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ISSN 0250 3638

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PREPRINT SERIES IN MATHEMATICS

No. 29/1985

BUCURESTI

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April 1985

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The Chern Classes of the Stable Rank 3 Vector Bundles on $\mathbb{P}^3$

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0. Introduction

Let k be an algebraically closed field of characteristic 0 and $P=\mathbb{P}_k^3$ the 3-dimensional projective space over k. The natural question of determining the triples of integers (c_1,c_2,c_3) which can be the Chern classes of a stable rank 3 vector bundle on P was formulated by R.Hartshorne in [6, Problem 14]. Since then, a number of results have been obtained which have limited the possible values of these triples. First of all, the Theorem of Riemann-Roch implies that $c_1c_2 \equiv c_3$ (mod 2). Then, for each c_1, c_2, there are bounds on c_3 which have been obtained by G.Elencwajg, O.Forster, M.Schneider and H.Spindler. Their results are summarized in [3]. We can normalize any rank 3 vector bundle on P by a suitable twist such that $c_1=0$, -1 or -2. Furthermore, by dualizing the bundle (and twisting with -1 if $c_1=-1$ or -2) we may suppose that $c_3 \geqq 0$ if $c_1=0$, $c_3 \geqq -c_2$ if $c_1=-1$ and $c_3 \geqq 0$ if $c_1=-2$. Now, according to [3, Sect.4] one has:

(i) <u>If</u> $c_1=0$ <u>then</u> $c_2 \geqq 2$ <u>and</u> $c_3 \leqq c_2^2 - c_2$

(ii) <u>If</u> $c_1=-1$ <u>then</u> $c_2 \geqq 1$ <u>and</u> $c_3 \leqq c_2^2-2c_2+2$

(iii) <u>If</u> $c_1=-2$ <u>then</u> $c_2 \geqq 2$ <u>and</u> $c_3 \leqq c_2^2-3c_2+2$

Furthermore, L.Ein, R.Hartshorne and H.Vogelaar have proved in [5, (7.4.1)] that one cannot have $c_1=0$, $c_2 \geqq 5$ and $c_2^2-5c_2+6 < c_3 < c_2^2-c_2$

Using [2] one finds further restrictions to be imposed to the Chern classes of a stable rank 3 vector bundle on P. Before stating these restrictions we introduce some notations. Put:

$$M_o(1,c_2) = \{c_2^2 - c_2\}$$

$$M_o(q,c_2) = [c_2^2-(2q-1)c_2,\ c_2^2-(2q-1)c_2 + 2(q^2-q+1)] \setminus \bigcup_{d=1}^{d_o(q)} (c_2^2-(2q-1)c_2+2(d-1)q,\ c_2^2-(2q-1)c_2+2dq-2d(d+1)),\ \text{for } q \geqq 2$$

$$M_1(q,c_2) = [c_2^2-2qc_2,\ c_2^2-2qc_2+2q^2] \setminus \bigcup_{d=1}^{d_1(q)} (c_2^2-2qc_2+2(d-1)q+2(d-1),\ c_2^2-2qc_2+2dq-2d(d+1)),\ \text{for } q \geqq 1$$

$$M_2(q,c_2) = [c_2^2-(2q+1)c_2+2q,\ c_2^2-(2q+1)c_2+2q^2] \setminus \bigcup_{d=1}^{d_2(q)} (c_2^2-(2q+1)c_2+2dq,\ c_2^2-(2q+1)c_2+2(d+1)q-2d(d+2)),\ \text{for } q \geqq 1$$

where $d_o(q)$ is the largest integer for which $d(d+1) < q-1$, $d_1(q)$ is the largest integer for which $(d+1)^2 \leqq q$ and $d_2(q)$ is the largest integer for which $(d+1)^2 < q$. It is easy to see that if $d(d+1) < q-1$ or if $(d+1)^2 \leqq q$ then $2dq-2d(d+1) \leqq \frac{1}{2}\cdot(q^2-2q)$, and if $(d+1)^2 < q$ then $2(d+1)q-2d(d+2) \leqq \frac{1}{2}\cdot(q^2+3)$. Now, according to [2, (2.7), (2.8) and (2.9)] one has:

(0) _If_ $c_1=0$ _then_ $c_2 \geqq 2$ _and_ $c_3 \leqq \frac{1}{2}\cdot c_2^2$ _or_ $c_3 \in M_o(q,c_2)$ _for some_ $1 \leqq q \leqq \frac{1}{2}\cdot(c_2+1)$

(1) _If_ $c_1= -1$ _then_ $c_2 \geqq 1$ _and_ $c_3 \leqq \frac{1}{2}\cdot c_2^2$ _or_ $c_3 \in M_1(q,c_2)$ _for some_ $1 \leqq q \leqq \frac{1}{2}\cdot(c_2-1)$

(2) _If_ $c_1= -2$ _then_ $c_2 \geqq 2$ _and_ $c_3 \leqq \frac{1}{2}\cdot(c_2-1)^2$ _or_ $c_3 \in M_2(q,c_2)$ _for some_ $1 \leqq q < \frac{1}{2}\cdot(c_2-1)$.

The aim of the present paper is to show that the above conditions suffice to assure the existence of a stable rank 3 vector bundle on P with the given Chern classes. We prove this assertion by producing various examples of stable rank 3 vector bundles.

With some exceptions, these bundles are realized as extensions:

$$0 \longrightarrow \mathcal{F} \longrightarrow E \longrightarrow I_{Y \cup S} \longrightarrow 0$$

where $\mathcal{F}$ is one of the stable rank 2 reflexive sheaves constructed by R. Miró in [7], S is the singular scheme of $\mathcal{F}$ and Y is a plane curve or the empty scheme. This kind of extension is described in [1, Sect.3].

Hence, we prove the following:

Theorem. c_1, c_2, c_3 can be the Chern classes of a stable rank 3 vector bundle on P if and only if $c_1 c_2 \equiv c_3$ (mod 2) and, after normalizations, c_1, c_2, c_3 satisfy one of the conditions (0), (1), (2).

I take this opportunity to express my thanks to C. Bănică for his consistent help and encouragement.

1. Complements about Extensions and Some Useful Examples

Let $\mathcal{F}$ be a rank 2 reflexive sheaf on P which can be realized as an extension:

$$0 \longrightarrow \mathcal{O}_P(a) \longrightarrow \mathcal{F} \longrightarrow I_Z(b) \longrightarrow 0 \qquad (1)$$

where Z is a closed subscheme of P, locally complete intersection (l.c.i.) of codimension 2. The extension is determined by a global section ξ of $\omega_Z(4+a-b)$ which generates this sheaf except at finitely many points. It follows, using the exact sequence (1), that $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_P) \cong \omega_Z(4-b)/\mathcal{O}_Z(-a)\cdot\xi$, hence there is a 0-dimensional closed subscheme S of P such that $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_P) \cong \mathcal{O}_S$. Let Y be a l.c.i. closed subscheme of P of codimension 2 such that $Y \cap S = \emptyset$. Let t be an integer. Suppose that $H^2(\mathcal{F}(-t))=0$ and that $\omega_Y(4-t)\otimes\mathcal{F}$ has a global section vanishing at no point of Y. Then by [1, Sect.3] there is an extension:

$$0 \longrightarrow \mathcal{F} \longrightarrow E \longrightarrow I_{Y \cup S}(t) \longrightarrow 0$$

with E a rank 3 vector bundle on P. The Chern classes of E are:

$$c_1(E) = c_1(\mathcal{F}) + t$$
$$c_2(E) = c_2(\mathcal{F}) + t\,c_1(\mathcal{F}) + \deg Y$$
$$c_3(E) = -c_3(\mathcal{F}) + t\,c_2(\mathcal{F}) + (c_1(\mathcal{F})-t+4)\deg Y - 2\chi(\mathcal{O}_Y)$$

However, the condition $H^2(\mathcal{F}(-t))=0$ is too strong for our purposes. In the next two propositions we shall consider two cases in which this condition is not necessarily fulfilled but a locally free extension still exists.

Proposition 1.1. Let $\mathcal{F}$ be a rank 2 reflexive sheaf on P which can be realized as an extension (1) and let S be the singular scheme of $\mathcal{F}$. Suppose that $\omega_Z(4+a)$ has a global section vanishing at no point of S.

