^2(0,1)$$

$$h(u) = \begin{cases} o & \text{if } u \in U \\ +\infty & \text{otherwise,} \end{cases}$$

$\Psi = -1$ and $f \equiv o$.

As noted above, this problem admits at least one solution.

We note that $\partial h(u)$ is the normal cane at U in u and it is the set of all $\xi \in L^2(0,1)$ such that

(1.17) $\xi(x) = \lambda$ in $\{x; -N < u(x) < o\}$, $\xi(x) \leq \lambda$ in $\{x; u(x) = -N\}$, $\xi(x) \geq \lambda$ in $\{x; u(x) = o\}$

where λ is some real number.

Thus according to Theorem 1 if u^* is an optimal control in problem (1.15) then $\exists\ \lambda \in R$ such that

(1.18) $$u^*(x) = \begin{cases} -N & \text{if } p(x) < \lambda \\ 0 & \text{if } p(x) > \lambda \end{cases}$$

where $p \in H_o^1(0,1)$ satisfies the equations

$$(1.19)\qquad p_{xx} + 1 = 0 \qquad \text{in } \{x \in (0,1); y^*(x) > 0\}$$

$$(1.20)\qquad p(x) = 0 \quad \text{a.e. in } \{x \in (0,1); y^*_{xx}(x) + u^*(x) \neq 0\}.$$

Since every solution y to problem (1.15) is concave and continuously differentiable (actually $y \in H^2(0,1)$) we conclude that the coincidence set $\{x \in [0,1]; y^*(x) = 0\}$ is an interval $[a,b]$, $0 < a \leq b < 1$ (if not empty). Then by (1.19) we see that p is in H^2 on $(0,a)$ and $(b,1)$, respectively. Moreover, one has

$$p_{xx} = -1 \quad \text{a.e. in } (0,a) \cup (b,1)$$

i.e.,

$$(1.21)\qquad p(x) = -\frac{x^2}{2} + C_1 x \qquad \text{for } 0 \leq x \leq a$$

$$(1.22)\qquad p(x) = -\frac{x^2}{2} + C_2 x - C_2 + \frac{1}{2} \qquad \text{for } b \leq x \leq 1$$

where C_1, C_2 are some constants.

We will prove first that $[a,b] \neq \emptyset$ and $a < b$. Indeed if $[a,b] = \emptyset$ then it follows by (1.19), (1.20) that

$$p(x) = -\frac{x^2}{2} + \frac{x}{2} \qquad \text{for } 0 \leq x \leq 1$$

and so by (1.18) it would follow that either $u^*(x) = 0$ for all $x \in [0,1]$ or $u^*(x) = -N$ for all $x \in [0,1]$ or $u^*(x) = -y^*_{xx}(x) = 0$ for $x \in [\eta, 1-\eta]$. Since $N > M$ and y^* is continuously differentiabl[e] on $[0,1]$ all these situations are impossible. If $a = b$ then clearl[y] $a = \frac{1}{2}$ and $y^*_x(a) = 0$. Since u^* takes the values $-N$ and 0 only it follows from (1.21), (1.22) that

$$y^*_{xx}(x) = -N \qquad \text{for } x \in [a-\varepsilon, a+\varepsilon]$$

$$y^*_{xx}(x) = 0 \qquad \text{for } x \in [0, a-\varepsilon] \cup [a+\varepsilon, 1].$$

Hence $y^*(x) = -\frac{N}{2}(x-a)^2$ for $|x-a| \leq \varepsilon$ and $y^*(x) = C_1x+1$ for $x \in [o, a-\varepsilon]$. Since $y^*(o) = 1$ we arrived at a contradiction. Hence $o < a < b < 1$.

Now we return back to Eq.(1.18) to show that $\lambda > o$. Assume first that $\lambda < o$ and argue from this to a contradiction. If $p(a) \geqslant \lambda$ then it follows by (1.18) that $u^* = o$ in $[o,a)$ and therefore $y^*(x) = \alpha x + \beta$ for $x \in [o,a]$ which surely contradicts the fact that $y^* \in H^2(0,1)$. If $p(a) < \lambda$ then again by (1.18) and continuity of p we see that $u^* = -N$ in a sufficiently small interval $[a-\varepsilon, a+\varepsilon]$ and by (1.20) $pu^* = o$ in $[a, a+\varepsilon]$. Hence $p = o$ in $[a, a+\varepsilon]$ contrary to our assumptions. Hence $\lambda \geqslant o$.
By a similar argument it follows that also the case $\lambda = o$ is impossible.

Now we shall prove that $p(a) = o$, $p(b) = o$.

Indeed if $p(a) > \lambda$ then by (1.18) we see that $u^* = o$ in a neighborhood of a which clearly contradicts the fact that $y^* \in H^2(0,1)$ and $y^* = o$ in (a,b). If $p(a) < \lambda$ then by continuity of p, $u^* = -N$ on some interval $[a, a+\varepsilon]$ and so by (1.20) $p = o$ on $[a, a+\varepsilon]$ which leads to a contradiction if $p(a) \neq o$. Finally, if $p(a) = \lambda$ then this means that $p(x) > \lambda$ and so $u^* = o$ an some left interval of a which as seen above contradicts the fact that y^* is in $H^1(0,1)$. Thus we must have $p(a) = o$ and by an identical argument it follows that $p(b) = o$.

To summarize, we have shown so far that $\lambda > o$ and the functions p in (1.21), (1.22) have the following precise form (see Fig.1 below)

(1.23) $p(x) = -\frac{x}{2}(x-a)$ for $o \leq x \leq a$; $p(x) = o$, $a \leq x \leq b$

(1.24) $p(x) = -\frac{x^2}{2} + \frac{1}{2}(b+1)x - \frac{b}{2}$ for $b \leq x \leq 1$

