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HYPONORMAL OPERATORS AND EIGENDISTRIBUTIONS

Mihai Putinar

Introduction

This paper deals with two-dimensional models for hyponormal operators. We link the canonical model described in a previous paper [12] to more familiar function spaces and then we relate it to other functional realizations of some special classes of hyponormal operators. The paper is centered around the space of globally defined eigendistributions of the adjoint of a hyponormal operator.

A hyponormal operator on a Hilbert space H is by definition a linear bounded operator $T \in \mathcal{L}(H)$ with the property $TT^* \leq T^*T$. The generic element of this class of operators was described by Daoxing-Xia [16] as a combination between multiplication operators with bounded measurable functions and the Hilbert transform, on the Hilbert space $L^2(a,b)$, where (a,b) is an interval of the real line. Xia's model and the cartesian decomposition of an operator into real and imaginary part were the principal methods in the theory of hyponormal operators. Recently, several authors (Xia [17], Clancey [4], Pincus-Xia-Xia [11]) have refered to hyponormal operators with one-dimensional self-commutator, and related objects to them, in complex coordinate terms. We adopt in this paper the same point of view. Clancey's report [4] was the motivation of the present paper, while the proofs below continues the technique developped in [12]. Let us recall the main construction from [12].

Let Ω be a bounded domain with smooth boundary of the complex plane $\mathbb{C}$, and let $T \in \mathcal{L}(H)$ be a hyponormal operator. For every smooth H-valued function $f \in \mathcal{E}(\overline{\Omega}, H)$, the Cauchy-Pompeiu formula gives rise to the inequality

$$(1) \quad \|(I-P)f\|_{2,\Omega} \leq C(\|(\bar{z}-T^*)\bar{\partial} f\|_{2,\Omega} + \|(\bar{z}-T^*)\bar{\partial}^2 f\|_{2,\Omega}),$$

where C is a positive constant depending only on Ω and $P: L^2(\Omega,H) \longrightarrow A^2(\Omega,H)$ denotes the Bergman projection. When the domain Ω contains the spectrum of T, $\Omega \supset \sigma(T)$, the linear map

$$V: H \longrightarrow H^2(\Omega,H)/\overline{(z-T)H^2(\Omega,H)},$$

where Vh represents the class $\widetilde{1\otimes h}$ of the constant function h on Ω ,is one to one and has closed range.Here $H^2(\Omega,H)$ stands for the Sobolev space of order 2 of H-valued functions .This fact is a consequence of the inequality (1) and of the Riesz-Dunford functional calculus.Then the operator $\widetilde{T}$ induced by the multiplication with z (the complex coordinate) on the quotient space above is (generalized)scalar in the terminology of Colojoară-Foiaş [5] and it extends T,that is $VT=\widetilde{T}V$.The existence of this natural scalar extension explains several spectral properties of hyponormal operators.

The present paper deals with the dual picture of the above construction. More precisely,the dual space of the quotient space where acts the scalar extension is the following space of distributions

$$W_T^{-2}(H) = \left\{ u \in H^{-2}(\Omega,H) \;\middle|\; (\bar{z}-T^*)u=0 \right\},$$

and the surjective map

$$V': W_T^{-2}(H) \longrightarrow H$$

acts by the formula

$$V'(u) = (u,\mathbb{1}) = \int u \ , \ u \in W_T^{-2}(H).$$

In other words ,the Hilbert space H is generated by the global eigendistributions of the operator T^*.

The space $W_T^{-2}(H)$ carries a Hilbert space norm which is independent of Ω , and this fact will be used in the sequel in order to define a canonical norm on the space of the scalar extension $\widetilde{T}$.

The surjectivity of the operator V' reminds of the characteristic property of a class of operators studied and classified by Cowen and Douglas [6] . Altough the case of a general hyponormal operator is more complicated,this analogy suggests a correspondence between a hyponormal operator T and an operator valued distribution kernel K_T which plays the role of the generalized Bergman kernel of Curto and Salinas [7]. This object offers a functional description of the initial Hilbert space, in which T^* becomes the multiplication operator with $\bar{z}$. The kernel K_T has a certain redundancy, and we were able to eliminate it and to describe a determining part of K_T only in a few well understood cases.

The content is the following:

In the first section we recall some facts concerning vector valued Sobolev spaces.

Since the natural scalar extension of a hyponormal operator relies on the multiplication operator with the complex coordinate on a Sobolev space, we collect in the second section a series of properties of this prototype operator.

The third section deals with the naturality problem for the Hilbert space structures introduced by various Sobolev space norms on the space of the natural scalar extension of a hyponormal operator. The list of the properties of the natural scalar extension is completed in the last part of this section with a spectral preserving theorem.

The fourth section is devoted to eigendistributions of cohyponormal operators. We prove that any cohyponormal operator possesses a global distribution resolvent in H^{-1}_{loc}, localized at an arbitrary vector. Then we associate in a natural way to a hyponormal operator T the distribution kernel $K_T \in H^{-2}(\mathbb{C}^2, \mathcal{L}(W_T^{-2}(H))$, which is a complete unitary invariant of T and behaves well to analytic changes of coordinates.

Thanks to the recent work of Clancey [3], [4] we determine in the fifth section a generating part (a compression) of the kernel K_T, in the case of irreducible hyponormal operators with one-dimensional self-commutator. As a byproduct we derive a concrete functional model for such operators, which diagonalizes T^* and is expressed only in terms of the principal function of T.

§1. PRELIMINARIES

In this section we recall some properties of the Sobolev spaces of vector valued functions. Altough most of the results listed below are more general, we concentrate on Sobolev spaces of order 2, on $\mathbb{R}^2$. A complete and optimal reference on that subject is Hörmander's book [9].

Let H be a complex Hilbert space and let $\mathcal{D}(\mathbb{C},H)$ be the $\mathcal{LF}$-space of smooth, compactly supported H-valued functions on the complex plane $\mathbb{C}$. Its topological dual is the space of H-valued distributions, denoted by $\mathcal{D}'(\mathbb{C},H)$. We shall use the nondegenerate sesquilinear pairing

$$\langle\cdot,\cdot\rangle : \quad \mathcal{D}'(\mathbb{C},H) \times \mathcal{D}(\mathbb{C},H) \longrightarrow \mathbb{C},$$

which extends the L^2-scalar product:

$$\langle\varphi,\psi\rangle_2 = \int \langle\varphi(z),\psi(z)\rangle_H \, d\mu(z) \quad ; \ \varphi,\psi \in \mathcal{D}(\mathbb{C},H).$$

Here, and throughout this paper, μ stands for the planar Lebesgue measure.

Let z denote as usually the complex coordinate on $\mathbb{C}$. One denotes: $\partial = \partial/\partial z$, $\bar{\partial} = \partial/\partial\bar{z}$ and $\Delta = 4\partial\bar{\partial}$.

The Hilbert space completion of $\mathcal{D}(\mathbb{C},H)$ with respect to the norm

$$\|\varphi\|_{H^2} = \|(1-\Delta)\varphi\|_2$$

is the Sobolev space $H^2(\mathbb{C},H)$. Its dual via the above sesquilinear form is the Sobolev space of order -2, denoted $H^{-2}(\mathbb{C},H)$. The norm of this space can be described in terms of the Fourier transform as follows:

$$\|u\|^2_{H^{-2}} = \int \|\hat{u}(\xi)\|^2 (1+|\xi|^2)^{-1} d\mu(\xi) .$$

We point out that for any $\varphi \in \mathcal{D}(\mathbb{C},H)$,

$$(2) \qquad \|(1-\Delta)\varphi\|^2_2 = \|\varphi\|^2_2 + 8\|\bar{\partial}\varphi\|^2_2 + 16\|\bar{\partial}^2\varphi\|^2_2 .$$

Let Ω be a complex domain with smooth boundary $\partial\Omega$. Then

$$H_0^2(\Omega, H) = \left\{ f \in H^2(\mathbb{C}, H) \mid \operatorname{supp}(f) \subset \overline{\Omega} \right\}$$

is a closed subspace of $H^2(\mathbb{C}, H)$. By the Sobolev embedding theorem the Hilbert space $H_0^2(\Omega, H)$ is continuously contained in the Banach space $C(\overline{\Omega}, H)$ of continuous functions on $\overline{\Omega}$, uniformly bounded in norm. The dual of $H_0^2(\Omega, H)$ with respect to the above pairing is denoted by $H^{-2}(\Omega, H)$ and it is a quotient Hilbert space of $H^{-2}(\mathbb{C}, H)$.