Then there is an extension:

$$0 \longrightarrow \mathcal{F} \longrightarrow E \longrightarrow I_S \longrightarrow 0$$

with E a rank 3 vector bundle on P.

Proof. Dualizing (1), one gets an exact sequence:

$$0 \longrightarrow \mathcal{O}_P(-b) \longrightarrow \mathcal{F}^* \longrightarrow I_Z(-a) \longrightarrow 0 \qquad (2).$$

Dualizing (2), one finds an exact sequence:

$$0 \to \mathcal{O}_P(a) \to \mathcal{F} \to \mathcal{O}_P(b) \to \mathcal{E}xt^1(I_Z(-a),\mathcal{O}_P) \to \mathcal{E}xt^1(\mathcal{F}^*,\mathcal{O}_P) \to 0$$

We have $\mathcal{E}xt^1(I_Z(-a),\mathcal{O}_P) \cong \omega_Z(4+a)$. Let η be a global section of $\omega_Z(4+a)$ vanishing at no point of S. η determines a global section η_o of $\mathcal{E}xt^1(\mathcal{F}^*,\mathcal{O}_P)$ which generates $\mathcal{E}xt^1(\mathcal{F}^*,\mathcal{O}_P)$ as an $\mathcal{O}_P$-module. From the commutative diagram:

$$\begin{array}{ccc} H^0(\mathcal{E}xt^1(I_Z(-a),\mathcal{O}_P)) & \longrightarrow & H^0(\mathcal{E}xt^1(\mathcal{F}^*,\mathcal{O}_P)) \\ \downarrow & & \downarrow \\ 0=H^2(\mathcal{H}om(I_Z(-a),\mathcal{O}_P)) & \longrightarrow & H^2(\mathcal{H}om(\mathcal{F}^*,\mathcal{O}_P)) \end{array}$$

it follows that by the canonical morphism $H^0(\mathcal{E}xt^1(\mathcal{F}^*,\mathcal{O}_P)) \longrightarrow H^2(\mathcal{F})$ η_o goes into 0. Hence η_o is the image of an element $e_0 \in \mathrm{Ext}^1(\mathcal{F}^*,\mathcal{O}_P)$.

Let:

$$0 \longrightarrow \mathcal{O}_P \longrightarrow E_o \longrightarrow \mathcal{F}^* \longrightarrow 0 \tag{3}$$

be the extension determined by e_o. Dualizing (3) one gets an exact sequence:

$$0 \longrightarrow \mathcal{F} \longrightarrow E_o^* \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{E}xt^1(\mathcal{F}^*, \mathcal{O}_P) \longrightarrow \mathcal{E}xt^1(E_o, \mathcal{O}_P) \longrightarrow 0$$

From the fact that η_o generates the $\mathcal{O}_P$-module $\mathcal{E}xt^1(\mathcal{F}^*, \mathcal{O}_P)$ it follows that $\mathcal{E}xt^1(E_o, \mathcal{O}_P) = 0$, hence E_o is locally free, and that $\mathrm{Im}\,(E_o^* \longrightarrow \mathcal{O}_P) = I_S$. $E = E_o^*$ is a rank 3 vector bundle on P and we have an exact sequence:

$$0 \longrightarrow \mathcal{F} \longrightarrow E \longrightarrow I_S \longrightarrow 0$$

Proposition 1.2. Let $\mathcal{F}$ be a rank 2 reflexive sheaf on P satisfying the hypothesis of (1.1) and let S be the singular scheme of $\mathcal{F}$. Let $d \geq 1$ and $e \geq 1$ be integers. Suppose that $\mathcal{F}(d+e)$ has a global section s vanishing in codimension ≥ 2.

If $Y \subset P$ is a complete intersection of two surfaces of degree d and e, respectively, such that s vanishes at no point of Y, then there is an extension

$$0 \longrightarrow \mathcal{F} \longrightarrow E \longrightarrow I_{Y \cup S} \longrightarrow 0$$

with E a rank 3 vector bundle on P.

Proof. We have an exact sequence:

$$\mathrm{Ext}^1(I_{Y \cup S}, \mathcal{F}) \longrightarrow H^0(\mathcal{E}xt^1(I_{Y \cup S}, \mathcal{F})) \longrightarrow H^2(\mathcal{H}om(I_{Y \cup S}, \mathcal{F}))$$

and isomorphisms: $\mathcal{H}om(I_{Y \cup S}, \mathcal{F}) \cong \mathcal{F}$, $\mathcal{E}xt^1(I_{Y \cup S}, \mathcal{F}) \cong \mathcal{E}xt^1(I_Y, \mathcal{F}) \oplus \mathcal{E}xt^1(I_S, \mathcal{F})$ and $\mathcal{E}xt^1(I_Y, \mathcal{F}) \cong \omega_Y(4) \otimes \mathcal{F} \cong \mathcal{F}(d+e)|Y$.

By (1.1) there is an $e_2 \in \mathrm{Ext}^1(I_S, \mathcal{F})$ which determines an extension:

$$0 \longrightarrow \mathcal{F} \longrightarrow E_2 \longrightarrow I_S \longrightarrow 0$$

with E_2 locally free. Let e_2' be the image of e_2 in $H^0(\mathcal{E}xt^1(I_S, \mathcal{F}))$.

Now, we want to describe the morphism $H^0(\mathcal{E}xt^1(I_Y,\mathcal{F})) \longrightarrow$ $\longrightarrow H^2(\mathcal{F})$. We consider the canonical locally free resolution of I_Y:

$$0 \longrightarrow \mathcal{O}_P(-d-e) \longrightarrow \mathcal{O}_P(-d) \oplus \mathcal{O}_P(-e) \longrightarrow I_Y \longrightarrow 0 \qquad (4)$$

Applying $\mathcal{H}om(-,\mathcal{F})$ to (4) one gets an exact sequence:

$$0 \longrightarrow \mathcal{H}om(I_Y,\mathcal{F}) \longrightarrow \mathcal{F}(d) \oplus \mathcal{F}(e) \longrightarrow \mathcal{F}(d+e) \longrightarrow \mathcal{E}xt^1(I_Y,\mathcal{F}) \longrightarrow 0$$

which decomposes into two short exact sequences:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}(d) \oplus \mathcal{F}(e) \longrightarrow (I_Y\cdot\mathcal{F})(d+e) \longrightarrow 0$$

$$0 \longrightarrow (I_Y\cdot\mathcal{F})(d+e) \longrightarrow \mathcal{F}(d+e) \longrightarrow \mathcal{F}(d+e)|Y \longrightarrow 0$$

The morphism $H^0(\mathcal{E}xt^1(I_Y,\mathcal{F})) \longrightarrow H^2(\mathcal{F})$ is equal to the composition of the morphisms $\delta_1 : H^0(\mathcal{F}(d+e)|Y) \longrightarrow H^1((I_Y\cdot\mathcal{F})(d+e))$ and $\delta_2 : H^1((I_Y\cdot\mathcal{F})(d+e)) \to H^2(\mathcal{F})$. The section $s \in H^0(\mathcal{F}(d+e))$ restricts to a section $e_1' \in H^0(\mathcal{F}(d+e)|Y)$ which vanishes at no point of Y and such that $\delta_1(e_1')=0$.

It follows that the element $(e_1',e_2') \in H^0(\mathcal{E}xt^1(I_{Y\cup S},\mathcal{F}))$ goes into 0 by the morphism $H^0(\mathcal{E}xt^1(I_{Y\cup S},\mathcal{F})) \longrightarrow H^2(\mathcal{F})$, hence it is the image of an element $e' \in \mathrm{Ext}^1(I_{Y\cup S},\mathcal{F})$. e' determines the extension we are looking for.

Next, we show that the stable rank 2 reflexive sheaves produced by R.Miró in [7] satisfy the hypotheses of (1.1) and (1.2) if they are constructed with some care.

