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ON THE REGULARITY OF THE PROBABILITIES ASSOCIATED WITH DIFFUSIONS

by

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On the regularity of the probabilities associated with diffusions

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Introduction

This paper concerns the absolute continuity property of the transition probabilities associated with stochastic integral equations.

Generally the existence of the smooth density for transition probabilities $P(t,y,.)$, $t>0$, $y\in R^n$, associated with diffusion processes can be obtained using the fundamental Hörmander's theorem on hypoelliptic operators (see [1]). In a remarkable paper ([2]) Malliavin has developped probabilistic techniques to prove the existence of the smooth density for the distributions associated with stochastic equations. Since Malliavin's article appeared there have been made various attempts to put Malliavin's calculus on firmer mathematical footing and to include the drift part in the probabilistic analysis; the theory and examples contained in [2]-[10] show that probabilistic techniques provide a method when Hörmander's theorem cannot be applied. The common features of all these papers are contained in [2] and the reasoning proceeds in the following way. The probability measure π on $C([0,\infty);R^n)$ generated by the solution of the stochastic equation is the image of the Wiener measure P on $C([0,\infty);R^m)$ by the mapping $\omega\to x(.,\omega)$. To prove that the distribution $P(t,y,.)$ of $x(t,.)$ has a smooth density it suffices to show that the differentials of $P(t,y,.)$ in the sense of distributions are bounded measures and to accomplish this goal an integration by parts formula is necessary.

The Malliavin calculus proves this formula on the infinite dimensional space $\Omega = C([0,\infty);R^m)$ endowed with the Wiener measure P and using a generalized Ornstein-Uhlenbeck operator which is an unbounded selfadjoint operator on $L_2(\Omega,P)$. A functional analytic approach of this subject is given by Stroock in [6].

A simpler approach is proposed by Bismut in [3] where the infinite dimensional aspect of the Malliavin calculus is replaced by Girsanov's theorem and the differential analysis becomes a finite dimensional one. Actually it is based on the embedding the original solution into a family of solutions $x^u(.)$, $u\in R^n$, which generate the same probability measure and the integration by parts formula is a direct consequence of the probability invariance.

This formula depends essentially on the matrix $[\partial x^u(T)/\partial u]_{u=0} = M(T)$

which is equivalent to the Malliavin's covariance matrix for diffusions and the nonsingularity of M(T) insures the existence of the density for the marginal distributions; the smoothness property of the density is a consequence of the property $(\det M(T))^{-1} \in \bigcap_{q \geq 1} L_q(\Omega, P)$.

In the Markovian case the above property has a nice description using Lie algebras of the vector fields defining stochastic equation and it was Stroock who proved the most complete form of this relations (see [7], [8]) which is the same with Hörmander's condition for second order parabolic differential operators.

The Malliavin calculus has reached a high level of generality both regarding the hypotheses in Markovian case and including a large class of problems in stochastic analysis (see [6]-[9]).

This paper is not intended to be more general than the above cited beautiful works. On the contrary it gives a more direct approach to the Malliavin's covariance matrix when dealing with stochastic integral equations and the main part which appeared in [11] was inspired by [5] and [14].

It is our conviction that it can make the subject more accessible.

It has the same spirit as in [3] but the Girsanov theorem is used in a different way. The variations of the given Wiener process are depending on a finite dimensional parameter $u \in R^n$ and a matrix of control functions U. As a result the matrix $M(T) = [\partial x^u(T)/\partial u]_{u=0}$ will depend on U and the property $(\det M(T))^{-1} \in \bigcap_{q \geq 1} L_q(\Omega, P)$ is obtained easier in some cases by choosing a suitable matrix U. There is a general way of taking U which corresponds to the Malliavin covariance matrix as it appears in [3] and [5] (see definition 18) and it works with minor changes not only for diffusions but also for conditioned diffusions and stochastic equations with delay.

The paper is divided into two parts. The first part contains the explanation of the embedding procedure and formulation of the main results stated in theorems 1 and 2.

In the second part we include all the auxiliary results and proofs of theorems. The result in theorem 1 uses only local conditions on the diffusion coefficients and it firstly appeared in [7]. Theorem 2 can be used for obtaining the regularity of the transition probabilities and estimates on the density when the diffusion matrix is degenerate (see [12]) and the drift part is a nonanticipating functional as in the following system

$$(*)\begin{cases} dx_1(t) = f^1(x_1(t))dt + \sum_{i=1}^{m} g_i^1(x_1(t))\,dw_i(t),\ x_1 \in R^m,\ x_1(0) = x_{10},\ t \geq 0,\ x_{10} \in R^m, \\ dx_2(t) = f^2(x_1(t-h), x_1(t))dt,\ x_2 \in R^m,\ x_2(s) = x_{20}(s),\ s \in [-h,0],\ t \geq 0, \\ \dots \\ dx_p(t) = f^p(x_{p-1}(t-h), x_{p-1}(t))dt,\ x_p \in R^m,\ x_p(s) = x_{p0}(s),\ s \in [-h,0],\ t \geq 0 \end{cases}$$

where $g_i^1, f^1 : R^m \to R^m$, $f^j : R^{2m} \to R^m$ are C^∞ and with partial derivatives $\partial g_i^1/\partial x_k(\cdot) \in C_b^\infty(R^m)$, $\partial f^1/\partial x_k(\cdot) \in C_b^\infty(R^m)$, $\partial f^j/\partial y_k(\cdot) \in C_b^\infty(R^{2m})$, $j = 2, \dots, p$, $y = (\tilde{x}, x) \in R^{2m}$.

It will be considered in more detail in the last part of the paper.

2. FORMULATION OF THE PROBLEM AND MAIN RESULTS

We consider the following Ito equation

$$dx(t) = f(x(t))dt + \sum_{j=1}^{m} g_j(x(t))dw_j(t),\ x(0) = x_0,\ t \geq 0,\ x \in R^n, \tag{1}$$

where $w(\cdot)$ is a standard m-dimensional Wiener process over the probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P\}$, and f, g_j are $C^\infty(R^n)$ fulfiling

$$\partial f/\partial x_k,\ \partial g_j/\partial x_k \in C_b^\infty(R^n),\ k = 1, \dots, n\ ; \tag{2}$$

and $C_b^\infty(R^n)$ consists of all $C^\infty(R^n)$ functions which are bounded along with all their partial derivaties. The regularity of the probability measures $P(t, x_0, \cdot)$, $t>0$, on R^n, generated by the Ito solution $x^0(\cdot)$ in (1), is obtained by using an embedding procedure of $x^0(\cdot)$, into a family of diffusion processes $x^u(\cdot)$, $u \in R^n$. The main tool of this procedure is the Girsanov theorem of changing the original Wiener process $w(\cdot)$ into a new one $w^u(\cdot)$, $u \in R^n$.

The final goal is to obtain estimates of the form

$$\left| E D_x^\alpha \varphi(x^0(T)) \right| \leq C_{\alpha, T} \sup_{x \in R^n} |\varphi(x)| \qquad (\forall)\ \varphi \in C_b^\infty(R^n) \tag{3}$$

where the constant $C_{\alpha,T}$ doesn't depend on φ, which ensure~~s~~ the existence of the density for probability measure, $P(T, x_o, dx) = p(T, x_o, x)dx$, with $P(T, x_o, \cdot) \in C_b^{\infty}(R^n)$ and fulfiling

$$(4) \qquad \|p(T, x_o, .)\|_{C_b^k(R^n)} \leq C_k \cdot T^{-r_k}, \text{ for some constants } C_k > 0 \text{ and } r_k \geq 0.$$

For getting (3) and (4) we need that $x^u(\cdot)$ is differentiable with respect to $u \in R^n$.

Every where in this paper the differentiability with respect to u is taken in the mean square in the space $L_2(\Omega, P)$ and it will not be specified later. Denote $D_u^{\alpha} = \dfrac{\partial^{|\alpha|}}{\partial u_1^{\alpha_1} \dots \partial u_n^{\alpha_n}}$, $|\alpha| = \sum_{i=1}^{n} \alpha_i$, for $\alpha_i \geq 0$, α_i integer.

In the following we shall describe more precisely the embedding procedure. We associate to (1) the following auxiliary system

$$(5) \qquad dx = \Big[f(x) + \sum_{i=1}^{m} \sum_{j=1}^{n} u_j U_j^i(z) g_i(x)\Big]dt + \sum_{i=1}^{m} g_i(x) dw_i(t), \quad x(0) = x_o ,$$

$$dX = \Big[A(x) + \sum_{i=1}^{m} \sum_{j=1}^{n} u_j U_j^i(z) B_i(x)\Big] X dt + \sum_{i=1}^{m} B_i(x) X dw_i(t), \quad X(0) = I$$

$$dY = -Y\Big[\tilde{A}(x) + \sum_{i=1}^{m} \sum_{j=1}^{n} u_j U_j^i(z) B_i(x)\Big]dt - \sum_{i=1}^{m} Y B_i(x) dw_i(t), \quad Y(0) = I$$

$$\frac{dN}{dt} = YG(x)U(z) \qquad\qquad N(0) = \Theta$$

where X, Y, N are $(n \times n)$ matrices, $z = (x, X, Y)$, $A(x) = \dfrac{\partial f}{\partial x}(x)$, $\tilde{A}(x) = A(x) - \sum_{i=1}^{m} B_i^2(x)$

$B_i(x) = \dfrac{\partial g_i}{\partial x}(x)$, $U = (U_j^i)_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$, $G = (g_1, \dots, g_m)$, Θ is the null matrix, I the identity matrix, and $U_1^i : R^{n+2n^2} \longrightarrow R$ are fixed in $C_b^{\infty}(R^{n+2n^2})$ such that $(YG(x)U(z))_{ij} \in C_b^{\infty}(R^{n+2n^2})$. The definition of (5) is justified later after getting (15).