Conversely, if one denotes by $H_0^{-2}(\Omega, H)$ the closed subspace of $H^{-2}(\mathbb{C}, H)$ of those distributions supported by $\overline{\Omega}$, then $H^2(\Omega, H)$ will denote its dual. It is convenient to identify $H^2(\Omega, H)$ with the orthogonal complement of $H_0^2(\mathbb{C} \setminus \overline{\Omega}, H)$:

$$H^2(\Omega, H) = H^2(\mathbb{C}, H) \ominus H_0^2(\mathbb{C} \setminus \overline{\Omega}, H).$$

We point out that the space $H_0^2(\Omega, H)$ is continuously contained in $H^2(\Omega, H)$.

By the definition of the H^2-norm, the operator $1 - \Delta \; : H_0^2(\Omega, H) \longrightarrow L^2(\Omega, H)$ is an isometry, which is not onto. The dual, in the sense of distributions, of the differential operator $1-\Delta$ has the same expression, hence the operator

$$(1-\Delta)^2 \; : \; H_0^2(\Omega, H) \longrightarrow H^{-2}(\Omega, H)$$

is unitary. We should remark at this point that the space $H^{-2}(\Omega, H)$ is naturally cantained in $\mathcal{D}'(\Omega, H)$, but not in $H^{-2}(\mathbb{C}, H)$. However, we may identify $H^{-2}(\Omega, H)$ with the range of the operator $(1-\Delta)^2 : H_0^2(\Omega, H) \longrightarrow H^{-2}(\mathbb{C}, H)$. In such a way $H^{-2}(\Omega, H)$ becomes a subspace of $H_0^{-2}(\Omega, H)$.

If we assume in addition that the domain Ω is bounded, the space $H_0^2(\Omega, H)$ carries the following equivalent norms:

$$\|f\|_{H^2} \sim \|\Delta f\|_2 \sim \|\bar{\partial}^2 f\|_2 \quad , \quad f \in H_0^2(\Omega, H).$$

Some formulae in this paper will be at hand in the third norm rather than in the first one. Consequently we denote throughout this paper by $W_0^2(\Omega, H)$ the space $H_0^2(\Omega, H)$ endowed with the following Hilbert space norm:

$$\|f\|_{W^2} = \|\bar{\partial}^2 f\|_2 = 1/4\|\Delta f\|_2.$$

Its isometric dual is denoted by $W^{-2}(\Omega,H)$, and it is endowed with the norm that makes the operator

$$(\partial\bar{\partial})^2 : W_0^2(\Omega,H) \longrightarrow W^{-2}(\Omega,H)$$

unitary.

At the level of local spaces we state the following.

LEMMA 1.1 <u>A locally integrable function</u> f <u>on</u> Ω <u>belongs to</u> $H^2_{loc}(\Omega,H)$ <u>iff</u> φf <u>belongs to</u> $W_0^2(\Omega,H)$ <u>for every</u> $\varphi\in\mathcal{D}(\Omega)$.

The space $W_0^2(\Omega,H)$ is again continuously embedded in $C(\bar{\Omega},H)$ and consequently the Dirac measures $\delta_\lambda\otimes h$ belong to $W^{-2}(\Omega,H)$, where $\lambda\in\Omega$ and $h\in H$. Moreover, these and only these are the elements of $W^{-2}(\Omega,H)$ supported by a single point.

§2. A SCALAR SUBNORMAL OPERATOR

In a previous paper, the multiplication operator with the complex coordinate on a Sobolev space of order 2 was the prototype in the functional model associated there to a hyponormal operator, see [12]. We present in this section some of the properties of that operator, though we will not make use of all of them in the sequel. We restrict ourselves to the scalar case dim(H)=1, the higher dimensional case being completely similar.

Let $\Omega\subset\mathbb{C}$ be a bounded domain with smooth boundary and let M denote the multiplication operator with z on the Hilbert space $W_0^2(\Omega)$. As we already remarked in [12], the operator M is scalar of order 2, in the sense of Colojoară and Foiaş [5], with the spectral distribution

$$\mathcal{U}: \mathcal{D}(\mathbb{C}) \longrightarrow \mathcal{L}(W_0^2(\Omega)),$$

$$\mathcal{U}(\varphi)f = \varphi f \quad , \ \varphi\in\mathcal{D}(\mathbb{C}),\ f\in W_0^2(\mathbb{C}).$$

The maximal spectral space associated to a closed suset F of $\mathbb{C}$ is

$$W_0^2(\Omega)_M(F) = \left\{ f\in W_0^2(\Omega) \;\middle|\; \mathrm{supp}(f)\subset F\cap\overline{\Omega} \right\} .$$

Let E_F denote the orthogonal projection of $W_0^2(\Omega)$ onto this space. Then E behaves like a spectral measure, with one exception-the countable additivity property. Indeed, let ω be a subdomain relatively compact in Ω, with smooth boundary.

LEMMA 2.1 <u>Let $\{K_n\}$ be an increasing compact exhaustion of ω. Then</u> $I-E_{\mathbb{C}\setminus\omega} \neq \text{s-lim}\, E_{K_n}$.

<u>Proof</u>. The orthogonal projections $P_1=E_{\mathbb{C}\setminus\omega}$ and $P_2=\text{s-lim}\, E_{K_n}$ are complementary and for every $f\in W_0^2(\Omega)$ the continuous functions $P_1 f$ and $P_2 f$ vanish on $\partial\omega$. Therefore $P_1+P_2\neq I$, q.e.d.

The operator M is subnormal because the operator

$$\bar{\partial}^2 : \ W_0^2(\Omega) \longrightarrow L^2(\Omega)$$

is an isometry which intertwines M with the multiplication operator with z, acting on $L^2(\Omega)$.

LEMMA 2.2 <u>The adjoint of</u> M <u>has the following expression</u>

$$(3) \qquad (M^*f)(z) = \bar{z}f(z) + 2/\pi \ p_\Omega\Big(\int \frac{\chi(z)f(\zeta)}{\zeta - z}\, d\mu(\zeta)\Big) \quad , \ f\in W_0^2(\Omega),$$

<u>where</u> $\chi\in\mathcal{D}(\Omega')$, $\chi\equiv 1$ <u>on</u> $\overline{\Omega}$, $\Omega\subset\subset\Omega'\subset\subset\mathbb{C}$ <u>are arbitrary and</u> p_Ω <u>denotes the orthogonal projection of</u> $W_0^2(\Omega')$ <u>onto</u> $W_0^2(\Omega)$.

<u>Proof</u>. The Cauchy transform is a linear bounded operator from $W_0^2(\Omega)$ into

$H^2(\Omega')$, hence the right side of (3) makes a good sense.

Let $\varphi, \psi \in \mathcal{D}(\Omega)$. Then one gets from the Cauchy-Pompeiu formula:

$$\langle \bar{z}\varphi, \psi \rangle_{W^2} + \langle 2/\pi \, p_\Omega (\chi \int \varphi(\zeta)/(\zeta - z) \, d\mu(\zeta)), \psi \rangle_{W^2} =$$

$$\langle \bar{\partial}^2(\bar{z}\varphi), \bar{\partial}^2\psi \rangle_2 + \langle 2/\pi \, \bar{\partial}^2 \int \varphi(\zeta)/(\zeta - z) \, d\mu(\zeta), \bar{\partial}^2\psi \rangle_2 =$$

$$\langle \bar{z}\,\bar{\partial}^2\varphi + 2\bar{\partial}\varphi, \bar{\partial}^2\psi \rangle_2 - \langle 2\bar{\partial}\varphi, \bar{\partial}^2\psi \rangle_2 = \langle \bar{z}\,\bar{\partial}^2\varphi, \bar{\partial}^2\psi \rangle_2 =$$

$$\langle \varphi, M\psi \rangle_{W^2} .$$

Because the space $\mathcal{D}(\Omega)$ is dense in $W_0^2(\Omega)$, the formula (4) is proved. As the operator M^* is unique, its formula doesn't depend on the choices of the domain Ω' and of the function χ, q.e.d.

The operator M^* is still scalar, with the spectral distribution $\mathcal{U}^*$. Its maximal spectral spaces are:

$$W_0^2(\Omega)_{M^*}(F) = \left\{ f \in W_0^2(\Omega) \mid \operatorname{supp}(\Delta^2 f) \subset F \right\} ,$$

where F is a closed subset of the complex plane. Indeed, $f \in W_0^2(\Omega)_{M^*}(F)$ iff $\mathcal{U}^*(\varphi)f = 0$ for every $\varphi \in \mathcal{D}(\mathbb{C} \setminus F)$. That is $\langle \mathcal{U}^*(\varphi)f, g \rangle = 0$ for every function $g \in W_0^2(\Omega)$. But we have

$$\langle \mathcal{U}^*(\varphi)f, g \rangle = \langle f, \mathcal{U}(\varphi)g \rangle = \langle \bar{\partial}^2 f, \bar{\partial}^2(\varphi g) \rangle_2 = \langle \partial^2\bar{\partial}^2 f, \varphi g \rangle =$$

$$4^{-1}\langle \Delta^2 f, \varphi g \rangle ,$$

and consequently $\Delta^2 f = 0$ on $\mathbb{C} \setminus F$.