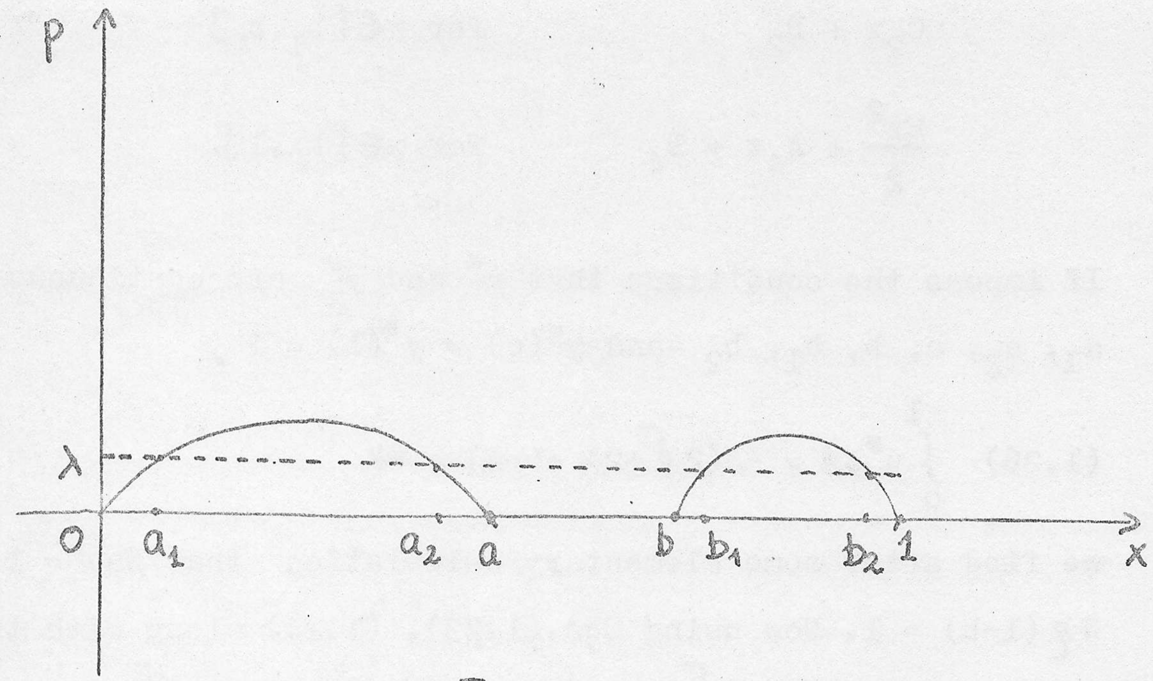

Fig. 1

Denote by $a_1 = \delta$, $a_2 = a-\delta$, $b_1 = b+\eta$, $b_2 = 1-\eta$ the solution of the equation $p(x) = \lambda$. Then by (1.18) we see that u^* is a bang-bang control having the following form

$$(1.25)\quad u^*(x) = \begin{cases} -N & \text{if } x \in (0,\delta) \cup (a-\delta,\ b+\eta) \cup (1-\eta,1) \\ 0 & \text{if } x \in (\delta, a-\delta) \cup (b+\eta, 1-\eta) \end{cases}$$

Since $y^* \in H^2(0,1)$ it is obvious that $0 < a_1 < a_2 < a < b < b_1 < b_2 < 1$.

Now by (1.15) and (1.25) we have

$$y^*(x) = \begin{cases} \frac{Nx^2}{2} + A_1 x & \text{for } x \in [a, a_1] \\ C_1 x + D_1 & \text{for } x \in [a_1, a_2] \\ \frac{Nx^2}{2} + A_2 x + B_2 & \text{for } x \in [a_2, a] \\ 0 & \text{for } x \in [a, b] \\ \frac{Nx^2}{2} + A_3 x + B_3 & \text{for } x \in [b, b_1]\ . \end{cases}$$

$C_2x + D_2$ for $x \in [b_1, b_2]$

$\frac{Nx^2}{2} + A_4x + B_4$ for $x \in [b_2, 1]$.

If impose the conditions that y^* and y_x^* are continuous in a_1, a_2, a, b, b_1, b_2 and $y^*(o) = y^*(1) = 1$,

$$(1.26) \quad \int_o^1 u^* dx = -N(2\delta + 2\eta + b - a) = -M$$

we find after some elementary calculation that $Na\delta = 1$ and $N\eta(1-b) = 1$. Now using Eqs.(1.23), (1.24) along with the obvious relation $p(\delta) = p(1-\eta)$ we find that $\delta = \eta$ and therefore $b = 1-a$.

Finally, using Eq.(1.26) we see that

$$(1.27) \quad \eta = \delta = (M-N+((M-N)^2+32N)^{1/2})(8N)^{-1}$$

$$(1.28) \quad a = 1-b = (M-N+((M-N)^2+32N)^{1/2})/8.$$

We have therefore proved that the <u>unique optimal control of problem</u> (1.24) <u>is the function</u> u^* <u>defined by</u> (1.25) <u>where</u> a, b, δ, η <u>are given by</u> (1.27), (1.28).

(At the same conclusion arrived Yaniro [18] by a different approach.)

One might expect to have a similar description of optimal control in two dimensions at least for some particular domains Ω.

2. OPTIMAL CONTROL OF SOME MOVING BOUNDARY PROBLEMS

Consider the following problem: <u>Minimize</u>

$$(2.1) \quad \int_o^T g(t, y(t))dt + h(u) + \varphi_o(y(T))$$

<u>on all</u> $y \in L^2(Q)$ <u>and</u> $u \in L^2(\Sigma_1)$ <u>subject to</u>

$$(2.2)\quad \begin{aligned} & y_t + A_o y = f_o \quad \text{in } \{(x,t) \in Q; y(x,t) > o\} \\ & y_t + A_o y \geqslant f_o, \quad y \geqslant o \quad \text{in } Q = \Omega \times (o,T) \\ & y(x,o) = y_o(x), \qquad x \in \Omega \\ & \frac{\partial y}{\partial \nu} + \alpha y = v \text{ in } \Sigma_1; \; y = o \text{ in } \Sigma_2. \end{aligned}$$

$$(2.3)\quad \frac{dv}{dt} + \Lambda v = Bu \qquad \text{for } t \in [o,T]; \; v(o) = o.$$

Here Ω is a bounded, open subset of R^n with a sufficiently smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Sigma_i = \Gamma_i \times (o,T)$, $i = 1,2$; A_o is a symmetric elliptic operator of the form (1.1), Λ is a linear continuous operator from $L^2(\Sigma_1)$ to itself, B is a linear continuous operator from a Hilbert space of controllers U to $L^2(\Sigma_1)$, $\alpha > o$ and

$$(2.4)\quad f_o \in W^{1,2}([o,T]; L^2(\Omega)).$$

$$(2.5)\quad y_o \in H^2(\Omega); \; y_o = o \text{ in } \Gamma_2, \; \frac{\partial y_o}{\partial \nu} + \alpha y_o = o \text{ in } \Gamma_1.$$

As regard functions $g: [o,T] \times L^2(\Omega) \longrightarrow R$, $\varphi_o: L^2(\Omega) \longrightarrow R$, $h: U \longrightarrow \overline{R}$ we will assume that

(i) h is convex, lower semicontinuous and

$$(2.6)\quad h(u) \geqslant \gamma \|u\|_U^2 + C \qquad \forall u \in U$$

for some $\gamma > o$ and $C \in R$.