The equation (5) fulfils the usual conditions of differentiability with respect to the parameter $u \in R^n$ and denoting by $y^u(t)$, $t \geq 0$, $y=(x,X,Y,N)$, the Ito solution in (5) we have

$$(6) \qquad E \max_{t\in[0,T]} |y^u(t)|^q < \infty \;, \; E \max_{t\in[0,T]} |D_u^{\alpha} y^u(t)|^q < \infty$$

for each $T>0$, $\alpha=(\alpha_1,\ldots,\alpha_n)$ and $q \geq 1$.

Applying Ito's differential rule to $X^u(.)$, $Y^u(.)$ in (5) it follows that $X^u(t)$ is nonsingular and

$$(7) \qquad (X^u(t))^{-1}=Y^u(t) \quad , \quad \text{for each } t\geq 0,\; u\in R^n$$

Define $w^u(.)$, $k^u(.)$ by

$$(8) \qquad w_i^u(t)=w_i(t)+\sum_{j=1}^{n} \int_o^t u_j U_j^i(z^u(s))ds, \quad i=1,\ldots,m, \; t\geq 0,$$

$$(9) \qquad k^u(t)=\exp-\left\{\sum_{i=1}^{m}\left[\int_o^t \sum_{j=1}^{n} u_j U_j^i(z^u(s))dw_i(s)+\frac{1}{2}\int_o^t \left(\sum_{j=1}^{n} u_j U_j^i(z^u(s))\right)^2 ds\right]\right\}, \; t\geq 0$$

where $z^u(.)$ is defined by $z^u(.)=(x^u(.),X^u(.),Y^u(.))$.

Let $T>0$ be fixed. Using Girsanov's theorem (see [2],[5]) it follows that $w^u(t)$, $t\in[0,T]$, is a standard m-dimensional Wiener process with respect to the new probability

$$(10) \qquad P^u=k^u(T)P$$

and $y^u(t)$, $t\in[0,T]$, the Ito solution in (5) fulfils the following Ito equation

$$(11) \qquad dx=f(x)dt+\sum_{i=1}^{m} g_i(x)dw_i^u(t) \;, \; x(0)=x_o \;,$$

$$dX=A(x)Xdt+\sum_{i=1}^{m}B_i(x)Xdw_i^u(t)\ ,\ X(0)=I,$$

$$dY=-Y\tilde{A}(x)dt-\sum_{i=1}^{m}YB_i(x)dw_i^u(t),\ Y(0)=I,$$

$$\frac{dN}{dt}=YG(x)U(z)\qquad N(0)=\Theta$$

over the new probability space $\{\Omega,\mathcal{F},\{\mathcal{F}_t\},P^u\}$, where $A,\tilde{A},B_i,G$ and U are defined as in (5). From (9) we have $k^o(t)\equiv 1$ and $k^u(.)$ fulfils

(12) $$dk=-k\sum_{i=1}^{m}\sum_{j=1}^{n}u_jU_j^i(z^u(t))dw_i(t),\quad k(0)=1,\quad t\geq 0$$

Combining (12) with (5) we get an Ito equation fulfiling the usual conditions of differentiability with respect to $u_1,\dots,u_n$. It follows

(13) $$k^u(T)\ \text{and}\ D_u^{\alpha}k^u(T)\in L_q(\Omega,P)\ \text{for any}\ \alpha=(\alpha_1,\dots,\alpha_n),\quad 1\leq q<\infty$$

In order to obtain estimates (3) we shall use the system (5) and it is required that the matrix $\left[\frac{\partial x^u}{\partial u}(t)\right]_{u=0}$ to be nonsingular for each $t>0$. Denote $M^o(t)=\left[\frac{\partial x^u}{\partial u}(t)\right]_{u=0}$, $t\geq 0$, and from (5) we get that $M^o(.)$ is the Ito solution of

(14) $$dM=\left[A(x^o(t))M+G(x^o(t))U(z^o(t))\right]dt+\sum_{i=1}^{m}B_i(x^o(t))Mdw_i(t);\quad M(0)=\Theta,$$

where $x^o(.)$, $z^o(.)$ define the solution $y^o(.)=(x^o(.),X^o(.),Y^o(.),N^o(.))$ in (5), for $u=0$. Since $X^o(t)$ is nonsingular and $(X^o(t))^{-1}=Y^o(t)$ (see (7)) from (14) we get

(15) $$M^o(t)=X^o(t)\int_0^t Y^o(s)G(x^o(s))U(z^o(s))ds=X^o(t)N^o(t)\ ,\quad t\geq 0$$

(16) $M^u(t)=X^u(t)N^u(t),\quad t\geq 0,\quad u\in R^n$,

where $y^u(.)=(x^u(.),\ X^u(.),Y^u(.),N^u(.))$ is the solution in (5). Now the definition of (5) becomes clearer and contains for u=0 the components defining the matrix $M^o(t)$. On the other hand the insertion of the parameter $u\in R^n$ is done in such a way that enables one to write (5) in the form (11), for each interval $[0,T]$, which has the property that $y^u(t)$, $t\in[0,T]$, the solution in (11) determines on $C([0,T];R^{n+3n^2})$ a probability Π^u fulfiling (see [13], th. 3.1)

(17) $\Pi^u=\Pi^o$ for any $u\in R^n$

Using (7) it follows that $M^u(T)$ in (16) is nonsingular iff $N^u(T)$ has this property and ~~the procedure of~~ the matrix $U(z)$ in (5) has to be taken in such a way that $N^u(T)$ is nonsingular . First of all there is a general way of choosing $U(z)$ in (5) which in a sense is equivalent with the general procedure used by Stroock in [5] in order to define the positive definite matrix $P^o(T)=X^o(T)\int_0^T Y^o(t)G(x^o(t))G^*(x^o(t))(Y^o(t))^*dt(X^o(T))^*$ where "M^*" indicates a transposed matrix.

This corresponds to the following U

(18) $U(z)=G^*(x)Y^*(1+|z-z_o|^{8n})^{-1}$, where $z_o=(x_o,I,I)$.

Denote $P(z)=YG(x)U(z)$. The factor $(1+|z-z_o|^{8n})^{-1}$ is introduced in order to have $P_{ij}(\cdot)$, $U_i^j(\cdot)\in C_b^\infty(R^{n+2n^2})$ but this do not affect in any way the strict positiveness of the matrix $N^u(T)$ if

$$\bar{N}^u(T)=\int_0^T Y^u(t)G(x^u(t))G^*(x^u(t))(Y^u(t))^*dt$$

is strictly positive (see remark below).

Remark 1

Assume $U(z)$ as in (18). Since $(\det N^u(T))^{-1} \leq (\inf_{|\lambda|=1} \lambda^* N^u(T)\lambda)^{-n} \leq$

$(1+\max_{t\in[0,T]} |y^u(t)-y_o|^{8n}).(\inf_{|\lambda|=1} \lambda^*\overline{N}^u(T)\lambda)^{-n}$, where $\lambda \in R^n$, it follows

that the symmetric matrix $N^u(T)$ is strictly positive definite and

$(\det N^u(T))^{-1} \in L_q(\Omega,P)$ for any $1\leq q<\infty$ if $(\inf_{|\lambda|=1} \lambda^*\overline{N}^u(T)\lambda)^{-1} \in L_q(\Omega,P)$

for any $1\leq q<\infty$.

Another possibility of choosing $U(z)$ in (5) is when $m=n$ and $g_1(x_o),\ldots$

$\ldots,g_n(x_o)$ are linearly independent. In this case we define U in (5) by

$$(19) \qquad U(z)=(1+|z-z_o|^{8n})^{-1}(\det X)(\det G(x))\overline{G}(x)\overline{Y},$$

where $\overline{M}$ denote the cofactor of M ($M\overline{M}=(\det M)I$). Since

$$|(\det X)(\det G(x))\overline{G}(x)\overline{Y}| \leq C(1+|z|^{4n})$$

the factor $(1+|z-z_o|^{8n})^{-1}$ in (19) is chosen to fulfil $P_{ij}(.), U^i_j(.)\in C_b^\infty(R^{n+2n^2})$.