The maximal spectral spaces of the operator M^* are not orthogonal for disjoit supports, as those of M, but one can estimate the angle between them, as follows. Let e_F denote the orthogonal projection of $W_0^2(\Omega)$ onto the space $W_0^2(\Omega)_{M^*}(F)$.

PROPOSITION 2.3 Let F,G be two disjoint closed subsets of $\bar{\Omega}$. Then there is a positive constant C, such that

$$\|e_F \cdot e_G\| \leqslant 1-C\left[\operatorname{dist}(F,G)\right]^2,$$

provided that dist(F,G) is small.

For a proof of Proposition 2.3 and related results we refer the reader to Simon's book [14, § III.4].

The operator $\tilde{M}$ -the dual of M on $W^{-2}(\Omega)$- will be of a certain interest in the next sections. As a first application of the duality between distributions and functions, we compute various spectra of the operator M. Altough some of the equalitis below are true for arbitrary scalar operators, we prove them in our context.

PROPOSITION 2.4 The operator M has the following spectra :

$$\sigma(M) = \sigma_{ess}(M) = \sigma_{ap}(M) = \bar{\Omega},$$

$$\sigma_r(M) = \Omega,$$

$$\sigma_c(M) = \partial\Omega.$$

Proof. The point spectrum of M is empty. Indeed, if $(M-\lambda)f = 0$ for a point $\lambda \in \bar{\Omega}$ and a function $f \in W_0^2(\Omega)$, then $\operatorname{supp}(f) \subset \{\lambda\}$ and, since f is continuous, f=0.

Let us assume that $0 \in \Omega \setminus \sigma_r(M)$, that is the operator M has dense range in $W_0^2(\Omega)$. On the other hand,

$$\operatorname{Ran}(M)^{\perp} = \left\{ u \in W^{-2}(\Omega) \,\middle|\, \langle u, zf\rangle = 0\ ,\ f \in W_0^2(\Omega) \right\} = \left\{ u \in W^{-2}(\Omega) \,\middle|\, \bar{z}u = 0 \right\}$$

$$= \mathbb{C}\cdot\delta,$$

a contradiction! Consequently $\Omega \subset \sigma_r(M)$.

Similarely $\partial\Omega \subset \sigma_c(M)$. The inclusions are in fact equalities, because $\bar{\Omega} = \sigma(M) = \sigma_r(M) \cup \sigma_c(M)$ and $\sigma_r(M) \cap \sigma_c(M) = \emptyset$.

Let us assume $0 \in \Omega \setminus \sigma_{ap}(M)$. Then the operator M has, by the above compution of $\mathrm{Ran}(M)^{\perp}$ a closed range of codimension 1 in $W_0^2(\Omega)$.Then every element $f \in W_0^2(\Omega)$, $f(0)=0$, would factorize as $f=zg$,with $g \in W_0^2(\Omega)$.But $\bar{z}/z$ doesn't belong to $H^2_{loc}(\Omega)$,which contradicts the assumption that M has closed range. In conclusion, $\sigma_{ess}(M) = \sigma_{ap}(M) = \bar{\Omega}$,and the proof is complete.

The dual $\bar{M}$ of the operator M,on $W^{-2}(\Omega)$,coincides with the multiplication with $\bar{z}$ .Its adjoit can be easily computed,as follows.

LEMMA 2.5 <u>The operator</u> $\bar{M}^* \in \mathcal{L}(W^{-2}(\Omega))$ <u>is unitarely equivalent with</u> M <u>and it is represented by the formula</u>

(4) $\bar{M}^* u = zu - 2/\pi\ (1/\bar{z} * u),\quad u \in W^{-2}(\Omega).$

Here the space $W^{-2}(\Omega)$ is embedded into $H_0^{-2}(\Omega)$ as the range of the operator $(\partial\bar{\partial})^2 : W_0^2(\Omega) \longrightarrow H^{-2}(\mathbb{C})$.This makes possible the convolution $1/\bar{z} * u$.

<u>Proof</u>. An element $u \in W^{-2}(\Omega)$ can be approximated with a smooth function of the form $(\partial\bar{\partial})^2\varphi$, $\varphi \in \mathcal{D}(\Omega)$.Since $\partial(1/\bar{z}) = -\pi$,one gets

$$1/\bar{z} * (\partial\bar{\partial})^2\varphi = \partial(1/\bar{z} * \partial\bar{\partial}^2\varphi) = -\pi\,\partial\bar{\partial}^2\varphi ,$$

hence the right part of the equality (4) represents a bounded operator on W^{-2}.

The relation $(\partial\bar{\partial})^2 M^* = \bar{M}(\partial\bar{\partial})^2$ implies $\bar{M}^*(\partial\bar{\partial})^2 = (\partial\bar{\partial})^2 M$,therefore the operator $\bar{M}^*$ is unitarely equivalent with M.Moreover,the same equality shows that in order to prove (4) it is enough to check that

$$\langle z(\partial\bar{\partial})^2\varphi, \psi\rangle - 2/\pi\ \langle 1/\bar{z} * (\partial\bar{\partial})^2\varphi, \psi\rangle = \langle (\partial\bar{\partial})^2 z\varphi, \psi\rangle ,$$

for any functions $\varphi, \psi \in \mathcal{D}(\Omega)$.But in view of the above computation we have

$$-2/\pi \langle 1/\bar{z} * (\partial\bar{\partial})^2\varphi, \psi\rangle = 2\langle \partial\bar{\partial}^2\varphi, \psi\rangle = 2\langle \partial\bar{\partial}^2\varphi, \psi\rangle ,$$

and the proof is complete.

§3. SCALAR EXTENSIONS OF HYPONORMAL OPERATORS

Let T be a hyponormal operator on the Hilbert space H. We describe in this section several Hilbert space structures on the space of the natural scalar extension of T.

Let Ω be a bounded domain in $\mathbb{C}$, with smooth boundary and wich contains the spectrum of T, $\Omega \supset \sigma(T)$. Then the Fréchet quotient space

$$\mathcal{H} = H^2_{loc}(\Omega, H)/\overline{(z-T)H^2_{loc}(\Omega, H)}$$

contains the space H, as classes of constant functions, via the embedding $V: H \longrightarrow \mathcal{H}$, $Vh = \widetilde{1\otimes h}$. The scalar operator $z\otimes I$ commutes on $H^2_{loc}(\Omega, H)$, as well as its spectral distribution, with the operator z-T, hence it induces a scalar operator on $\mathcal{H}$, denoted by $\tilde{T}$. Moreover, $VT=\tilde{T}V$, see [12].

The Fréchet space topology of $\mathcal{H}$ is compatible with various Hilbert space norms, as for instance that used in [12], by relation (2) and Lemma 1.1. Most of these Hilbert space norms depend on a choice, e.g. of the domain Ω. For example, every domain Ω endows the space $\mathcal{H}$ with a Hilbert space structure, by identifying $\mathcal{H}$ with the space

$$\mathcal{H}_0(\Omega) = W^2_0(\Omega, H) \ominus (z-T)W^2_0(\Omega, H).$$

The bad behaviour of these norms when comparing them for different domains can be easily illustrated by the following.

<u>Example</u>. Hyponormal operators on finite dimensional spaces.

We assume that dim(H) is finite. In that case T is a normal operator. Let consider two domains with the property $\sigma(T) \subset \Omega' \subset \Omega$. Then the operator $A: \mathcal{H}_0(\Omega') \longrightarrow \mathcal{H}_0(\Omega)$ induced by the natural extension map $W^2_0(\Omega') \subset W^2_0(\Omega)$ is not, in general, unitary. Notice that the operator A is invertible.

Indeed, let us assume that $f \in \mathcal{H}_0(\Omega)$ satisfies $\|f\| = \|A^{-1}f\|$. Then the function f vanishes on $\Omega \setminus \Omega'$, because the space $W^2_0(\Omega')$ is isometrically contained in $W^2_0(\Omega)$. As the spectrum $\sigma(T)$ is finite and the space $\mathcal{H}_0(\Omega)$

contains only continuous functions, it is not possible that A would be unitary for an arbitrary small neighbourhood Ω' of the spectrum.

If the space is finite dimensional, then $\tilde{T}$ coincides with T, that is dim(H)= dim($\mathcal{K}$). Indeed, since the operator T is diagonalizable, it sufficies to prove the equality of dimensions in the case dim(H)=1. Let us assume then in addition that $\sigma(T) = \{0\}$. By the proof of Proposition 2.4 we obtain

$$\mathcal{K}_0(\Omega) \cong \mathrm{Ran}(M)^{\perp} = \left\{ u \in W^{-2}(\Omega) \,\middle|\, \bar{z}u = 0 \right\} = \mathbb{C}\cdot\delta ,$$

and therefore dim($\mathcal{K}$) = 1.