(ii) g is measurable in t, $g(t,o) \in L^\infty(o,T)$ and there exists $C \in R$ such that

$$(2.7)\quad g(t,y) + \varphi_o(y) \geq C\,(\|y\|_{L^2(\Omega)} + 1) \quad \forall\, y \in L^2(\Omega).$$

For every $r > o$ there exists $L_r \geq o$ such that

$$|g(t,y) - g(t,z)| + |\varphi_o(y) - \varphi_o(z)| \leq L_r \|y-z\|_{L^2(\Omega)}$$

for all $t \in [o,T]$ and $\|y\|_{L^2(\Omega)} + \|z\|_{L^2(\Omega)} \leq r$.

Under assumptions (2.4), (2.5) the boundary value problem (2.2), (2.3) has for every $u \in U$ a unique solution $y \in W^{1,2}([o,T];V) \cap W^{1,\infty}([o,T];H)$. (see [5], p.157).

Here $V = \{y \in H^1(\Omega); y = o \text{ in } \Gamma_2\}$, $H = L^2(\Omega)$ and $W^{1,p}([o,T];X) = \{y \in L^p(o,T;X),\ \frac{dy}{dt} \in L^p(o,T;X)\}$. Moreover, $y = \lim_{\varepsilon \to o} y_\varepsilon$ strongly in $C([o,T];H)$ and weakly in $W^{1,\infty}([o,T];H) \cap W^{1,2}([o,T];V)$ where y_ε is the solution to approximating equation

$$y_t + A_o y + \beta^\varepsilon(y) = f_o \quad \text{in } Q = \Omega \times (o,T)$$

$$(2.8)\quad y(x,o) = y_o(x)$$

$$\frac{\partial y}{\partial \nu} + \alpha\, y = v \quad \text{in } \Sigma_1;\ y = o \text{ in } \Sigma_2.$$

$$(2.9)\quad \frac{dv}{dt} + \Lambda v = Bu \quad \text{in } [o,T];\ v(o) = o$$

where β^ε is defined by (1.13).

The following estimate holds:

$$(2.10)\quad \|y_\varepsilon\|_{W^{1,\infty}([o,T];H) \cap W^{1,2}([o,T];V)} \leq C(1 + \|u\|_U).$$

By standard device it follows that optimal control problem (2.1) admits at least one optimal control u^*. As regards the characterization of optimal controllers we have the following result ([5], p.239.)

THEOREM 2 *Let* (y^*,u^*) *be an optimal pair in problem* (2.1) $\sim$ (2.3). *Then there exists* $p \in L^2(0,T;V) \cap L^\infty(0,T;L^2(\Omega)) \cap BV([0,T];(V \cap H^s(\Omega))')$, $s > n/2$ *such that* $p_t - A_0 p \in (L^\infty(Q))^*$ *and*

(2.11) $(p_t-A_0p)_a \in \partial g(t,y^*)$ a.e. in $\{(x,t) \in Q; y^*(x,t) > 0\}$.

(2.12) $p(T) + \partial \varphi_0(y^*(T)) \ni 0$ in Ω.

(2.13) $\dfrac{\partial p}{\partial \nu} + \alpha p = 0$ in Σ_1; $p = 0$ in Σ_2.

(2.14) $p = 0$ a.e. in $\{(x,t) \in Q; y^*(x,t) = 0,\ f_0(x,t) \neq 0\}$.

(2.15) $B^* \int_t^T e^{-\Lambda^*(s-t)} p(s)ds \in \partial h(u^*)$

(Λ^* *is the adjoint of* Λ .)

If $n = 1$ *then* $y^* \in C(\overline{Q})$ *and* Eq.(2.11) *becomes*

(2.11)' $p_t - A_0 p = \xi \in \partial g(t,y^*)$ in $\{(x,t) \in Q; y^*(x,t) > 0\}$

where $\xi \in L^2(Q)$.

Here $BV([0,T];(V \cap H^s(\Omega))')$ is the space of functions with bounded variation from $[0,T]$ to $(V \cap H^s(\Omega))'$ and $(p_t-A_0p)_a$ is the absolutely continuous part of the measure p_t-A_0p (p_t is the distributional derivative of $p:[0,T] \longrightarrow L^2(\Omega)$ $\partial \varphi_0$, ∂g are the generalized gradient of φ_0 and $y \longrightarrow g(t,y)$.

Proof. Since the proof is essentially the sqme as that of Theorem 6.3 in [5] it will be sketched only. Also for the sake of simplicity we will assume that g and φ_0 are differentiable on $L^2(\Omega)$.

For every $\varepsilon > 0$ consider the approximating control problem: *Minimize*

(2.16) $\int_0^T (g(t,y(t))dt + h(u) + \frac{1}{2} \| u-u^* \|^2_U + \varphi(y(T))$

on all $(y,u) \in (W^{1,\infty}([o,T];H) \cap W^{1,2}([o,T];V)) \times L^2(\Sigma_1)$ subject to Eqs.(2.8), (2.9).

