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by

Gh. Ştefănescu

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Gh. Ştefănescu*)

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*) Department of Mathematics, National Institute for Scientific and Technical Creation, Bd.Pacii 220, 79622 Bucharest, Romania.

ON FLOWCHART THEORIES II (nondeterministic case)

By Gh. Stefănescu

To my wife

Abstract This part of the paper givesan algebraic specification of the semantic of simple nondeterministic program, i.e. with sequential composition, parallel composition(random choice) and feedback(random iteration).

Introduction

Many years, as a man, the computer was able to do only one linear string of operations in its consciousness. In the last ten years, due to some theoretical and practical progresses, there was possible to construct a computer which work as a lot of people, i.e. more than one devil pass through a program and work together on the same problem. In order to do correctly their job, they must to synchronize, to communicate one with the other, Dijkstra [4], Hoare [8] and Milner [9]. Of course, we want efficient programs. Since the programmer can not expect the exact order in which different devils finish their jobs, we have to allow the computer to have a nondeterministic behaviour, namely a devil may wait signals from many others and choice one, from the existent ones, in a random way. At the actual stage of our(my) knowledge it seems to be quite difficult to have an exact, mathematical semantic (one way in which we can simplify the problem is to give an operational semantic by interliving, i.e. to put one devil to do the entire job: it choice in a random way a devil x, skip there and do an atomic step for x).

Here we give an algebraic semantic for the clasical nondeterministic programs (those which has,as possible behaviour,a linear, but nondeterministic string of operations).

In the previous paper [10] we defined a common generalization of iterative algebraic theories [5] and ω-continuous algebraic theories [1] (stronger than iteration theories [6]), namely the so-called theories with strong iterate. The main result was to show that (a quotient of) the abstract Σ-flowcharts over such a theory T is the theory with strong iterate freely generated by adding Σ to T. Here we look for an analogous result, in the nondeterministic case. In this more general case the things seem to be more natural. The corresponding structure is, roughly speaking, a bit more strong than a theory with strong iterate for which the dual category is also a theory with strong iterate. But there is a more direct way to give this structure:

> We call a <u>repetitive reticulum</u> T the set of matrices over a semiring with $1+1=1$, which has, beside $+$ and $\cdot$, a $*$-operation that fulfils four natural axioms (an axiomatization of $A^* = 1 + A + A^2 + \ldots$).

If Σ denote a set of atomic flowcharts, we define the theory $Fl_{\Sigma,T}$ of abstract Σ-flowcharts over T (the flowcharts with vertices from Σ connected by morphisms from T) with natural $+,\cdot,*$ operations (union, sequential composition and repetition). Every Σ-flowchart over T may be represented as a matrix $f = e\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, where e is a string of its atomic elements (labels of its vertices) and the morphisms A,B,C,D from T, respectively give the out-of-the-box behaviour of f, the inputs from exterior into the box, the outputs from the box to exterior and the into-the-box behaviour of f. As in the paper of Goguen and Messeguer [7] we have access only to the visible behaviour of the flowchart (there to some sorts of a multi-sorted algebra), namely to its inputs and outputs. Hence we allow the flowchart to be changed (for example, to minimize the number of its vertices) by deletion or adding inaccessible or coinaccessible parts or by folding or unfolding some vertices, if its input-output relation do not change.

Mathematically, we define an equivalence $\equiv$ on $Fl_{\Sigma,T}$ (stronger than the bisimulation of Park [3] and which preserves the computing paths) and show that the quotient $RFl_{\Sigma,T} = Fl_{\Sigma,T}/\equiv$ is the repetitive reticulum freely generated by adding Σ to T (as X-polinomials over a ring R , with equivalence = reduction of similar terms, is the ring freely generated by adding X to R ; in a more categorial language this is the coproduct of a given structure with the free one generated by a set), eachtime when $RFl_{\Sigma,T}$ is a repetitive reticulum. The main technical result show that $RFl_{\Sigma,T}$ is a repetitive reticulum if the simulation relation (the basic relation for defining $\equiv$) has the confluent (Church-Rosser) property. For arbitrary Σ, in that case we are if T is $M_{\{0,1\}}$, the repetitive reticulum of matrices over $\{0,1\}$. Hence the usual nondeterministic flowcharts $RFl_{\Sigma,M_{\{0,1\}}}$ is the repetitive reticulum freely generated by Σ.

It remains an open problem to see if $RFl_{\Sigma,T}$ is always a repetitive reticulum, eachtime when T is.

We point here some limits in the application of our results. Particularly, our equivalence $\equiv$ says that every $\sigma \in \Sigma$ is isomorphic with the matrix of its components (its behaviour is known if we know for every i,j the behaviour of σ when we restrict σ to its i-input and j-output). This is not always true. For example, the interliving operator $\| \in \Sigma_{2,1}$ (which make one devil from two) has the matrix of its components equal to $\begin{bmatrix}1\\1\end{bmatrix}$ (as $w\|1 = w$, the first component is $(1+0)\| = 1$).

Part I: Algebraic foundations

1. Repetitive reticula

The aim of this part is to give an algebraic structure, called repetitive reticula, which in our view models the connections between vertices in a flowchart. To make the things more readable we restrict ourselves to the onesorted case.

A <u>semireticulum</u> M_R is a category in which the objects are the natural numbers and the set of morphisms, with usual matrix multiplication as composition, are

$$M_R(m,n) = \text{the set of } m \times n \text{ matrices over } R,$$

$(R,+,0,\cdot,1)$ being a semiring (i.e. $+$ is an associative and commutative addition with 0 neutral element, $\cdot$ is an associative multiplication with 1 neutral element and $\cdot$ is left and right distributive with respect to $+$).

