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ANISOTROPIC REGULARIZATIONS OF SADDLE FUNCTIONS

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E.KRAUSS and D.TIBA

1. INTRODUCTION

Let $K: X \times Y \to [-\infty, +\infty]$ be a closed, proper saddle (i.e. concave - convex) function defined on the reflexive Banac spaces X and Y. For the necessary background in the theory of saddle functions, we refer to Rockafellar [6], [7], [8], Barbu-Precupanu [3], McLinden [5], Gossez [4].

We define the anisotropic regularization of K:

$$(1.1)\quad K_{\mu}^{\lambda}(x,y) = \sup_{u \in X} \inf_{v \in Y} \left\{ - \frac{|x-u|^2}{2\lambda} + \frac{|y-v|^2}{2\mu} + K(u,v) \right\},$$

and the partial regularizations of K:

$$(1.2)\quad K^{\lambda}(x,y) = \sup_{u \in X} \left\{ - \frac{|x-u|^2}{2\lambda} + K(u,y) \right\},$$

$$(1.3)\quad K_{\mu}(x,y) = \inf_{v \in Y} \left\{ \frac{|y-v|^2}{2\mu} + K(x,v) \right\}.$$

Here $|\cdot|$ stands for the norm in both X and Y. Thes constructions were previously considered by Attouch and Wets [1], [2] Tiba [9], Tiba and Krauss [10] and are extensions of the usual Moreau - Yosida approximation of a convex proper lowe semicontinuous function.

In this paper we study regularity results for the above regularizations, their behaviour when λ, μ tend to zero, as well as the computation and the properties of the Gâteaux derivative of K_μ^λ.

2. REGULARITY

In this section we investigate the continuity and closedness of the different regularizations of K.

For any $\lambda, \mu > 0$, K^λ, K_μ, K_μ^λ are saddle functions and obviously we have

(2.1) $(K_\mu)^\lambda = (K^\lambda)_\mu = K_\mu^\lambda$.

Moreover, they only depend on the equivalence class of K, i.e.

Proposition 2.1

(2.2) $K_\mu = (cl_i K)_\mu$, $K^\lambda = (cl_i K)^\lambda$, $K_\mu^\lambda = (cl_i K)_\mu^\lambda$, $i=1,2$.

Proof

$$(cl_1 K)_\mu(x,y) = \inf_v \left\{ \frac{|y-v|^2}{2\mu} + cl_1 K(x,v) \right\} =$$

$$= \inf_v \left\{ \frac{|y-v|^2}{2\mu} + cl_2 cl_1 K(x,v) \right\} = \inf_v \left\{ \frac{|y-v|^2}{2\mu} + cl_2 K(x,v) \right\} =$$

$$= \inf_v \left\{ \frac{|y-v|^2}{2\mu} + K(x,v) \right\},$$ by the closedness of K and

since the infimum of a convex function equals the infimum of its closure.

The last two terms are $(cl_2 K)_\mu$, respectively K_μ and the first relation of (2.2) follows. The remaining part may be derived similarly.

Theorem 2.2. All saddle functions K^{λ}, K_{μ}, K^{λ}_{μ} are closed, proper and satisfy

(2.3) $K_{\mu}=cl_1(K_{\mu})$

(2.4) $K^{\lambda}=cl_2(K^{\lambda})$

(2.5) $K^{\lambda}_{\mu}=cl_1(K^{\lambda}_{\mu})=cl_2(K^{\lambda}_{\mu})$.

Proof

By (2.1) we can restrict ourselves to the study of K_{μ}. For each $y \in Y$, the concave function $x \to (cl_1 K)_{\mu}(x,y)$ is closed as the infimum of a family of closed concave functions. By (2.2), $K_{\mu}=(cl_1 K)_{\mu}=cl_1\left[(cl_1 K)_{\mu}\right]=cl_1(K_{\mu})$. This proves (2.3).

Denote by F_{μ}, F the partial Fenchel conjugate of K_{μ}, K with respect to the convex variable. Then:

$$(2.6)\quad F_{\mu}(x,\cdot)=K_{\mu}(x,.)^{*}=\left\{K(x,.)\,\Box\,\frac{|\cdot|^2}{2\mu}\right\}^{*}=K(x,.)^{*}+$$

$$+\frac{\mu}{2}|\cdot|_{*}^{2}=F(x,.)+\frac{\mu}{2}|\cdot|_{*}^{2},$$

where $\Box$ stands for the infimal convolution and $|.|_{*}$ for the dual norm on Y^{*}. According to Rockafellar [7], [8], under our assumptions on K, F is a proper convex lower semicontinuous function. By (2.6), F_{μ} is also proper, convex, lower semicontinuous. Using the result of Rockafellar in the opposite directio[n] we see that K_{μ} is closed and proper.