Let $(y_\varepsilon, u_\varepsilon)$ be a solution to problem (2.15). By estimate (2.10) and assumptions (i), (ii) we see that for $\varepsilon \longrightarrow o$

$$u_\varepsilon \longrightarrow u^* \quad \text{strongly in } U$$

$$(2.17) \quad y_\varepsilon \longrightarrow y^* \quad \text{strongly in } C([o,T];H)$$
$$\text{weakly in } W^{1,\infty}([o,T];H) \cap W^{1,2}([o,T];V)$$

$$v_\varepsilon \longrightarrow v^* \quad \text{strongly in } W^{1,2}([o,T];L^2(\Gamma_1))$$

where

$$(2.18) \quad \frac{dv_\varepsilon}{dt} + \Lambda v_\varepsilon = Bu_\varepsilon \quad \text{a.e. in } [o,T];\ v_\varepsilon(o) = o.$$

On the other hand, we have

$$(2.19) \quad \int_o^T (\nabla_y g(t,y_\varepsilon(t)), z(t))dt + h'(u_\varepsilon,w) + \langle u_\varepsilon - u^*, w\rangle \geq o \quad \forall\, w \in U$$

where h' is the directional derivative of h, $(\cdot,\cdot)$ and $\langle \cdot,\cdot \rangle$ are the scalar products in $L^2(\Omega)$ and U, respectively, whilst z is the solution to

$$z_t + A_o z + \dot{\beta}^\varepsilon(y_\varepsilon)z = o \quad \text{in } Q$$

$$z(x,o) = o \quad \text{in } \Omega$$

$$\frac{\partial z}{\partial \nu} + \alpha z = \tilde{v} \text{ in } \Sigma_1,\ z = o \text{ in } \Sigma_2$$

$$\frac{d\tilde{v}}{dt} + \Lambda \tilde{v} = Bw \quad \text{in } [o,T],\ \tilde{v}(o) = o.$$

Let $p_\varepsilon \in W^{1,2}([o,T];L^2(\Omega)) \cap L^2(o,T;V)$ be the solution to boundary value problem

$$(p_\varepsilon)_t - A_0 p_\varepsilon - p_\varepsilon \dot{\beta}^\varepsilon(y_\varepsilon) = \nabla_y g(t,y_\varepsilon) \text{ in } Q$$

$$(2.20)\quad p_\varepsilon(x,T) + \nabla\varphi_0(y_\varepsilon(T))(x) = 0, \quad x \in \Omega$$

$$\frac{\partial p_\varepsilon}{\partial \nu} + \alpha p_\varepsilon = 0 \quad \text{in } \Sigma_1;\ p_\varepsilon = 0 \quad \text{in } \Sigma_2.$$

After some calculation involving (2.19) and (2.20) we see that

$$(h'(u_\varepsilon,w) + \langle u_\varepsilon - u^*, w\rangle - \int_{\Sigma_1} p_\varepsilon(\sigma,t)\Big(\int_0^t e^{-\Lambda(t-s)}(Bw)(s)ds\Big)d\sigma\, dt \geq 0$$

$$\forall\ w \in U.$$

This yields

$$(2.21)\quad B^*\Big(\int_t^T e^{-\Lambda^*(s-t)} p_\varepsilon(s)ds\Big) + u_\varepsilon - u^* \in \partial h(u_\varepsilon).$$

Next we multiply Eq.(2.20) by p_ε and sgn p_ε and integrate on Q. We obtain the estimate

$$\|p_\varepsilon(t)\|^2_{L^2(\Omega)} + \int_0^T \|p_\varepsilon(t)\|^2_{H^1(\Omega)} dt + \int_Q |\dot{\beta}^\varepsilon(y_\varepsilon)p_\varepsilon|\, dxdt \leq C.$$

Hence $\{(p_\varepsilon)_t\}$ is bounded in $L^1(0,T;L^1(\Omega)) + L^2(0,T;V') \subset L^1(0,T;(H^s(\Omega)\cap V)')$ for $s > n/2$ (by the Sobolev's imbedding theorem). Thus on a subsequence, we have

$$p_\varepsilon \longrightarrow p \quad \text{weakly in } L^2(0,T;V) \text{ and weak star in } L^\infty(0,T;L^2(\Omega))$$

and by the Helly theorem

$$p_\varepsilon(t) \longrightarrow p(t) \quad \text{strongly in } (H^s(\Omega)\cap V)' \ \forall\ t \in [0,T].$$

Now since the injection of V into $L^2(\Omega)$ is compact, for every $\lambda > 0$ we have

$$\|p_\varepsilon(t) - p(t)\|_{L^2(\Omega)} \leq \lambda \|p_\varepsilon(t) - p(t)\|_V + \delta(\lambda)\|p_\varepsilon(t) - p(t)\|_{(H^s(\Omega)\cap V)'}$$

Hence

$$p_\varepsilon \longrightarrow p \quad \text{strongly in } L^2(Q)$$

$$p_\varepsilon(t) \longrightarrow p(t) \quad \text{weakly in } L^2(\Omega) \quad \forall\, t \in [0,T]$$

and on a generalized sequence

(2.22) $\dot{\beta}^{\varepsilon}(y_\varepsilon)p_\varepsilon \longrightarrow \mu$ weak star in $(L^\infty(Q))^*$.

Finally, arguing as in the proof of Theorem 5.2 in [5] we see that (on a subsequence.)

(2.23) $p_\varepsilon \beta^{\varepsilon}(y_\varepsilon) \longrightarrow p(f_o - y_t^* - A_o y^*)$ strongly in $L^1(Q)$

(2.24) $p_\varepsilon \dot{\beta}^{\varepsilon}(y_\varepsilon) \longrightarrow o$ strongly in $L^1(Q)$.

Combining the above relations we conclude that p satisfies Eqs.(2.11) ~ (2.15).

If n = 1 then it follows by (2.17) that

$$y_\varepsilon \longrightarrow y^* \quad \text{in} \quad C(\overline{Q})$$

and by (2.22), (2.24) we infer that

$$\mu y^* = (p_t - A_o p - \xi)\mu = o \quad \text{in } Q$$

where $\xi = \lim_{\varepsilon \to o} \nabla_y g(t, y_\varepsilon)$ (in $L^2(Q)$) and μy^* stands for the product of μ with y^*.

<u>The controlled one-phase Stefan problem</u> consider the model of the metting of a body of ice $\Omega \subset R^3$ maintained at 0^oC in contact with a region of water. The boundary Γ of Ω is composed of two disjoint parts Γ_1 and Γ_2. The boundary Γ_1 is in contact with a heating medium with temperature θ_1 while the temperature on boundary Γ_2 is zero.