Introducing (19) in (5) and using (16) we obtain

$$(20) \qquad M^u(t)=X^u(t)\int_0^t \ell(z^u(s))ds, \quad N^u(t)=(\int_0^t \ell(z^u(s))ds)I$$

where $\ell:R^{n+2n^2} \longrightarrow [0,\infty)$, $\ell(.)\in C_b^\infty(R^{n+2n^2})$ is given by

$$(21) \qquad \ell(z)=(1+|z-z_o|^{8n})^{-1}(\det G(x))^2$$

Remark 2

The matrix $M^u(T)$ in (20) is nonsingular and the components of $(M^u(T))^{-1}$

are in $L_q(\Omega,P)$ for any $1\leq q$, iff $p^u(T)=(\int_0^T \ell(z^u(t))dt)^{-1}$ fulfils

$p^u(T)\in L_q(\Omega,P)$ for any $1\leq q<\infty$. Now we are in position to state the main

results

Theorem 1

Assume $m = n$ and $g_1(x_0),\dots,g_n(x_0)$ are linearly independent (in (1). Let $p(t,x_0,.)$ be the probability measure generated on R^n by $x^0(t)$, where $x^0(.)$ is the solution in (1). Then $P(t,x_0,.)$ has a density with respect to Lebesgue measure, $p(t,x_0,dx) = p(t,x_0,x)dx$, with $p(t,x_0,.) \in C_b^\infty(R^n)$ for each $t>0$.

In addition, there exist r_0, T_0, $C_k>0$ such that

$$\|p(t,x_0,.)\|_{C_b^k} \leq C_k \cdot t^{-n(n+k+2)} \qquad (\forall) 0<t\leq T_0 ;$$

where C_k depends only on the bound of f,g_i on $S(x_0,r_0)$, $(k_0)^{-1}$ and the bounds on the derivatives of f,g_i where $(\det G(y))^2 \geq k_0 > 0$ $(\forall) y \in S(x_0,r_0)$.

Theorem 2

Let $x^0(.)$ be the solution in (1) and $P(t,x_0,.)$ the probability measure generated on R^n by $x^0(t)$.

Suppose that there exists the matix U in (5) such that for each $T>0$ is fulfilled $d^0(T) = (\det N^0(T))^{-1} \in L_q(\Omega,P)$ $(\forall)$ $1\leq q<\infty$. Then $P(t,x_0,.)$ has a density with respect to Lebesgue measure, $P(t,x_0,dx) = p(t,x_0,x)dx$, with $p(t,x_0,.) \in C_b^\infty(R^n)$ for each $t>0$.

In addition, if there exists $T_0>0$ such that

$$E\,|d^0(t)|^q \leq \tilde{C}_q\, t^{-pq} \qquad (\forall)\ 0<t\leq T_0, \qquad 1\leq q<\infty$$

for some constants $\tilde{C}_q>0$ and $p\geq 0$, then

$$\|p(t,x_0,.)\|_{C_b^k} \leq C_k \cdot t^{-p(n+k+2)} \qquad (\forall)\ 0<t\leq T_0$$

where C_k depends on $\tilde{C}_{n+k+1}$ and the bounds on the derivatives of f and g_i.

Denote $d^u(T)=(\det N^u(T))^{-1}$ or $d^u(T) = (\det M^u(T))^{-1}$ where $N^u(.)$ is defined in (5) and $M^u(.)$ in (16).

Proposition

If $d^0(T) \in L_q(\Omega, P)$ $(\forall)$ $1 \leq q < \infty$ then $d^u(T) \in L_q(\Omega, P)$ $(\forall)$ $1 \leq q < \infty$, $u \in R^n$.

Proof

It is obvious that in the place of $u=0$ can be taken any $u_0 \in R^n$. The proof will be given for $d^u(T)=(\det N^u(T))^{-1}$ and it is the same for $M^u(T)$ replacing N by M.

By hypothesis

$$E|d^0(T)|^q = \int_{\mathcal{Y}} (\det N(T))^{-q} d\Pi^0 < \infty$$

where the metric space $\mathcal{Y} \subseteq C([0,T]; R^{n+3n^2})$ is defined by $y(0)=(x_0, I, I, \Theta)$ and $\det N(T) \neq 0$, where $y=(x,X,Y,N)$ and Π^0 fulfils (17).

Approximate $h(r)\equiv 1$ for $r \in [0,\infty)$ by an increasing sequence of continuous functions $h_n: [0,\infty) \to [0,1]$ defined by $h_n(r) = 1$ for $r \leq n$, $h_n(r) = 0$ for $r > n+1$, $h_n(r) \in [0,1]$ for $n < r \leq n+1$. Define

$\varphi_n: \mathcal{Y} \to R$ continuous and bounded by

$$\varphi_n(y) = h_n(|\det N(T)|^{-1}) |\det N(T)|^{-q}$$

Using (17) it follows

$$E\, k^u(T) h_n(|d^u(T)|) |d^u(T)|^q = \int_{\mathcal{Y}} \varphi_n(y(\cdot))\, d\Pi^u = \int_{\mathcal{Y}} \varphi_n(y(\cdot))\, d\Pi^0$$

$$= E\, h_n(|d^0(T)|)\, |d^0(T)|^q$$

where $k^u(.)$ is defined in (9) and fulfil (10) and (12). Since $h(r)=1=\lim_{n\to\infty} h_n(r)$, $r \in [0,\infty)$, using Lebesgue's convergence theorem we get

$$E\, k^u(T) |d^u(T)|^q = E\, |d^0(T)|^q \quad (\forall) \quad 1 \leq q < \infty \; ; \; u \in R^n$$

On the other hand $l^u(t) = (k^u(t))^{-1}$ is the solution of

$$dl = l \sum_{i=1}^{m} \Big[\sum_{j=1}^{n} u_j\, l l_j^i (x^u(t)) \Big] dw_i^u(t), \quad l(0) = 1, \quad t \geq 0$$

and fulfils $l^u(T) \in L_q(\Omega, P^u)$, $1 \leq q < \infty$, where $P^u = k^u(T) P$,

It follows $\quad E\, |d^u(T)|^q = E\, |d^u(T)|^q\, k^u(T)\, l^u(T) =$

$$E^u |d^u(T)|^q\, l^u(T) \leq \left(E^u |d^u(T)|^{2q} \right)^{1/2} \cdot \left(E^u (l^u(T))^2 \right)^{1/2} < \infty$$

and the proof is complete.

3. AUXILIARY RESULTS AND PROOFS OF THEOREMS 1 AND 2

In order to get estimates of the form (3) and (4) we need to use Girsanov theorem not only for (5) but also for some "derivated systems" obtained from (5) by successive differentiation with respect to the parameter u. As a model we consider the following Ito equation

$$dy=F(y)dt+\sum_{i=1}^{m} G_i(y)dw_i(t), \quad y(0)=y_o\in R^k, \quad t\geq 0, \quad k\geq n, \tag{22}$$

where $w(.)$ is the original Wiener process in (1) and $F,G_i\in C^{\infty}(R^k)$ with $\frac{\partial F}{\partial x}, \frac{\partial G_i}{\partial x}\in C_b^{\infty}(R^k)$. Let $T>0$ be fixed and for each $u\in R^n$ denote $w^u(t)$, $t\in[0,T]$, a new Wiener process over the probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P^u\}$, with $w^o(t)=w(t)$, $t\geq 0$, and $P^u=k^u(T)P$, where $k^u(T)\in L_q(\Omega,P)$ for any $1\leq q<\infty$ and $k^o(t)\equiv 1$. Using theorem 3.5 (see [13]) it follows that $y^u(t)$, $t\in[0,T]$, the Ito solution of

$$dy=F(y)dt+\sum_{i=1}^{m} G_i(y)dw_i^u(t), \quad y(0)=y_o, \quad t\in[0,T] \tag{23}$$

defines on $C([0,T];R^k)$ a probability π^u fulfiling

$$\pi^u=\pi^o \quad \text{for any} \quad u\in R^n \tag{24}$$

Denote by $\mathcal{Y}$ a metric subspace of $C([0,T];R^k)$ such that $y^u(.,\omega)\in\mathcal{Y}$ a.e. (P) in $\omega\in\Omega$, for each $u\in\mathcal{O}$, where $\mathcal{O}\subseteq R^n$ is an open set.

First we shall state a lemma which is a consequence of (24). Let $k^u(T)$ be defined as above and denote by E^u,E the expectation with respect to P^u and P respectively.