We shall denote the component-vise addition of matrices also by $+$, by 0 and 1 the zero matrices and the identity matrices of adequate dimensions, by $x_i^n \in M_R(1,n)$ the zero row vector with a 1 on place i and by $y_i^n \in M_R(n,1)$ the transposition of x_i^n. $\{0,1\}$ will denote the boolean semiring, i.e. $+$ and $\cdot$ are the usual boolean operations <u>or</u> and <u>and</u>.

A semireticulum M_R is a <u>reticulum</u> if in R holds $1+1=1$.

A <u>repetitive reticulum</u> is a reticulum T endowed with an operation, called repetition,

$$* \; ; \; T(n,n) \longrightarrow T(n,n)$$

(which intuitively means to nondeterministicaly repeat the application of a morphism zero or more times), which fulfils the following axioms:

(R1) $A^* = AA^*+1_n = A^*A+1_n$, if $A \in T(n,n)$;

(R2) $(A+B)^* = (A^*B)^*A^*$, if $A,B \in T(n,n)$;

(R3) $A(BA)^* = (AB)^*A$, if $A \in T(m,n)$ and $B \in T(n,m)$;

(R4) if $A\varrho = \varrho B$ then $A^*\varrho = \varrho B^*$, for every $A \in T(m,m)$, $B \in T(n,n)$ and $\varrho \in M_{\{0,1\}}(m,n)$.

The following proposition shows how one can compute the $*$ of a morphism when he know the $*$ of its components.

Proposition 1.1. In a repetitive reticulum

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} A^* + A^*BWCA^* & A^*BW \\ WCA^* & W \end{bmatrix},$$

where $W = (CA^*B + D)^*$.

Proof. Firstly we shall prove that

$$\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^* = \begin{bmatrix} A^* & A^*B \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix}^* = \begin{bmatrix} 1 & 0 \\ D^*C & D^* \end{bmatrix}.$$

Indeed,

$$\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} \right)^* \begin{bmatrix} 1 \\ 0 \end{bmatrix} \underset{(R3)}{=} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot A^*$$

$$= \begin{bmatrix} A^* \\ 0 \end{bmatrix}.$$

Hence

$$\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^* = \begin{bmatrix} A^* & X \\ 0 & Y \end{bmatrix}.$$

Using (R1) one can see that

$$\begin{bmatrix} A^* & X \\ 0 & Y \end{bmatrix} = \begin{bmatrix} A^* & X \\ 0 & Y \end{bmatrix} \cdot \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} AA^*+1 & A^*B \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A^* & A^*B \\ 0 & 1 \end{bmatrix}.$$

In a similar way one can prove the second equality.

With these identities we may return to the starting equality and compute:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix} \right)^* \underset{\substack{(R2) \\ \&(R3)}}{=} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^* \left(\begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix} \cdot \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^* \right)^*$$

$$= \begin{bmatrix} A^* & A^*B \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ CA^* & CA^*B+D \end{bmatrix}^* = \begin{bmatrix} A^* & A^*B \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ WCA^* & W \end{bmatrix}$$

$$= \begin{bmatrix} A^* + A^*BWCA^* & A^*BW \\ WCA^* & W \end{bmatrix}. \quad \square$$

<u>Basic example.</u> Let S be a set (of states). Then

$\text{Relations}_S(m,n)$ = the $m \times n$ matrices with elements relations on S

with usual operations (+ = union, $\cdot$ = composition and $A^* = 1 \cup A \cup A^2 \cup \ldots$) is a repetitive reticulum. Particularly, if S has only one element then $\text{Relations}_S = M_{\{0,1\}}$.

Part II: Flowchart theories

2. Definitions

Notations: $[n] = \{1,\dots,n\}$ and $\omega = \{0,1,\dots\}$.

If $(T, \cup, 0, \cdot, 1, *)$ is a repetitive reticulum and Σ a double ranked alphabet for the set of atomic flowcharts ($r_{in}, r_{out} : \Sigma \to \omega$ give the in- and out-ranks; their monoid extensions to Σ^* will be denoted by r_{in}, r_{out}, too), then a <u>Σ-flowchart over T</u> with m inputs and n outputs is a double $f = (1,e)$, where $e \in \Sigma^*$ and $1 \in T(m+r_{out}(e),\ n+r_{in}(e))$.

In a matrix representation this means (typical, every string $e \in \Sigma^*$ is $e = e_1 \dots e_{|e|}$, where $|e|$ is the length of e and $e_i \in \Sigma$ for $i \in [|e|]$)

$$
\begin{array}{ll}
 & \begin{array}{cccc} n & \overbrace{r_{in}(e_1)}^{e_1} & \dots & \overbrace{r_{in}(e_{|e|})}^{e_{|e|}} \end{array} \\
\begin{array}{l} m \\ e_1 \left\{ r_{out}(e_1) \right. \\ \vdots \\ e_{|e|} \left\{ r_{out}(e_{|e|}) \right. \end{array} &
\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]
\end{array}
$$

where the left-up corner A gives the visible behaviour of the flowchart (out of the box), the right-down corner D gives the nonvisible(internal) behaviour of the flowchart (into the box), the B-part gives the inputs from exterior into the box and the C-part gives the outputs from the box to exterior. For example the $\{\sigma, \tau\}$-flowchart over $M_{\{0,1\}}$ from figure 1 may be represented as it is shown there.

$$
\begin{array}{c}
\\ 2 \\ \\ \sigma\{ \\ \tau - \\ \sigma\{
\end{array}
\begin{array}{c|cccccc}
3 & \sigma & & \tau & \sigma & \\
\hline
0\,0\,0 & 1 & 0 & 1 & 0 & 0 \\
0\,0\,1 & 0 & 0 & 0 & 1 & 0 \\
\hline
0\,0\,0 & 0 & 0 & 0 & 0 & 0 \\
1\,0\,0 & 0 & 0 & 0 & 0 & 0 \\
0\,0\,0 & 0 & 1 & 0 & 0 & 0 \\
0\,1\,1 & 0 & 0 & 1 & 0 & 0
\end{array}
$$