Let $T > o$ and $\theta(x,t)$ be the water temperature at point $x \in \Omega$ and time t. Initially the water occupies the domain $\Omega_o \subset \Omega$

and is at temperature η (see Fig.2 below)

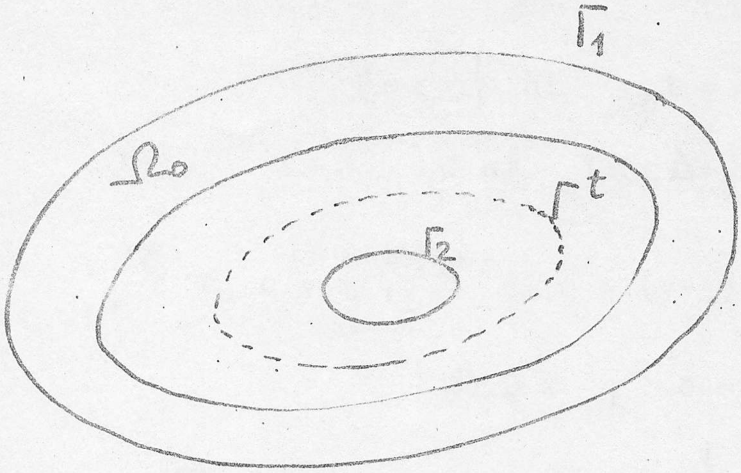

Fig. 2

If $t = \ell(x)$ is the equation of water-ice interface then temperature distribution θ satisfies the classical Stefan problem

$$\theta_t - \Delta\theta = 0 \quad \text{in } \{(x,t) \in Q;\ \ell(x) < t < T\}$$

$$\theta = 0 \quad \text{in } \{(x,t) \in Q;\ \ell(x) \geq t\}$$

(2.25) $$\nabla_x \theta \cdot \nabla \ell(x) = -\rho \quad \text{in } \{(x,t) \in Q;\ t = \ell(x)\}$$

$$\frac{\partial\theta}{\partial\nu} + \alpha(\theta - \eta) = 0 \text{ in } \Sigma_1, \quad \theta = 0 \text{ in } \Sigma_2$$

$$\theta(x,0) = \theta_0(x) \text{ for } x \in \Omega_0; \quad \theta(x,0) = 0 \quad \forall\, x \in \Omega \setminus \Omega_0.$$

Of course $\Omega_0 = \{x \in \Omega;\ \ell(x) < 0\}$.

Using a well-known device (see for instance [5], [9], [11]) we may reduce the free boundary problem (2.25) to a parabolic variational inequality of the form (2.2). More precisely, the function y defined by

$$(2.26)\quad y(x,t) = \int_0^t \theta(x,\tau)\chi(x,\tau)\,d\tau\,,\quad (x,t)\in Q$$

where $\chi(x,t) = 1$ if $\ell(x)\leq t$; $\chi(x,t) = o$ if $\ell(x) > t$

is the solution to

$$(2.27)\quad y_t - \Delta y = f_o \quad \text{in } \{y > o\}$$

$$y \geq o,\ y_t - \Delta y \geq f_o \quad \text{in } Q$$

$$\frac{\partial y}{\partial \nu} + \alpha(y-v) = o \text{ in } \Sigma_1,\ y = o \text{ in } \Sigma_2$$

$$y(x,o) = o\quad,\quad x\in\Omega$$

where $v(\sigma,t) = \int_0^t \eta(\sigma,\tau)\,d\tau\,,\ (\sigma,t)\in\Sigma_1$ and

$$(2.28)\quad f_o(x) = \theta_o(x) \text{ for } x\in\Omega_o;\ f_o(x) = -\rho \text{ for } x\in\Omega\setminus\Omega_o.$$

Obviously $\{(x,t)\in Q;\ \ell(x) > t\} = \{(x,t)\in Q;\ y(x,t) > o\}$.

Consider the following problem: Maximize

$$(2.29)\quad \int_Q \theta(x,t)\,dxdt$$

for θ and $u\in U_o$ satisfying Eqs.(2.25) where

$$(2.30)\quad \eta(\sigma,t) = g_o(\sigma)u(t)\quad \forall\,\sigma\in\Gamma_1,\ t\in[o,T]$$

$$(2.31)\quad U_o = \{u\in L^2(o,T);\ o\leq u(t)\leq N,\ \int_0^T u(t)dt = M\}.$$

Here $g_o\in L^2(\Gamma_1)$ is a given nonnegative function on Γ_1 and M, N are positive constants such that $T > M/N$.

In terms of y defined by transformation (2.26) problem (2.29) becomes

$$(2.32)\quad \mathrm{Max}\left\{\int_\Omega y(x,T)dx:\ u\in U_o\right\}$$

where the maximum is taken on all y satisfying Eqs.(2.27), (2.28), (2.30).

We will apply Theorem 2 where $\Lambda \equiv 0$, $Bu = g_0 u$, $U = L^2(0,T)$, $g \equiv 0$, $\varphi_0(y) = \int_\Omega y(x)dx$ and

$$(2.33) \quad h(u) = \begin{cases} 0 & \text{if } u \in U_0 \\ +\infty & \text{otherwise.} \end{cases}$$

Hence every optimal control u^* is given by

$$(2.34) \quad u^*(t) = (\partial h)^{-1}\left(\int_{\Gamma_1} g_0(\sigma) \int_t^T p(\sigma,s)ds\right) \quad \text{a.e. } t \in [0,T]$$

where $p \in L^2(0,T;H^1(\Omega))$ satisfies the following systems

$$(2.35) \quad \begin{aligned} &(p_t + p)_a = 0 \quad \text{in} \quad \{(x,t); y^*(x,t) > 0\} \\ &p(x,T) = 1 \quad , \quad x \in \Omega \\ &\frac{\partial p}{\partial \nu} + \alpha p = 0 \quad \text{in } \Sigma_1, \ p = 0 \text{ in } \Sigma_2 . \\ &p = 0 \quad \text{in } \{(x,t) \in Q; y^*(x,t) = 0\}. \end{aligned}$$

Let us observe that $p \geq 0$ a.e. in Q. This can be seen recalling that $p = \lim_{\varepsilon \to 0} p_\varepsilon$ (strongly in $L^2(Q)$) where p_ε is the solution to system (2.22) ($g \equiv 0, \nabla \varphi_0 = -1$). By the maximum principle, $p_\varepsilon \geq 0$ in Q and so $p \geq 0$ as claimed.

