INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

REGULARITY PROPERTIES FOR THE SOLUTIONS OF A CLASS OF VARIATIONAL INEQUALITIES

by

Marius COCU and Anca RADOSLOVESCU

PREPRINT SERIES IN MATHEMATICS

No. 30/1984

BUCURESTI

REGULARITY PROPERTIES FOR THE SOLUTIONS OF A CLASS OF VARIATIONAL INEQUALITIES

by

Marius COCU*) and Anca RADOSLOVESCU**)

April 1985

*) IFTM, Department of Solid Mechanics, str. C.Mille 15, 70701 Bucharest, ROMANIA

**) The National Institute for Scientific and Tehnical Creation Bdul Pacii 220, 79622 Bucharest, ROMANIA

REGULARITY PROPERTIES FOR THE SOLUTIONS OF A CLASS OF VARIATIONAL INEQUALITIES

Marius Cocu

IFTM, Department of Solid Mechanics, Str.C.Mille 15,
70701 Bucharest, Romania

and

Anca Radoslovescu

INCREST, Department of Mathematics, Bd.Păcii 220,
79622 Bucharest, Romania

Key words and phrases: Regularity properties, variational inequalities, Signorini problem.

1. INTRODUCTION

The aim of this paper is to study the regularity of the solutions of a class of variational inequalities of second kind.

The regularity of solutions of variational inequalities for a scalar second order elliptic operator has been studied by many authors. For contributions in this area see [1-4].

In Section 2 of this paper we recall some standard results.

In Section 3 we state the variational inequality for which we obtain local and global regularity results. Our proof is based on the method of translation as Brezis did in his thesis [3] for a scalar second order elliptic operator.

Finally, in Section 4 we apply the results of Section 3 to the Signorini problem with a non local friction law.

2. NOTATIONS AND PRELIMINARY RESULTS

Let Ω be an open set in the Euclidean space R^p. We denote by $H^m(\Omega)$, $m \geq 0$ an integer, the Sobolev space of functions in $L^2(\Omega)$ with derivatives of order less than or equal to m in $L^2(\Omega)$.

For a function $\underset{\sim}{v}$ defined on R^p we denote by $\underset{\sim}{v}_h^i$ the function defined by

$$\underset{\sim}{v}_h^i(\underset{\sim}{x}) = \underset{\sim}{v}(\underset{\sim}{x}+h\underset{\sim}{e}_i) ,$$

where e_i is the unit vector $(\delta_{1i}, \delta_{2i}, \ldots, \delta_{pi})$, δ_{ij} being Kronecker's symbol and h is a real number.

We shall use the following standard results, see e.g. [5] for proofs.

PROPOSITION 2.1. Let Ω be an open set in R^p. If $v \in H^1(\Omega)$ and $\varphi \in C^1(\bar{\Omega})$ then $v\varphi \in H^1(\Omega)$ and

$$\frac{\partial}{\partial x_i}(v\varphi) = \frac{\partial v}{\partial x_i}\varphi + v\frac{\partial \varphi}{\partial x_i}, \quad i=1,\ldots,p.$$

In the following we denote by C, C_i positive constants which we distinquish by subscripts, if necessary.

PROPOSITION 2.2. Let Ω be a domain in R^p with the segment property. If $v \in H^m(\Omega)$, $m \geq 0$ an integer, and if there exists a number $C > 0$ such that

$$\left\| \frac{v_h^i - v}{h} \right\|_{H^m(\Omega')} \leq C, \tag{2.1}$$

for every $\bar{\Omega}' \subset \Omega$ and for all $h \neq 0$ with $|h|$ sufficiently small, then

$$\left\| \frac{\partial v}{\partial x_i} \right\|_{H^m(\Omega)} \leq C.$$

If (2.1) holds for every $i \in \{1, \ldots, p\}$ then $v \in H^{m+1}(\Omega)$.

PROPOSITION 2.3. Let Ω be an open set in R^p. Suppose that $v \in H^m(\Omega)$, $m \geq 1$, and let $\bar{\Omega}' \subset \Omega$. Then

$$\left\| \frac{v_h^i - v}{h} \right\|_{H^{m-1}(\Omega')} \leq \|v\|_{H^m(\Omega)},$$

for all $h \neq 0$ such that $\mathrm{dist}(\bar{\Omega}', \partial\Omega) > |h|$.

From proposition 2.3 we derive the following

COROLLARY 2.1. Let $\eta \in C^\infty(S)$ with $\mathrm{supp}\,\eta \subset S \cup \Sigma$, where $S = \{\xi = (\xi_1, \ldots, \xi_p) \in R^p \; ; |\xi| < 1, \; \xi_p > 0\}$, $p \geq 2$ and $\Sigma = \{\xi \in R^p \; ; |\xi| < 1, \; \xi_p = 0\}$. Then, for every $\underset{\sim}{v} \in [H^1(S)]^p$ we have

$$\left\| \frac{\eta(\underset{\sim}{v}_h^i - \underset{\sim}{v})}{h} \right\|_{[L^2(S)]^p} \le C \|\underset{\sim}{v}\|_{[H^1(S)]^p} ,$$

for all $h \neq 0$ with $|h| < \mathrm{dist}(\partial S \setminus \Sigma, \mathrm{supp}\,\eta)$ and $i=1,2,\dots,p-1$, where supp w denotes the support of w i.e. the closure of the set $\{ \underset{\sim}{x} ; w(\underset{\sim}{x}) \neq 0 \}$.