Lemma 1

Let $\varphi \in C_b^{\infty}(R^k)$ and h be a polynomial of k+1 variables. Consider $p:\mathcal{Y} \to R$ continuous and denote $p^u(T)=p(y^u(.))$, where $y^u(.)$ is the solution in (6) for $u \in \mathcal{O}$. Assume $p^u(T) \in L_q(\Omega,P)$ $(\forall)$ $1 \leq q < \infty$. Then

a) $$E^u \varphi(y^u(T))h(y^u(T),p^u(T))=E\,\varphi(y^u(T))h(y^u(T),p^u(T))k^u(T)=$$
$$=E\,\varphi(y^o(T))h(y^o(T),p^o(T)) \quad \text{for any } u \in \mathcal{O}$$

In addition, let $y^u(T)$, $k^u(T)$ and $p^u(T)$ be differentiable of any order with respect to $u \in \mathcal{O}$ and assume $l^u(T)$, $D_u l^u(T) \in L_q(\Omega,P)$ $(\forall)$ $1 \leq q < \infty$, for any $\alpha=(\alpha_1,\dots,\alpha_n)$, where $l=y,k,p$ respectively. Then

b) $$0=D_u^{\alpha}\{E\,\varphi(y^u(T))h(y^u(T),\,p^u(T))k^u(T) =$$
$$=ED_u^{\alpha}[\varphi(y^u(T))h(y^u(T),p^u(T))k^u(T)], \qquad u \in \mathcal{O}$$

Proof

The conclusion (b) is a direct consequence of (a). Let $h:R^{k+1} \longrightarrow R$ be continuous and bounded. Using (24) it follows

$$\int_{\mathcal{Y}} \varphi(y(T))h(y(T),p(y(.))d\pi^u=\int_{\mathcal{Y}} \varphi(y(T))h(y(T),p(y(.))d\pi^o$$

and

$$E^u \varphi(y^u(T))h(y^u(T),p^u(T))=E\,\varphi(y^o(T))h(y^o(T),p^o(T)) \text{ for any } u \in \mathcal{O}$$

Since $P^u=k^u(T)P$ we obtain (a) for h continuous and bounded. Generally, h is a pointwise limit of a sequence of continuous and bounded functions $\{h_n\}_{n\geq 1}$ with $|h_n(y,p)| \leq |h(y,p)|$. By hypothesis,

$$E|\varphi(y^u(T))h(y^u(T),p^u(T))| < \infty$$

and using Lebesgue's convergence theorem we get $\lim_{n\to\infty} E\varphi(y^u(T))h_n(y^u(T),p^u(T))k^u(T)$

$=E\varphi(y^u(T))h(y^u(T),p^u(T))=\text{const.}$ for $u\in\mathcal{O}$, and the proof is complete.

Lemma 2

Suppose that $y^u(.)=(x^u(.),X^u(.),Y^u(.),N^u(.))$ is the solution in (5), $k^u(.)$ in (9) and the family of matrices $\{N^u(T),\ u\in\mathcal{O}\}$ fulfils $(\det N^u(T))^{-1}\in L_q(\Omega,P)$ for any $1\le q<\infty$ and $u\in\mathcal{O}$, where $\mathcal{O}\subseteq R^n$ is an open set containing the origin and $T>0$ is fixed. Then

$$E\frac{\partial\varphi}{\partial x_i}(x^o(T))=-E\varphi(x^o(T))\sum_{j=1}^{n}\frac{\partial}{\partial u_j}\left[L^u_{ij}(T)k^u(T)\right]_{u=0}\quad\text{for any }\varphi\in C_b^\infty(R^n)$$

where $(L^u_{ij}(T))=L^u(T)=(M^u(T))^{-1}$, and $M^u(T)$ is defined in (16).

Proof

We shall apply Lemma 1. Take $k=n+3n^2$ and define the metric subspace $\mathcal{Y}\subseteq C([0,T];R^k)$ by $y(0)=y_o=(y_o,I,I,\Theta)$ and $\det N(T)\neq 0$, $\det X(T)\neq 0$ where the elements $y(.)$ in $C([0,T];R^k)$ are written in the form $y(t)=(x(t),X(t),Y(t),N(t))$ and $X(t)$, $Y(t)$, $N(t)$ are $(n\times n)$ matrices. It is obvious that $p(y(.))=$ $=(\det Y(T)\det N(T))^{-1}$ is a continuous functional on Y. Using (6) and (7) and the hypothesis we get that $p^u(T)=p(y^u(.))$ fulfils the hypotheses in Lemma 1.

Using (6) and (13) we see that $y^u(T)$, $k^u(T)$ fulfil the differentiability properties required in Lemma 1.

The Lemma 1 is applied with $h=Z_{ij}p$, $i,j=1,\dots,n$ where $Z=(Z_{ij})$ is the cofactor matrix of XN $(Z.XN=(\det XN)I)$. By definition $h(y^u(T,p^u(T))=L^u_{ij}(T)$, and from (b) in Lemma 1 we get

$$0=\left[\sum_{j=1}^{n}\frac{\partial}{\partial u_j}E\varphi(x^u(T))L^u_{ij}(T)k^u(T)\right]_{u=0}=$$

$$=E\frac{\partial\varphi}{\partial x}(x^o(T))\left(\sum_{j=1}^{n}\frac{\partial x^u}{\partial u_j}(T)L^u_{ij}(T)\right)_{u=0}+E\varphi(x^o(T))\sum_{j=1}^{n}\frac{\partial}{\partial u_j}\left[L^u_{ij}(T)k^u(T)\right]_{u=0}$$

Since $\left(\left[\frac{\partial x^u}{\partial u}(T)\right]_{u=0}\right)^{-1}=L^o(T)$ it follows $\left(\sum_{j=1}^{n}\frac{\partial x^u}{\partial u_j}(T)L^u_{ij}(T)\right)_{u=0}=e_i$, where $e_1,\ldots,e_n\in R^n$ is the canonical base and the proof is complete.

The next lemma is a consequence of the exponential martingale inequality and a similar result is contained in [3] [5].

Lemma 3

Let $\tau=\inf\{t\geq 0: |y(t)-y_o|\geq r_o\}$, where $y(.)$ is the solution in (22) and $r_o>0$ is fixed. Then $(\tau)^{-1}\in L_q(\Omega,P)$ for any $1\leq q<\infty$ and there exists $T_o>0$ such that

$$E(\tau\wedge t)^{-q}\leq C_q\, t^{-q} \quad \text{for} \quad 0<t\leq T_o$$

where the constant C_q depends only on q, r_o and the upper bounds for $|F|$, $|G_i|$ on $\{y\in R^k: |y-y_o|\leq r_o\}$.

Proof

From (22) we get

$$(25)\quad y(t\wedge\tau)-y_o=\int_0^t c(s)F(y(s))ds+\sum_{i=1}^{m}\int_0^t c(s)G_i(y(s))dw_i(s),\quad t\geq 0,$$

where $c(s)=1$ if $s<\tau$ and $c(s)=0$ if $s\geq\tau$.

From (25), using Ito's differential rule, it follows

$$(26) \qquad |y(t\wedge\tau)-y_o|^2 = \int_o^t a(s)ds + \sum_{i=1}^m \int_o^t b_i(s)dw_i(s), \quad t\geq 0,$$

where a, b_i are bounded measurable scalar functions. We take $T_o>0$ sufficiently small such that

$$(27) \qquad T_o \sup_{s\geq 0} |a(s)| < \frac{r_o}{2}.$$

In order to prove that $(\tau)^{-1}\in L_q(\Omega,P)$ it is enough to verify that $(\tau\wedge T_o)^{-1}\in L_q(\Omega,P)$ as the following inequality shows

$$(\tau)^{-1}\leq (\tau\wedge T_o)^{-1} I_{\{\tau<T_o\}} + \frac{1}{T_o} I_{\{\tau\geq T_o\}}$$

For each $T\leq T_o$, using (26) and (27) we get

$$(28) \qquad P\{\tau<T\} = P\left\{\int_o^T a(s)ds + \sum_{i=1}^m \int_o^T b_i(s)dw_i(s)\geq r_o\right\}\leq$$

$$P\left\{\max_{t\leq T} \sum_{i=1}^m \int_o^t b_i(s)dw_i(s) > \frac{r_o}{2}\right\}$$

On the other hand, the exponential martingale inequality gives

$$(29) \qquad P\left\{\max_{t\leq T}\left[\sum_{i=1}^m \int_o^t b_i(s)dw_i(s) - \frac{\alpha}{2}\sum_{i=1}^m \int_o^t (b_i(s))^2 ds\right] > \beta\right\}\leq \exp(-\alpha\beta)$$

and choosing $\alpha = \frac{r_o}{2TM}$, $\beta = \frac{r_o}{4}$, where $M = \sup_{s\geq 0} \sum_{i=1}^m (b_i(s))^2$, we get

$$(30) \qquad P\left\{\max_{t\leq T} \sum_{i=1}^m \int_o^t b_i(s)dw_i(s) > \frac{r_o}{2}\right\}\leq$$

$$P\left\{\max_{t\leq T}\left[\sum_{i=1}^m \left(\int_o^t b_i(s)dw_i(s) - \frac{\alpha}{2}\int_o^t (b_i(s))^2 ds\right)\right] > \frac{r_o}{4}\right\}\leq \exp(-\rho T^{-1})$$

where $\rho = \frac{r_o^2}{8M}$.

(31) $P\{\tau^{-1}>T^{-1}\}=P\{\tau<T\}\leq \exp(-\rho T^{-1})$ for any $T\leq T_o$

Denote $A_n=\{T_o 2^{-(n+1)}\leq \tau<T_o 2^{-n}\}$, $n=0,1,2,\ldots$, and we have

(32) $$(\tau\wedge T_o)^{-q}\leq (2T_o^{-1})^q\sum_{n=0}^{\infty}(2^n)^q I_{A_n}$$

Taking expectation in (32) and using (31) we get

(33) $$E(\tau\wedge T_o)^{-q}\leq (2T_o^{-1})^q\sum_{n=0}^{\infty}(2^n)^q PA_n\leq (2T_o^{-1})^q\sum_{n=0}^{\infty}(2^n)^q\exp(-\rho 2^n T_o^{-1})\leq$$

$$K(2T_o^{-1})^q\sum_{n=0}^{\infty}2^{-n}=K2^{q+1}\,T_o^{-q},$$

where $K>0$ is taken such that $u^{q+1}\exp(-\rho u T_o^{-1})\leq K$, $(\forall)\ u\in[0,\infty)$.