We set $\psi(t) = \int_t^T ds \int_{\Gamma_1} g_0(\sigma)p(\sigma,s)d\sigma$. Then Eq.(2.34) is equivalent to the following

$$\int_0^T u^*(t)\psi(t)dt \geq \int_0^T u(t)\psi(t)dt \quad \forall \ u \in U_0$$

i.e.,

$$(2.36) \quad \int_0^T u^*(t)(\psi(T)-\psi(t))dt \leq \int_0^T u(t)(\psi(T)-\psi(t))dt \quad \forall \ u \in U_0$$

Inasmuch as the function $t \longrightarrow \psi(T)-\psi(t)$ is nonnegative and monotone increasing we infer by (2.30)(see Lemma 1 below) that

$$(2.37) \quad u^*(t) = \begin{cases} N & \text{if } 0 \leq t \leq t_1 \\ 0 & \text{if } t_1 < t \leq T \end{cases}$$

where $t_1 = N/M$.

We have therefore proved

COROLLARY 1 <u>The optimal control problem</u> (2.32) <u>has a unique solution</u> u^* <u>given by</u> (2.37).

REMARK 1 Problem (2.29) has been studied in a special case by E.N.Barron [6] (see also [12]).

LEMMA 1 <u>Let</u> $\varphi : [0,T] \longrightarrow R$ <u>be a nonnegative and monotone increasing function and let</u> $U^0 = \{u \in L^\infty(0,T)\ ;$ $0 \leq \alpha \leq u(t) \leq \beta,\ \int_0^T u(t)dt = \gamma$ <u>where</u> $\alpha T \leq \gamma \leq \beta T$.

<u>Then the function</u> $u \longrightarrow \int_0^T u(t)\varphi(t)dt$ <u>attains its maximum on</u> U^0 <u>in a unique point</u> u^* <u>given by</u>

$$(2.38) \quad u^*(t) = \begin{cases} \alpha & \text{for } t \in [0,t_1] \\ \beta & \text{for } t \in (t_1,T] \end{cases}$$

<u>where</u> $t_1 = (T\beta - \gamma)(\beta - \alpha)^{-1}$.

The proof is elementary. However, we outline it for readear's convenience. Indeed it is readily seen that

$$\int_0^T u(t)\varphi(t)dt \leq \int_0^T u^*(t)\varphi(t)dt \quad \forall\ u \in U^0.$$

To prove that u^* is unique we assume that there is some $u \in U^0$ such that

$$\int_o^{t_1} (u(t)-\alpha)\varphi(t)dt + \int_{t_1}^{T} (u(t)-\beta)\varphi(t)dt = o.$$

Then by the mean theorem we have for some $\tau_1 \in [o,t_1], \tau_2 \in [t_1,T]$

$$\varphi(\tau_1)\int_o^{t_1} (u(t)-\alpha)dt + \varphi(\tau_2)\int_{t_1}^{T} (u(t)-\beta)dt = o$$

i.e.,

$$(2.39)\quad (\varphi(\tau_1)-\varphi(\tau_2))\int_o^{t_1} (u(t)-\alpha)dt = o.$$

If φ is nonidentically constant then we may take τ_1 and τ_2 such that $\varphi(\tau_1) < \varphi(\tau_2)$. Then by (2.39) we see that $u(t) = \alpha$ for $t \in [o,t_1]$ as claimed.

3. A BILINEAR OPTIMAL CONTROL PROBLEM

We will study here the following model problem: <u>Minimize</u>

$$(3.1)\quad \int_o^T \int_o^1 g_o(y(x,t))dxdt + \int_o^1 \gamma_o(y(x,T))dx$$

<u>on all</u> $u \in U_o$ <u>and</u> $y \in L^2(Q)$ <u>subject to</u>

$$y_t - y_{xx} + uy + \beta(y) \ni f \quad \text{in } Q = (o,1)\times(o,T)$$

$$(3.2)\quad y(x,o) = y_o(x), \qquad x \in (o,1)$$

$$y_x(o,t) = o, \quad y(1,t) = o, \qquad t \in [o,T].$$

Here $U = \{u \text{ monotone increasing}, o \le u(x) \le N; \int_o^1 u(x)dx = M\}$ and β is a maximal monotone graph in $R \times R$ such that

$$(3.3)\quad \beta_\varepsilon \text{ is concave and } \beta_\varepsilon(r)r^- \le C(r^-)^2 \quad \forall\, r \in R$$

where $\beta_\varepsilon(r) = \varepsilon^{-1}(r-(1+\varepsilon\beta)^{-1}r) \quad \forall\, r \in R; \ \varepsilon > o$

(3.4) $g_o, \psi_o \in C^2(R)$, $g_o^{(k)}(x) \geq o$, $\psi_o^{(k)}(x) \geq o$

for $x \in [o,1]$ and $k = 1,2$.

Finally,

(3.5) $f\ f_x \in L^2(Q)$, $y_o \in H^1(o,1)$, $y_o(1) = o$.

(3.6) $f, f_x \geq o$ a.e. in Q, $y_o(x) \geq o$, $y_o'(x) \leq o$ a.e. $x \in (o,1)$.

In linear case, i.e., $\beta \equiv o$ a problem of this type was solved by A.Friedman [10]. Problem (3.2) serves as model for the controlled cooling of an inhomogeneous rod of length by dissipation effect in the presence of a thermostat control process. In this case the control u is the heat transfer coefficient and β is a multivalued graph of the form

$$(3.7)\qquad \beta(r) = \begin{cases} o & \text{if } a < r < \infty \\ R^- & \text{if } r = a \\ \emptyset & \text{if } r < a . \end{cases}$$

In the case of a black body radiation of the rod , $\beta(r) = \alpha(r^4 - r_o^4)$ while in the presence of natural heat convection $\beta(r) = \alpha r^{5/4}$ for $r \geq o$, $\beta(r) = o$ for $r < o$. Note that in all these situation, hypothesis (3.3) holds.

By standard existence theory (see for instance [1], Chap.4) we know that under assumptions (3.5), problem (3.2) has for every $u \in L^\infty(o,1)$ a unique solution $y \in W^{1,2}([o,T];L^2(o,1)) \cap L^2(o,T;H^2(o,1)) \subset C([o,T];H^1(o,1)) \subset C(\bar{Q})$. Moreover, the map $u \longrightarrow y$ is bounded from $L^\infty(o,1)$ to $W^{1,2}([o,T];\ L^2(o,1)) \cap L^2(o,T;H^2(o,1))$ and therefore compact from $L^\infty(o,1)$ to $L^2(Q)$. Then by a standard argument we infer that problem (3.1) has at least one solution.

THEOREM 3 *There exists a unique solution* u^* *to optimal problem* (3.1) *given by*

(3.8) $u^*(x) = M \quad \forall\ x \in [0,1]$.