Lemma 1.3. _Let_ Z_1, Z_2 _be nonsingular (connected) curves in_ P _such that the scheme_ $Z_1 \cap Z_2$ _is nonempty and consists of finitely many simple points, and let_ $Z=Z_1 \cup Z_2$. _Let_ $D=Z_1 \cap Z_2$ _considered as a divisor on_ Z_1 _or_ Z_2. _Then:_

(i) Z _is l.c.i. in_ P

(ii) $\omega_Z|Z_i \cong \omega_{Z_i} \otimes \mathcal{O}_{Z_i}(D)$, $i=1,2$

(iii) _If, for some_ $n \in \mathbb{Z}$, _the restriction_ $H^0(\mathcal{O}_Z(n)) \longrightarrow$ $\longrightarrow H^0(\mathcal{O}_{Z_1}(n))$ _is surjective then the restriction_ $H^0(\omega_Z(-n)) \longrightarrow$

$\longrightarrow H^0(\omega_Z(-n)|Z_2)$ is surjective.

(iv) $\omega_Z(1)$ is generated by its global sections.

Proof. (i) Let $x \in Z_1 \cap Z_2$. One can choose a regular system of parameters u,v,w of $\mathcal{O}_{P,x}$ such that $I_{Z_1,x} = (u,v)$ and $I_{Z_2,x} = (u,w)$. Then $I_{Z,x} = I_{Z_1,x} \cap I_{Z_2,x} = (u,vw)$.

(ii) We start with the exact sequence:

$$0 \longrightarrow I_{Z_1}/I_Z \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_{Z_1} \longrightarrow 0$$

But $I_{Z_1}/I_Z = I_{Z_1}/(I_{Z_1} \cap I_{Z_2}) \cong (I_{Z_1} + I_{Z_2})/I_{Z_2} = \mathcal{O}_{Z_2}(-D)$. Hence we obtain an exact sequence of $\mathcal{O}_P$-modules:

$$0 \longrightarrow \mathcal{O}_{Z_2}(-D) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_{Z_1} \longrightarrow 0 \qquad (5)$$

Applying $\mathcal{E}xt^2(-,\omega_P)$ to (5) one gets an exact sequence :

$$0 \longrightarrow \omega_{Z_1} \longrightarrow \omega_Z \longrightarrow \omega_{Z_2} \otimes \mathcal{O}_{Z_2}(D) \longrightarrow 0 \qquad (6)$$

Restricting to Z_2, we get an epimorphism $\omega_Z|Z_2 \longrightarrow \omega_{Z_2} \otimes \mathcal{O}_{Z_2}(D)$.

But this is an epimorphism of invertible $\mathcal{O}_{Z_2}$-modules, hence it is an isomorphism.

(iii) We consider the exact cohomology sequence associated to (6) twisted with $-n$:

$$H^0(\omega_Z(-n)) \rightarrow H^0(\omega_{Z_2}(-n) \otimes \mathcal{O}_{Z_2}(D)) \longrightarrow H^1(\omega_{Z_1}(-n)) \rightarrow H^1(\omega_Z(-n))$$

It follows that we have to show that the morphism $H^1(\omega_{Z_1}(-n)) \rightarrow$ $\rightarrow H^1(\omega_Z(-n))$ is injective. If $\varphi \in H^0(\mathcal{O}_Z(n))$ then the diagram:

$$\begin{array}{ccc} H^1(\omega_{Z_1}) & \longrightarrow & H^1(\omega_Z) \\ \downarrow {\scriptstyle H^1(\varphi|Z_1)} & & \downarrow {\scriptstyle H^1(\varphi)} \\ H^1(\omega_{Z_1}(-n)) & \longrightarrow & H^1(\omega_Z(-n)) \end{array}$$

is commutative. We have an exact sequence:

$$H^1(\omega_{Z_1}) \longrightarrow H^1(\omega_Z) \longrightarrow H^1(\omega_{Z_2} \otimes \mathcal{O}_{Z_2}(D))$$

But $H^1(\omega_{Z_2} \otimes \mathcal{O}_{Z_2}(D)) \cong H^0(\mathcal{O}_{Z_2}(-D))' = 0$ and $H^1(\omega_{Z_1}) \cong k$. It follows that the morphism $H^1(\omega_{Z_1}) \longrightarrow H^1(\omega_Z)$ is an isomorphism.

We have proved that the morphism $H^1(\omega_{Z_1}(-n)) \to H^1(\omega_Z(-n))$ is the dual of the morphism $H^0(\mathcal{O}_Z(n)) \to H^0(\mathcal{O}_{Z_1}(n))$. Now, the assertion follows from our hypothesis.

(iv) $H^0(\mathcal{O}_{Z_i}(-1)) = 0$ hence, by (iii), the morphism $H^0(\omega_Z(1)) \to H^0(\omega_Z(1)|Z_i)$ is surjective, $i=1,2$. Using (ii) it follows that $\deg(\omega_Z(1)|Z_i) \geqq 2g(Z_i)$ hence $\omega_Z(1)|Z_i$ is generated by its global sections, $i=1,2$. It follows that $\omega_Z(1)$ is generated by its global sections.

<u>Example</u> 1.4. Let $q \geqq 1$ and $c_2 \geqq 1$ be integers such that $c_2 \geqq 2q-2$. Let Σ_1, Σ_2 be nonsingular surfaces in P of degree $q-1$ and q, respectively, intersecting transversally. Put $Z_2 = \Sigma_1 \cap \Sigma_2$. Let H be a plane in P which intersects transversally Σ_1, Σ_2 and Z_2. Put $C_i = H \cap \Sigma_i$, $i=1,2$. Let $Z_1 \subset H$ be a nonsingular curve of degree c_2 passing through r of the points of $H \cap Z_2$, $0 \leqq r \leqq q(q-1)$, and let $Z = Z_1 \cup Z_2$. One can construct a stable rank 2 reflexive sheaf $\mathcal{F}$ on P using an extension:

$$0 \longrightarrow \mathcal{O}_P(-q) \longrightarrow \mathcal{F} \longrightarrow I_Z(q-1) \longrightarrow 0$$

The Chern classes of $\mathcal{F}$ are: $c_1(\mathcal{F}) = -1$, $c_2(\mathcal{F}) = c_2$, $c_3(\mathcal{F}) = c_2^2 - 2(q-1)c_2 + 2r$ (see [7, Sect.2] for details).

Firstly, we investigate under which conditions $\mathcal{F}$ satisfies the hypothesis of (1.1). The extension (7) is determined by a global section ξ of $\omega_Z(5-2q)$ vanishing only at finitely many points. If L is an invertible $\mathcal{O}_Z$-module then we have an exact sequence:

$$0 \to H^0(L) \to H^0(L|Z_1) \times H^0(L|Z_2) \to H^0(L|Z_1 \cap Z_2).$$

Put $D = Z_1 \cap Z_2$. By (1.3), $\omega_Z(5-2q)|Z_1 \cong \mathcal{O}_{Z_1}(c_2-(2q-2)) \otimes \mathcal{O}_{Z_1}(D)$ and $\omega_Z(5-2q)|Z_2 \cong \mathcal{O}_{Z_2}(D)$. Choose $s_i \in H^0(\mathcal{O}_{Z_i}(D))$ such that the divisor of zeros of s_i is D, $i=1,2$, and let $t_1 \in H^0(\mathcal{O}_{Z_1}(c_2-(2q-2)))$ be a nonzero section. We may take $\mathcal{S} = (t_1 \otimes s_1, s_2)$.

It follows that, in order to verify the hypothesis of (1.1), it suffices to find a global section of $\omega_Z(4-q)$ vanishing at no point of D. This happens, for example, if $D = \emptyset$. Now, suppose that $D \neq \emptyset$.