Proof. Let $S' = \{ \xi \in S ; \eta(\xi) \neq 0 \}$. We set $\tilde{S} = \{ \xi \in R^p ; |\xi| < 1 \}$ and

$$\tilde{S}' = S' \cup (\Sigma \cap \bar{S}') \cup \{ \xi = (\xi_1,\dots,\xi_p) \in R^p ; (\xi_1,\dots,-\xi_p) \in S' \} .$$

For any function $\underset{\sim}{w}$ we define the following function

$$\tilde{\underset{\sim}{w}}(\xi) = \begin{cases} \underset{\sim}{w}(\xi) & \text{if } \xi_p \geq 0, \\ \underset{\sim}{w}(\xi_1,\dots,-\xi_p) & \text{if } \xi_p < 0 . \end{cases} \tag{2.2}$$

It is easily to see that if $\underset{\sim}{w} \in [H^1(S)]^p$ then $\tilde{\underset{\sim}{w}} \in [H^1(\tilde{S})]^p$ and

$$\|\tilde{\underset{\sim}{w}}\|^2_{[H^m(\tilde{S})]^p} = 2 \|\underset{\sim}{w}\|^2_{[H^m(S)]^p} \qquad \text{for } m=0,1. \tag{2.3}$$

Let $\tilde{\eta}$, $\tilde{\underset{\sim}{v}}$ and $\tilde{\underset{\sim}{v}}_h^i$ $(i=1,\dots,p-1)$ be defined as in (2.2). Then $\mathrm{supp}\,\tilde{\eta} = \tilde{\bar{S}}' \subset \tilde{S}$ and

$$\left\| \frac{\eta(\underset{\sim}{v}_h^i - \underset{\sim}{v})}{h} \right\|_{[L^2(S)]^p} = \left\| \frac{\eta(\underset{\sim}{v}_h^i - \underset{\sim}{v})}{h} \right\|_{[L^2(S')]^p} =$$

$$= \frac{1}{\sqrt{2}} \left\| \frac{\tilde{\eta}(\tilde{\underset{\sim}{v}}_h^i - \tilde{\underset{\sim}{v}})}{h} \right\|_{[L^2(\tilde{S}')]^p} \le \frac{1}{\sqrt{2}} C \left\| \frac{\tilde{\underset{\sim}{v}}_h^i - \tilde{\underset{\sim}{v}}}{h} \right\|_{[L^2(\tilde{S}')]^p} . \tag{2.4}$$

If we apply proposition 2.3 , we deduce from (2.3) and (2.4) that

$$\left\| \frac{\eta(\underset{\sim}{v}_h^i - \underset{\sim}{v})}{h} \right\|_{[L^2(S)]^p} \le \frac{1}{\sqrt{2}} C \left\| \frac{\tilde{\underset{\sim}{v}}_h^i - \tilde{\underset{\sim}{v}}}{h} \right\|_{[L^2(\tilde{S}')]^p} \le$$

$$\le \frac{1}{\sqrt{2}} C \|\tilde{\underset{\sim}{v}}\|_{[H^1(\tilde{S})]^p} = C \|\underset{\sim}{v}\|_{[H^1(S)]^p} ,$$

which proves the corollary.

3. VARIATIONAL FORMULATION OF THE PROBLEM AND REGULARITY RESULTS

Let us consider a bounded domain Ω in R^p with Γ an open subset of its boundary $\partial\Omega$. Let $x_o \in \Gamma$. We suppose that Ω is C^3- smooth in x_o i.e. there exists a neighborhood I of x_o such that the set $\bar{\Omega} \cap \bar{I}$ can be mapped C^3-homeomorphically onto $\bar{S}$ where $S = \{\xi \in R^p ; |\xi| < 1, \xi_p > 0\}$, such that the set $\partial\Omega \cap \bar{I}$ is mapped onto the set $\bar{\Sigma}$ where $\Sigma = \{\xi = (\xi_1, \dots, \xi_p) \in R^p ; |\xi| < 1, \xi_p = 0\}$. Without loss of generality we may assume that $\partial\Omega \cap I \subset \Gamma$.

Let θ be the C^3-homeomorphism from $\bar{\Omega} \cap \bar{I}$ to $\bar{S}$. If w is a function defined on $\Omega \cap I$, we shall denote by $\tilde{w}$ the function

$$\tilde{w}(\xi) = w(\theta^{-1}(\xi)) \qquad \forall \xi \in S .$$

Let $v \in [H^1(\Omega)]^p$. For $\eta \in \mathcal{D}(I)$ and h a real number, we set

$$\bar{v}_h^i(x) = \begin{cases} v(x) + \eta(x)[\tilde{v}_h^i(\theta(x)) - v(x)] & \text{if } x \in \operatorname{supp}\eta \cap \Omega \\ v(x) & \text{if } x \in \Omega \setminus \operatorname{supp}\eta \end{cases} ,$$

where $i = 1, \dots, p-1$ and $\tilde{v}_h^i = (\tilde{v})_{h_i}^i$. Note that, for $|h|$ sufficiently small, $\bar{v}_h^i$ is well defined and $\bar{v}_h^i \in [H^1(\Omega)]^p$.