Using (33) we obtain $(\tau)^{-1}\in L_q(\Omega,P)$ and replacing T_o in (33) by $0<T\leq T_o$ we get

$$E(\tau\wedge T)^{-q}\leq C_q\,T^{-q},\text{ where } C_q=K\cdot 2^{q+1}$$

and the proof is complete.

Remark 3

The conclusion in Lemma 3 remains unchanged if F, G_i in (22) are replaced by Borel measurable functions $h:[0,\infty)\times R^k\to R^k$ which are C^1 in $y\in R^k$ and fulfiling

$$|h(t,y)|\leq M(1+|y|),\qquad t\in[0,\infty),\qquad y\in R^k$$

Lemma 4

Let $m=n$ and assume that $g_1(x_o),\ldots,g_n(x_o)$ are linearly independent in

(1). Define $U(z)$ as in (19) and $p^u(T)$ as in remark 2. Then there exists $D_u^\alpha p^u(T)$ for any $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\alpha_i\geq 0$, and $D_u^\alpha p^u(T)\in L_q(\Omega,P)$ $(\forall)$ $1\leq q<\infty$, for each $u\in R^n$ and $T>0$.

In addition, with τ^u defined in (34), there exists $T_o>0$ such that $E(p^u(T))^q\leq C_q E(T\wedge\tau^u)^{-q}$ for each $T>0$, where $C_q>0$ depends only λ_o, q and $(k_o)^{-1}$ and

$$E(t\wedge\tau^u)^{-q}\leq C_q^u\, t^{-q} \qquad (\forall)\quad 0<t\leq T_o$$

where the constants $C_q^u>0$, $u\in\mathcal{O}\subseteq R^n$, are uniformly bounded if $\mathcal{O}$ is a bounded set, and C_q^o depends only on q, λ_o, and the bounds for $|f|$, $|g_i|$, $|\frac{\partial f}{\partial x}|$, $|\frac{\partial g_i}{\partial x}|$ on $\{x\in R^n: |x-x_o|\leq\lambda_o\}$.

Proof

Define

$$(34)\qquad \tau^u=\inf\left\{t\geq 0: |z^u(t)-z_o|\geq\lambda_o\right\}$$

where $\lambda_o>0$ is sufficiently small such that $(\det G(x))^2\geq k_o>0$, for x in the ball $S(x_o,\lambda_o)$, and $y^u(.)=(z^u(.),N^u(.))$ is the solution in (5). For each $u\in R^n$, the equation fulfilled by $z^u(.)$ is of type (22) and the hypotheses in Lemma 3 are verified. It follows that there exists $T_o>0$ corresponding to $u=0$, such that

$$(35)\qquad E(\tau^o\wedge t)^{-q}\leq C_q^o\, t^{-q} \qquad (\forall)\quad 0<t\leq T_o\ ,$$

$$(\tau^u)^{-1}\in L_q(\Omega,P) \qquad (\forall)\quad 1\leq q<\infty \text{ for each } u\in R^n$$

where C_q^o depends only on q, λ_o and the bounds, for $|f|$, $|g_i|$, $|\frac{\partial f}{\partial x}|$, $|\frac{\partial g_i}{\partial x}|$ on $\{x: |x-x_o|\leq\lambda_o\}$.

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If we change τ^o with τ^u in (35) the constant T_o can be preserved as in (35) as the following remark shows. For each $u \in R^n$ the solution $z^u(t)$, $t \in [0,T_o]$, in (5) has the same trajectories as the solution $z^u(.)$ in (11) and the coefficients in (11) are those in (5) for u=0.

The equation verified by $z^u(t)$, $t \in [0,T_o]$, in (11), fulfils the hypotheses in Lemma 3 and we get that there exists $T_o > 0$ such that

$$(36) \qquad E^u(t \wedge \tau^u)^{-q} \leq \tilde{C}_q . t^{-q} \qquad (\forall) \quad 0 < t \leq T_o \quad \text{and} \quad 1 \leq q < \infty .$$

for each u, where $\tilde{C}_q$ depends on q, r_o and the bounds for $|f|$, $|g_i|$, $|\frac{\partial f}{\partial x}|$, $|\frac{\partial g_i}{\partial x}|$ on $\{x: |x-x_o| \leq r_o\}$.

Since $E(t \wedge \tau^u)^{-q} = E^u[(t \wedge \tau^u)^{-q}(k^u(T_o))^{-1}]$ for any $0 < t \leq T_o$, where $k^u(.)$ is in (9), we obtain

$$(37) \qquad E(t \wedge \tau^u)^{-q} \leq C_q^u \, t^{-q} \qquad (\forall) \quad 0 < t \leq T_o ,$$

$$C_q^u = (\tilde{C}_{2q})^{1/2} [E^u(k^u(T_o))^{-2}]^{1/2}$$

where $\tilde{C}_{2q}$ is the corresponding constant in (36).

On the other hand, using $(\det G(x^u(t))^2 \geq k_o > 0$ for any $0 < t \leq \tau^u$ we get

$$(38) \quad p^u(T) \leq \left(\int_0^{T \wedge \tau^u} (1+|z^u(t)-z_o|^{8n})^{-1} (\det G(x^u(t)))^2 dt \right)^{-1} \leq$$

$$(k_o)^{-1} (1+ \max_{t \leq \tau^u \wedge T} |z^u(t)-z_o|^{8n}) (\tau^u \wedge T)^{-1} \leq$$

$$(k_o)^{-1} (1+ \lambda_o^{8n}) (\tau^u \wedge T)^{-1}$$

Using (36) and (37) in (38) it follows

$$(39) \qquad p^u(T) \in L_q(\Omega, P) \qquad (\forall) \quad 1 \leq q < \infty,$$

$$E(p^u(T))^q \le C_q E(\tau^u \wedge T)^{-q} \text{ , where } C_q=(k_o)^{-1}(1+\ell_o^{8n})$$

Since $y^u(.)=(z^u(.),\ N^u(.))$ fulfils (6) it follows that there exists $D_u^{\alpha} p^u(T)$ for each $\alpha=(\alpha_1,\dots,\alpha_n)$ and

(40) $$D_u^{\alpha} p^u(T) \in L_q(\Omega,P) \qquad (\forall)\quad 1\le q<\infty,\quad u\in R^n$$

The proof is complete.

In order to get estimates (3) we need to use Lemma 2 and to write $\sum_{j=1}^{n} \frac{\partial}{\partial u_j}(L_{ij}^u(T)k^u(T))_{u=0}$ in a more explicit form. Denote $D_j y^o(t)=\left[\frac{\partial y^u}{\partial u_j}(t)\right]_{u=0}$ $D_j k^o(t)=\left[\frac{\partial k^u}{\partial u_j}(t)\right]_{u=0}$ and

(41) $$q^u(T)=(\det N^u(T)) \text{ , } d^u(T)=(q^u(t))^{-1} \text{ ,}$$

where $y^u(.)=(z^u(.),N^u(.))$ and $k^u(.)$ are the solutions in (5) and (12) respectively.

By differentiating with respect to u_j in (5) and (12) at u=0, we get that $D_j y^o(t)$, $D_j k^o(t)$, $t\ge 0$, is the solution of the system

(42) $$dy=F(y)dt+\sum_{i=1}^{n} G_i(y)dw_i(t),\quad y(0)=(z_o^*,\Theta)^*$$

$$dD_jy=\left[\frac{\partial F}{\partial y}(y)D_jy+\sum_{i=1}^{n} U_j^i(y)G_i(y)\right]dt+\sum_{i=1}^{n}\frac{\partial G_i}{\partial y}(y)D_jydw_i(t),\quad D_jy(0)=0,$$

$$dD_jk=-\sum_{i=1}^{n} U_j^i(z)dw_i(t) \text{ ,} \qquad D_jk(0)=0,$$

$j=1,\dots,n$, where $F(y)=\left[f^*(x),(A(x)X)^*,\ -(YA(x))^*,\ (YG(x)U(z))^*\right]^*$ $G_i(y)=(g_i^*(x),\ (B_i(x)X)^*,\ -\ (YB_i(x))^*,\Theta)^*$ and Θ is the null matrix in $R^{n\times n}$.

From (16) and under the hypotheses in Lemma 2 we have $(M^u(T))^{-1}=$ $=d^u(T)\overline{N}^u(T)Y^u(T)$ and using (42) we obtain

$$(43)\quad \sum_{j=1}^{n} \frac{\partial}{\partial u_j}(L^u_{ij}(T)k^u(T))_{u=0} = -(d^o(T))^2 \sum_{j=1}^{n} (\overline{N}^o(T)Y^o(T))_{ij} D_j q^o(T) +$$

$$d^o(T)\Big(\sum_{j=1}^{n} D_j (\overline{N}^o(T)Y^o(T))_{ij} + (\overline{N}^o(T)\overline{Y}^o(T))_{ij} D_j k^o(T)\Big),$$

where $\overline{M}$ is the cofactor matrix of M ($M\overline{M}=(\det M)\mathbb{I}$).