The morphism $H^0(\mathcal{O}_P(q-4)) \to H^0(\mathcal{O}_{Z_1}(q-4))$ is surjective. By (1.3 iii), the morphism $H^0(\omega_Z(4-q)) \longrightarrow H^0(\omega_Z(4-q)|Z_2)$ is surjective, hence it suffices to find a global section of $\omega_Z(4-q)|Z_2$ vanishing at no point of D. Put $D_2 = (H \cap Z_2) \setminus D$. We have:

$$\omega_Z(4-q)|Z_2 \cong \mathcal{O}_{Z_2}(q-1) \otimes \mathcal{O}_{Z_2}(D) \cong \mathcal{O}_{Z_2}(q) \otimes \mathcal{O}_{Z_2}(-D_2)$$

One can identify $H^0(\mathcal{O}_{Z_2}(q) \otimes \mathcal{O}_{Z_2}(-D_2))$ with the global sections of $\mathcal{O}_{Z_2}(q)$ vanishing at any point of D_2. Hence we must find a global section of $\mathcal{O}_{Z_2}(q)$ vanishing at any point of D_2 but at no point of D. The morphisms $H^0(\mathcal{O}_P(q)) \longrightarrow H^0(\mathcal{O}_{Z_2}(q))$ and $H^0(\mathcal{O}_P(q)) \longrightarrow H^0(\mathcal{O}_{C_1}(q))$ being surjective, it suffices to find a global section of $\mathcal{O}_{C_1}(q)$ vanishing at any point of D_2 but at no point of D. Now, $\mathcal{O}_{C_1}(q) \otimes \mathcal{O}_{C_1}(-D_2) \cong \mathcal{O}_{C_1}(D)$. Hence we must find a global section of $\mathcal{O}_{C_1}(D)$ vanishing at no point of D. Such a section exists if and only if:

$$h^0(\mathcal{O}_{C_1}(D-x)) = h^0(\mathcal{O}_{C_1}(D))-1 \text{ for any } x \in D \text{ (see [5; IV, 3.1]).}$$

By the Theorem of Riemann-Roch this is equivalent to:

$$h^0(\omega_{C_1}(-D+x)) = h^0(\omega_{C_1}(-D)) \quad \text{for any } x \in D.$$

Now, we show how one can choose $D \subseteqq H \cap Z_2 = C_1 \cap C_2$ such that $h^o(\omega_{C_1}(-D+x))=0$ for any $x \in D$. We have $\omega_{C_1} \cong \mathcal{O}_{C_1}(q-4)$. Using the exact sequence:

$$0 \longrightarrow \mathcal{O}_H(-2q+1) \longrightarrow \mathcal{O}_H(-q) \oplus \mathcal{O}_H(-q+1) \longrightarrow I_{C_1 \cap C_2} \longrightarrow 0$$

one finds that $h^o(I_{C_1 \cap C_2}(q-4))=0$, hence if $\sigma \in H^o(\omega_{C_1})$ vanishes at any point of $C_1 \cap C_2$ then $\sigma=0$. It follows that there is a set $T \subset C_1 \cap C_2$ consisting of $h^o(\omega_{C_1}) = \frac{1}{2} \cdot (q-2)(q-3)$ points such that, if $\sigma \in H^o(\omega_{C_1})$ vanishes at any point of T then $\sigma=0$.

Let $y_1, \dots, y_g$ be the points of T and let $\sigma_i \in H^o(\omega_{C_1})$ be the unique (up to scalar) section vanishing at $y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_g$ but not at y_i. $\sigma_1, \dots, \sigma_g$ is a basis of $H^o(\omega_{C_1})$ hence for any $x \in C_1$, there is an i such that $\sigma_i(x) \neq 0$. It follows that for any $x \in C_1 \setminus T$ there is an $f \in H^o(\mathcal{O}_{C_1}(q))$ which vanishes at any point of T but not at x. Indeed, choose σ_i such that $\sigma_i(x) \neq 0$ and $\lambda \in H^o(\mathcal{O}_{C_1}(1))$ such that $\lambda(y_i)=0$ and $\lambda(x) \neq 0$. We may take $f = \lambda^4 \cdot \sigma_i$.

We have proved that the base locus of the linear system of the curves of degree q in H passing through the points of T is T. It follows that, moving C_2 (in fact $\sum_2$) if necessary, we may suppose that, for any i, σ_i vanishes at no point of $(C_1 \cap C_2) \setminus T$.

Now, suppose that $T \subset D \subseteqq C_1 \cap C_2$. Let $x \in D$. If $\sigma \in H^o(\omega_{C_1})$ vanishes at any point of $D \setminus \{x\}$ then $\sigma=0$, hence $H^o(\omega_{C_1}(-D+x)) = 0$.

We have proved that if $r=0$ or if $\frac{1}{2}(q-2)(q-3)+1 \leqq r \leqq q(q-1)$ then one can construct $\mathcal{F}$ such that it satisfies the hypothesis of (1.1).

One can similarly prove, using the curve C_2 instead of C_1, that if $r=0$ or if $\frac{1}{2} \cdot (q-1)(q-2)+1 \leq r \leq q(q-1)$ then one can construct $\mathcal{F}$ such that $\mathcal{F}(-1)$ satisfies the hypothesis of (1.1).

Next, we show that if $q \geqq 2$ then, for any $n \geqq 1$, $\mathcal{F}(n)$ has a global section vanishing in codimension $\geqq 2$. Firstly, we show that for any $d \geqq q-1$ there is an irreducible surface of degree d in P containing Z_2 but not Z_1. We may suppose $d \geqq q+1$. Then the base locus of the linear system of the surfaces of degree d containing Z_2 is Z_2 and this linear system separates the points of $P \setminus Z_2$. By the Theorem of Bertini, the general surface of degree d containing Z_2 is irreducible.

Now, let $n \geqq 1$ be an integer. Let $h=0$ be an equation of the plane H and let $g=0$ be an equation of an irreducible surface of degree $n+q-2$ containing Z_2 but not Z_1. Let s be a global section of $\mathcal{F}(n)$ which goes into $h \cdot g \in H^0(I_Z(n+q-1))$. If s vanishes in codimension 1 then there is an $m<n$, an $s' \in H^0(\mathcal{F}(m))$ and an $f \in H^0(\mathcal{O}_P(n-m))$ such that $s=f \cdot s'$. Let g' be the image of s' in $H^0(I_Z(m+q-1))$. We have $f \cdot g' = h \cdot g$. By unique factorization, $g'=h$ or $g'=g$, but none of them vanishes on Z and this is a contradiction.

Example 1.5. Let $q \geqq 1$ and $c_2 \geqq 2$ be integers such that $c_2 \geqq 2q-1$. Let $Z_2 \subset P$ be a complete intersection of two surface of degree q and let Z_1 be a plane curve of degree c_2 such that Z_1 meets Z_2 at r simple points, $0 \leqq r \leqq q^2$. Put $Z = Z_1 \cup Z_2$. One can construct a stable rank 2 reflexive sheaf $\mathcal{F}$ on P as an extension:

$$0 \longrightarrow \mathcal{O}_P(-q) \longrightarrow \mathcal{F} \longrightarrow I_Z(q) \longrightarrow 0$$

The Chern classes of $\mathcal{F}$ are: $c_1(\mathcal{F})=0$, $c_2(\mathcal{F})=c_2$, $c_3(\mathcal{F}) =$ $= c_2^2-(2q-1)c_2+2r$ (see [7, Sect.2] for details).

One can show, as in (1.4), that if $r=0$ or if $\frac{1}{2}\cdot(q-1)(q-2)+$ $+1 \leqq r \leqq q^2$ then one can construct $\mathcal{F}$ such that $\mathcal{F}(-1)$ satisfies the hypothesis of (1.1). Also, if $n \geqq 1$ then $\mathcal{F}(n)$ has a global section vanishing in codimension $\geqq 2$.

We end the section with an example of a semistable rank 3 vector bundle on P with $c_1=0$, which will be used in the sections 3 and 4.

Example 1.6. Let $q \geq 1$ and $c_2 \geq 2q$ be integers. Let Z_1, Z_2 be plane curves in P of degree c_2-q and q, respectively, contained in different planes H_1 and H_2 and such that Z_1 meets Z_2 at s simple points, $0 \leq s \leq q$. Put $Z=Z_1 \cup Z_2$. Let H be a plane which intersects transversally Z_1 and Z_2 and which does not contain any point of $H_1 \cap H_2 \cap Z$. Put $L_i = H \cap H_i$, $i=1,2$.