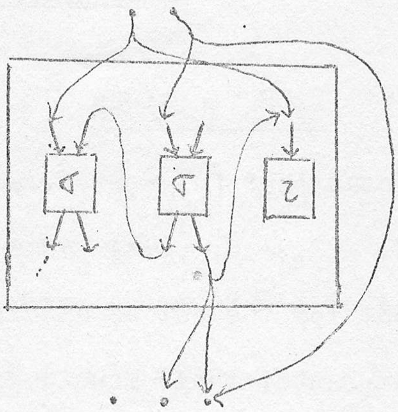


Figure 1.

We denote by $Fl_{\Sigma,T}(m,n)$ the set of Σ-flowcharts over T with m inputs and n outputs. Every $f \in T(m,n)$ may be represented as the visible flowchart (f, λ) and every $\sigma \in \Sigma$ as the flowchart

$$
\begin{array}{c}
\\ r_{in}(\sigma) \\ \sigma
\end{array}
\begin{array}{c|c}
r_{out}(\sigma) & \sigma \\
\hline
0 & 1 \\
\hline
1 & 0
\end{array}
.
$$

The fundamental operations on flowchart, in the matrix representation, look as follows (the sharing of rows and columns with respect to ranks of atomic flowchart from Σ is preserved and we omit its writing):

<u>union</u>

$$
\begin{array}{c} m \\ e \end{array}
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\,n \;\; e}
\cup
\begin{array}{c} m \\ e' \end{array}
\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}^{\,n \;\; e'}
=
\begin{array}{c} \\ m \\ e \\ e' \end{array}
\left(\begin{array}{c|cc}
n & e & e' \\
\hline
A \cup A' & B & B' \\
\hline
C & D & 0 \\
C' & 0 & D'
\end{array}\right)
;
$$

<u>composition</u>

$$\begin{array}{c|cc} & n & e \\ \hline m & A & B \\ e & C & D \end{array} \circ \begin{array}{c|cc} & p & e' \\ \hline n & A' & B' \\ e' & C' & D' \end{array} = \begin{array}{c|ccc} & p & e & e' \\ \hline m & AA' & B & AB' \\ e & CA' & D & CB' \\ e' & C' & 0 & D' \end{array} ;$$

and <u>repetition</u>

$$\left(\begin{array}{c|cc} & m & e \\ \hline m & A & B \\ e & C & D \end{array}\right)^{*} = \begin{array}{c|cc} & m & e \\ \hline m & A^{*} & A^{*}B \\ e & CA^{*} & CA^{*}B \cup D \end{array} .$$

From now on we shall not write the sharing of columns with respect to Σ as it is a direct consequence of the sharing of rows.

We shall define a derived operation which shows how can be construct an $m \times n$ matrix of flowcharts $[f_{ij}]_{i,j}$,

$$[f_{ij}]_{i,j} = \bigcup_{i \in [m]} \bigcup_{j \in [n]} y_i^m f_{ij} x_j^n .$$

In the matrix representation this is

$$[f_{ij}] = \begin{array}{c} \\ \\ \\ e_{11} \\ \vdots \\ e_{1n} \\ \vdots \\ e_{m1} \\ \vdots \\ e_{mn} \end{array} \left[\begin{array}{ccc|ccccc} A_{11} & \dots & A_{1n} & B_{11} \dots B_{1n} & & & 0 \dots 0 \\ \dots & & & \dots & & \dots & \dots \\ A_{m1} & \dots & A_{mn} & 0 \dots 0 & & & B_{m1} \dots B_{mn} \\ \hline C_{11} & \dots & 0 & D_{11} \dots 0 & & & 0 \dots 0 \\ \dots & & & \dots & & \dots & \dots \\ 0 & \dots & C_{1n} & 0 & D_{1n} & & 0 \dots 0 \\ \vdots & & & \vdots & & & \vdots \\ C_{m1} & \dots & 0 & 0 \dots 0 & & & D_{m1} \dots 0 \\ \dots & & & \dots & & \dots & \dots \\ 0 & \dots & C_{mn} & 0 \dots 0 & & & 0 \dots D_{mn} \end{array}\right]$$

3. Equivalent flowcharts

Denote by $Rel_\Sigma(e,e')$, for $e,e' \in \Sigma^*$ the set of relations ϱ from $[|e|]$ to $[|e'|]$ which preserves labels, i.e.

$$\text{if} \quad (i,j) \in \varrho \quad \text{then} \quad e_i = e'_j \ .$$

Every $\varrho \in Rel_\Sigma(e,e')$ has a natural extension to inputs

$$\varrho_{in} \in M_{\{0,1\}}(r_{in}(e),\ r_{in}(e'))$$

obtained by replacing every 0,1 in the matrix ϱ by the zero, identity matrix of appropriate dimensions (if this 0,1 is on i - row and j - column, then these dimensions are $r_{in}(e_i)$, $r_{in}(e'_j)$). Similar with ϱ_{out}-extension.

<u>The basic relation.</u> Let $f=(l,e)$, $f'=(l',e') \in Fl_{\Sigma,T}(m,n)$ and $\varrho \in Rel_\Sigma(e,e')$. We say that f is <u>simulated via ϱ</u> by f' , and write $f \xrightarrow[\varrho]{} f'$, if

$$\left[\begin{array}{c|c} A & B\varrho_{in} \\ \hline C & D\varrho_{in} \end{array}\right] = \left[\begin{array}{c|c} A' & B' \\ \hline \varrho_{out}C' & \varrho_{out}D' \end{array}\right].$$

Perhaps the following figure gives some points for its intuitive understanding.