The same duality argument shows that the operator $z-T: W_0^2(\Omega,H) \longrightarrow W_0^2(\Omega,H)$ has not, in general, closed range.

Let us come back to an arbitrary hyponormal operator T. A simpler and a more canonical picture is obtained by dualizing the space $\mathcal{K}$ with respect to the sesquilinear form

$$\langle \cdot,\cdot \rangle : H_{co}^{-2}(\Omega,H) \times H_{loc}^{2}(\Omega,H) \longrightarrow \mathbb{C} ,$$

defined in the preliminaries. Here $H_{co}^{-2}(\Omega,H)$ denotes the space of those distributions $u \in H^{-2}(\mathbb{C},H)$ with compact support in Ω . Let us define the space

$$W_T^{-2}(H) = \mathcal{K}' = \left\{ u \in H_{co}^{-2}(\Omega,H) \,\middle|\, (\bar{z}-T^*)u = 0 \right\} .$$

The space $W_T^{-2}(H)$ inherits a Hilbert space norm from $H_0^{-2}(\Omega,H)$, because the support of a distribution $u \in W_T^{-2}(H)$ is contained in $\sigma(T)$. Moreover, since the natural inclusion $H_0^{-2}(\Omega',H) \subset H_0^{-2}(\Omega,H)$ is isometric, whenever $\Omega' \subset \Omega$, this Hilbert space norm on $W_T^{-2}(H)$ doesn't depend on Ω .

On the other hand, the space $H^{-2}(\Omega,H)$ contains $W_T^{-2}(H)$, but not isometrically, so that the unitary operator $(1-\Delta)^2 : H_0^2(\Omega,H) \longrightarrow H^{-2}(\Omega,H)$ defines by pull back a universal norm on the space $[(1-\Delta)^2]^{-1} W_T^{-2}(H)$. But a straightforward computation shows that

$$\mathcal{K} \cong H_0^2(\Omega,H) \ominus (z-T)H_0^2(\Omega,H) = [(1-\Delta)^2]^{-1} W_T^{-2}(H).$$

In conclusion we state the following.

PROPOSITION 3.1 Let Ω be a domain which contains the spectrum of a hyponormal operator T. The differential operator

$$\mathcal{H} \cong H_0^2(\Omega,H) \ominus (z-T)H_0^2(\Omega,H) \xrightarrow{(1-\Delta)^2} W_T^{-2}(H)$$

is invertible and the Hilbert space norm on $\mathcal{H}$ which makes $(1-\Delta)^2$ unitary doesn't depend on Ω .

Let $W_T^2(H)$ denote the space $\mathcal{H}$ endowed with this canonical norm.

We point out that, altough an element $f \in W_T^2(H)$, realized in virtue of the above proposition as a function in $H_0^2(\Omega,H)$, has a trivial extension F to $\mathbb{C}$, as well as every distribution $u \in W_T^{-2}(H)$, the relation $(1-\Delta)^2 f = u$ holds only on Ω. When extending f and u to $\mathbb{C}$, the distribution $(1-\Delta)^2 F-u$ is not necesarely identically zero, being supported by $\partial\Omega$.

PROPOSITION 3.2 The natural map

$$\rho : H^2(\Omega,H) \ominus (z-T)H^2(\Omega,H) \longrightarrow W_T^2(H)$$

is an isometric isomorphism, whenever $\Omega \supset \sigma(T)$.

Proof. Let Ω be a domain with smooth boundary, which contains the spectrum of the operator T. By the definition of the Sobolev spaces, the operator

$$H^2(\Omega,H) = H^2(\mathbb{C},H) \ominus H_0^2(\mathbb{C}\setminus\overline{\Omega},H) \xrightarrow{(1-\Delta)^2} H_0^{-2}(\Omega,H)$$

is unitary. Let P_Ω denote the orthogonal projection of $H^2(\mathbb{C},H)$ onto $H^2(\Omega,H)$, and let $\chi \in \mathcal{D}(\mathbb{C})$, $\chi \equiv 1$ on $\overline{\Omega}$. Then $(\chi z-T)(I-P_\Omega) = (I-P_\Omega)(\chi z-T)(I-P_\Omega)$.

For any functions $f \in H^2(\Omega,H) \ominus (z-T)H^2(\Omega,H)$ and $g \in H^2(\mathbb{C},H)$ we have

$$\langle(\bar{z}-T^*)(1-\Delta)^2 f,g\rangle = \langle(1-\Delta)^2 f,(\chi z-T)g\rangle = \langle f,(\chi z-T)g\rangle_{H^2} =$$

$$\langle f,(z-T)P_\Omega g\rangle + \langle f,(I-P_\Omega)(\chi z-T)(I-P_\Omega)g\rangle = 0 .$$

And conversely, the same computations show finally that

$$(1-\Delta)^2\left[H^2(\Omega,H)\ominus(z-T)H^2(\Omega,H)\right] = W_T^{-2}(H).$$

Let Q denote the orthogonal projection of $H^2(\Omega,H)$ onto $H_0^2(\Omega,H)$. Then $Q[H^2(\Omega,H)\ominus(z-T)H^2(\Omega,H)] = H_0^2(\Omega,H)\ominus(z-T)H_0^2(\Omega,H)$, as easily follows from the equality $\langle Qf,(z-T)g\rangle = \langle f,(z-T)g\rangle$, where $f\in H^2(\Omega,H)$ and $g\in H_0^2(\Omega,H)$.

In conclusion, the map ρ in the statement, which coincides with Q when the space $W_T^2(H)$ is realized inside $H_0^2(\Omega,H)'$, is unitary, q.e.d.

The second part of this section is devoted to a spectral property of the scalar extension $\tilde{T}\in\mathcal{L}(W_T^2(H))$ of a hyponormal operator T.

THEOREM 3.3 <u>Let T be a hyponormal operator and let $\tilde{T}$ be its natural scalar extension. Then</u> $\sigma(\tilde{T}) = \sigma(T)$.

<u>Proof.</u> Since the spectral distribution $\tilde{\mathcal{U}}$ of the scalar operator $\tilde{T}$ is supported by $\sigma(T)$, the inclusion of local spectra $\sigma_{\tilde{T}}(Vh)\subset\sigma_T(h)$ holds true for every $h\in H$. We recall that the operator $V: H\longrightarrow W_T^2(H)$ intertwines T and $\tilde{T}$.

Let $h\in H$ be fixed. In order to prove the converse inclusion, $\sigma_T(h)\subset\sigma_{\tilde{T}}(Vh)$, we identify $W_T^2(H)$ with the Hilbert space $H^2(\Omega,H)\ominus(z-T)H^2(\Omega,H)$, where Ω is a bounded domain which contains the spectrum of T.

Let $\lambda\notin\sigma_{\tilde{T}}(Vh)$. Then there exists an open neighbourhood ω of λ and an analytic function $\tilde{g}\in\mathcal{O}(\omega,W_T^2(H))$, such that

$$(\zeta-T)\tilde{g}(\zeta) = Vh\quad,\quad \zeta\in\omega.$$

Let $g\in\mathcal{O}(\omega,H^2(\Omega,H))$ be a holomorphic lifting of $\tilde{g}$. Then for a fixed $\zeta\in\omega$,

$$h-(\zeta-z)g(\zeta,z)\in\overline{(z-T)H^2(\Omega,H)}\quad.$$

But the dense range property of a Hilbert space operator is preserved by the topological tensor multiplication with a nuclear space. Therefore there is a sequence $f_n'\in\mathcal{O}(\omega,H^2(\Omega,H))$, so that

$$\lim_n \left(h - (\zeta - z)g(\zeta, z) - (z-T)f'_n(\zeta, z)\right) = 0$$

in the Fréchet topology of the space $\mathcal{O}(\omega, H^2(\Omega, H))$.

Let ω' be another open neighbourhood of the pount λ ,relatively compact in ω .Let m denote the unique continuous linear extension

$$m : \mathcal{O}(\omega)\hat{\otimes} H^2(\Omega, H) \longrightarrow H^2(\omega', H)$$

of the map $a\otimes b \mapsto (a.b)|\omega'$ .Then $m(h-(\zeta-z)g(\zeta,z) -(z-T)f'_n(\zeta,z)) = h-(z-T)f_n(z)$,where $f_n(z) = f'_n(z,z)$ for $z \in \omega'$ .Therefore $h = \lim(z-T)f_n$ in $H^2(\omega',H)$.

In view of the inequality (1) we obtain $\lim \| f_n - Pf_n \|_{2,\omega'} = 0$ and $\lim \| h-(z-T)Pf_n \|_{2,\omega'} = 0$,which in turn implies $h \in \overline{(z-T)\mathcal{O}(\omega',H)}$. But the operator T satisfies Bishop's property (β),see [15] or for instance as a subscalar operator,so that the operator (z-T) has closed range on $\mathcal{O}(\omega',H)$.