There are elements $t_1, t_2 \in H^0(\mathcal{O}_Z(1))$ which generate $\mathcal{O}_Z(1)$ and such that t_1 vanishes at any point of $H \cap Z_2$ and t_2 vanishes at any point of $H \cap Z_1$. $\omega_Z(3)$ is generated by its global sections, hence we can find $\sigma_1, \sigma_2 \in H^0(\omega_Z(3))$ such that $\xi_1 = t_1 \cdot \sigma_1$ and $\xi_2 = t_2 \cdot \sigma_2$ generate $\omega_Z(4)$. ξ_1 and ξ_2 determine an extension:

$$0 \longrightarrow \mathcal{O}_P^2 \longrightarrow E \longrightarrow I_Z \longrightarrow 0$$

with E a semistable rank 3 vector bundle on P with Chern classes: $c_1(E)=0$, $c_2(E)=c_2$, $c_3(E)=c_2^2-(2q-1)c_2+2q^2+2s$.

We assert that for any $n \geq 1$ there is an epimorphism $E_H \longrightarrow \mathcal{O}_H(n)$. Indeed, dualizing the exact sequence:

$$0 \longrightarrow \mathcal{O}_H^2 \longrightarrow E_H \longrightarrow I_{Z \cap H} \longrightarrow 0$$

one gets an exact sequence:

$$0 \longrightarrow \mathcal{O}_H \longrightarrow E_H^* \longrightarrow \mathcal{O}_H^2 \longrightarrow \omega_{Z \cap H}(4) \longrightarrow 0$$

The morphism $\mathcal{O}_H^2 \longrightarrow \omega_{Z \cap H}(4)$ is determined by $\xi_1|H$ and $\xi_2|H$. It follows that the image of the morphism $H^0(E_H^*(n)) \longrightarrow H^0(\mathcal{O}_H(n))^2$ consists of the pairs (f_1, f_2) such that f_i vanishes at any point of $H \cap Z_i$, $i=1,2$. Let $\lambda_1, \lambda_2 \in H^0(\mathcal{O}_H(1))$ be such that $\lambda_i=0$ is an equation of L_i, $i=1,2$, and let $\lambda_0 \in H^0(\mathcal{O}_H(1))$ be a linear form which does not vanish at the point x where L_1 and L_2 intersect. Let φ_0 be the epimorphism $E_H \longrightarrow I_{Z\cap H}$.

Let φ be a global section of $E_H^*(n)$ whose image in $H^0(\mathcal{O}_H(n))^2$ is $(\lambda_1^n,\lambda_2^n)$. $(\lambda_1^n,\lambda_2^n)$ vanishes only at the point x. It follows that φ might vanish only at x. If φ vanishes at x then the global section $\varphi+\lambda_0^n\cdot\varphi_0$ of $E_H^*(n)$, whose image in $H^0(\mathcal{O}_H(n))^2$ is still $(\lambda_1^n,\lambda_2^n)$, vanishes at no point of H.

2. Examples of Stable Rank 3 Vector Bundles with $c_1=0$.

With some exceptions, the bundles considered in this section are constructed as extensions:

$$0 \longrightarrow \mathcal{F} \longrightarrow E \longrightarrow I_{Y\cup S}(1) \longrightarrow 0 \qquad (1)$$

where $\mathcal{F}$ is a stable rank 2 reflexive sheaf on P as in (1.4) and Y is a plane curve of degree $d\geq 2$. One can verify the stability of E as it follows. In every case one has $S\neq\emptyset$, hence we may suppose $H^0(I_{Y\cup S}(1))=0$. It follows that $H^0(E)=0$. Dualizing (1) one gets an exact sequence:

$$0 \longrightarrow \mathcal{O}_P(-1) \longrightarrow E^* \longrightarrow \mathcal{F}^* \longrightarrow \omega_Y(3) \longrightarrow 0$$

If $c_2\geq 3$ then $\mathcal{F}^*\cong\mathcal{F}(1)$ has only one (up to scalar) global section s. We may suppose that $s|Y\neq 0$, hence the morphism $H^0(\mathcal{F}^*) \longrightarrow H^0(\mathcal{F}^*|Y)$ is injective. $\omega_Y(3)\cong\mathcal{O}_Y(d)$ and $\det\mathcal{F}^*\cong\mathcal{O}_P(1)$, hence we have an exact sequence:

$$0 \longrightarrow \mathcal{O}_Y(1-d) \longrightarrow \mathcal{F}^*|Y \longrightarrow \omega_Y(3) \longrightarrow 0$$

It follows that the morphism $H^0(\mathcal{F}^*|Y) \longrightarrow H^0(\omega_Y(3))$ is injective, hence $H^0(E^*)=0$.

Example 2.1. Let $q\geq 3$ and $c_2\geq 2q-2$ be integers. Let $\mathcal{F}$ be a stable rank 2 reflexive sheaf on P constructed as an extension:

$$0 \longrightarrow \mathcal{O}_P(-q+1) \longrightarrow \mathcal{F} \longrightarrow I_Z(q-2) \longrightarrow 0$$

where $Z=Z_1\cup Z_2$, with Z_1 a plane curve of degree c_2-1 and Z_2 a complete intersection of two surfaces of degree q-2 and q-1, respectively, such that Z_1 meets Z_2 at r simple points.

We construct a stable rank 3 vector bundle E on P as an extension:

$$0 \longrightarrow \mathcal{F} \longrightarrow E^* \longrightarrow I_{Y \cup S}(1) \longrightarrow 0$$

where Y is a conic. According to (1.4), if r=0 or if $\frac{1}{2}\cdot(q-2)(q-3)+1 \leqq r \leqq (q-1)(q-2)$ then the construction is possible. The Chern classes of E are: $c_1(E)=0$, $c_2(E)=c_2$, $c_3(E)=c_2^2-(2q-1)c_2+2(q-2)+2r$.

This example covers all the values of c_3 with $c_3 = c_2^2 - (2q-1)c_2+2(q-2)$ or with $c_2^2-(2q-1)c_2+(q-1)(q-2)+2 \leqq c_3 \leqq c_2^2 - (2q-1)c_2+2q(q-2)$.

Example 2.2. Let $\mathcal{F}$ be a stable rank 2 reflexive sheaf on P constructed as an extension:

$$0 \longrightarrow \mathcal{O}_P(-1) \longrightarrow \mathcal{F} \longrightarrow I_L \longrightarrow 0$$

where L is a line. The Chern classes of $\mathcal{F}$ are: $c_1(\mathcal{F})=-1$, $c_2(\mathcal{F})=1$, $c_3(\mathcal{F})=1$. We have $H^2(\mathcal{F}(-1))=0$, and $\mathcal{F}(1)$ is generated by its global sections. It follows that if Y is a l.c.i. curve in P with $\omega_Y(2)$ generated by its global sections then $\omega_Y(3)\otimes\mathcal{F}$ has a section vanishing at no point of Y. In this case there is an extension:

$$0 \longrightarrow \mathcal{F} \longrightarrow E \longrightarrow I_{Y \cup S}(1) \longrightarrow 0$$

with E a rank 3 vector bundle on P. Let $Y_1,\ldots,Y_n$ be the connected components of Y. If there is an i such that $h^0(\omega_{Y_i}(2)^*)=0$ or if $h^0(\omega_{Y_1}(2)^*)=1$ and $n \geqq 2$ then E is stable (see [1; Sect.3, Example 2] for details).

Now, let $q \geqq 1$ and $c_2 \geqq 2$ be integers such that $c_2 \geqq 2q-2$. Let $Y=Y_1 \cup Y_2$, where Y_1 is a plane curve of degree c_2-q+1 and Y_2 a plane curve of degree q-1, situated in different planes and such that Y_1 meets Y_2 in s simple points, $0 \leqq s \leqq q-1$. In this case the Chern classes of E are: $c_1(E)=0$, $c_2(E)=c_2$, $c_3(E)=c_2^2-$ $-(2q-1)c_2+2q(q-2)+2(s+1)$.

This example covers all the values of c_3 with:

$$c_2^2-(2q-1)c_2+2q(q-2)+2 \leqq c_3 \leqq c_2^2-(2q-1)c_2+2q(q-1)$$

Example 2.3. Let $d \geqq 1$, $q \geqq \max(2d,3)$ and $c_2 \geqq q+1$ be interges. Let $\mathcal{F}$ be a stable rank 2 reflexive sheaf on P constructed as an extension:

$$0 \longrightarrow \mathcal{O}_P(-d-1) \longrightarrow \mathcal{F} \longrightarrow I_Z(d) \longrightarrow 0$$

where $Z=Z_1 \cup Z_2$, with Z_1 a plane curve of degree q and Z_2 a complete intersection of two surface of degree d and d+1, respectively, such that Z_1 meets Z_2 at r simple points.