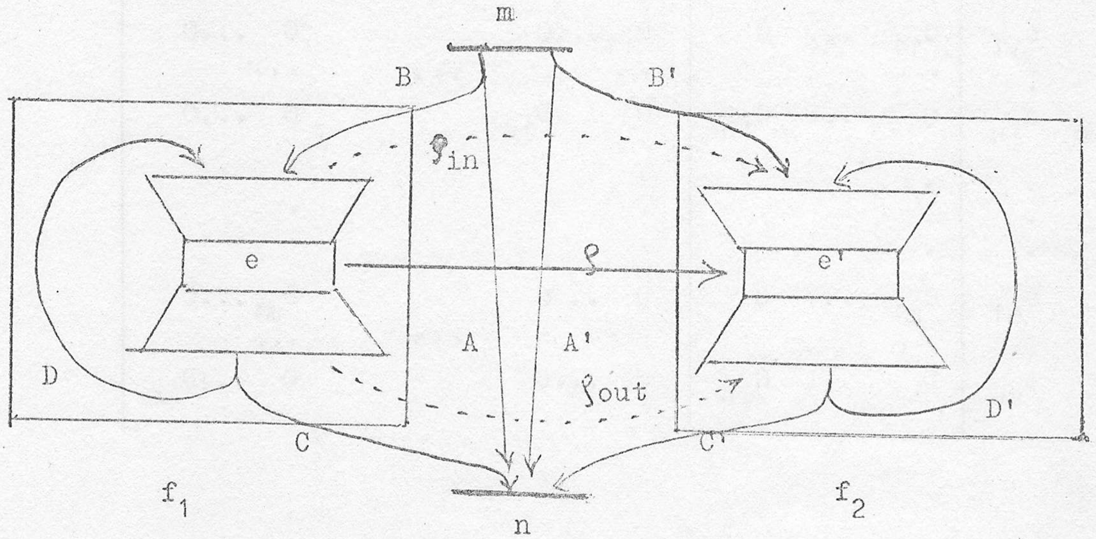


Figure 2.

It may easily be seen that this is a transitive, reflexive relation, but not a transitive one. Our equivalence on flowcharts is that generated by simulation. This means that f is <u>equivalent</u> with f', write this $f \equiv f'$, if there exist f_i, ϱ_i such that

$$f \xrightarrow[\varrho_1]{} f_1 \xleftarrow[\varrho_2]{} f_2 \xrightarrow[\varrho_3]{} f_3 \ldots f_{2k-1} \xleftarrow[\varrho_{2k}]{} f' \, .$$

Remark. If the simulation relation has the confluent(or Church-Rosser) property (namely, for every $f \xrightarrow[\varrho]{} f_1$, $f \xrightarrow[\tau]{} f_2$ there exist $\overline{f}, \varrho_1, \tau_1$ such that $f_1 \xrightarrow[\varrho_1]{} \overline{f} \xleftarrow[\tau_1]{} f_2$), then this equivalence may be written as

$f \equiv f'$ iff there exist $\overline{f}, \varrho, \tau$ such that $f \xrightarrow[\varrho]{} \overline{f} \xleftarrow[\tau]{} f'$. □

The following lemma shows the compatibility of simulation with flowchart operations.

<u>Lemma 3.1.</u> If $f_1 \xrightarrow[\varrho]{} f_1'$, $f_2 \xrightarrow[\tau]{} f_2'$ then the following relations hold, whenever the operations make sense,

a) $f_1 \cup f_2 \xrightarrow[\begin{bmatrix} \varrho & 0 \\ 0 & \tau \end{bmatrix}]{} f_1' \cup f_2'$;

b) $f_1 \circ f_2 \xrightarrow[\begin{bmatrix} \varrho & 0 \\ 0 & \tau \end{bmatrix}]{} f_1' \circ f_2'$;

c) $f_1^* \xrightarrow[\varrho]{} f_1'^*$.

<u>Proof.</u> Obvious. □

<u>Proposition 3.2.</u> The equivalence $\equiv$ is compatible with union, composition and repetition.

<u>Proof.</u> By using trivial simulations given by $\varrho = 1_e$ we can suppose that different chains of simulations have the same length. The conclusion directly follows from lemma 3.1. □

If $RFl_{\Sigma,T} = Fl_{\Sigma,T}/_{\equiv}$, then the above proposition shows that $\cup$, $\cdot$ and $*$ are well-defined in $RFl_{\Sigma,T}$.

Examples of equivalent flowcharts:

$$\begin{array}{c} \\ e_1 \\ e_2 \end{array}\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ 0 & 0 & D_{22} \end{array}\right] \xrightarrow[\left[\begin{array}{c} 1_{e_1} \\ 0_{e_2,e_1} \end{array}\right]]{} \begin{array}{c} \\ e_1 \end{array}\left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right] ;$$

$$\begin{array}{c} \\ e \\ e \end{array}\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & D_{11} & D_{12} \\ C & D_{21} & D_{22} \end{array}\right] \xrightarrow[\left[\begin{array}{c} 1_{e} \\ 1_{e} \end{array}\right]]{} \begin{array}{c} \\ e \end{array}\left[\begin{array}{c|c} A & B_1 \cup B_2 \\ \hline C & D_{11} \cup D_{12} \end{array}\right] \text{ if } D_{21} \cup D_{22} = D_{11} \cup D_{12};$$

$$\begin{array}{c} \\ e_1 \end{array}\left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right] \xrightarrow[\left[\begin{array}{cc} 1_{e_1} & 0_{e_1,e_2} \end{array}\right]]{} \begin{array}{c} \\ e_1 \\ e_2 \end{array}\left[\begin{array}{c|cc} A & B_1 & 0 \\ \hline C_1 & D_{11} & 0 \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

and

$$\begin{array}{c} \\ e \end{array}\left[\begin{array}{c|c} A & B \\ \hline C_1 \cup C_2 & D_{11} \cup D_{21} \end{array}\right] \xrightarrow[\left[\begin{array}{cc} 1_{e} & 1_{e} \end{array}\right]]{} \begin{array}{c} \\ e \\ e \end{array}\left[\begin{array}{c|cc} A & B & B \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \text{ if } D_{11} \cup D_{21} = D_{12} \cup D_{22}.$$