Denote $\tilde{y}=(y,D_1y,\dots,D_ny,\ D_1k,\dots,D_nk)$ and (43) shows that

$$(44)\quad \sum_{j=1}^{n} \frac{\partial}{\partial u_j}(L^u_{ij}(T)k^u(T))_{u=0} = h_i(\tilde{y}^o(T),d^o(T))$$

where h_i is a polynomial of degree $(n+3)$ with respect to the components of $\tilde{y}^o(T)$ and $d^o(T)$ but of second degree with respect to $d^o(T)$ where $\tilde{y}(t)$, $t\in[0,T]$, is the solution in (42)..

Using the above remark and Lemma 2 we have

Lemma 5

Suppose that the hypotheses in Lemma 2 are fulfilled. Then

$$\sum_{j=1}^{n} \frac{\partial}{\partial u_j}(L^u_{ij}(T)k^u(T))_{u=0} \in L_q(\Omega,P),\quad (V)\quad 1\le q<\infty \text{ and}$$

$$\Big| E\frac{\partial \varphi}{\partial x_i}(x^o(T))\Big| \le C_i \sup_{x\in R^n}|\varphi(x)| \quad \text{for any } \varphi\in C_b^\infty(R^n)$$

where $C_i=E|h_i(\tilde{y}^o(T),d^o(T))|$, and h_i is the polynomial in (44).

In the Lemmas 2 and 5 we described how to get estimates (3) for the particular case $\alpha=e_i$, $i=1,\dots,n$, where $e_1,\dots,e_n\in R^n$ is a canonical base. In general, to obtain (3) for an arbitrary α we need to repeat the arguments used for $\alpha=e_i$ but for new equations obtained from (42) in the following manner.

We replace $w(t)$, $t\in[0,T]$, in (42) by $w^u(t)$, $t\in[0,T]$, defined in (8), and denote by S^u the new equation. Let $\tilde{y}^u(t)$, $t\in[0,T]$, be the solution in

S^u over the probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},P^u)$ where $P^u=k^u(T)P$ and $k^u(.)$ is given in (9).

Denote

$$\tilde{P}_i^u(T)=h_i(\tilde{y}^u(T),d^u(T)) \tag{45}$$

where h is the polynomial in (44).

We have

$$\tilde{P}_i^o(T)=P_i^o(T),\ i=1,\dots,n \tag{46}$$

Denote by D the transformation applied in (42) *obtained by* changing $w(t)$ *into* $w^u(t)$, $t\in[0,T]$, the differentiation $\left[\frac{\partial\tilde{y}^u}{\partial u}(t)\right]_{u=0}$, $t\in[0,T]$, of the solution $\tilde{y}^u(.)$ in S^u and adding at (42) the Ito equation fulfilled by $\left[\frac{\partial\tilde{y}^u}{\partial u}(t)\right]_{u=0}$, $t\in[0,T]$.

<u>Definition</u>

The Ito equation obtained from (42) by p transformations D is called a derived system of order p and denote it by D^pS. [Let $p\geq 0$, p integer be fixed and $\psi(t)$, $t\in[0,T]$, the Ito solution in D^pS. It is obvious that the Wiener process in D^pS is the original one $w(t)$, $t\in[0,T]$, and changing it into $w^u(t)$, $t\in[0,T]$, we denote by D^pS^u the new system so obtained. Let $\psi^u(t)$, $t\in[0,T]$, the solution in D^pS^u. We have $\psi^o(t)=\psi(t)$, $t\in[0,T]$.

<u>Remark 4</u>

The solution $\psi^u(t)$, $t\in[0,T]$ in D^pS^u is differentiable with respect to $u\in R^n$ and $D_u^\alpha\,\psi^u(T)\in L_q(\Omega,P)$ $(\forall)$ $1\leq q<\infty$, $\alpha=(\alpha_1,\dots,\alpha_n)$. It is clear using that the solution $\psi^u(.)$ in D^pS^u is obtained from a system of the form (23) which can be rewritten in the form

$$(46) \qquad d\psi=\Big[F(\psi)+\sum_{j=1}^{n} u_j \sum_{i=1}^{m} U_i^j(z)G_i(\psi)\Big]\, dt+\sum_{i=1}^{m} G_i(\psi)dw_i(t)\ ,\quad \psi(0)=\psi_o\ ,\quad t\in[0,T]$$

where $\psi=(z,\dots)$, and $U_i^j(z)$, z are as in (5). The differentiability properties of the solution $\psi^u(.)$ with respect to u in (46) follows from $F,G_i\in C^\infty(R^k)$ and $\frac{\partial F}{\partial x}, \frac{\partial G_i}{\partial x}\in C_b^\infty(R^k)$.

Denote by P_k^r the set of polynomials $Q(\psi,d)$ of k+1 variables, of degree r+n+2 and of (r+1) degree with respect to d, where $k=\dim\psi$.

Let R_k^r be the set of functions $\psi:R^n\times\Omega\longrightarrow R$ obtained by $\psi(u,\omega)=$ $=Q(\psi^u(T),d^u(T))$, where $Q\in P_k^r$ and $\psi^u(t)$, $t\in[0,T]$ is the solution in a system D^pS^u.

Define $\tilde{P}_i(T):R_k^r$ by

$$(47) \qquad \big[\tilde{P}_i(T)(\psi)\big](u)=\psi(u)\tilde{P}_i^u(T)+\sum_{j=1}^{n} L_{ij}^u(T)k^u(T)\frac{\partial}{\partial u_j}\psi(u)$$

Lemma 6

Let $T>0$ be fixed and suppose that $d^u(T)=(\det N^u(T))^{-1}$ fulfils $d^u(T)\in L_q(\Omega,P)$ $(\forall)$ $1\le q<\infty$ for $u\in\mathcal{O}\subseteq R^n$, $\mathcal{O}$ open, $0\in\mathcal{O}$, where $y^u(.)=(z^u(.),N^u(.))$ is the solution in (5).

Then

a) $\quad ED_x^\alpha\varphi(x^o(T))=(-1)^{|\alpha|}\quad E\varphi(x^o(T))\big[(\tilde{P}_n(T))^{\alpha_n}\dots(\tilde{P}_1(T))^{\alpha_1}(1)\big](0)$

for any $\varphi\in C_b^\infty(R^n)$.

There exists $1\le k<\infty$, k integer, depedning only on $|\alpha|$ and n such that

b) $\quad \psi_\alpha=(\tilde{P}_n(T))^{\alpha_n}\dots(\tilde{P}_1(T))^{\alpha_1}(1)\in R_k^{|\alpha|}$ and $\psi_\alpha(0)\in L_q(\Omega,P)$

$(\forall)$ $1\le q<\infty$ for each $\alpha=(\alpha_1,\dots,\alpha_n)$.

In addition, if there exists $T_o>0$ and $p\ge 0$ such that

$E|d^o(t)|^q \leq \tilde{C}_q\, t^{-pq}$ $(\forall)$ $0 < t \leq T_o$, for some constant $\tilde{C}_q > 0$, $(\forall)$ $1 \leq q < \infty$, then

c) $E\ |\psi_\alpha(0)| \leq C_\alpha\, T^{-p(|\alpha|+1)}$ $(\forall)$ $0 < T \leq T_o$

where the constant $C_\alpha > 0$ depends only on $\tilde{C}_q$ and the bounds of derivatives of f and g_i.

Proof

The proof is given by induction with respect to $|\alpha|$. For $|\alpha|=1$, we have $\alpha = e_i$ for some $i \in \{1,\dots,n\}$, where $e_1,\dots,e_n$ is the canonical base in R^n.

The conclusion (a) for $\alpha = e_i$ is the same with the statement in Lemma 2 via (45) and (47).

Using (45) we have $\tilde{P}^u_i(T) = h(\tilde{y}^u(T), d^u(T))$, where h is a polynomial of degree (n+3) and $y^u(t)$, $t \in [0,T]$, is the solution in S^u. The conclusion (b) for $\alpha = e_i$ means $\tilde{P}^u_i(T) \in R^1_k$, where k=dim $\tilde{y}$ and $\tilde{P}^u_i(T) \in L_q(\Omega,P)$ $(\forall)$ $1 \leq q < \infty$.

Since $\tilde{y}^u(T) \in L_q(\Omega,P^u)$ and $(k^u(T))^{-1} \in L_q(\Omega,P)$ $(\forall)$ $1 \leq q < \infty$ it follows $z^u(T) \in L_q(\Omega,P)$ $(\forall)$ $1 \leq q < \infty$.

By hypothesis $d^u(T) \in L_q(\Omega,P)$ $(\forall)$ $1 \leq q < \infty$, and we obtain that $h(\tilde{y}^u(T), d^u(T)) \in L_q(\Omega,P)$ $(\forall)$ $1 \leq q < \infty$, for each $u \in \mathcal{U}$, which proves (b) for $\alpha = e_i$.

Since h is a polynomial of second degree in the variable d we obtain

$$(48) \quad E|\tilde{P}^o_i(T)| = E|h(y^o(T), d^o(T))| \leq (E(d^o(T))^4)^{1/2} \cdot (E\, h_1(\tilde{y}^o(T)))^2)^{1/2}$$

where h_1 is a polynomial depending only on N, Y, k and D_jN, D_jY, D_jk, $j=1,\dots,n$.