Finally $h \in (z-T)\mathcal{O}(\omega',H)$,or,in other terms, $\lambda \notin \sigma_T(h)$, q.e.d.

In fact we have proved more,namely.

COROLLARY 3.4 <u>The local spectra $\sigma_T(h)$ and $\sigma_{\tilde{T}}(Vh)$ coincide for every $h \in H$.</u>

The above spectral behaviour of a minimal scalar extension of a hyponormal operator is different from that of the normal extension of a subnormal operator.In particular,the natural scalar extension $\tilde{S}$ of a subnormal operator S doesn't coincide,in general,with the normal extension of S.On the essential resolvent set of S,the operator $\tilde{S}$ has not,in general,closed range.

§4. A DISTRIBUTION KERNEL

This section deals with the relationship between hyponormal operators and operator valued distribution kernels on $\mathbb{C}^2$.The existence of a scalar extension of a hyponormal operator makes possible the analogy with the generalized Bergman kernels theory of Curto and Salinas [7] ,Altough the general framework developped in the sequel leads to rather tautological results,when applying it to particular hyponormal operators the fine invariants fit naturally into this scheme.

Let T be a hyponormal operator on the Hilbert space H.With the notations of the preceding section,the dual V': $W_T^{-2}(H) \longrightarrow H$ of the embedding V is onto. We recall that $W_T^{-2}(H)$ denotes the set of those distributions $u \in H^{-2}(\mathbb{C},H)$ which are annihilated by $\bar{z}-T^*$.The operator V' acts by the formula

$$V'(u) = (u, \mathbb{1}) \quad , u \in W_T^{-2}(H) ,$$

where (,) stands for the natural <u>bilinear</u> pairing

$$(\, , \,) : \quad \mathcal{E}'(\mathbb{C},H) \times \mathcal{E}(\mathbb{C}) \longrightarrow H.$$

We shall use the following continuity property of this bilinear form:

(5) $\quad \| (u,\varphi)\| \leq \|u\|_{H^{-2}} \cdot \|\varphi\|_{H^2} \quad , u \in \mathcal{E}'(\mathbb{C},H), \ \varphi \in \mathcal{E}(\mathbb{C}),$

which can be proved by a Fourier transform argument.

For the beginning we prove for its own interest the following.

PROPOSITION 4.1 <u>A cohyponormal operator</u> $T^* \in \mathcal{L}(H)$ <u>has a generalized global resolvent</u>,<u>localized at an arbitrary vector</u> $h \in H$: <u>i.e</u>. <u>there exists a distribution</u> $v_h \in H^{-1}_{loc}(\mathbb{C},H)$ <u>so that</u> $(\bar{z}-T^*)v_h = h$ <u>on</u> $\mathbb{C}$.

The vectors of the range of the operator $[T^*,T]^{1/2}$ have even a global resolvent of a function type,[13].

Proof. We choose for a fixed $h \in H$ a distribution $u \in W_T^{-2}(H)$ with the property $V'(u) = -\pi h$. Because the operator $\partial : H^{-1}_{loc}(\mathbb{C},H) \longrightarrow H^{-2}_{loc}(\mathbb{C},H)$ is onto, there exists a distribution $u_1 \in H^{-1}_{loc}(\mathbb{C},H)$ such that $\partial u_1 = u$. In particular, the restriction of u_1 to the open set $\mathbb{C} \setminus \sigma(T)$ is an atiholomorphic function:

$$u_1(z) = \sum_{n=-\infty}^{\infty} a_n \bar{z}^n ,$$

where the series converges for $|z| > \|T\|$ and $a_n \in H$. Then the expression

$$f(z) = \sum_{n=0}^{\infty} a_n \bar{z}^n$$

defines an antiholomorphic function on the whole complex plane.

The desired global resolvent is $v_h = u_1 - f$. Indeed, $\partial(\bar{z}-T^*)v_h = (\bar{z}-T^*)u = 0$, and the estimate

$$\|(\bar{z}-T^*)(u_1(z)-f(z))\| = O(1) \quad \text{for } |z| \to \infty ,$$

holds true. Then by Liouville's Theorem $(\bar{z}-T^*)v_h \equiv a_{-1}$.

It remains to compute the constant a_{-1}. Let $\chi \in \mathcal{D}(\mathbb{C})$, $\chi \equiv 1$ on a neighbourhood of $\sigma(T)$. By Stokes Theorem we obtain

$$-\pi h = (u, 1\!\!1) = (u, \chi) = (\partial v_h, \chi) = -(v_h, \partial\chi) =$$
$$-\sum_{n=1}^{\infty} a_{-n} \int \bar{z}^{-n} \partial\chi \, (z) d\mu(z) = -\pi a_{-1} ,$$

and the proof is complete.

Throughout this section we denote for the sake of simplicity $L = W_T^{-2}(H)$ for a fixed hyponormal operator T.

DEFINITION 4.2 The <u>distribution kernel</u> $K_T \in \mathcal{D}'(\mathbb{C}^2, \mathcal{L}(L))$ associated to the hyponormal operator T is

(6) $\langle K_T(\varphi \otimes \psi)u, v \rangle_L = \langle (u,\psi), (v, \bar{\varphi}) \rangle$,

where $\varphi, \psi \in \mathcal{D}$ and $u, v \in L$.

The distribution K_T is completely determined by the relation (6) because of the following estimate derived from (5):

$$(7) \quad |\langle K_T(\varphi \otimes \psi)u, v\rangle_L| \leq \|u\|_L \cdot \|v\|_L \cdot \|\varphi\|_{H^2} \cdot \|\psi\|_{H^2} ,$$

with the same notations as above.

PROPOSITION 4.3 <u>The kernel K_T has the following properties:</u>

a) $K_T(\varphi \otimes \psi)^* = K_T(\bar{\psi} \otimes \bar{\varphi})$, $\varphi, \psi \in \mathcal{D}(\mathbb{C})$;

b) <u>For every finite sequence</u> $(\varphi_i)_{i=1}^n$, $\varphi_i \in \mathcal{D}(\mathbb{C})$, $n \geq 1$, <u>the operator matrix</u> $(K_T(\varphi_i \otimes \bar{\varphi}_j))_{i,j}$ <u>is positive</u> ;

c) $\mathrm{Supp}(K_T) \subset \sigma(T) \times \sigma(T)$;

d) $K_T \in H^{-2}(\mathbb{C}^2, \mathcal{L}(H))$.

<u>Proof.</u> a) Let $u, v \in L$. Directly by the definition (6),

$$\langle K_T(\varphi \otimes \psi)^* u, v\rangle = \langle u, K_T(\varphi \otimes \psi)v\rangle = \langle (u, \bar{\varphi}), (v, \psi)\rangle = \langle K_T(\bar{\psi} \otimes \bar{\varphi})u, v\rangle .$$

b) If $(\varphi_i)_{i \in I}$, $\varphi_i \in \mathcal{D}(\mathbb{C})$ and $(u_i)_{i \in I}$, $u_i \in L$, are finite systems, then

$$\sum_{i,j} \langle K_T(\varphi_i \otimes \bar{\varphi}_j)u_j, u_i\rangle = \sum_{i,j} \langle (u_j, \bar{\varphi}_j), (u_i, \bar{\varphi}_i)\rangle = \|\sum_i (u_i, \bar{\varphi}_i)\|^2 .$$

c) It suffices to recall that $\mathrm{supp}(u) \subset \sigma(T)$ whenever $u \in L$.

d) The distribution K_T belongs to $H^{-2}_{loc}(\mathbb{C}^2, \mathcal{L}(L))$ by (7), and it has compact support, hence $K_T \in H^{-2}(\mathbb{C}^2, \mathcal{L}(L))$.

The continuity and the positivity properties of the kernel K_T insure the existence of a scalar product on the space $\mathcal{D}(\mathbb{C}, L)$, which extends continuously the following form

$$\langle K_T f, g\rangle = \int \langle K_T(z,w)f(w), g(z)\rangle = \sum_{i,j} \langle K_T(\bar{\psi}_j \otimes \varphi_i)u_i, v_j\rangle_L ,$$

where $f = \sum_{i=1}^{n} \varphi_i \otimes u_i \in \mathcal{D}(\mathbb{C},L)$, $g = \sum_{j=1}^{m} \psi_j \otimes v_j \in \mathcal{D}(\mathbb{C},L)$ and the integers n,m are finite.

Let us define the continuous evaluation map

$$\Psi : \mathcal{D}(\mathbb{C},L) \longrightarrow H,$$

which acts on simple functions of the form $\varphi \otimes u$ by the formula

$$\Psi(\varphi \otimes u) = (u,\varphi) \quad , \varphi \in \mathcal{D}(\mathbb{C}),\ u \in L.$$

Then,for the above finite dimensional valued functions f and g,the relation

$$(8) \quad \langle K_T f,g \rangle = \langle \Psi(f), \Psi(g) \rangle_H$$

holds true.The right side of (8) makes a good sense for arbitrary functions $f,g \in \mathcal{D}(\mathbb{C},L)$,and it can be taken as a second definition of the kernel K_T.