4. The main result

We want to have a formula for the semantic of $\begin{array}{c} \\ e \end{array}\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ in a repetitive reticulum Q , when we know the meaning of T in Q by $\varphi_T : T \longrightarrow Q$ and of Σ in Q by $\varphi_{\Sigma} : \Sigma \longrightarrow Q$. The extension of φ_{Σ} to Σ^* , also denoted by φ_{Σ} , is its monoid extension, when Q is a monoid with the operation

$$\begin{array}{c} \\ m \\ e \end{array}\left[\begin{array}{c|c} \overset{n}{A} & B \\ \hline C & D \end{array}\right] \,||\, \begin{array}{c} \\ m' \\ e' \end{array}\left[\begin{array}{c|c} \overset{n'}{A'} & B' \\ \hline C' & D' \end{array}\right] = \begin{array}{c} \\ m \\ m' \\ e \\ e' \end{array}\left[\begin{array}{cc|cc} \overset{n}{A} & \overset{n'}{0} & B & 0 \\ 0 & A' & 0 & B' \\ \hline C & 0 & D & 0 \\ 0 & C' & 0 & D' \end{array}\right] .$$

The desired formula is

$$\varphi^{\#}\left(e\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]\right) = \varphi_T(A) \cup \varphi_T(B)\,\varphi_{\Sigma}(e)\,(\varphi_T(D)\,\varphi_{\Sigma}(e))^{*}\,\varphi_T(C) .$$

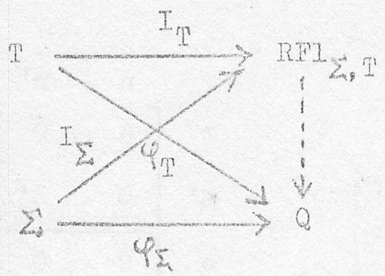
The right expression show what can be heard from outside after zero (or directly, $\varphi_T(A)$), one ($\varphi_T(B)\varphi_{\Sigma}(e)\varphi_T(C)$), or more inward racks.

<u>MAIN THEOREM 4.1.</u> $RFl_{\Sigma,T}$ is the repetitive reticulum frely generated by adding Σ to the repetitive reticulum T (in a more categorial language $RFl_{\Sigma,T}$ is the coproduct, in the category of repetitive reticula, of T and the free repetitive reticulum generated by Σ), eachtime when $RFl_{\Sigma,T}$ is a repetitive reticulum.

<u>Proof.</u>

We have to show the property of universality, namely that there are a repetitive reticulum morphism $I_T: T \longrightarrow RFl_{\Sigma,T}$ and a rank-preserving function $I_{\Sigma}: \Sigma \longrightarrow RFl_{\Sigma,T}$ such that for every repetitive reticulum $(Q,\cup,0,\cdot,1,*)$ and every repetitive reticulum morphism $\varphi_T: T \longrightarrow Q$ and rank-preserving function $\varphi_{\Sigma}: \Sigma \longrightarrow Q$ there exists exactly one repetitive reticulum morpfism $\varphi^{\#}: RFl_{\Sigma,T} \longrightarrow Q$ such that $I_T\varphi^{\#} = \varphi_T$ and

$I_\Sigma \varphi^{\#} = \varphi_\Sigma$.



I_T and I_Σ are the above embeding of T and Σ in $RFl_{\Sigma,T}$, namely

$$I_T(A) = [A] \quad \text{and} \quad I_\Sigma(\sigma) = {}_\sigma\left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right] .$$

We define $\varphi^{\#}$ on $Fl_{\Sigma,T}$ as was shown

$$\varphi^{\#}\left({}_e\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \right) = \varphi_T(A) \cup \varphi_T(B)\varphi_\Sigma(e)\,(\varphi_T(D)\varphi_\Sigma(e))^* \varphi_T(C) .$$

The first problem is to show that this is a well-defined function on $RFl_{\Sigma,T}$, namely

$$\text{if } f \equiv f' \text{ then } \varphi^{\#}(f) = \varphi^{\#}(f') .$$

In fact it is enough to prove this implication only for the simulation

$$\text{if } f \xrightarrow[\rho]{} f' \text{ then } \varphi^{\#}(f) = \varphi^{\#}(f') .$$

Remark that for $\rho \in Rel_\Sigma(e,e')$ we have $\varphi_\Sigma(e)\rho_{out} = \rho_{in}\varphi_\Sigma(e')$.