We construct a stable rank 3 vector bundle E on P as an extension:

$$0 \longrightarrow \mathcal{F} \longrightarrow E \longrightarrow I_{Y \cup S}(1) \longrightarrow 0$$

where Y is a plane curve of degree c_2-q+1. According to (1.4), if $\frac{1}{2}\cdot d(d-1)+1 \leqq r \leqq d(d+1)$ then the construction is possible. The Chern classes of E are: $c_1(E)=0$, $c_2(E)=c_2$, $c_3(E)=c_2^2-(2q-1)c_2 +$ $+ 2dq-2r$.

Example 2.4. Let $d \geqq 1$, $q \geqq d+2$ and $c_2 \geqq q+d$ be integers. Let $\mathcal{F}$ be a stable rank 2 reflexive sheaf on P constructed as an extension:

$$0 \longrightarrow \mathcal{O}_P(-d-1) \longrightarrow \mathcal{F} \longrightarrow I_Z(d) \longrightarrow 0$$

where $Z=Z_1 \cup Z_2$, with Z_1 a plane curve of degree $c_2-q+d+1$ and Z_2 a complete intersection of two surfaces of degree d and d+1, respectively, such that Z_1 meets Z_2 at r simple points.

We construct a stable rank 3 vector bundle E on P an extension:

$$0 \longrightarrow \mathcal{F} \longrightarrow E^* \longrightarrow I_{Y \cup S}(1) \longrightarrow 0$$

where Y is a plane curve of degree q-d. According to (1.4), if

$\frac{1}{2}\cdot d(d-1)+1 \leqq r \leqq d(d+1)$ then the construction is possible. The Chern classes of E are: $c_1(E)=0$, $c_2(E)=c_2, c_3(E)=c_2^2-(2q-1)c_2 +$ $+ 2dq-2d(d+1)+2r$.

The examples (2.3) and (2.4) cover all the values of c_3 with $c_2^2-(2q-1)c_2+2dq-2d(d+1) \leqq c_3 \leqq c_2^2-(2q-1)c_2+2dq$. Now, let $\tilde{d}$ be the largest integer for which $2\tilde{d} \leqq q$. We have $2\tilde{d}q \geqq (q-1)(q-2)$. It follows that the examples from (2.1) to (2.4) cover all the values of c_3 with $c_3 \in M_o(q,c_2)$, except $c_3=c_2^2-(2q-1)c_2+2(q^2-q+1)$ and $c_3=c_2^2-(2q-1)c_2$.

<u>Example</u> 2.5. Let $q \geqq 2$ and $c_2 \geqq 3$ be integers such that $c_2 \geqq 2q-2$. Let E' be a stable rank 3 vector bundle on P with Chern classes $c_1'= -1$, $c_2'=q$, $c_3'=q^2-2q+2$ (see [8, (5.8)]). Let $H \subset P$ be a plane which contains a generic line of E'. By [8,(5.8)], E'(q-1) is generated by its global sections, hence the same is true for $E'_H(c_2-q+1)$. It follows that $E'_H(c_2-q+1)$ has a global section vanishing at no point of H, hence there is an epimorphism $\alpha_H : E'^*_H \longrightarrow \mathcal{O}_H(c_2-q+1)$. Composing with the morphism $E'^* \longrightarrow E'^*_H$ we get an epimorphism $\alpha : E'^* \longrightarrow \mathcal{O}_H(c_2-q+1)$. Let $E^*= \mathrm{Ker}\,\alpha$. E is a rank 3 vector bundle on P and, by definition, there is an exact sequence:

$$0 \longrightarrow E^* \longrightarrow E'^* \xrightarrow{\alpha} \mathcal{O}_H\ (c_2-q+1) \longrightarrow 0$$

Dualizing, we get an exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow \mathcal{O}_H\ (-c_2+q) \longrightarrow 0$$

The Chern classes of E are $c_1(E)=0$, $c_2(E)=c_2, c_3(E)=c_2^2-(2q-1)c_2 +$ $+ 2(q^2-q+1)$. In order to show that E is stable it suffices to show that $H^o(\alpha)$ is injective. We have $H^o(E'^*(-1))=0$, hence the morphism $H^o(E'^*) \longrightarrow H^o(E'^*_H)$ is injective. Now, let $F_H=\mathrm{Ker}\,\alpha_H$.

Dualizing the exact sequence:

$$0 \longrightarrow F_H \longrightarrow E'^*_H \xrightarrow{\alpha_H} \mathcal{O}_H\ (c_2-q+1) \longrightarrow 0$$

and using the fact that $H^0(E'_H(-1))=0$ one finds that $H^0(F^*_H(-1))=0$. But $F^*_H(-1)\cong F_H(c_2-q-1)$, hence $H^0(F_H)=0$. It follows that $H^0(\alpha_H)$ is injective.

<u>Example</u> 2.6. Let $q\geqq 1$ and $c_2\geqq 2$ be integers such that $c_2\geqq 2q-1$. Let E' be a stable rank 3 vector bundle on P with Chern classes $c_1'=-1$, $c_2'=q$, $c_3'=-q^2$. By [8, Sect.5], there is an exact sequence

$$0 \longrightarrow E' \longrightarrow \mathcal{O}_P^3 \longrightarrow \mathcal{O}_{H_0}(q) \longrightarrow 0$$

for some plane $H_0 \subset P$. Let $F_{H_0} = \mathrm{Ker}(\mathcal{O}^3_{H_0} \longrightarrow \mathcal{O}_{H_0}(q))$. Using the Snake lemma, as in [2,(1.1)], one gets an exact sequence:

$$0 \longrightarrow \mathcal{O}_P(-1)^3 \longrightarrow E' \longrightarrow F_{H_0} \longrightarrow 0$$

$F^*_{H_0}$ is generated by its global sections. But $F^*_{H_0}\cong F_{H_0}(q)$. It follows that $E'(q)$ is generated by its global sections.

Let $H\subset P$ be a plane which contains a generic line of E'. One can construct, as in (2.5), a stable rank 3 vector bundle E on P as an extension:

$$0 \longrightarrow E' \xrightarrow{\sim} E \longrightarrow \mathcal{O}_H(-c_2+q) \longrightarrow 0$$

The Chern classes of E are $c_1(E)=0$, $c_2(E)=c_2$, $c_3(E)=c_2^2-(2q-1)c_2$.

Now, let $\bar{q}$ be the largest integer for which $2\bar{q}-1\leqq c_2$ and $\bar{d}=d_0(\bar{q})$. $c_2^2-(2\bar{q}-1)c_2+2(\bar{q}^2-\bar{q}+1)$ is equal to $\frac{1}{2}\cdot c_2^2+2$ if c_2 is even and to $\frac{1}{2}(c_2^2+3)$ if c_2 is odd. It follows that the examples from (2.1) to (2.6) cover all the possible values of c_3 with $c_3\geq c_2^2-(2\bar{q}-1)c_2+2\bar{d}\bar{q}-2\bar{d}(\bar{d}+1):=m_0(c_2)$.

The remaining values of c_3 are coverd by:

<u>Example</u> 2.7. Let $c_2\geqq 2$ be an integer. Let $\mathcal{F}$ be a stable rank 2 reflexive sheaf on P as in (2.2). We construct a stable rank 3 vector bundle E on P as an extension:

$$0 \longrightarrow \mathcal{F} \longrightarrow E \longrightarrow I_{Y\cup S}(1) \longrightarrow 0$$

where Y is a l.c.i. curve in P satisfying the conditions stated at the beginning of (2.2).

Let d be an integer with $0 \leqq d \leqq c_2 - 1$. If Y is a disjoint union of c_2-d-1 lines and of a rational curve of degree d+1 then the Chern classes of E are: $c_1(E)=0$, $c_2(E)=c_2$, $c_3(E)=2d$.