Theorem 2.3. K_{μ}^{λ} is finite and locally Lipschitz on $X \times Y$. Moreover, one has

$$(2.7) \quad K_{\mu}^{\lambda}(x,y) = \max_{u \in X} \min_{v \in Y} \left\{ -\frac{|x-u|^2}{2\lambda} + \frac{|y-v|^2}{2\mu} + K(u,v) \right\} =$$

$$= \min_{v \in Y} \max_{u \in X} \left\{ -\frac{|x-u|^2}{2\lambda} + \frac{|y-v|^2}{2\mu} + K(u,v) \right\}.$$

Proof

The identity (2.7) follows from Rockafellar's minimax theorem in [8]. Since $\dot{K}$ is proper the saddle value defining K_{μ}^{λ} is finite. The proof is finished by the remark that closed saddle functions are locally Lipschitz on the interior of their domain (Barbu-Precupanu [3], p.134).

Remark

In the last section we even show that K_{μ}^{λ} is Lipschitz continuous on bounded sets.

3. CONVERGENCE

In this section we ask for the behaviour of the regularizations of K if the parameters tend to zero.

Theorem 3.1. One has the identities:

$$(3.1) \quad \sup_{\mu > 0} K_{\mu} = cl_2 K, \quad \inf_{\lambda > 0} K^{\lambda} = cl_1 K$$

(3.2) $\sup_{\mu>0} K_\mu^\lambda = K^\lambda$, $\inf_{\lambda>0} K_\mu^\lambda = K_\mu$

(3.3) $\inf_{\lambda>0} \sup_{\mu>0} K_\mu^\lambda = cl_1 K$, $\sup_{\mu>0} \inf_{\lambda>0} K_\mu^\lambda = cl_2 K$

<u>In each of these expression one can replace</u>

$$\sup_{\mu>0} \text{ (}\underline{\text{or}}\text{ } \inf_{\lambda>0}\text{) } \underline{\text{by}} \lim_{\mu\searrow 0} \text{ (}\underline{\text{or}}\text{ } \lim_{\lambda\searrow 0}\text{)}.$$

<u>Proof.</u>

Clearly, K_μ, K^λ, K_μ^λ increase as λ increases or μ decreases. Thus one can substitute $\sup_{\mu>0}$ by $\lim_{\mu\searrow 0}$ and $\inf_{\lambda>0}$ by $\lim_{\lambda\searrow 0}$. Recalling (2.2), we get

$$\sup_{\mu>0} K_\mu = \lim_{\mu\searrow 0} K_\mu = \lim_{\mu\searrow 0} (cl_2 K)_\mu = cl_2 K.$$

Except for the trivial case $cl_2(K(x,.) \equiv -\infty$, the last identity is just the wellknown convergence result for the Moreau-Yosida approximation of the closed proper convex functio[n] $cl_2 K(x,.)$.

To finish the proof one has to apply the above argument to K_μ, K^λ, instead of K, and to use (2.3), (2.4).

<u>Theorem 3.2. It holds the estimation</u>

(3.4) $cl_2 K \leq \underline{\lim}_{\lambda,\mu\searrow 0} K_\mu^\lambda \leq \overline{\lim}_{\lambda,\mu\searrow 0} K_\mu^\lambda \leq cl_1 K$.

<u>Proof</u>

We remark that:

(3.5) $K_\mu(x,y) \leq K_\mu^\lambda(x,y) \leq K^\lambda(x,y)$.

Then $\underline{\lim}_{\lambda,\mu \searrow 0} K_\mu^\lambda \geq \underline{\lim}_{\mu \searrow 0} K_\mu = \sup_{\mu>0} K_\mu = cl_2 K$.

The second inequality follows similarly.

Remark. By a different approach, Attouch-Wets [1], [2] showed that the estimation (3.3) and the local Lipschitz continuity of K_μ^λ remain valid for non concave-convex functions on metric spaces.

4. DIFFERENTIABILITY

We now investigate the regularity of the subdifferential of K_μ^λ.

Throughout this section we suppose that the spaces X, Y and their duals are strictly convex.

Theorem 4.1. K_μ^λ is Gâteaux differentiable on XxY. The differential coincides with the subdifferential ∂K_μ^λ and

$$[x, y] \to \partial K_\mu^\lambda(x, y)$$

is demicontinuous and maps bounded sets into bounded sets.

Remark. By the mean value theorem we obtain that K_μ^λ is Lipschitz continuous on bounded sets.

Under additional assumptions on XxY we obtain a stronger version of this result.

Theorem 4.2. If X,Y and their duals are locally uniformly convex then K_μ^λ is continuously Fréchet differentiable. This differential is Lipschitz continuous if X and Y are Hilbert spaces.

Proof of the Theorems 4.1 and 4.2.