Hence we can apply to

$$\varphi_T(D)\varphi_\Sigma(e)y_{out} = \varphi_T(D)y_{in}\varphi_\Sigma(e') = y_{out}\varphi_T(D')\varphi_\Sigma(e')$$

the axiom (R4), which gives

$$(\varphi_T(D)\varphi_\Sigma(e))^* \, y_{out} = y_{out}\,(\varphi_T(D')\varphi_\Sigma(e'))^* .$$

Now since $A = A'$, $C = y_{out}C'$ and $By_{in} = B'$, one can easy see that

$$\varphi^{\#}(f) = \varphi_T(A) \cup \varphi_T(B)\varphi_\Sigma(e)\cdot(\varphi_T(D)\varphi_\Sigma(e))^* \varphi_T(C)$$

$$= \varphi_T(A') \cup \varphi_T(B')\varphi_\Sigma(e')\cdot(\varphi_T(D')\varphi_\Sigma(e'))^* \varphi_T(C') = \varphi^{\#}(f') .$$

The second problem is to show that $\varphi^{\#}$ preserves the operations. For the sake of simplicity we shall dropp the writing of $\varphi_T, \varphi_\Sigma$ and use typical notations: a for $\varphi_T(A)$, b for $\varphi_T(B)\varphi_\Sigma(e)$, c for $\varphi_T(C)$ and d for $\varphi_T(D)\varphi_\Sigma(e)$. The main technical result to be use is that from Proposition 1.1 .

Union:

$$\varphi^{\#}(f \cup f') = a \cup a' \cup [b \quad b'] \begin{bmatrix} d & 0 \\ 0 & d' \end{bmatrix}^{*} \begin{bmatrix} c \\ c' \end{bmatrix}$$

$$= a \cup a' \cup bd^{*}c \cup b'd'^{*}c' = \varphi^{\#}(f) \cup \varphi^{\#}(f') .$$

Composition:

$$\varphi^{\#}(f \cdot f') = aa' \cup [b \quad ab'] \begin{bmatrix} d & cb' \\ 0 & d' \end{bmatrix}^{*} \begin{bmatrix} ca' \\ c' \end{bmatrix}$$

$$= aa' \cup [b \quad ab'] \begin{bmatrix} d^{*} & d^{*}cb'd'^{*} \\ 0 & d'^{*} \end{bmatrix} \begin{bmatrix} ca' \\ c' \end{bmatrix}$$

$$= (a \cup bd^{*}c) \cdot (a' \cup b'd'^{*}c') = \varphi^{\#}(f) \cdot \varphi^{\#}(f') .$$

Repetition:

$$\varphi^{\#}(f) = a^{*} \cup a^{*}b(ca^{*}b \cup d)^{*}ca^{*} \underset{(R2)}{=} a^{*}(1 \cup bd^{*}(ca^{*}bd^{*})^{*}ca^{*})$$

$$\underset{(R3)}{=} a^{*}(1 \cup bd^{*}ca^{*}(bd^{*}ca^{*})^{*}) \underset{(R1)}{=} a^{*}(bd^{*}ca^{*})^{*} \underset{(R2)}{=} (a \cup bd^{*}c)^{*} = \varphi^{\#}(f)^{*}.$$

The third problem is to show that the extension $\varphi^{\#}$ is the unique repetitive reticulum morphism such that $I_T\varphi^{\#} = \varphi_T$ and $I_\Sigma\varphi^{\#} = \varphi_\Sigma$. It is enough to see that every flowchart f has an equivalent representation as

$$f' = I_T(A) \cup I_T(B)I_\Sigma(e) \, (I_T(D)I_\Sigma(e))^{*} \, I_T(C) .$$

Since

$$I_\Sigma(e) = {}_e\left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right] , \quad I_T(B)I_\Sigma(e) = {}_e\left[\begin{array}{c|c} 0 & B \\ \hline 1 & 0 \end{array}\right] , \quad (I_T(D)I_\Sigma(e))^{*} = {}_e\left[\begin{array}{c|c} 1 & D \\ \hline 1 & D \end{array}\right]$$

we are done

$$f' = [A] \cup \begin{array}{c} \\ e \end{array}\left[\begin{array}{c|c} 0 & B \\ \hline 1 & 0 \end{array}\right] \cdot \begin{array}{c} \\ e \end{array}\left[\begin{array}{c|c} 1 & D \\ \hline 1 & D \end{array}\right] \cdot [C] = \begin{array}{c} \\ e \\ e \end{array}\left[\begin{array}{c|c|c} A & B & 0 \\ \hline C & 0 & D \\ C & 0 & D \end{array}\right] \xrightarrow[{[1_e \;\; 1_e]}]{}$$

$$\begin{array}{c} \\ e \end{array}\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = f \;. \;\; \square$$

As a direct consequence of this fundamental result, the main technical result 5.3 and the appendix A we have the following

<u>Corollary 4.2.</u> $RFl_{\Sigma, M_{\{0,1\}}}$ is the free repetitive reticulum generated by Σ . $\square$

5. The main technical result

Proposition 5.1. $RFl_{\Sigma,T}(1,1)$ with addition = union, multiplication = composition and $0, 1$ the visible flowcharts $(0,\lambda)$, $(1,\lambda)$ has a structure of semiring with $1 \cup 1 = 1$.

Proof. In $Fl_{\Sigma,T}(1,1)$ the union is associative, with 0 neutral element and with $1 \cup 1 = 1$ and the composition is associative, with 1 neutral element, hence in $RFl_{\Sigma,T}(1,1)$ too. Moreover,

$$f \cup f' = \begin{array}{c} \\ e \\ e' \end{array}\left[\begin{array}{c|cc} A \cup A' & B & B' \\ \hline C & D & 0 \\ C' & 0 & D' \end{array}\right] \xrightarrow[\begin{bmatrix} 0 & 1^e \\ 1_{e'} & 0 \end{bmatrix}]{} \begin{array}{c} \\ e' \\ e \end{array}\left[\begin{array}{c|cc} A' \cup A & B' & B \\ \hline C' & D' & 0 \\ C & 0 & D \end{array}\right] = f' \cup f$$