In the sequel we use the summation convention.

For all $u, v \in [H^1(\Omega)]^p$, define the following bilinear form

$$b(u,v) = \int_\Omega \left[a_{ij}^{kl}(x) \frac{\partial u_k}{\partial x_i} \frac{\partial v_l}{\partial x_j} + b_i^{kl}(x) \frac{\partial u_k}{\partial x_i} v_l + c_i^{kl}(x) \frac{\partial v_l}{\partial x_i} u_k + d^{kl}(x) u_k v_l\right] dx,$$

where $a_{ij}^{kl}, b_i^{kl}, c_i^{kl}, d^{kl} \in C^1(\bar{\Omega})$ and $a_{ij}^{kl} = a_{ji}^{kl}$. $\forall\, i,j,k,l = 1, \dots, p$.

In the matrix form we write

$$b(u,v) = \int_\Omega \left[a_{ij}(x) u_{,i} v_{,j} + b_i(x) u_{,i} v + c_i(x) u v_{,i} + d(x) u v\right] dx ,$$

where

$$w_{,i} = \left(\frac{\partial w_1}{\partial x_i}, \dots, \frac{\partial w_p}{\partial x_i}\right), \quad a_{ij} = (a_{ij}^{kl})_{k,l}, \; b_i = (b_i^{kl})_{k,l}, \; c_i = (c_i^{kl})_{k,l}, \; d = (d^{kl})_{k,l}.$$

In the following we denote by $\|\cdot\|_{m,\Omega}$ the norm on the product space $[H^m(\Omega)]^p$.

We suppose that there exists a constant $m > 0$ such that

$$b(\underset{\sim}{u},\underset{\sim}{v}) \geq m \|\underset{\sim}{v}\|^2_{1,\Omega}, \forall \underset{\sim}{v} \in [H^1(\Omega)]^p \text{ with supp } \underset{\sim}{v} \subset \bar{\Omega} \cap I. \quad (3.1)$$

Let $J : [H^1(\Omega)]^p \longrightarrow R$ be the function defined by

$$J(\underset{\sim}{v}) = \int_\Gamma g(x)\,\psi(\underset{\sim}{v}(\underset{\sim}{x}))\,ds \;, \forall \underset{\sim}{v} \in [H^1(\Omega)]^p \;,$$

where $g \in H^1(\Gamma)$ with $g \geq 0$ a.e. on Γ and ψ is a seminorm on R^p.

We denote by Q a non-empty closed convex subset of $[H^1(\Omega)]^p$ such that

$$\left.\begin{array}{l} \text{if } \underset{\sim}{v} \in Q \text{ then } \bar{\underset{\sim}{v}}{}^l_h \in Q, \forall\, l=1,\dots,p-1, \forall \eta \in \mathcal{D}(I) \text{ with } 0 \leq \eta \leq 1, \\ \quad\text{and } \forall\, h \neq 0 \text{ with } |h| < \text{dist}(\partial S \setminus \Sigma, \text{supp}\,\tilde{\eta}), \end{array}\right\} \quad (3.2)$$

where $\tilde{\eta}(\xi) = \eta(\Theta^{-1}(\xi))$, $\forall \xi \in S$.

With the above preliminaries now established, we consider the following variational inequality

$$\left.\begin{array}{l} \underset{\sim}{u} \in Q \\ b(\underset{\sim}{u},\underset{\sim}{v}-\underset{\sim}{u}) + J(\underset{\sim}{v}) - J(\underset{\sim}{u}) \geq (\underset{\sim}{F},\underset{\sim}{v}-\underset{\sim}{u}) \quad , \forall \underset{\sim}{v} \in Q \end{array}\right\} \quad (3.3)$$

where $\underset{\sim}{F} \in [L^2(\Omega)]^p$ and

$$(\underset{\sim}{F},\underset{\sim}{v}-\underset{\sim}{u}) = \int_\Omega F_i(v_i - u_i)\,dx \quad .$$

We can now state our local regularity result.

THEOREM 3.1. Let us suppose that there exists a solution $\underset{\sim}{u}$ of the variational inequality (3.3). Then, for any open set I' containing $\underset{\sim}{x}_o$ such that $\bar{I}' \subset I$, we have $\underset{\sim}{u} \in [H^2(\Omega \cap I')]^p$ and

$$\|\underset{\sim}{u}\|_{2,\Omega\cap I'} \leq C(\|\underset{\sim}{u}\|_{1,\Omega\cap I} + \|g\|_{H^1(\Gamma\cap I)} + \|\underset{\sim}{F}\|_{0,\Omega\cap I}) \;. \quad (3.4)$$

Proof. Let $S' = \Theta(\Omega \cap I')$. We will prove that

$$\|\tilde{\underset{\sim}{u}}\|_{2,S'} \leq C\,(\|\tilde{\underset{\sim}{u}}\|_{1,S} + \|\tilde{g}\|_{H^1(\Sigma)} + \|\tilde{F}\|_{0;S}) \quad .$$