The separate completion of the space $\mathcal{D}(\mathbb{C},L)$ with respect to the seminorm derived from the scalar product (8) is isometrically isomorphic with the space H,through the linear extension of the operator Ψ.In this description, as a vector valued function space,of the Hilbert space H,the operator T^* becomes the multiplication with $\bar{z}$.Moreover,the reproducing kernel K_T of this function space is a complete unitary invariant of the operator T,in the following sense.

LEMMA 4.4 <u>Two hyponormal operators</u> $T \in \mathcal{L}(H)$ <u>and</u> $T' \in \mathcal{L}(H')$ <u>are unitarely equivalent iff there exists a unitary operator</u> $U: W_T^{-2}(H) \longrightarrow W_{T'}^{-2}(H')$ <u>such that</u> $UK_TU^* = K_{T'}$.

<u>Proof</u>. Let K denote the separate completion of the space $\mathcal{D}(\mathbb{C},W_T^{-2}(H))$ in the norm $\langle K_T f,f \rangle$,so that the operator $\Psi : K \longrightarrow H$ is unitary and has the property $\Psi^* T^* \Psi = \bar{z}$.Analogously,K' and Ψ' denote the corresponding objects associated to T'.

If U is a unitay operator as in the statement,then it induces a unitary transform $I\otimes U : K \longrightarrow K'$ with the property $(I\otimes U)\bar{z} = \bar{z}(I\otimes U)$.Consequently $(I\otimes U)\Psi^* T^* \Psi(I\otimes U) = \Psi'^* T' \Psi'$,and the proof is complete.

There are examples which show that only a "thin" subspace of $W_T^{-2}(H)$ is necessary, in order to classify the hyponormal operator T up to unitary equivalence,as in Lemma 4.4.Thus we adopt the next.

DEFINITION 4.5 A closed subspace G of $L(=W_T^{-2}(H))$ is called generating for the operator T if $\Psi\mathcal{D}(\mathbb{C},G)$ is a dense subspace of H.

In other words $G\subset L$ is generating if the linear space

$$\{(u,\varphi) \quad u\in G, \varphi\in\mathcal{D}(\mathbb{C})\}$$

is dense in H.

Let P_G denote the orthogonal projection of L onto a generating subspace G. Then the compression of the kernel K_T to G,

$$K_T^G := P_G K_T P_G \in \mathcal{D}'(\mathbb{C}^2, \mathcal{L}(G))$$

has all the properties listed in Proposition 4.3.Therefore the restriction Ψ_G of the operator Ψ to $\mathcal{D}(\mathbb{C},G)$ is related to the kernel K_T^G by the relation:

(9) $\langle K_T^G f,g\rangle = \langle \Psi_G(f), \Psi_G(g)\rangle \quad , \quad f,g\in\mathcal{D}(\mathbb{C},G).$

Now we may conclude with the following.

THEOREM 4.6 <u>Let T be a hyponormal operator on the Hilbert space H and let G be a generating space of eigendistributions of T^*.</u>

<u>Let $\mathcal{H}$ denote the separate completion of the space $\mathcal{D}(\mathbb{C},G)$ with respect to the seminorm $\langle K_T f,f\rangle$ and let $U: \mathcal{H}\longrightarrow H$ be the continuous linear extension of the operator Ψ_G.</u>

<u>Then U is unitary and $U(\bar{z}f) = T^*U(f)$ for every $f\in\mathcal{H}$.</u>

Proof. The operator U is an isometry, which is onto because G was supposed a generating subspace of L. For a simple function of the form $\varphi \otimes u \in \mathcal{D}(\mathbb{C}, G)$, we have

$$T^{*} \psi_{G}(\varphi \otimes u) = T^{*}(u, \varphi) = (\bar{z}u, \varphi) = \psi_{G}(\bar{z}\varphi \otimes u),$$

because $u \in W^{-2}_{T}(H)$. Then the proof is over by an approximation argument.

Lemma 4.4 and Theorem 4.6 lead to the next.

COROLLARY 4.7 Let $T \in \mathcal{L}(H)$ and $T' \in \mathcal{L}(H')$ be hyponormal operators and let G, G' be generating subspaces of eigendistributions of T^{*}, respectively of T'^{*}.

If there exists a unitary operator $U: G \rightarrow G'$, so that $UK^{G}_{T}U = K^{G'}_{T'}$, then T is unitarely equivalent with T'.

The necessary condition for unitary equivalence stated in Corollary 4.7 is also sufficient whenever the space G is canonically related to the operator T. In many cases dim(G) is finite, so that the classification of the related operators is a finite dimensional problem. We illustrate this statement with two examples.

Example 1. Normal irreducible operators.

Let N be a normal operator on the Hilbert space H, with a cyclic vector ξ of norm one. We denote as usually by $\varphi(N)$ the continuous functional calculus of $N, \varphi \in C(\sigma(N))$.

Let Ω be a bounded domain which contains $\sigma(N)$. In virtue of the continuity of the functional calculus and of the Sobolev embedding theorem, the following estimate

$$\|\varphi(T)\xi\| \leq \|\varphi\|_{\infty,\Omega} \leq C\|\varphi\|_{H^{2}(\Omega)}$$

holds true, with a positive constant C depending only on Ω. Thus the relation

$$(u,\varphi)=\varphi(T)\xi\ , \qquad \varphi\in\mathcal{D}(\mathbb{C}),$$

defines a H-valued distribution $u\in H_0^{-2}(\Omega,H)$. Moreover, $u\in W_N^{-2}(H)$. Indeed, for every $\varphi\in\mathcal{D}(\mathbb{C})$,

$$((N^*-\bar{z})u,\varphi) = N^*\varphi(N)\xi - (\bar{z}\varphi)(N)\xi = 0.$$

Since the linear space $\{\varphi(N)\xi\ ,\ \varphi\in\mathcal{D}(\mathbb{C})\}$ is dense in H, the one dimensional subspace $\mathbb{C}u$ of $W_N^{-2}(H)$ is generating for N. The compression k of the kernel K_N to this space is a scalar distribution which can be easily computed, as follows: let $\varphi,\psi\in\mathcal{D}(\mathbb{C})$,

$$k(\varphi\otimes\psi) = \langle K_N(\varphi\otimes\psi)u,u\rangle = \langle(u,\psi),(u,\bar{\varphi})\rangle = \langle\psi(N)\xi,\bar{\varphi}(N)\xi\rangle =$$

$$\langle\varphi(N)\psi(N)\xi,\xi\rangle = \int\varphi\psi\ d\mu_\xi\ ,$$

where $d\mu_\xi = \langle dE\,\xi,\xi\rangle$ and E is the spectral measure of N. The norm of the space $\mathbb{C}u$ was choosen so that $\|u\|=1$. Concluding, we have proved the formula

$$k(z,w) = \mu_\xi(z)\,\delta(z-w),$$

where $\delta(z-w)$ stands for the Dirac measure supported by the diagonal of the space $\mathbb{C}^2$.

The completion of the space $\mathcal{D}(\mathbb{C})$ with respect to the seminorm given by the kernel k coincides with the space $L^2(\mu_\xi)$ and the unitary $U:L^2(\mu_\xi)\longrightarrow H$ is the operator which diagonalizes N.

<u>Example</u> 2. The class $A(\Omega)$ of Curto and Salinas [7].

The class $A(\Omega)$ of operators provides the localization of the class $B(\Omega)$ of Cowen and Douglas [6]. We restrict for the beginning to a particular element of $A(\Omega)$.

Let Ω be a bounded domain in the complex plane and let S be a hyponormal operator which has the properties:

(i) Ran(z-S) is closed for all $z \in \Omega$;

(ii) span $\{ \mathrm{Ker}(\bar{z}-S^*) \mid z \in \Omega \}$ is dense in H ; and

(iii) There exists a <u>bounded</u> coanalytic function $\Gamma : \Omega \longrightarrow \mathcal{L}(\mathbb{C}^n, H)$, such that Ran $\Gamma(z) = \mathrm{Ker}(\bar{z}-S^*)$ for every $z \in \Omega$.

For a relation between hyponormal operators and the class $B(\Omega)$ see [1].

Under the assumptions (i)-(iii) the space $W_S^{-2}(H)$ contains the following priviledged eigendistributions.Let $e_i, i=1,n$, be the canonical basis in $\mathbb{C}^n$. Since the operatorial function Γ is uniformly bounded,the formula

$$(u_i, \varphi) = \int_\Omega \varphi(z)\, \Gamma(z) e_i \, d\mu(z) \quad , \quad \varphi \in \mathcal{D}(\mathbb{C})$$

defines for every i=1,n a distribution $u_i \in H^{-2}(\mathbb{C}, H)$.It is plain to check that $(\bar{z}-S^*)u_i = 0$,in the sense of distributions.