Proof. Let u^* be any optimal control of problem (3.1). Consider the approximating control problem

$$\text{(3.9)}\quad \inf\left\{ \int_Q g_0(y)\,dxdt + \int_0^1 \psi_0(y(x,T))\,dx + \frac{1}{2}\int_0^1 |u(x)-u^*(x)|^2 d;u\in$$

where y is the solution to

$$y_t - y_{xx} + \beta^{\varepsilon}(y) + uy = f \quad \text{in } Q$$

$$\text{(3.10)}\quad y(x,0) = y_0(x) \quad , \quad x \in \Omega \quad .$$

$$y_x(0,t) = 0,\ y(1,t) = 0 \ ,\quad t \in [0,T].$$

Here β^{ε} is a smooth approximation of β_{ε}. For in stance we may define β^{ε} as a mollifier of β_{ε} , i.e.,

$$\text{(3.11)}\quad \beta^{\varepsilon}(r) = \int_{-\infty}^{\infty} \beta_{\varepsilon}(r-\varepsilon^2\theta)\,\rho(\theta)\,d\theta \ ,\quad r \in R$$

where $\rho \in C_0^{\infty}(R)$, $\int \rho\, d\theta = 1$ and support $\rho \subset [-1,1]$. We see that $(\beta^{\varepsilon})' \geq 0$, $(\beta^{\varepsilon})'' \geq 0$ and by (3.3) we may assume that

$$\text{(3.12)}\quad \beta^{\varepsilon}(r)r^- \leq C(r^-)^2 \quad \forall\ r \in R.$$

For every $\varepsilon > 0$ problem (3.9) has at least one solution u_{ε} . Letting ε tend to zero in the obvious inequality

$$\int_0^T\int_0^1 g_0(y_{\varepsilon})\,dxdt + \int_0^1 \psi_0(y_{\varepsilon})\,dx + \frac{1}{2}\int_0^1 |u_{\varepsilon} - u^*|^2\,dx \leq$$

$$\int_0^T\int_0^1 g_0(y_{\varepsilon}^*)\,dxdt + \int_0^1 \psi_0(u^*)\,dx$$

where y_{ε}^* is the solution to Eq.(3.10) where $u = u^*$ we see that

$$u_\varepsilon \longrightarrow u^* \text{ strongly in } L^2(0,1)$$

(3.13)

$$y_\varepsilon \longrightarrow y^* \text{ strongly in } C([0,T];H^1(0,1)) \text{ and weakly in } L^2(0,T;H^2(0,1)) \; W^{1,2}([0,T];L^2(0,1)).$$

Let $p_\varepsilon \in L^2(0,T;H^1(0,1))$ be the solution to

$$(p_\varepsilon)_t+(p_\varepsilon)_{xx}-\dot{\beta}^\varepsilon(y_\varepsilon)p_\varepsilon-u_\varepsilon p_\varepsilon = g_0'(y_\varepsilon) \text{ in } Q$$

(3.14) $$p_\varepsilon(x,T) = -\gamma_0'(y_\varepsilon(x,T)) \quad \forall\; x\in\Omega$$

$$(p_\varepsilon)_x(0,t) = 0,\; p_\varepsilon(1,t) = 0 \;,\quad t\in[0,T].$$

It is well known that $(T-t)^{1/2}p_\varepsilon \in L^2(0,T;H^1(0,1))$ and $(T-t)^{1/2}(p_\varepsilon)_t \in L^2(Q)$.

Since u_ε is optimal in problem (3.1) we find after some calculation involving Eqs.(3.14) that

$$\int_0^1 v(x)dx\int_0^T p_\varepsilon(x,t)y_\varepsilon(x,t)dt + I_U'(u_\varepsilon,v) + $$

$$+\int_0^T (u_\varepsilon-u^*)vdt \geq 0 \quad\forall\; v\in L^2(0,1)$$

where $I_U'(u_\varepsilon,v) = \lim_{\lambda\to 0}(I_U(u_\varepsilon+\lambda v)-I_U(u_\varepsilon))/\lambda$ and $I_U(u) = 0$ if $u\in U$ and $I_U(u) = +\infty$ if $u\bar{\in} U$.

Hence

$$-\int_0^T p_\varepsilon(x,t)y_\varepsilon(x,t)dt + u^* - u_\varepsilon \in \partial I_U(u_\varepsilon) \text{ a.e. in } (0,1).$$

In other words,

(3.15) $$u_\varepsilon(x) = (\partial I_U)^{-1}(\varphi_\varepsilon+u^*-u_\varepsilon)(x) \quad \text{a.e. } x\in(0,1)$$

where

(3.16) $$\varphi_\varepsilon(x) = -\int_0^T p_\varepsilon(x,t)y_\varepsilon(x,t)dt \;,\quad x\in[0,1].$$

Now multiply Eqs.(3.14) by p_ε and sgn p_ε respectively, and integrate on $(o,1)\times(t,T)$. We get the following estimates

$$(3.17)\quad \int_o^1 |p_\varepsilon(x,t)|^2 dx+\int_t^T\int_o^1 (p_\varepsilon)_x^2(x,s)dxds+2\int_t^T\int_o^1 u_\varepsilon(x)p_\varepsilon^2(x,s)dxds \leq$$

$$(3.18)\quad \int_o^T\int_o^1 |p_\varepsilon(x,t)\dot{\beta}^\varepsilon(y_\varepsilon(x,t))|dxdt \leq C \qquad \forall \varepsilon > o.$$

Hence $(p_\varepsilon)_t$ is bounded in $L^1(o,T;L^1(o,1)+L^2(o,T;(H^1(o,1))')\subset L^1(o,T;(H^1(o,1))')$. Thus selecting a subsequence (again denoted ε) we have

$$(3.19)\quad p_\varepsilon \longrightarrow p \quad \text{weakly in } L^2(o,T;H^1(o,1)) \text{ and}$$
$$\text{weak star in } L^\infty(o,T;L^2(o,1))$$

and by Helly's theorem $p\in BV([o,T];(H^1(o,1))')$ and

$$(3.20)\quad p_\varepsilon(t) \longrightarrow p(t) \quad \text{strongly in } (H^1(o,1))'$$
$$\text{for every } t\in[o,T].$$