To begin with, we observe that if we choose in (3.3) $\underset{\sim}{v} = \underset{\sim}{u}^l_h$ and, respectively, $\underset{\sim}{v} = \bar{\underset{\sim}{u}}{}^l_h$, $l=1,\dots,p-1$, we obtain by using the local coordinates

$$\tilde{b}(\tilde{u},\tilde{\eta}(\tilde{u}_h^1-\tilde{u}))+\int_{\Sigma}\tilde{g}\tilde{\eta}\psi(\tilde{u}_h^1)\,d\sigma-\int_{\Sigma}\tilde{g}\tilde{\eta}\psi(\tilde{u})\,d\sigma\geq(\tilde{F},\tilde{\eta}(\tilde{u}_h^1-\tilde{u})) \qquad (3.5)$$

$$\tilde{b}(\tilde{u},\tilde{\eta}(\tilde{u}_{-h}^1-\tilde{u}))+\int_{\Sigma}\tilde{g}\tilde{\eta}\psi(\tilde{u}_{-h}^1)\,d\sigma-\int_{\Sigma}\tilde{g}\tilde{\eta}\psi(\tilde{u})\,d\sigma\geq(\tilde{F},\tilde{\eta}(\tilde{u}_{-h}^1-\tilde{u})) \qquad (3.6)$$

where

$$\tilde{b}(\tilde{u},\tilde{v}) = \int_S(a_{ij}\,\tilde{u}_{,i}\tilde{v}_{,j}+b_i\tilde{u}_{,i}\tilde{v}+c_i\tilde{u}\tilde{v}_{,i}+d\tilde{u}\tilde{v})\,d\xi ,$$

$$(\tilde{F},\tilde{v}) = \int_S F_i\tilde{v}_i\,d\xi ,$$

and, for the sake of simplicity, we do not change the notations of a_{ij}, b_i, c_i, d, F_i.

Let $\varphi\in\mathcal{D}(I)$ be such that $0\leq\varphi\leq 1$ and $\varphi\equiv 1$ on I'. Taking $\tilde{\eta}=\tilde{\varphi}^2$ in (3.5) and $\tilde{\eta}=\tilde{\varphi}^2_{-h}$ in (3.6), by adding the two inequalities we get, for $|h|$ sufficiently small

$$0\leq b(u,\varphi^2(u_h-u))-b(u,\varphi^2_{-h}(u-u_{-h}))+\int_{\Sigma}g\varphi^2[\psi(u_h)-\psi(u)]\,d\sigma+$$

$$+\int_{\Sigma}g\varphi^2_{-h}[\psi(u_{-h})-\psi(u)]\,d\sigma+(F,\varphi^2_{-h}(u-u_{-h}))-(F,\varphi^2(u_h-u)), \qquad (3.7)$$

where, for simplicity, we omitted the "~" sign and the index 1. Therefore, we deduce from (3.7)

$$b(\varphi(u_h-u),\varphi(u_h-u))\leq b(\varphi(u_h-u),\varphi(u_h-u))-b(u,\varphi^2_{-h}(u-u_{-h}))+$$

$$+b(u,\varphi^2(u_h-u))+\int_{\Sigma}g\varphi^2[\psi(u_h)-\psi(u)]\,d\sigma+\int_{\Sigma}g\varphi^2_{-h}[\psi(u_{-h})-\psi(u))]\,d\sigma+$$

$$+(F,\varphi^2_{-h}(u-u_{-h}))-(F,\varphi^2(u_h-u)) . \qquad (3.8)$$

We now estimate the right-hand side of (3.8). First, we have

$$b(\varphi(u_h-u),\varphi(u_h-u))-b(u,\varphi^2_{-h}(u-u_{-h}))+b(u,\varphi^2(u_h-u))=$$

$$=\int_S\Big\{a_{ij}[\varphi(u_h-u)]_{,i}[\varphi(u_h-u)]_{,j}+b_i[\varphi(u_h-u)]_{,i}[\varphi(u_h-u)]+$$

$$+c_i[\varphi(u_h-u)]\cdot[\varphi(u_h-u)]_{,i}+d\varphi^2(u_h-u)(u_h-u)-a_{ij}u_{,i}[\varphi^2_{-h}(u-u_{-h})]_{,j}-$$

$$-b_iu_{,i}[\varphi^2_{-h}(u-u_{-h})]-c_iu[\varphi^2_{-h}(u-u_{-h})]_{,i}-d\cdot u\varphi^2_{-h}(u-u_{-h})+$$

$$+a_{ij}u_{,i}[\varphi^2(u_h-u)]_{,j}+b_i u_{,i}[\varphi^2(u_h-u)]+c_i u[\varphi^2(u_h-u)]_{,i}+$$

$$+d\,u\,\varphi^2(u_h-u)\Big\}\,d\xi=\int_S\Big\{a_{ij}\varphi_{,i}\varphi_{,j}(u_h-u)(u_h-u)+$$

$$+a_{ij}\varphi_{,i}\varphi(u_h-u)(u_h-u)_{,j}+a_{ij}\varphi\varphi_{,j}(u_h-u)_{,i}(u_h-u)+$$

$$+a_{ij}\varphi^2(u_h-u)_{,i}(u_h-u)_{,j}+b_i[\varphi(u_h-u)]_{,i}\varphi(u_h-u)+$$