and

$$\bar{f}(f \cup f') = \begin{array}{c} \\ \bar{e} \\ e \\ e' \end{array}\left[\begin{array}{c|ccc} \bar{A}(A \cup A') & \bar{B} & \bar{A}B & \bar{A}B' \\ \hline \bar{C}(A \cup A') & \bar{D} & \bar{C}B & \bar{C}B' \\ C & 0 & D & 0 \\ C' & 0 & 0 & D' \end{array}\right] \xrightarrow[\begin{bmatrix} 1_{\bar{e}} & 0 & 1_{\bar{e}} & 0 \\ 0^e & 1 & 0^e & 0 \\ 0 & 0^e & 0 & 1_{e'} \end{bmatrix}]{}$$

$$\begin{array}{c} \\ \bar{e} \\ e \\ \bar{e} \\ e' \end{array}\left[\begin{array}{c|cccc} \bar{A}A \cup \bar{A}A' & \bar{B} & \bar{A}B & \bar{B} & \bar{A}B' \\ \hline \bar{C}A & \bar{D} & \bar{C}B & 0 & 0 \\ C & 0 & D & 0 & 0 \\ \bar{C}A' & 0 & 0 & \bar{D} & \bar{C}B' \\ C' & 0 & 0 & 0 & D' \end{array}\right] = \bar{f} \cdot f \cup \bar{f} \circ f'. \quad \square$$

Proposition 5.2. $RFl_{\Sigma,T}(m,n)$ is isomorphic with the set of $m \times n$ matrices over the above semiring $RFl_{\Sigma,T}(1,1)$. Hence $RFl_{\Sigma,T}$ is a reticulum.

Morover, the isomorphism is natural, namely

flowchart union = matrix addition

and flowchart composition = matrix multiplication.

<u>Proof.</u> The components of $f \in Fl_{\Sigma,T}(m,n)$ are $x_i^m f y_j^n$ and their matrix is equivalent with f,

$$
[x_i^m f y_j^n] = \begin{array}{c} \\ \\ \\ e \\ \vdots \\ e \\ \vdots \\ e \\ \vdots \\ e \end{array}
\left[\begin{array}{ccc|ccc}
x_1^m A y_1^n \ldots x_1^m A y_n^n & & & x_1^m B \ldots x_1^m B & & 0 \ldots 0 \\
\ldots & & & \ldots & \ldots & \ldots \\
x_m^m A y_1^n \ldots x_m^m A y_n^n & & & 0 \ldots 0 & & x_m^m B \ldots x_m^m B \\
\hline
C y_1^n & \ldots & 0 & D \ldots 0 & & 0 \ldots 0 \\
\vdots & & & \vdots & \ldots & \vdots \\
0 & \ldots & C y_n^n & 0 \ldots D & & 0 \ldots 0 \\
\vdots & & & \vdots & \ldots & \\
C y_1^n & \ldots & 0 & 0 \ldots 0 & & D \ldots 0 \\
\vdots & & & \vdots & \ldots & \vdots \\
0 & \ldots & C y_n^n & 0 \ldots 0 & & 0 \ldots D
\end{array}\right]
$$

$$
\longleftarrow
\begin{bmatrix} 1_e \ldots 1_e & \ldots & 0 \ldots 0 \\ \ldots & & \\ 0 \ldots 0 & \ldots & 1_e \ldots 1_e \end{bmatrix}
\; e \;
\begin{array}{c} \\ \\ \\ e \\ \vdots \\ e \end{array}
\left[\begin{array}{c|c}
 & x_1^m B \ldots 0 \\
A & \ldots \\
 & 0 \ldots x_m^m B \\
\hline
C & D \ldots 0 \\
\vdots & \vdots \\
C & 0 \ldots D
\end{array}\right]
\longrightarrow
[1_e \ldots 1_e] \; e \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = f .
$$

Reciproc, if $f_{ij} \in Fl_{\Sigma,T}(1,1)$, then they are the components of their matrix,

$$
x_i^m \cdot [f_{ij}] \cdot y_j^n = \begin{array}{c} \\ e_{11} \\ \vdots \\ e_{1j} \\ \vdots \\ e_{1n} \\ \cdots \\ e_{i1} \\ \vdots \\ e_{ij} \\ \vdots \\ e_{in} \\ \cdots \\ e_{m1} \\ \vdots \\ e_{mj} \\ \vdots \\ e_{mn} \end{array}
\left[\begin{array}{c|cccccc}
A_{ij} & 0 \ldots 0 & \ldots B_{i1} & \ldots B_{in} & \ldots & 0 & \ldots 0 \\ \hline
0 & D_{11} \ldots 0 & \ldots 0 & \ldots 0 & \ldots & 0 & \ldots 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
C_{1j} & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \ldots D_{1n} \ldots 0 & \ldots 0 & \ldots & 0 & \ldots 0 \\
\cdots & \cdots & & & & & \\
0 & 0 \ldots 0 & \ldots D_{i1} & \ldots 0 & \ldots & 0 & \ldots 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
C_{ij} & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 \ldots 0 & \ldots 0 & \ldots D_{in} & \ldots & 0 & \ldots 0 \\
\cdots & \cdots & & & & & \\
0 & 0 \ldots 0 & \ldots 0 & \ldots 0 & \ldots & D_{m1} & \ldots 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
C_{mj} & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 \ldots 0 & \ldots 0 & \ldots 0 & \ldots & 0 & \ldots D_{mn}
\end{array}\right]
$$

$$
\overset{\longleftarrow}{\left[\begin{array}{ccc} 0..0 \; \ldots 1_{e_{i1}} \ldots 0 & \ldots 0..0 \\ \ldots & \\ 0..0 \; \ldots 0 \; \ldots 1_{e_{in}} & \ldots 0..0 \end{array}\right]}
\begin{array}{c} \\ e_{i1} \\ \vdots \\ e_{ij} \\ \vdots \\ e_{in} \end{array}
\left[\begin{array}{c|c}
A_{ij} & B_{i1} \ldots B_{in} \\ \hline
0 & D_{i1} \ldots 0 \\
\vdots & \vdots \\
C_{ij} & \vdots \\
\vdots & \vdots \\
0 & 0 \ldots D_{in}
\end{array}\right]
\overset{\longrightarrow}{\left[\begin{array}{c} 0 \\ \vdots \\ 1_{e_{ij}} \\ \vdots \\ 0 \end{array}\right]}
\; e_{ij} \left[\begin{array}{c|c} A_{ij} & B_{ij} \\ \hline C_{ij} & D_{ij} \end{array}\right] = f_{ij}
$$