If Y is a nonsingular curve of degree c_2 and of genus g then $c_1(E)=0$, $c_2(E)=c_2$, $c_3(E)=2c_2-2+2g$. According to [4], there are such curves for all g with $0 \leqq g \leqq \frac{1}{6} \cdot c_2(c_2-3)+1$.

It follows that this example covers all the values of c_3 with $0 \leqq c_3 \leqq \frac{1}{3} \cdot c_2(c_2+3)$. One can easily see that $\frac{1}{3} \cdot c_2(c_2+3) \geqq$ $\geqq m_o(c_2)$ for all $c_2 \geqq 2$.

3. Examples of Stable Rank 3 Vector Bundles with $c_1 = -1$

Example 3.1. Let $q \geqq 1$ and $c_2 \geqq 2q$ be integers. Let $\mathcal{F}$ be a stable rank 2 reflexive sheaf on P constructed as an extension:

$$0 \longrightarrow \mathcal{O}_P(-q) \longrightarrow \mathcal{F} \longrightarrow I_Z(q) \longrightarrow 0$$

where $Z = Z_1 \cup Z_2$, with Z_1 a plane curve of degree c_2 and Z_2 a complete intersection of two surfaces of degree q such that Z_1 meets Z_2 at r simple points. We construct a stable rank 3 vector bundle E on P as an extension:

$$0 \longrightarrow \mathcal{F}(-1) \longrightarrow E^*(-1) \longrightarrow I_S \longrightarrow 0$$

According to (1.5), if r=0 or if $\frac{1}{2} \cdot (q-1)(q-2)+1 \leqq r \leqq q^2$ then the construction is possible. The Chern classes of E are $c_1(E)= -1$, $c_2(E)=c_2$, $c_3(E)=c_2^2-2qc_2+2r$.

This example covers all the value of c_3 with $c_3=c_2^2-2qc_2$ or with $c_2^2-2qc_2+(q-1)(q-2)+2 \leqq c_3 \leqq c_2^2-2qc_2+2q^2$.

Example 3.2. Let $d \geqq 1$, $q \geqq 2d$ and $c_2 \geqq q$ be integers. Let $\mathcal{F}$ be a stable rank 2 reflexive sheaf on P constructed as an extension:

$$0 \longrightarrow \mathcal{O}_P(-d-1) \longrightarrow \mathcal{F} \longrightarrow I_Z(d) \longrightarrow 0$$

where $Z=Z_1 \cup Z_2$, with Z_1 a plane curve of degree q and Z_2 a complete intersection of two surfaces of degree d and d+1, respectively, such that Z_1 meets Z_2 at r simple points. We construct a stable rank 3 vector bundle E on P as an extension:

$$0 \longrightarrow \mathcal{F} \longrightarrow E \longrightarrow I_{Y \cup S} \longrightarrow 0$$

where Y is a plane curve of degree c_2-q. According to (1.4), if $\frac{1}{2}\cdot(d-1)(d-2)+1 \leqq r \leqq d(d+1)$, then the construction is possible. The Chern classes of E are $c_1(E)=-1$, $c_2(E)=c_2$, $c_3(E)=c_2^2-2qc_2+2dq-2r$.

<u>Example</u> 3.3. Let $d \geqq 1$, $q \geqq d$ and $c_2 \geqq q+d$ be integers. Let $\mathcal{F}$ be a stable rank 2 reflexive sheaf on P constructed as an extension:

$$0 \longrightarrow \mathcal{O}_P(-d) \longrightarrow \mathcal{F} \longrightarrow I_Z(d) \longrightarrow 0$$

where $Z=Z_1 \cup Z_2$, with Z_1 a plane curve of degree c_2-q+d and Z_2 a complete intersection of two surfaces of degree d such that Z_1 meets Z_2 at r simple points. We construct a stable rank 3 vector bundle E on P as an extension:

$$0 \longrightarrow \mathcal{F}(-1) \longrightarrow E^*(-1) \longrightarrow I_{Y \cup S} \longrightarrow 0$$

where Y is a plane curve of degree q-d. According to (1.5), if $\frac{1}{2}\cdot(d-1)(d-2)+1 \leqq r \leqq d^2$ then the construction is possible. The Chern classes of E are: $c_1(E)=-1$, $c_2(E)=c_2$, $c_3(E)=c_2^2-2qc_2+2dq-$ $-2d^2+2r$.

<u>Example</u> 3.4. Let $d \geqq 1$, $q \geqq d+1$ and $c_2 \geqq q+d$ be integers. Let E' be a semistable rank 3 vector bundle on P constructed, as in (1.6), as an extension:

$$0 \longrightarrow \mathcal{O}_P^2 \longrightarrow E' \longrightarrow I_Z \longrightarrow 0$$

where $Z=Z_1 \cup Z_2$, with Z_1 a plane curve of degree c_2-q and Z_2 a plane curve of degree d such that Z_1 meets Z_2 at s simple points, $0 \leqq s \leqq d$. Let H be the plane considered in (1.6). The generic splitting type of E' is (0,0,0) and H contains a generic line of E'. By (1.6), there is an epimorphism $E' \longrightarrow \mathcal{O}_H(q-d)$. Let E be the kernel of this epimorphism. We have, by definition, an exact sequence :

$$0 \longrightarrow E \longrightarrow E' \longrightarrow \mathcal{O}_H(q-d) \longrightarrow 0$$

It follows, as in (2.5), that E is a stable rank 3 vector bundle on P with Chern classes: $c_1(E) = -1$, $c_2(E) = c_2$, $c_3(E) = c_2^2 - 2qc_2 + 2dq + 2s$.

The examples from (3.2) to (3.4) cover all the values of c_3 with $c_2^2 - 2qc_2 + 2dq - 2d(d+1) \leqq c_3 \leqq c_2^2 - 2qc_2 + 2dq + 2d$. Now, let $\tilde{d}$ be the largest integer for which $2\tilde{d} \leqq q$. Then $2\tilde{d}q + 2\tilde{d} \geqq (q-1)(q-2)$. It follows that the examples from (3.1) to (3.4) cover all the values of c_3 with $c_3 \in M_1(q, c_2)$.

Let $\bar{q}$ be the largest integer for which $2\bar{q} \leqq c_2$ and $\bar{d} = d_1(\bar{q})$. $c_2^2 - 2\bar{q}c_2 + 2\bar{q}^2$ is equal to $\frac{1}{2} \cdot c_2^2$ if c_2 is even and to $\frac{1}{2} \cdot (c_2^2 + 1)$ if c_2 is odd. It follows that the examples from (3.1) to (3.4) cover all the possible values of c_3 with $c_3 \geqq c_2^2 - 2\bar{q}c_2 + 2\bar{d}\bar{q} - 2\bar{d}(\bar{d}+1) :=$ $:= m_1(c_2)$.

The remaining values of c_3 are covered by:

Example 3.5. Let $c_2 \geqq 1$ be an integer. We construct a rank 3 vector bundle E on P as an extension:

$$0 \longrightarrow \mathcal{O}_P(-1)^2 \longrightarrow E \longrightarrow I_Y(1) \longrightarrow 0$$

where Y is a l.c.i. curve in P with $\omega_Y(2)$ generated by its global sections and such that $H^0(I_Y(1)) = 0$. The extension is determined by two global sections ξ_1, ξ_2 of $\omega_Y(2)$ which generate this sheaf. If ξ_1 and ξ_2 are linearly independent over k then E is stable (as one can easily see dualizing the extension).

Let $2 \leqq d \leqq c_2 + 1$ be an integer. If Y is a disjoint union of d-1 lines and of a rational curve of degree $c_2 - d + 2$ then the Chern classes of E are: $c_1(E) = -1$, $c_2(E) = c_2$, $c_3(E) = c_2 + 2 - 2d$.

If $c_2 \geq 2$ and Y is a rational curve of degree c_2+1 then: $c_1(E) = -1$, $c_2(E)=c_2$, $c_3(E)=c_2$.

Now, suppose that $c_2 \geq 3$. If Y is a nonsingular curve of degree c_2+1 and of genus g contained in no plane then the Chern classes of E are : $c_1(E) = -1$, $c_2(E)=c_2$, $c_3(E)=c_2+2g$. According to [4], there are such curves for all g with $0 \leq g \leq \frac{1}{6}\cdot(c_2+1)(c_2-2)+1$.