By hypothesis

$$(E(d^o(T))^4)^{1/2} \leq (\tilde{C}_4)^{1/2}\, t^{-2p} \quad (\forall)\ 0 < t \leq T_o$$

and the conclusion (c) for $\alpha = e_i$ follows from (48) with

$c_i=(\tilde{C}_u)^{1/2}\left[E \max_{0\le T\le T_0} |h_1(\tilde{y}^o(T))|^2\right]^{1/2}$.

Let (a), (b), (c) be true for $|\tilde{\alpha}|=\ell$ and let α be such that $|\alpha|=\ell+1$. Then $\alpha=\tilde{\alpha}+e_i$ with $|\tilde{\alpha}|=\ell$ and

$$(49)\qquad (\tilde{P}_n(T))^{\alpha_n}\dots(\tilde{P}_1(T))^{\alpha_1}(1)=\tilde{P}_i(T)\left[(\tilde{P}_n(T))^{\tilde{\alpha}_n}\dots(\tilde{P}_1(T))^{\tilde{\alpha}_1}(1)\right]$$

By hypothesis $(\tilde{P}_n(T))^{\tilde{\alpha}_n}\dots(\tilde{P}_1(T))^{\tilde{\alpha}_1}(1)\in R_k^{\ell}$ for some $k\ge 1$, and

$$(50)\qquad \left[(\tilde{P}_n(T))^{\tilde{\alpha}_n}\dots(\tilde{P}_1(T))^{\tilde{\alpha}_1}(1)\right](u)=Q(\psi^u(T),d^u(T)),\ Q\in P_k^{\ell},\ k=\dim\psi^u,$$

where $\psi^u(t)$, $t\in[0,T]$, is the solution in a system D^pS^u. Using (49) and (50) the conclusion (a) for $|\alpha|=l+1$ can be written as

$$(51)\qquad ED_x^{\tilde{\alpha}}\frac{\partial\varphi}{\partial x_i}(x^o(T))=(-1)^{\ell+1}E\,\varphi(x^o(T))\left[\tilde{P}_i(T)Q(\psi^u(T),d^u(T))\right](0)$$

Since (a) is fulfilled for $\tilde{\alpha}$ and φ replaced by $\frac{\partial\varphi}{\partial x_i}$ we get

$$(52)\qquad ED_x^{\tilde{\alpha}}\frac{\partial\varphi}{\partial x_i}(x^o(T))=(-1)^{\ell}E\frac{\partial\varphi}{\partial x_i}(x^o(T))Q(\psi^o(T),d^o(T))$$

Now we are in position to apply Lemma 1 for the equation defining D^pS^u, which is of the form (23), and for the polynomial $h(y,p)=L_{ij}Q(\psi,d)$, where $\psi=y$, $p=d$, $L_{ij}=d(\bar{N}Y)_{ij}$, $\dim\psi=k$, and $(x,X,Y,N)=(z,N)$ is the vector in (5), and $\bar{N}N=(\det N)I$.

By definition of D^pS, the system (42) is included in any system D^pS and any system D^pS^u contains (11). So, first $n+3n^3$ components of $\psi^u(.)$ are defined by $(x^u(.),X^u(.),Y^u(.),N^u(.))$, the solution in (11). The matrices $X^u(t)$, $Y^u(t)$ are nonsingular and $X^u(t)Y^u(t)=I$, $t\in[0,T]$, (see (7)), and by hypothesis $N^u(T)$ is nonsingular and $d^u(T)\in L_q(\Omega,P)$ (V) $1\le q<\infty$.

Define a metric subspace $\mathcal{Y}\subseteq C([0,T];R^k)$ by the following conditions

(53) $\psi(0)=(x_o,I,I,\Theta,0,\ldots,0)$, where $y_o=(x_o,I,I,\Theta)$ is given in (5) and $0\in R$ is the null element,

(54) the matrices $X(T)$, $Y(T)$, $N(T)$ are nonsingular and the firs $n+3n^2$ components of $\psi(.)\in C([0,T];R^k)$ are $x(.)$, $X(.)$, $Y(.)$, $N(.)$.

It is obvious that $d(\psi(.))=(\det N(T))^{-1}$ is a continuous functional on $\mathcal{Y}$.

Using Remark 4 it follows that $\psi^u(T)$, $k^u(T)$ and $d^u(T)$ fulfil the hypothesis in Lemma 1 and from (b) for $u=0$ we get

(55)
$$0=\sum_{j=1}^{n}\frac{\partial}{\partial u_j}\Big[E\,\varphi(x^u(T))L^u_{ij}(T)Q(\psi^u(T),d^u(T))k^u(T)\Big]_{u=0}=$$

$$=E\Big[\Big\langle\frac{\partial\varphi}{\partial x}(x^o(T)),\ \sum_{j=1}^{n}\Big[\frac{\partial x^u}{\partial u_j}(T)\Big]_{u=0}L^o_{ij}(T)\Big\rangle\, Q(\psi^o(T),d^o(T))\Big]+$$

$$+E\Big\{\varphi(x^o(T))\sum_{j=1}^{n}\frac{\partial}{\partial u_j}\Big[L^u_{ij}(T)k^u(T)\Big]_{u=0}Q(\psi^o(T),d^o(T))\Big\}+$$

$$+E\,\varphi(x^o(T))\ \sum_{j=1}^{n}L^o_{ij}(T)\frac{\tilde{\partial}}{\partial u_j}Q(\psi^u(T),d^u(T))\ .$$

Since $\Big[\frac{\partial x^u}{\partial u}(T)\Big]^{-1}_{u=0}=L^o(T)$, we have $\sum_{j=1}^{n}\Big[\frac{\partial x^u}{\partial u_j}(T)\Big]_{u=0}L^o_{ij}(T)=e_i$,

and using (47), from (55) we obtain

(56)
$$E\frac{\partial\varphi}{\partial x_i}(x^o(T))Q(\psi^o(T),d^o(T))=-E\,\varphi(x^o(T))\Big[\tilde{P}_i(T)Q(\psi^u(T),d^u(T))\Big](0)$$

Therefore using (52) and (56) we get

(57)
$$E D^{\ell}_x\frac{\partial\varphi}{\partial x_i}(x^o(T))=(-1)^{\ell+1}E\,\varphi(x^o(T))\Big[\tilde{P}_i(T)Q(\psi^u(T),d^u(T))\Big](0)$$

which represents (51) and the proof of (a) is complete.

Since $Q \in P_k^1$ from (47) we obtain $\tilde{P}_i(T)Q(\psi^u(T), d^u(T)) \in R_{k_1}^{l+1}$ for some $k_1 > k$ and using the same argument as for $\alpha = e_i$ the proof of (b) is complete.

The conclusion (c) for $|\alpha| = l+1$ is a consequence of (b) and the estimate (48).

The proof is complete.

Now we are in position to give the proofs of theorems 1 and 2.

<u>Proof of Theorem 2</u>

By hypothesis the conditions in Proposition and Lemma 6 are fulfilled. From (a) and (b) in Lemma 6 we get that for each $\alpha = (\alpha_1, \dots, \alpha_n)$ and $t > 0$ there exists $C_{\alpha,t} > 0$ such that

$$\left| \int_{R^n} D_x^\alpha \varphi(x) P(t, x_o, dx) \right| \leq C_{\alpha,t} \sup_{x \in R^n} |\varphi(x)| \qquad (\forall)\ \varphi \in C_b^\infty(R^n)$$

Using Lemm 3.1 in ([5], part I) we obtain that $P(t, x_o, \cdot)$ has a density with respect to Lebesgue measure, $P(t, x_o, dx) = p(t, x_o, x)dx$, with $p(t, x_o, \cdot) \in C_b^\infty(R^n)$ for each $t > 0$.

In addition using (c) in Lemma 6 we get that there exists $T_o > 0$ such that

$$C_{\alpha,t} \leq C_\alpha\, t^{-p(|\alpha|+1)} \qquad (\forall)\quad 0 < t \leq T_o$$

where $C_\alpha > 0$ is a constant which depends only on the bounds of the derivatives on f, g_i and $\tilde{C}_{|\alpha|}$, where $\tilde{C}_q$ are given.

Applying Lemma 3.1 in ([5], part I) for $|\alpha| \leq n+k+1$ we obtain that

$$\|p(t, x_o, \cdot)\|_{C_b^k} \leq C_k\, t^{-p(n+k+2)} \qquad (\forall)\quad 0 < t \leq T_o$$

and the proof is complete.

Proof of Theorem 1

By hypothesis the conditions in Lemma 4 are fulfilled. Take $U(z)$ in (1) as it is defined in (19). In this case $d^u(T)=(\det N^u(T))^{-1}=(p^u(T))^{-n}$ where $p^u(T)$ is defined in remark 2. Using Lemma 4 we obtain that $d^u(T)$ fulfils the hypotheses in Lemma 6 with $p=n$ and $T_o > 0$ deduced from Lemma 4.

Therefore the hypotheses in Theorem 2 are fulfilled with $p=n$ and using theorem 2 we get the statement.

The proof is complete.

4. ON THE REGULARITY OF THE PROBABILITIES ASSOCIATED WITH A CLASS OF STOCHASTIC EQUATIONS WITH DELAY

The equations we consider are defined in the first part (see (*)) and can be assimilated with a class of nonaticipating coefficients equations for which the existence of the smooth density is analyzed in [8] and [9]. Generally as it appears in [8] and [9], the problem is solved assuming that the diffusion matrix is a global nondegenerate one.