Let $\lambda,\mu>0$ be fixed and set $\eta := \frac{\lambda+\mu}{2}$. On X and Y we introduce equivalent norms both denoted by $\|\cdot\|$:

(4.1) $\|x\|^2 := \frac{\eta}{\lambda}|x|^2$ for $x\in X$,

(4.2) $\|y\|^2 := \frac{\eta}{\mu}|y|^2$ for $y\in Y$.

A simple calculation shows

(4.3) $K^{\lambda}_{\mu} = \hat{K}^{\eta}_{\eta}$,

where $\hat{K}^{\eta}_{\eta}$ denotes the regularization with respect to the new norms. But the case of isotropic regularizations was already treated in Tiba [9] (Theorem 4.1) and Krauss - Tiba [10] (Theorem 4.2).

We conclude the paper with a formula for $\partial K^{\lambda}_{\mu}$. Since X and Y are strictly convex, the minimax identity (2.7) defines exactly one saddle point $[x^{\lambda}_{\mu}, y^{\lambda}_{\mu}]$. By J_1 and J_2 we denote the duality mapping of $(X,|.|)$ and $(Y,|.|)$, respectively.

Theorem 4.3. We have

(4.4) $\partial K^{\lambda}_{\mu}(x,y) = \left[\frac{1}{\lambda}J_1(x^{\lambda}_{\mu}-x), \frac{1}{\mu}J_2(y-y^{\lambda}_{\mu})\right]$.

Proof

The identity (4.3) yields

$$\partial K^{\lambda}_{\mu} = \partial\hat{K}^{\eta}_{\eta}, \quad \text{with } \eta = \frac{\lambda+\mu}{2}.$$

and $(Y, \|.\|)$, respectively (cf. (4.1), (4.2)).

One checks easily

(4.5) $$F_1 = \frac{\eta}{\lambda} J_1 , \qquad F_2 = \frac{\eta}{\mu} J_2$$

The isotropic regularization $\hat{K}^{\eta}_{\eta}$ satisfies (cf. Tiba[9])

(4.6) $$\partial \hat{K}^{\eta}_{\eta}(x,y) = \left[\frac{F_1(\tilde{x}-x)}{\eta} , \frac{F_2(y-\tilde{y})}{\eta}\right],$$

with $[\tilde{x},\tilde{y}]$ as the unique solution to

(4.7) $$\begin{cases} -\partial_1 K(\tilde{x},\tilde{y}) + \frac{1}{\eta} F_1(\tilde{x}-x) \ni 0 \\ \partial_2 K(\tilde{x},\tilde{y}) + \frac{1}{\eta} F_2(\tilde{y}-y) \ni 0 \end{cases}$$

Using the description of saddle points by subdifferentials we conclude from (4.5) and (4.7) $\tilde{x}=x^{\lambda}_{\mu}$, $\tilde{y}=y^{\lambda}_{\mu}$. Now the desired formula follows from (4.5) and (4.6).

Remark

For a closed proper saddle function $L: X \times Y \to [-\infty,+\infty]$, Rockafellar [7], [8] introduced a maximal monotone operator R_L,

$$[f, g] \in R_L(x,y) \text{ if } [-f,g] \in \partial K(x,y).$$

It is easy to see that (4.4) can be reformulated as

$$R_{K^{\lambda}_{\mu}} = [R_K]_{\eta} , \qquad \eta = \frac{\lambda+\mu}{2} ,$$

where $[R_K]_{\eta}$ denotes the Yosida approximation with respect to the norms $\|.\|$ on X and Y.

REFERENCES

1. Attouch, H., Wets, R.J. - B., A convergence theory for saddle functions, Transactions A.M.S. 280(1983), 1-41.
2. Attouch, H., Wets, R.J.-B., A convergence for bivariate functions aimed at the convergence of saddle values, Leture Notes in Mathematics 979, Springer Verlag (1983), 1-43
3. Barbu, V., Precupanu, Th., Convexity and optimization in Banach Spaces, Ed.Academiei (Bucureşti) - Sijthoff and Nordhoff Int.Publ.Hause (Leyden), (1978).
4. Gossez, J.-P., On the subdifferential of a saddle function, Journal Funct. Anal.11 (1972), 220-230.
5. McLinden, L., Dual operations on saddle functions, Transactions A.M.S. 179(1973), 363-381.
6. Rockafellar, R.T., Convex Analysis, Princeton Univ. Press, Princeton (N.J.) (1970).
7. Rockafellar, R.T., Monotone operators associated with saddl functions and minimax problems, in Nonlinear functional ana lysis, Part I, Proc. of Symposia in Pure Math.18(1970), 241-250.
8. Rockafellar, R.T., Saddle points and convex analysis, in: Differential games and related topics, Am.Elsevier, New Yor (1971), 109-127.
9. Tiba, D., Regularization of saddle functions, Bolletino U.M.I.(5), 15-A(1980), 420-427.
10. Tiba, D., Krauss, E., Regularization of saddle functions an the Yosida approximation of monotone operators (to appear).

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