$$+c_i\varphi(u_h-u)[\varphi(u_h-u)]_{,i}-[(a_{ij})_h u_{h,i}-a_{ij}u_{,i}]\cdot[\varphi^2(u_h-u)]_{,j}-$$

$$-[(b_i)_h u_{h,i}-b_i u_{,i}]\varphi^2(u_h-u)-[(c_i)_h u_h-c_i u]\cdot[\varphi^2(u_h-u)]_{,i}-$$

$$-(d_h-d)u_h\varphi^2(u_h-u)\Big\}\,d\xi=\int_S\Big\{a_{ij}\varphi_{,i}\varphi_{,j}(u_h-u)(u_h-u)-$$

$$-[(a_{ij})_h-a_{ij}]u_{h,i}[\varphi^2(u_h-u)]_{,j}+b_i[\varphi(u_h-u)]_{,i}\varphi(u_h-u)+$$

$$+c_i\varphi(u_h-u)[\varphi(u_h-u)]_{,i}-[(b_i)_h u_{h,i}-b_i u_{,i}]\varphi^2(u_h-u)-$$

$$-[(c_i)_h u_h-c_i u]\cdot[\varphi_{,i}\varphi(u_h-u)+\varphi(\varphi(u_h-u))_{,i}]-\text{[illegible]}$$

$$-(d_h-d)u_h\varphi^2(u_h-u)\Big\}\,d\xi\,. \tag{3.9}$$

In the above relations we used proposition 2.1. Now we can apply corollary 2.1 by taking $\eta=\varphi$ and $\eta=\varphi_{,i}$. Therefore, for all $h\neq 0$ such that $|h|<\operatorname{dist}(\partial S\setminus\Sigma,\operatorname{supp}\eta)$, we obtain

$$\frac{1}{|h|}\Big[b(\varphi(u_u-u),\varphi(u_h-u))-b(u,\varphi^2_{-h}(u-u_{-h}))+b(u,\varphi^2(u_h-u))\Big]\leq$$

$$\leq c_1\|u\|_{1,S}\|\varphi(u_h-u)\|_{1,S}\,. \tag{3.10}$$

On the other hand, by proposition 2.3 , we have

$$\int_\Sigma g\varphi^2[\psi(u_h)-\psi(u)]\,d\sigma+\int_\Sigma g\varphi^2_{-h}[\psi(u_{-h})-\psi(u)]\,d\sigma=$$

$$=\int_\Sigma(g-g_h)\varphi^2[\psi(u_h)-\psi(u)]\,d\sigma\leq C_2|h|\,\|g\|_{H^1(\Sigma)}\|\varphi(u_h-u)\|_{1,S} \tag{3.11}$$

and

$$(F,\varphi^2_{-h}(u-u_{-h}))-(F,\varphi^2(u_h-u))=-(F,\varphi^2(u_h-u)+\varphi^2_{-h}(u_{-h}-u))\leq$$

$$\leq C_3 |h| \|F\|_{0,S} \|\varphi(u_h - u)\|_{1,S} \quad . \tag{3.12}$$

Combining (3.10)-(3.12) and using (3.1), the inequality (3.8) implies

$$\left\| \frac{\varphi(u_h - u)}{h} \right\|_{1,S} \leq C_4 (\|u\|_{1,S} + \|g\|_{H^1(\Sigma)} + \|F\|_{0,S}) ,$$

from which

$$\left\| \frac{u_h - u}{h} \right\|_{1,S'} \leq C_4 (\|u\|_{1,S} + \|g\|_{H^1(\Sigma)} + \|F\|_{0,S}) . \tag{3.13}$$

We conclude therefore from proposition 2.2 that

$$\frac{\partial^2 u}{\partial \xi_i \partial \xi_j} \in \left[L^2(S')\right]^p \text{ for } i=1,\ldots,p-1 \text{ and } j=1,\ldots,p.$$

Let us remark that u solves the system

$$-\frac{\partial}{\partial \xi_j} (a_{ij} \frac{\partial u}{\partial \xi_i}) + b_i \frac{\partial u}{\partial \xi_i} - \frac{\partial}{\partial \xi_i}(c_i u) + d \cdot u = F \quad \text{in } S'. \tag{3.14}$$

From condition (3.1) it follows that $\det(a_{pp}(\xi)) \neq 0, \forall \xi \in S'$, which implies that $\partial^2 u/\partial \xi_p^2$ can be calculated from (3.14). Thus $\partial^2 u/\partial \xi_p^2 \in \left[L^2(S')\right]^p$ and, by (3.13),

$$\|u\|_{2,S'} \leq C(\|u\|_{1,S} + \|g\|_{H^1(\Sigma)} + \|F\|_{0,S}) .$$

Then, by transformation back to the $x_1,\ldots,x_p$ coordinates, we obtain the estimate (3.4), q.e.d.