It follows that, for $c_2 \geq 3$, this example covers all the values of c_3 with $-c_2 \leq c_3 \leq \frac{1}{3}\cdot(c_2+1)^2+1$. One can easily see that $\frac{1}{3}\cdot(c_2+1)^2+1 \geq m_1(c_2)$ for all $c_2 \geq 3$.

4. Examples of Stable Rank 3 Vector Bundles with $c_1 = -2$

Example 4.1. Let $q \geq 1$ and $c_2 \geq 2q$ be integers. Let $\mathcal{F}$ be a stable rank 2 reflexive sheaf on P constructed as an extension:

$$0 \longrightarrow \mathcal{O}_P(-q) \longrightarrow \mathcal{F} \longrightarrow I_Z(q-1) \longrightarrow 0$$

where $Z=Z_1 \cup Z_2$, with Z_1 a plane curve of degree c_2-1 and Z_2 a complete intersection of two surfaces of degree q-1 and q, respectively, such that Z_1 meets Z_2 at r simple points. We construct a stable rank 3 vector bundle E on P as an extension:

$$0 \longrightarrow \mathcal{F} \longrightarrow E^*(-1) \longrightarrow I_S \longrightarrow 0$$

According to (1.4), if r=0 or if $\frac{1}{2}\cdot(q-2)(q-3)+1 \leq r \leq q(q-1)$ then the construction is possible. The Chern classes of E are: $c_1(E) = -2$, $c_2(E)=c_2$, $c_3(E)=c_2^2-(2q+1)c_2+2q+2r$.

This example covers all the values of c_3 with $c_3=c_2^2-(2q+1)c_2+2q$ or with $c_2^2-(2q+1)c_2+2q+(q-2)(q-3)+2 \leq c_3 \leq c_2^2-(2q+1)c_2+2q^2$.

Example 4.2. Let $d \geq 1$, $q \geq \max(2d,3)$ and $c_2 \geq q$ be integers. Let $\mathcal{F}$ be a stable rank 2 reflexive sheaf on P constructed as an extension:

$$0 \longrightarrow \mathcal{O}_P(-d) \longrightarrow \mathcal{F} \longrightarrow I_Z(d) \longrightarrow 0$$

where $Z=Z_1 \cup Z_2$, with Z_1 a plane curve of degree q-1 and Z_2 a complete intersection of two surfaces of degree d such that Z_1 meets Z_2 at r simple points. We construct a stable rank 3 vector bundle E on P as an extension:

$$0 \longrightarrow \mathcal{F}(-1) \longrightarrow E \longrightarrow I_{Y \cup S} \longrightarrow 0$$

where Y is a plane curve of degree c_2-q. According to (1.5), if $\frac{1}{2}\cdot(d-1)(d-2)+1 \leqq r \leqq d^2$ then the construction is possible. The Chern classes of E are: $c_1(E)= -2$, $c_2(E)=c_2$, $c_3(E)=c_2^2-(2q+1)c_2+$ $+2(d+1)q-2d-2r$.

Example 4.3. Let $d \geqq 1$, $q \geqq d+1$ and $c_2 \geqq q+d$ be integers. Let $\mathcal{F}$ be a stable rank 2 reflexive sheaf on P constructed as an extension:

$$0 \longrightarrow \mathcal{O}_P(-d-1) \longrightarrow \mathcal{F} \longrightarrow I_Z(d) \longrightarrow 0$$

where $Z=Z_1 \cup Z_2$, with Z_1 a plane curve of degree c_2-q+d and Z_2 a complete intersection of two surfaces of degree d and d+1, respectively, such that Z_1 meets Z_2 at r simple points. We construct a stable rank 3 vector bundle E on P as an extension:

$$0 \longrightarrow \mathcal{F} \longrightarrow E^*(-1) \longrightarrow I_{Y \cup S} \longrightarrow 0$$

where Y is a plane curve of degree q-d-1. According to (1.4), if $\frac{1}{2}\cdot(d-1)(d-2)+1 \leqq r \leqq d(d+1)$ then the construction is possible. The Chern classes of E are: $c_1(E)= -2$, $c_2(E)=c_2$, $c_3(E)=c_2^2-(2q+1)c_2$ + + $2(d+1)q-2d(d+1)+2r$.

Example 4.4. Let $d \geqq 1$, $q \geqq 2d+1$ and $c_2 \geqq q+1$ be integers. Let E' be a semistable rank 3 vector bundle on P constructed, as in (1.6), as an extension:

$$0 \longrightarrow \mathcal{O}_P^2 \longrightarrow E' \longrightarrow I_Z \longrightarrow 0$$

where $Z=Z_1 \cup Z_2$, with Z_1 a plane curve of degree q-d-1 and Z_2 a plane curve of degree d such that Z_1 meets Z_2 at s simple points, $0 \leqq s \leqq d$. Let H be the plane considered in (1.6). By (1.6), there is an epimorphism $E' \longrightarrow \mathcal{O}_H(c_2-q)$. Let $E^*(-1)$ be the kernel of this epimorphism. We have, by definition, an exact sequence:

$$0 \longrightarrow E^*(-1) \longrightarrow E' \longrightarrow \mathcal{O}_H(c_2-q) \longrightarrow 0$$

It follows that E is a stable rank 3 vector bundle on P with Chern classes: $c_1(E)=-2$, $c_2(E)=c_2$, $c_3(E)=c_2^2-(2q+1)c_2+2(d+1)q-2d(d+1)-2s$.

The examples from (4.2) to (4.4) cover all the values of c_3 with $c_2^2-(2q+1)c_2+2(d+1)q-2d(d+2) \leqq c_3 \leqq c_2^2-(2q+1)c_2+2(d+1)q$.

Now, let $\tilde{d}$ be the largest integer for which $2\tilde{d}+1 \leqq q$. Then $2(\tilde{d}+1)q \geqq 2q+(q-2)(q-3)$. It follows that the examples from (4.1) to (4.4) cover all the values of c_3 with $c_3 \in M_2(q,c_2)$.

Let $\bar{q}$ be the largest integer for which $2\bar{q} \leqq c_2$ and $\bar{d}=d_2(\bar{q})$. $c_2^2-(2\bar{q}+1)c_2+2\bar{q}^2$ is equal to $\frac{1}{2}\cdot(c_2^2-2c_2)$ if c_2 is even and to $\frac{1}{2}\cdot(c_2-1)^2$ if c_2 is odd. It follows that the examples from (4.1) to (4.4) cover all the values of c_3 with $c_3 \geqq c_2^2-(2\bar{q}+1)c_2+2(\bar{d}+1)\bar{q}-2\bar{d}(\bar{d}+2) := m_2(c_2)$.

The remaining values of c_3 are covered by:

<u>Example</u> 4.5. Let $c_2 \geqq 2$ be an integer. We construct a rank 3 vector bundle E on P as an extension:

$$0 \longrightarrow \mathcal{O}_P(-1)^2 \longrightarrow E \longrightarrow I_Y \longrightarrow 0$$

where Y is a l.c.i. curve in P with $\omega_Y(3)$ generated by its global sections. The extension is determined by two global sections ξ_1, ξ_2 of $\omega_Y(3)$ which generate this sheaf. If ξ_1 and ξ_2 are linearly independent over k then E is stable.

Let $1 \leqq d \leqq c_2-1$ be an integer. If Y is a disjoint union of d-1 lines and of a rational curve of degree c_2-d then the Chern classes of E are: $c_1(E)=-2$, $c_2(E)=c_2$, $c_3(E)=2c_2-2-2d$.

If Y is a nonsingular curve of degree c_2-1 and of genus g then the Chern classes of E are: $c_1(E)=-2$, $c_2(E)=c_2$, $c_3(E)=$

According to [4], there are such curves for all g with $0 \leqq g \leqq$ $\leqq \frac{1}{6} \cdot (c_2-1)(c_2-4)+1$.

It follows that this example covers all the values of c_3 with $0 \leqq c_3 \leqq \frac{1}{3} \cdot (c_2-1)(c_2+2)$. One can easily see that : $\frac{1}{3} \cdot (c_2-1)(c_2+2) \geqq m_2(c_2)$ for all $c_2 \geqq 2$.

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