The linear span G of $u_i, i=1,n$,is a generating subspace of $W_S^{-2}(H)$,because of the assumption (ii).Let us compute the corresponding distribution kernel.

We renorm the space G so that $(u_1, \ldots, u_n)$ becomes an orthonormal basis.Let $f, g \in \mathcal{D}(\mathbb{C}, G)$,so that

$$f = \sum_{i=1}^{n} \varphi_i \otimes u_i \quad \text{and} \quad g = \sum_{i=1}^{n} \psi_j \otimes u_j \ ,$$

where $\varphi_k, \psi_k \in \mathcal{D}(\mathbb{C})$,k=1,n.Then,

$$\langle K_S^G f, g \rangle = \sum_{i,j} \langle K_S^G(\varphi_i \otimes u_i), \psi_j \otimes u_j \rangle = \sum_{i,j} \langle K_S(\bar{\psi}_j \otimes \varphi_i) u_i, u_j \rangle =$$

$$= \sum \langle (u_i, \varphi_i), (u_j, \psi_j) \rangle = \sum \int_{\Omega \times \Omega} \langle \varphi_i(z) \Gamma(z) e_i, \psi_j(w) \Gamma(w) e_j \rangle \, d\mu \times d\mu$$

$$= \int_{\Omega \times \Omega} \langle \Gamma(w)^* \Gamma(z) f(z), g(w) \rangle \, d\mu(z) d\mu(w) .$$

In conclusion the distribution kernel K_S^G coincides on $\Omega \times \Omega$ with the generalized Bergman kernel of the function Γ , $K_\Gamma(z,w) = \Gamma(w)^* \Gamma(z)$.

In the general case,when the function Γ is not necessarely bounded,the Lebesgue measure must be multiplied by a weight,in order to annihilate the growth of $\|\Gamma(z)\|$ when $z \rightarrow \partial\Omega$.

The last subject of this section is a naturality formula of the kernel K_T, to analytic changes of coordinates.

Let $f:U \longrightarrow V$ be a biholomorphic map between two domains of the complex plane. We recall for the beginning the operation of change of coordinates introduced by f at the level of distributions.

Let $\alpha \in \mathcal{D}'(V)$ and let $\varphi \in \mathcal{D}(U)$. The distribution $\alpha \circ f \in \mathcal{D}'(U)$ is defined by the formula

$$(\alpha \circ f, \varphi) = (\alpha, \psi),$$

where

$$\psi(\zeta) = \varphi(f^{-1}(\zeta))\,|\partial f^{-1}(\zeta)|^2 \qquad , \zeta \in V.$$

The same definition applies to vector valued distributions.

LEMMA 4.8 <u>Let</u> $T \in \mathcal{L}(H)$ <u>be a hyponormal operator with</u> $\sigma(T) \subset U$. <u>If</u> $u \in W_T^{-2}(H)$, <u>then</u> $u \circ f^{-1} \in W_{f(T)}^{-2}(H)$.

<u>Proof</u>. The spectrum of the operator f(T) is, by the spectral mapping theorem, contained in V. Let us assume $u \in W_T^{-2}(H)$. For every function $\varphi \in \mathcal{D}(V,H)$ there exists by standard arguments a function $\psi \in \mathcal{D}(U,H)$, so that

$$(z-T)\psi(z) = (f(z) - f(T))\,\varphi(f(z))\,|\partial f(z)|^2 \qquad , z \in U.$$

Then

$$\langle(\bar{\zeta} - f(T))(u \circ f^{-1}), \varphi\rangle = \langle u \circ f^{-1}, (\zeta - f(T))\varphi\rangle =$$

$$= \langle u, (f - f(T))(\varphi \circ f)|\partial f|^2\rangle = \langle u, (z-T)\psi\rangle = 0,$$

and therefore $u \circ f^{-1} \in W_{f(T)}^{-2}(H)$, q.e.d.

Since the support of a distribution $u \in W_T^{-2}(H)$ is contained in the compact subset $\sigma(T)$ of U, the linear operator

$$C_f : W^{-2}_{f(T)}(H) \longrightarrow W^{-2}_T(H) \quad , C_f(v) = v\circ f,$$

is bounded. Notice that $C_{f^{-1}}$ is a two-sided inverse of C_f.

THEOREM 4.9 <u>Let $T\in\mathcal{L}(H)$ be a hyponormal operator and let $f:U \longrightarrow V$ be a biholomorphic map, defined on a domain which contains the spectrum of T. Then</u>

$$K_{f(T)}\circ(f\times f) = C^*_f K_T C_f .$$

<u>Proof</u>. Let $\varphi,\psi\in\mathcal{D}(U,H)$ and let $u,v\in W^{-2}_{f(T)}(H)$. By Definition 4.2 we have

$$\langle K_{f(T)}\circ(f\times f)(\varphi\otimes\psi)u,v\rangle =$$

$$\langle K_{f(T)}((\varphi\circ f^{-1})|\partial f^{-1}|^2\otimes(\psi\circ f^{-1})|\partial f^{-1}|^2)u,v\rangle =$$

$$\langle (u,(\psi\circ f^{-1})|\partial f^{-1}|^2),(v,(\bar{\varphi}\circ f^{-1})|\partial f^{-1}|^2)\rangle =$$

$$\langle(u\circ f,\psi),(v\circ f,\bar{\varphi})\rangle =$$

$$\langle(C_f u,\psi),(C_f v,\bar{\varphi})\rangle =$$

$$\langle K_T(\varphi\otimes\psi)C_f u,C_f v\rangle =$$

$$\langle C^*_f K_T(\varphi\otimes\psi)C_f u,v\rangle \quad ,$$

and the proof is complete.

The compression of the kernel K_T to a generating subspace G of $W^{-2}_T(H)$ has a similar formula, if one takes the compression of the kernel $K_{f(T)}$ to the generating subspace $C_{f^{-1}}(G)\subset W^{-2}_{f(T)}(H)$.

§5. OPERATORS WITH ONE-DIMENSIONAL SELF-COMMUTATOR

The compression of the distribution kernel of a hyponormal operator with one-dimensional self-commutator to a space generated by a single eigendistribution of its adjoint coincides with a scalar kernel already existing in the literature, cf. [3], [11], [18]. We use this kernel in order to obtain a concrete functional description of the operator in terms of its principal function.

Let us recall for the beginning some facts and notations concerning the operators with rank one self-commutator. The reader is refered to [4], [8] and [10] for details.

Let T be a bounded linear operator on the Hilbert space H, so that

$$[T^*,T] = \xi \otimes \xi ,$$

where $\xi \otimes \xi$ is a rank one, positive, operator: $(\xi \otimes \xi)h = \langle h, \xi \rangle \xi$, $h \in H$.

The complete invariant to unitary equivalence of an irreducible operator with one-dimensional self-commutator was discovered by Pincus [10] as a scalar function on the spectrum, called the principal function. The <u>principal function</u> g_T of the operator T is a compactly supported, real valued measurable function on $\mathbb{C}$, characterized by Helton and Howe formula [8]:

$$\operatorname{tr}[p(T,T^*),q(T,T^*)] = 1/\pi \int_{\mathbb{C}} (\bar{\partial} p \partial q - \partial p \bar{\partial} q) g_T \, d\mu ,$$

where p, q are polynomials in two variables.

The order of the factors T and T^* in the monomials of p and q doesn't affect the trace of the commutator $[p(T,T^*),q(T,T^*)]$, which exists whenever $[T^*,T]$ is trace-class. In our case, when T is hyponormal and $\operatorname{rk}[T^*,T] = 1$, the principal function satisfies the additional condition $0 \leq g_T \leq 1$.

Putnam proved in [13] that for an arbitrary complex number $z \in \mathbb{C}$, the equation

$$(\bar{z}-T^*)\, x = \xi$$

has a unique solution $x \in \operatorname{Ker}(\bar{z}-T^*)^{\perp}$. Moreover, this solution depends weakly continuous on $z\in\mathbb{C}$, and it will be denoted in the sequel by $(\bar{z}-T^*)^{-1}\xi$.

Recently, Clancey [3] proved the identity

$$(10)\qquad 1-\langle(\bar{w}-T^*)^{-1}\xi\ ,(\bar{z}-T^*)^{-1}\xi\rangle = \exp\left\{-1/\pi\int\frac{g_T(\zeta)}{(\zeta-z)(\bar{\zeta}-\bar{w})}\,d\mu(\zeta)\right\},$$

which holds on the whole $\mathbb{C}^2$, see also [11]. The integral of the right side of (10) has removable singularities off the diagonal of $\mathbb{C}^2$. In case $z=w$ and $\int |\zeta-z|^2 g_T(\zeta)\,d\mu(\zeta)=\infty$, the right side of (10) is taken to be zero. The kernel

$$\mathcal{T}_*(z,w) = 1-\langle(\bar{w}-T^*)^{-1}\xi,(\bar{z}-T^*)^{-1}\xi\rangle$$

was used in [4] in connection with a distributional model for T. We use the next.