Remark 3.1. Using a similar technique as in theorem 3.1 we can obtain the (known) regularity of the solution of the variational inequality (3.3) in a neighborhood of an interior point of Ω. Indeed, for any point $x \in \Omega$, by taking $I = \{y \in \Omega; |y-x| < R\}$ with R sufficiently small such that $\bar{I} \subset \Omega$, one can repeat the proof of theorem 3.1 with no need of using local coordinates, by requiring that

$$b(v,v) \geq m\|v\|^2_{1,\Omega} , \forall v \in \left[H^1(\Omega)\right]^p \text{ with supp } v \subset I,$$

and

$$\left.\begin{array}{l}\text{if } \underset{\sim}{v} \in Q \text{ then } (1-\eta)\underset{\sim}{v}+\eta \underset{\sim}{v}^1_{h} \in Q, \forall\, l=1,\ldots,p, \forall \eta \in \mathcal{D}(I) \\ \text{with } 0 \leq \eta \leq 1, \forall\, h \neq 0 \text{ with } |h| < \text{dist}(\partial I, \text{supp}\, \eta)\end{array}\right\} \quad (3.15)$$

If Ω is C^3-smooth in $\underset{\sim}{x} \in \partial\Omega$, we shall denote by $I_{\underset{\sim}{x}}$ the corresponding neighborhood of $\underset{\sim}{x}$ and, if $\underset{\sim}{x} \in \Omega$ we shall denote by $I_{\underset{\sim}{x}}$ the following set $\{\underset{\sim}{y} \in R^p; |\underset{\sim}{y}-\underset{\sim}{x}| < R\}$ with R sufficiently small such that $I_{\underset{\sim}{x}} \subset \Omega$.

From the local regularity result discussed in this section we can easily get a global regularity theorem for the inequality (3.3).

THEOREM 3.2. Suppose that Ω is C^3-smooth in any point $\underset{\sim}{x}$ of $\partial\Omega$ and $\Gamma = \partial\Omega$. Let (3.2) and (3.15) hold for every $\underset{\sim}{x} \in \partial\Omega$ and for every $\underset{\sim}{x} \in \Omega$, respectively. Assume that there exists a solution $\underset{\sim}{u}$ of the variational inequality(3.3). If the bilinear form b satisfies

$$b(\underset{\sim}{v},\underset{\sim}{v}) \geq m\|\underset{\sim}{v}\|^2_{1,\Omega}, \forall\, \underset{\sim}{v} \in [H^1(\Omega)]^p \text{ with supp } \underset{\sim}{v} \subset \bar{\Omega} \cap I_{\underset{\sim}{x}}, \forall\, \underset{\sim}{x} \in \bar{\Omega},$$

then $\underset{\sim}{u} \in [H^2(\Omega)]^p$ and

$$\|\underset{\sim}{u}\|_{2,\Omega} \leq C(\|\underset{\sim}{u}\|_{1,\Omega} + \|g\|_{H^1(\Gamma)} + \|\underset{\sim}{F}\|_{0,\Omega}) \quad . \quad (3.16)$$

Proof. For every $\underset{\sim}{x} \in \bar{\Omega}$, let $I'_{\underset{\sim}{x}}$ be an open set containing $\underset{\sim}{x}$ such that $\bar{I}'_{\underset{\sim}{x}} \subset I_{\underset{\sim}{x}}$. Since $\bar{\Omega}$ is compact, we can extract from $\{I'_{\underset{\sim}{x}}\}_{\underset{\sim}{x} \in \bar{\Omega}}$ a finite open covering $\{I_{\underset{\sim}{x}_i}\}_{i=1,\ldots,n}$. By theorem 3.1. and the above remark we have

$$\underset{\sim}{u} \in [H^2(\Omega \cap I'_{\underset{\sim}{x}})]^p$$

and

$$\|\underset{\sim}{u}\|_{2,\Omega\cap I'_{\underset{\sim}{x}_i}} \leq C_i(\|\underset{\sim}{u}\|_{1,\Omega\cap I_{\underset{\sim}{x}_i}} + \|g\|_{H^1(\Gamma\cap I_{\underset{\sim}{x}_i})} + \|\underset{\sim}{F}\|_{0,\Omega\cap I_{\underset{\sim}{x}_i}}) \quad (3.17)$$

for all $i \in \{1,\ldots,n\}$. It follows that $\underset{\sim}{u} \in [H^2(\Omega)]^p$ and summing (3.17) over all i we obtain (3.16).

Note that theorems 3.1 and 3.2 still hold with the same proofs, under a less restrictive assumption about Ω.

Remark 3.2. In more restrictive assumptions on b and Q and for

4. AN APPLICATION TO SIGNORINI PROBLEMS

Let $\Omega \subset R^p$ ($p=2$ or 3) be a bounded domain, the boundary of which is decomposed as $\partial\Omega = \bar\Gamma_o \cup \bar\Gamma_1 \cup \bar\Gamma_2$, where Γ_l, $l=o,1,2$, are disjoint and open in $\partial\Omega$ and $\Gamma_2 \neq \emptyset$.

Let us introduce

$$V = [H^1(\Omega)]^p$$

and

$$K = \{ \underset{\sim}{v} \in V;\ \underset{\sim}{v}=0 \text{ on } \Gamma_o,\ v_n \leq 0 \text{ on } \Gamma_2 \}.$$

K is a non-empty closed convex subset of V.