Now since the injection of $H^1(o,1)$ into $L^2(o,1)$ is compact, for every $\lambda > o$, $\exists\, \delta(\lambda)$ such that

$$\|p_\varepsilon(t)-p(t)\|_{L^2(o,1)} \leq \lambda\|p_\varepsilon(t)-p(t)\|_{H^1(o,1)} + \delta(\lambda)\|p_\varepsilon(t)-p(t)\|_{(H^1(o,1))'}$$

Then by (3.19), (3.20) we conclude that

$$(3.21)\quad p_\varepsilon \longrightarrow p \quad \text{strongly in } L^2(o,T;L^2(o,1)) = L^2(Q).$$

Now coming back to the sequence $\{y_\varepsilon\}$ we see that $y_\varepsilon \geqslant o$ in Q. Indeed if multiply Eq.(3.10) by y_ε^- and use (3.12) and the fact that $f \geqslant o$ in Q and $y_o \geqslant o$ in Ω we get after some calculation that

$$\int_o^1 (y_\varepsilon^-)^2(x,t)dx + \int_o^t\int_o^1 (y_\varepsilon^-)_x^2 dxds + 2\int_o^t\int_o^1 u_\varepsilon(y_\varepsilon^-)^2 dxds \leq$$

$$\leq \quad C \int_0^t \int_0^1 (y_\varepsilon^-)^2 \, dxds$$

which yields $y_\varepsilon^- = o$ as claimed. Similarly it follows by (3.14) and hypotheses (3.4) that

$$p_\varepsilon(x,t) \leq o \qquad \text{a.e. } (x,t) \in Q \ .$$

We will prove further that

$$x \longrightarrow y_\varepsilon(x,t) \text{ is monotone decreasing}$$

$$x \longrightarrow p_\varepsilon(x,t) \text{ is monotone increasing}$$

for almost all $t \in [o,T]$.

Without loss of generality we may assume that u_ε is smooth (Otherwise we approximate it by a smooth function and pass to limit.) Then the function $\theta = (y_\varepsilon)_x$ is the solution to the equation

$$\theta_t - \theta_{xx} + \dot{\beta}^\varepsilon(y_\varepsilon)\theta + u_\varepsilon' y_\varepsilon + u_\varepsilon \theta = f_x \text{ in } Q$$

(3.22) $$\theta(x,o) = y_o'(x) \quad , \quad x \in (o,1)$$

$$\theta(o,t) = o \ .$$

Moreover, since $y_\varepsilon \geq o$ in $(o,1)$ and $y_\varepsilon(1,t) = o$ we infer that $(y_\varepsilon)_x(1,t) \leq o$. Hence $\theta(1,t) \leq o$ and by the maximum principle we see by (3.22) and (3.6) that $\theta(x,t) \leq o \quad \forall \ (x,t) \in Q$.

Similarly, $q = (p_\varepsilon)_x$ satisfies the equation

$$q_t + q_{xx} - (\beta^\varepsilon)''(y_\varepsilon)p_\varepsilon(y_\varepsilon)_x - \dot{\beta}^\varepsilon(y_\varepsilon)q - u_\varepsilon q - u_\varepsilon' p_\varepsilon = g_o''(y_\varepsilon)(y_\varepsilon)_x \text{ in } Q$$

$$q(x,T) = -\varphi_o''(y_\varepsilon(x,T))(y_\varepsilon)_x(x,T) \geq o \text{ in } (o,1)$$

$$q(o,t) = o, \ q(1,t) \geq o \ , \ t \in [o,T]$$

and by (3.3), (3.4) it follows that $q \geq o$ in Q.

We have shown in particular that

(3.23) $\varphi_\varepsilon(x) \geq o \quad \forall \ x \in [o,1]$; φ_ε is monotone decreasing for every $\varepsilon > o$.

On the other hand, it follows by (3.13) and (3.21) that for $\varepsilon \longrightarrow o$

(3.24) $$\varphi_\varepsilon \longrightarrow \varphi = - \int_o^T y^*(x,t)p(x,t)dt \text{ strongly in } L^2(o,1).$$

Thus letting ε tend to zero in Eq.(3.15), i.e.,

$$\int_o^1 \varphi_\varepsilon(x)(u_\varepsilon(x)-v(x))dx + \int_o^1 (u^*(x)-u_\varepsilon(x))(u_\varepsilon(x)-v(x))dx \geq o \quad \forall \ v \in U$$

yields

(3.25) $$\varphi \in \partial I_U(u^*).$$

By (3.24) we see that the function φ is continuous on $[o,1]$ (because y^* and p belong to $L^2(o,T;H^1(o,1))$. Moreover, by (3.23) φ is nonnegative and monotone decreasing. Then the conclusion of Theorem 3 is an immediate consequence of Eq.(3.25) and of Lemma 2 below.

LEMMA 2 <u>Let</u> $\varphi : [o,1] \longrightarrow R$ <u>be a continuous, nonnegative and monotone decreasing function</u>. <u>Then</u>

$$\sup \left\{ \int_o^1 u(x)\varphi(x)dx;\ u \in U \right\} = M \int_o^1 \varphi(x)dx$$

<u>and if</u> φ <u>is no identically constant the suppremum is attained in the unique point</u> $u^* = M$.

<u>Proof</u>. We set v = u - M and decompose the interval $[o,1]$ into $[o,a) \cup [a,b] \cup (b,1]$ where $v(x) < o$ for $x \in [o,a)$, $v(x) = o$ for $x \in [a,b]$ and $v(x) > o$ for $x \in (b,1]$. If φ is $\not\equiv$ constant

we may assume that $\varphi(a) < \varphi(b)$. Then we have (if $v \not\equiv o$)

$$\int_o^1 \varphi(x)v(x)dx \leq \varphi(a)\int_o^a v(x)dx + \varphi(b)\int_b^1 v(x)dx < o$$

as claimed.

REMARK 2 Clearly Theorem 3 remains valid for nonsmooth functions g_o and φ_o provided they are monotone increasing, convex and locally Lipschitzian.

Moreover, the same method applies to optimal control problems governed by the equation

$$y_t - y_{xx} + \beta(y) \ni f \qquad \text{in } Q$$

$$y(x,o) = y_o(x), \qquad x \in \Omega$$

$$y_x(o,t) + u(t)y(o,t) = o, \quad y_x(1,t) = o$$

where the control u belongs to the constraint set $U = \{u$ monotone increasing on $[o,T]$, $o \leq u(t) \leq N$, $\int_o^T u(t)dt = M\}$.

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