THEOREM 5.1 (Clancey [4]) <u>An operator</u> T <u>with</u> $[T^*,T]=\xi\otimes\xi$ <u>is irreducible iff the values of the function</u> $(\bar{z}-T^*)^{-1}\xi$ <u>span</u> H.

The theorem has, with the terminology of the preceding section, the following.

COROLLARY 5.2 <u>Let</u> T <u>be an irreducible hyponormal operator with</u> $[T^*,T]=\xi\otimes\xi$. <u>The distribution</u> $\partial(\bar{z}-T^*)^{-1}\xi$ <u>belongs to</u> $W_T^{-2}(H)$ <u>and it spans a one-dimensional generating space for</u> T.

<u>Proof</u>. The function $(\bar{z}-T^*)^{-1}\xi$ belogs to $L^2_{loc}(\mathbb{C},H)$ and

$$(\bar{z}-T^*)\,\partial(\bar{z}-T^*)^{-1}\xi = \partial\left[(\bar{z}-T^*)(\bar{z}-T^*)^{-1}\xi\right] = \partial\xi = 0,$$

therefore $\partial(\bar{z}-T^*)^{-1}\xi\in W_T^{-2}(H)$.

Furtheron we prove that the space

$$M=\left\{(\partial(\bar{z}-T^*)^{-1}\xi\ ,\varphi)\ \middle|\ \varphi\in\mathcal{D}(\mathbb{C})\right\}$$

is dense in H.

The Cauchy (anti)transform of a test function $\varphi \in \mathcal{D}(\mathbb{C})$ is

$$f(z) = 1/\pi \int \varphi(\zeta)/(\bar{\zeta}-\bar{z})\, d\mu(\zeta) .$$

It is a antianalytic function off supp(φ), vanishing at ∞, and $\partial f = -\varphi$. Let Ω be a bounded complex domain with smooth boundary, which contains supp(φ) and $\sigma(T)$. Then by Stokes Theorem

$$\int_{\mathbb{C}} \varphi(z)(\bar{z}-T^*)^{-1}\xi\, d\mu(z) = -\int_{\Omega} \partial f(z)(\bar{z}-T^*)^{-1}\xi\, d\mu(z) =$$

$$1/2i \int_{\partial\Omega} f(z)(\bar{z}-T^*)^{-1}\xi\, d\bar{z} + (\partial(\bar{z}-T^*)^{-1}\xi, f).$$

The contour integral is zero since $f(z)(\bar{z}-T^*)^{-1}$ is a antianalytic function on $\mathbb{C}\setminus\Omega$, vanishing of second order at ∞. Therefore $\int \varphi(\bar{z}-T^*)^{-1}\xi\, d\mu \in M$.

By Theorem 5.1 the vectors $\int \varphi(\bar{z}-T^*)^{-1}\xi\, d\mu$, $\varphi \in \mathcal{D}(\mathbb{C})$, span the Hilbert space H, hence the space M is also dense in H, and the proof is over.

The above corollary and Theorem 4.6 imply that the compression k of the distribution kernel K_T to the space generated in $W_T^{-2}(H)$ by the distribution $u = \partial(\bar{z}-T^*)^{-1}\xi$ is a scalar kernel which reproduces H and diagonalizes T^*. Let us compute this kernel. Let consider the test functions $\varphi, \psi \in \mathcal{D}(\mathbb{C})$. Then,

$$k(\varphi\otimes\psi) = \langle K_T(\varphi\otimes\psi)u,u\rangle = \langle (u,\psi),(u,\bar{\varphi})\rangle =$$

$$\int \langle (\bar{w}-T^*)^{-1}\xi\, \partial\psi(w), (\bar{z}-T^*)^{-1}\xi\, \partial\bar{\varphi}(z)\rangle\, d\mu(z)d\mu(w) =$$

$$\int \langle (\bar{w}-T^*)^{-1}\xi, (\bar{z}-T^*)^{-1}\xi\rangle\, \partial\psi(w)\, \bar{\partial}\varphi(z)\, d\mu(z)d\mu(w) =$$

$$-\int \bar{\partial}_z \partial_w \mathcal{T}_*(z,w)\, \psi(w)\varphi(z) d\mu(z)d\mu(w) .$$

In conclusion, we have proved the equality

$$k(z,w) = -\bar{\partial}_z \partial_w \mathcal{T}_*(z,w).$$

THEOREM 5.3 Let $T \in \mathcal{L}(H)$ be an irreducible hyponormal operator with $[T^*,T] = \xi \otimes \xi$, and let g be the principal function of T.

The separate completion $\mathcal{H}$ of the space $\mathcal{D}(\mathbb{C})$ with respect to the seminorm

$$\|\varphi\|_g^2 = -\int \exp\left\{-1/\pi \int \frac{g(\zeta)}{(\zeta - z)(\bar{\zeta} - \bar{w})} d\mu(\zeta)\right\} \partial\varphi(w)\, \bar{\partial}\bar{\varphi}(z)\, d\mu(z) d\mu(w)$$

is unitarely equivalent with the space H, via the operator $U: \mathcal{H} \longrightarrow H$,

$$U(\varphi) = \int \partial\varphi(z)(\bar{z} - T^*)^{-1}\xi \quad d\mu(z) \,;$$

The operators T and T^* become in that identification:

(11) $$U^*TU(\varphi)(z) = z\varphi(z) - 1/\pi \int \varphi(\zeta)g(\zeta)/(\bar{\zeta} - \bar{z})\, d\mu(\zeta) \text{ ,and}$$

$$U^*T^*U(\varphi)(z) = \bar{z}\varphi(z).$$

The right side of (11) should be taken as the class of the respective function in $\mathcal{H}$.

Proof. Theorem 4.5 and the preceding computation of the kernel k prove the assertions of the statement, with the exception of the relation (11).

In order to prove (11), we recall from Proposition 7 of [4] the identity

(12) $$T\partial(\bar{z} - T^*)^{-1}\xi = z\partial(\bar{z} - T^*)^{-1}\xi - g(z)(\bar{z} - T^*)^{-1}\xi \;,$$

which holds at the level of distributions.

Let $\chi \in \mathcal{D}(\mathbb{C})$, so that $\chi \equiv 1$ on a neighbourhood of $\sigma(T)$. Because the kernel k is supported by $\sigma(T) \times \sigma(T)$, it is sufficient to prove that

$$U(z\varphi - \chi(z)/\pi \int \varphi(\zeta)g(\zeta)/(\bar{\zeta} - \bar{z})\, d\mu(\zeta)) = TU(\varphi),$$

for every function $\varphi \in \mathcal{D}(\mathbb{C})$. But the explicit form of the operator U gives

$$U(z\varphi - \chi/\pi \int \varphi(\zeta)g(\zeta)/(\bar{\zeta} - \bar{z})\, d\mu(\zeta)) =$$

$$= \int \partial \Big[z\varphi - \chi/\pi \int \varphi(\zeta) g(\zeta)/(\bar\zeta - \bar z)\, d\mu(\zeta) \Big] (\bar z - T^*)^{-1} \xi \, d\mu(z)$$

$$= -(\partial(\bar z - T^*)^{-1}\xi, z\varphi) + ((\bar z - T^*)^{-1}\xi, g\varphi) +$$

$$+ \int \partial \Big[(1-\chi)/\pi \int \varphi(\zeta) g(\zeta)/(\bar\zeta - \bar z)\, d\mu(\zeta) \Big] (\bar z - T^*)^{-1} \xi \, d\mu(z)$$

$$= -(T\partial(\bar z - T^*)^{-1}\xi, \varphi) + I = TU(\varphi) + I,$$

where we have used (12) and we have denoted the last singular integral by I.

Stokes Theorem and the observation that the antianalytic functions $\int \varphi(\zeta) g(\zeta)/(\bar\zeta - \bar z)\, d\mu(\zeta)$ and $(\bar z - T^*)^{-1}\xi$ vanish at infinity, imply $I=0$, and the proof is complete.

Final remarks.

a) The formulae of T and T^* obtained in Theorem 5.3 are dual to those given by Pincus-Xia-Xia [11] on their analytic model.

b) Because the vector ξ has a priviledged place in the Hilbert space H, relative to the operator T, Theorem 4.6 shows that the scalar kernel k is a complete unitary invariant of the operator T.

c) Conversely, a compactly supported, measureable function g on $\mathbb{C}$, with $0 \le g \le 1$, produces by the formulae of Theorem 5.3 an irreducible hyponormal operator with rank one self-commutator and with principal function $g_T = g$.

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