Let consider the following variational formulation of the Signorini problem with non-local friction (see [7], [8]): find $\underset{\sim}{u} \in K$ such that

$$a(\underset{\sim}{u},\underset{\sim}{v}-\underset{\sim}{u})+j(\underset{\sim}{u},\underset{\sim}{v})-j(\underset{\sim}{u},\underset{\sim}{u}) \geq (\underset{\sim}{L},\underset{\sim}{v}-\underset{\sim}{u}) \ , \ \forall \underset{\sim}{v} \in K \tag{4.1}$$

where

$$a(\underset{\sim}{u},\underset{\sim}{v}) = \int_\Omega a_{ijkl}\ \varepsilon_{ij}(\underset{\sim}{u})\ \varepsilon_{kl}(\underset{\sim}{v})\ d\underset{\sim}{x} \ ,$$

with $a_{ijkl} \in C^1(\bar\Omega)$, $a_{ijkl}=a_{jikl}=a_{klij}$,

$$\varepsilon_{ij}(\underset{\sim}{w}) = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right),$$

$$j(\underset{\sim}{u},\underset{\sim}{v}) = -\int_{\Gamma_2} \mu\, \sigma_n^*(\underset{\sim}{u})\ |\underset{\sim}{v}_t|\ ds \ ,$$

with $\mu \in C^1(\Gamma_2)$, $0 \leq \mu \leq 1$ a.e. on Γ_2,

and

$$(\underset{\sim}{L},\underset{\sim}{v}) = \int_\Omega f_i v_i\ d\underset{\sim}{x} + \int_{\Gamma_1} t_i v_i\ ds \ ,$$

with $f=(f_i) \in [L^2(\Omega)]^p$, $t=(t_i) \in [L^2(\Gamma_1)]^p$.

Here v_n, σ_n denote the normal components of the displacement and of the stress vector, respectively, $v_{ti}=v_i - v_n n_i$ where $\underset{\sim}{n}=(n_i)$ is the outward normal unit vector to the boundary of Ω and $\sigma_n^*(\underset{\sim}{u})$ represents a regularized stress such that $\sigma_n^*(\underset{\sim}{u}) \in C^1(\Gamma_2)$ and $\sigma_n^*(\underset{\sim}{u}) \geq 0$.

The existence of the solutions of (4.1) is proved in [7], [8] for $\mathrm{mes}(\Gamma_o) > 0$ and in [9] for $\mathrm{mes}(\Gamma_o) = 0$ and under additional assumptions on K and L.

Let us suppose that there exists a solution $\underset{\sim}{u}$ of (4.1).

THEOREM 4.1. If Ω is C^3-smooth in all $\underset{\sim}{x} \in \Gamma_2$, then for every open set U such that $\bar{U} \subset \Omega \cup \Gamma_2$ we have

$$\underset{\sim}{u} \in [H^2(U)]^p.$$

Proof. Let $\underset{\sim}{x} \in \Gamma_2$ and I be a corresponding neighborhood of $\underset{\sim}{x}$ from the definition of the C^3-smoothness of Ω in $\underset{\sim}{x}$. We may assume that $\partial\Omega \cap I \subset \Gamma_2$.

Observe that if $\underset{\sim}{u}$ is a solution of inequality (4.1) then $\underset{\sim}{u}$ satisfies the following variational inequality

$$\int_{\Omega\cap I} a_{ijkl}\, \varepsilon_{ij}(u)\, \varepsilon_{kl}(\underset{\sim}{v})\, dx + \int_{\Gamma_2\cap I} g|\underset{\sim}{v}_t|\, ds - \int_{\Gamma_2\cap I} g|\underset{\sim}{u}_t|\, ds \geqslant$$

$$\geqslant \int_{\Omega\cap I} f_i(v_i - u_i)\, d\underset{\sim}{x}\,, \qquad \forall\, \underset{\sim}{v} \in K_{\underset{\sim}{u}} \tag{4.2}$$

where $g(x) = -\mu(\underset{\sim}{x})\sigma_n^*(u(\underset{\sim}{x}))$ and

$$K_{\underset{\sim}{u}} = \{\underset{\sim}{w} \in [H^1(\Omega\cap I)]^p;\, \underset{\sim}{w} = \underset{\sim}{u} \text{ on } \Omega\cap\partial I,\ w_n \leq 0 \text{ on } \Gamma_2\cap I\}.$$

Indeed, for every $\underset{\sim}{w} \in K_{\underset{\sim}{u}}$ we have $\underset{\sim}{w}' \in K$ where

$$\underset{\sim}{w}' = \begin{cases} \underset{\sim}{w} & \text{on } \Omega\cap I\,, \\ \underset{\sim}{u} & \text{on } \Omega\setminus I\,. \end{cases}$$

By taking $\underset{\sim}{v} = \underset{\sim}{w}'$ in (4.1) we obtain (4.2).