The problem we consider has the particular feature that the diffusion matrix is a degenerate one and the main tool in getting the result is a simple version of the Malliavin Calculus as it is stated in Theorem 2.

The strong solution in (*) exists and it is constructed succesively on the intervals of the length h supposing that the initial condition $x_{2o}(s),\dots,x_{po}(s)$, $s\in[-h,0]$, is a continuous function.

As in Theorem 2 the original solution $x(.)=(x_1(.),\dots,x_p(.))$ in (*) is embedded into a family of solutions $x^u(.)$, $u\in R^{mp}$, which for each $T>0$ fixed fulfil the following properties

a) $x^u(.)$ generate the same probability measure π on $C([0,T];R^{mp})$ for any $u\in R^{mp}$,

b) $x^u(T)$ is differentiable in u of any order in $L_2(\Omega,P)$ and $D_u^{\alpha}x^u(T)\in\bigcap_{q\geq 1}L_q(\Omega,P)$,

c) the matrix $[\partial x^u(T)/\partial u]_{u=0}=M(T)$ is a nonsingular one and $(\det M(T))^{-1}\in\bigcap_{q\geq 1}L_q(\Omega,P)$.

The embedding is based on the changing the original Wiener process $w(.)$ into a new one $w^u(.)$, and it is performed separately on each interval $[0,h]$, $[h,2h]$,..., $[kh,(k+1)h]$,... . On each fixed interval $[kh,(k+1)h]$ the equations (*) are replaces by the equations of the type (5) where $n=mp$, $f=(f^1,f^2,\dots,f^n)$, $g_i=(g_i^1,\underbrace{\theta,\dots,\theta}_{(p-1)})$, θ-the null element in R^m. This time $f:R^{2m}\longrightarrow R^n(f(\tilde{x},x))$, $A(\tilde{x},x)=\partial f/\partial x(\tilde{x},x)$, $g_i:R^m\longrightarrow R^n$, $B_i(x)=$ $=\partial g_i/\partial x_1(x_1)$, $\tilde{A}(\tilde{x},x)=A(x,x)-\sum_{i=1}^{m}B_i^2(x)$, $z=(x,X,Y)$; and the matrix $\mathcal{U}(z)=(\mathcal{U}_j^i(z))$, $i=1,\dots,m$, $j=1,\dots,n$, has its elements in $C_b^\infty(R^{n+2n^2})$ such that $(YG(x)\mathcal{U}(z))$ is positive definite $(\geqslant 0)$ and $(YG(x)\mathcal{U}(z))_{ij}\in C_b(R^{n+2n^2})$, where $G=(g_1\dots g_m)$. The corresponding system (5) is written for $\tilde{x}=x(t-h)$ and the solution for $t\in[kh,(k+1)h]$ is constructed by considering that $x(t)$, $t\in[0,kh]$, is the solution in (*) and the initial conditions are $x_o(s)=x(s)$, $s\in[(k-1)h,kh]$, $X(kh)=Y(kh)=I$; denote this system by (**) and let $z^u(t)=(x^u(t),X^u(t),Y^u(t))$ be the solution in (**) for $t\in[kh,(k+1)h]$. Using Ito's differential rule it follows $(X^u(t))^{-1}=Y^u(t)$, $t\in[kh,(k+1)h]$, and by the usual rule of derivation we get

$$(***)\quad M^o(T)=[\partial x^u(T)/\partial u]_{u=0}=X^o(T)\int_{kh}^{T}Y^o(t)G(x^o(t))\,\mathcal{U}(z^o(t))dt$$

where $z^o(t)$ is the solution in (**) for $u=0$.

Denote $G_1(x_1)=(g_1^1(x_1)\dots g_m^1(x_1))$, $\mathcal{A}_1=G_1G_1^*$ and assume

I_1) $\det\mathcal{A}_1(x_1)\geqslant k_o>0$, for any $x_1\in R^m$

Denote $A^{j+1}(\tilde{x}_j,x_j)=\partial f^{j+1}/\partial x_j\ (\tilde{x}_j,x_j)$, $j=1,\dots,p-1$ and assume

I_2) the matrices A^j, $j=2,\dots,p$, are constant and $|\det A^j| \geq k_1 > 0$.

By a direct inspection we get that the matrices $X^o(.)$, $Y^o(.)$ have a triangular form

$$X^o(.) = \begin{bmatrix} X_{11}(.) & 0 & \dots & 0 \\ X_{21}(.) & I & 0 & 0 \\ \vdots & & & \\ X_{p1}(.) & \dots & X_{pp-1}(.) & I \end{bmatrix}, \quad Y^o(.) = \begin{bmatrix} Y_{11}(.) & 0 & \dots & 0 \\ Y_{21}(.) & I & 0 \dots & 0 \\ \vdots & & & \\ Y_{p1}(.) & \dots & Y_{pp-1}(.) & I \end{bmatrix}$$

where $Y_{11}(t)=(X_{11}(t))^{-1}$, X_{ij}, Y_{ij} are (mxm) matrices, I is the identity matrix and 0 is the zero matrix.

It is easily seen that $Y_{j1}(t)$, $j=2,\dots,p$, $t \in [kh,(k+1)h]$, are given by $Y_{21}(t)=M_2(t)Y_{11}(t)$, $Y_{31}(t)=M_3(t)Y_{11}(t),\dots$, $Y_{p1}(t)=M_p(t)Y_{11}(t)$, where

$$M_j(t)=(t-kh)^{j-1}A^2\dots A^j\tilde{M}_j(t) \text{ and } \tilde{M}_j(t)=\frac{(-1)^{j-1}}{(j-2)!}(t-kh)^{1-j}.$$

$$\cdot \int_{kh}^{t}(t_1-kh)^{j-2}X_{11}(t_1)dt_1 \qquad \tilde{M}_j(0)=(-1)^{j-1}/(j-2)!$$

We define

$$U(z)=(1+|z|^{8n})^{-1}\left[G_1^* 0_1^*\right]\begin{bmatrix}\bar{\mathcal{A}}_1 \\ 0_1\end{bmatrix} X_{11}\left[IM_2^*\dots M_p^*\right]$$

where $\bar{\mathcal{A}}_1\mathcal{A}_1=(\det \mathcal{A}_1)I$ and 0_1 is the zero matrix with (p-1)m lines and m columns.

The corresponding matrix in (***) becomes

$$M^o(T)=X^o(T)\int_{kh}^{T}(1+|z(t)|^{8n})^{-1}\det\mathcal{A}_1(x_1(t))\cdot\begin{bmatrix} I \\ M_2(t) \\ \vdots \\ M_p(t)\end{bmatrix}\left[IM_2^*(t)\ldots M_p^*(t)\right]dt$$

and $(\det M^o(T))^{-1}\in\bigcap_{q\geq 1}L_q(\Omega,P)$ if $(\det \tilde{N}(T))^{-1}\in\bigcap_{q\geq 1}L_q(\Omega,P)$,

where

$$\tilde{N}(T)=\int_{kh}^{T}\begin{bmatrix} I \\ M_2(t) \\ \vdots \\ M_p(t)\end{bmatrix}\left[IM_2^*(t)\ldots M_p^*(t)\right]dt$$

Using a stopping time $\tau=\inf\{t\in(kh,(k+1)h] : |X_{11}(t)-I|\geq\rho\}$

we get $\max_{t\in[0,\tau]}|\tilde{M}_j(t)|\leq c_1$, $\max_{t\in[0,\tau]}|\tilde{M}_j(t)-\tilde{M}_j(0)|\leq c_1\rho$ and

$\det\tilde{N}(T\wedge\tau)\geq\frac{1}{2}(T\wedge\tau-kh)^{p^2m}\Delta_o$ for ρ sufficiently small where

$$0<\Delta_o=\det\begin{bmatrix} I & \frac{1}{2}N_2^*(0) & \ldots & \frac{1}{p}N_p^*(0) \\ 1/2N_2(0) & \ldots & 1/(p+1)N_2(0)N_2^*(0) & \\ \vdots & & & \\ 1/pN_p(0) & \ldots & 1/(p-1)N_p(0)N_p^*(0) & \end{bmatrix},\quad N_j(t)=A^2\ldots A^j\tilde{M}_j(t)$$

Since $(\tau-kh)^{-1}\in\bigcap_{q\geq 1}L_q(\Omega,P)$ we obtain $(\det\tilde{N}(T\wedge\tau))^{-1}\in\bigcap_{q\geq 1}L_q(\Omega,P)$

for any $kh<T\leq(k+1)h$ and using Proposition and Lemma 6 as in the proof of Theorem 2 we get

THEOREM 3

Assume that I_1 and I_2 are fulfilled for (*). Then $(\det M^o(T))^{-1} \in \bigcap_{q \geq 1} L_q(\Omega, P)$ and the probability measure $P(T,.)$ generated on R^{mp} by the solution $x(T,.)$ in (*) has a smooth density $P(T,dx)=p(T,x)dx$ with $p(T,.) \in C_b^{\infty}(R^{pm})$ for any $T > 0$.

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