In order to apply theorem 3.1 we shall use an argument due to Fichera [6]. The C^3-smoothness of Ω in $\underset{\sim}{x}$ implies that in every $y \in \Omega\cap I$ there exists an orthogonal system of unit-vectors $\underset{\sim}{w}^1(y), \ldots, \underset{\sim}{w}^p(y)$ such that $\underset{\sim}{w}^i \in [C^2(\bar{\Omega}\cap\bar{I})]^p$, $i=1,\ldots,p$ and $\underset{\sim}{w}^p(y) = \underset{\sim}{n}(y)$ for $y \in \Gamma_2\cap\bar{I}$. Hence, for each $\underset{\sim}{v} \in [H^1(\Omega\cap I)]^p$ we have

$$\underset{\sim}{v}(y) = \bar{v}_i(y)\, \underset{\sim}{w}^i(y) \quad , \ \forall\, y \in \Omega\cap I.$$

If we denote $\underset{\sim}{\bar{v}} = (\bar{v}_1, \ldots, \bar{v}_p)$ and by Q we denote the following closed and convex subset of $[H^1(\Omega\cap I)]^p$

$$Q=\{\bar{v}\in[H^1(\Omega\cap I)]^p;\ \bar{v}=\bar{u} \text{ on } \Omega\cap\partial I \text{ and } \bar{v}_p\leq 0 \text{ on } \Gamma_2\cap I\},$$

then $v\in K_u$ iff $\bar{v}\in Q$.

Let us define the following forms

$$b(\bar{v},\bar{v}')=\int_{\Omega\cap I}\Big[a^{kl}_{ij}(y)\frac{\partial\bar{v}_k}{\partial y_i}\frac{\partial\bar{v}'_l}{y_j}+b^{kl}_i(y)\frac{\partial\bar{v}_k}{\partial y_i}v'_l+$$

$$+c^{kl}(y)\frac{\partial\bar{v}'_l}{\partial y_i}\bar{v}_k+d^{kl}(y)\,\bar{v}_k\bar{v}'_l\Big]dy\quad,\forall\,\bar{v},\bar{v}'\in Q$$

$$J(\bar{v})=\int_{\Gamma_2\cap I}g(y)\,\psi(\bar{v})\,ds\quad,\ \forall\,\bar{v}\in Q$$

$$(F,\bar{v})=\int_{\Omega\cap I}F_j\bar{v}_j\,dy\quad,\ \forall\,\bar{v}\in Q\ ,$$

where $a^{kl}_{ij}=a_{ijqr}w^k_q w^l_r$, $b^{kl}_i=a_{ijqr}w^k_q(w^l_r)_{,j}$, $c^{kl}_i=a_{ijqr}(w^k_q)_{,j}w^l_r$, $d^{kl}=a_{ijqr}(w^k_q)_{,i}(w^l_r)_{,j}$, $\psi(\bar{v})=|(\bar{v}_1,\dots,\bar{v}_{p-1},0)|$ and $F_j=f_iw^j_i$.

We observe that

$$b(\bar{v},\bar{v}')=\int_{\Omega\cap I}a_{ijkl}\,\varepsilon_{ij}(v)\,\varepsilon_{kl}(v')\,dx\quad,$$

$J(\bar{v})=j(v)$ and $(F,\bar{v})=(f,v)$. Thus, the inequality (4.2) becomes

$$b(\bar{u},\bar{v}-\bar{u})+J(\bar{v})-J(\bar{u})\geq (F,\bar{v}-\bar{u})\ ,\ \forall\,\bar{v}\in Q\ .$$

One can easily verify that J, F and Q satisfy the hypothesis of theorem 3.1 with Ω replaced by $\Omega\cap I$ and that, by Korn's inequality, condition (3.1) holds for any $\bar{v}\in[H^1(\Omega\cap I)]^p$ with supp $\bar{v}\subset\bar{\Omega}\cap I$. Therefore $u\in[H^2(\Omega\cap I'_x)]^p$, $\forall\, I'_x\ni x$ with $\bar{I}'_x\subset I$.

Arguing as in the proof of theorem 3.2 , we conclude that $u\in[H^2(U)]^p$.

Acknowledgement - The authors would like to express sincere thanks to Dr. Eugen Soós for his encouragement and consistent support. Furthermore we are indebted to Dr. Mihaela Suliciu, Dr. Horia Ene and Prof.dr. Viorel Barbu for helpful discussions.

REFERENCES

1. LIONS J.L., Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod and Gauthier-Villars, Paris (1969).
2. BRÉZIS H. & STAMPACCHIA G., Sur la régularité de la solution d'inéquations elliptiques, Bull.Soc.math.Fr.96,153 (1968).
3. BRÉZIS H., Problèmes unilatéraux, J.de Math. Pures et Appliquées 51,1 (1972).
4. NEČAS J., On regularity of solutions to nonlinear variational inequalities for second-order elliptic system, Rend.di Matematica Serie VI,8,481 (1975).
5. AGMON S., Lectures on Elliptic Boundary Value Problems, Van Nostrand, New York-Toronto-London-Melbourne (1965).
6. FICHERA G., Boundary Value Problems of Elasticity with Unilateral Constraints, Encyclopedia of Physics (Edited by S.Flügge), Vol.VI a/2,pp.391-424, Springer-Verlag,Berlin (1972).
7. DUVAUT G., Equilibre d'un solide élastique avec contact unilateral et frottement de Coulomb, C.R.Acad.Sc.Paris Série A,290,263 (1980).
8. ODEN J.T. & PIRES E., Contact problems in elastostatics with non-local friction laws, TICOM Report 81-12, Austin (1981).
9. COCU M., Existence of solutions of Signorini problems with friction, Int.J.Engng.Sci.,22,567, (1984).