The proof of

$$[f_{ij}] \cup [f'_{ij}] = [f_{ij} \cup f'_{ij}]$$

and

$$[f_{ij}] \circ [f'_{jk}] = \Big[\bigcup_{j \in [n]} f_{ij} \circ f'_{jk} \Big]_{i,k}$$

is as easy as the above proofs and is left as an exercise to the reader. □

<u>Theorem 5.3.</u> If T is a repetitive reticulum then $RFl_{\Sigma,T}$ is a repetitive reticulum, eachtime when T,Σ are such that the simulation relation has the confluent property.

<u>Proof.</u> By Proposition 4.2 only the axioms of repetition are to be shown.

$$\text{(R1)}\quad f^*f \cup 1 = \begin{array}{c} \\ e \\ e \end{array}\left[\begin{array}{c|cc} A^*A\cup 1 & A^*B & A^*B \\ \hline CA^* & CA^*B\cup D & CA^*B \\ C & 0 & D \end{array}\right] \xleftarrow[\begin{bmatrix}1_e & 1_e\end{bmatrix}]{} \begin{array}{c} \\ e \end{array}\left[\begin{array}{c|c} A^* & A^*B \\ \hline C(A^*\cup 1) & CA^*B\cup D \end{array}\right] = f^*$$

and

$$f\cdot f^*\cup 1 = \begin{array}{c} \\ e \\ e \end{array}\left[\begin{array}{c|cc} AA^*\cup 1 & B & AA^*B \\ \hline CA^* & D & CA^*B \\ CA^* & 0 & CA^*B\cup D \end{array}\right] \xrightarrow[\begin{bmatrix}1_e \\ 1_e\end{bmatrix}]{} \begin{array}{c} \\ e \end{array}\left[\begin{array}{c|c} A^* & (1\cup AA^*)B \\ \hline CA^* & CA^*B\cup D \end{array}\right] = f^*.$$

(R2) Denote by $Y = (A^*A')^*A^*$ and $Z = A^*A'Y \cup A^*$. Remark that $Y = Z = (A \cup A')^*$.

$$(f^*f')^*f^* = \begin{array}{c} \\ e \\ e' \\ e \end{array}\left[\begin{array}{c|ccc} Y & YB & YB' & YB \\ \hline CA^*A'Y & CA^*A'YB\cup CA^*B\cup D & CA^*A'YB'\cup CA^*B' & CA^*A'YB \\ C'Y & C'YB & C'YB'\cup D' & C'YB \\ CA^* & 0 & 0 & CA^*B\cup D \end{array}\right] \xleftarrow[\begin{bmatrix}1_e & 0 & 1_e \\ 0 & 1_{e'} & 0\end{bmatrix}]{}$$

$$\begin{array}{c} \\ e \\ e' \end{array}\left[\begin{array}{c|cc} Y & YB & YB' \\ \hline CZ & CZB\cup D & CZB' \\ C'Y & C'YB & C'YB'\cup D' \end{array}\right] = (f \cup f')^*.$$

(R3) Suppose $X = A(A'A)^*$, $Y = (AA')^*A$, $S = (A'A)^*$ and $T = (AA')^*$.

$$f(f'f)^{*} = \begin{array}{c} \\ e \\ e' \\ e \end{array}\left[\begin{array}{c|ccc} X & B & XB' & XA'B \\ \hline CS & D & CSB' & CSA'B \\ C'X & 0 & C'XB' \cup D' & C'(XA' \cup 1)B \\ C & 0 & CSB' & CSA'B \cup D \end{array}\right] \xrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ 0^{e} & 1 \\ 1_{e} & 0^{e'} \end{smallmatrix}\right]} \begin{array}{c} \\ e \\ e' \end{array}\left[\begin{array}{c|cc} X & (XA' \cup 1)B & XB' \\ \hline CS & CSA'B \cup D & CSB' \\ C'X & C'(XA' \cup 1)B & C'XB' \cup D' \end{array}\right]$$

and

$$(ff')^{*}f = \begin{array}{c} \\ e \\ e' \\ e \end{array}\left[\begin{array}{c|ccc} Y & TB & YB' & TB \\ \hline CA'Y & CA'TB \cup D & C(A'Y \cup 1)B' & CA'TB \\ C'Y & C'TB & C'YB' \cup D' & C'TB \\ C & 0 & 0 & D \end{array}\right] \xleftarrow{\left[\begin{smallmatrix} 1_{e} & 0 & 1_{e} \\ 0 & 1_{e'} & 0 \end{smallmatrix}\right]}$$

$$\begin{array}{c} \\ e \\ e' \end{array}\left[\begin{array}{c|cc} Y & TB & YB' \\ \hline C(A'Y \cup 1) & CA'TB \cup D & C(A'Y \cup 1)B' \\ C'Y & C'TB & C'YB' \cup D' \end{array}\right]$$

Since $X = Y$, $A'Y \;\; 1 = S$ and $XA' \;\; 1 = T$ the flowcharts are equivalent.

(R4) Suppose $f\varrho \equiv \varrho f'$. Due to the confluent property the chain of simulations for this equivalence can be replaced by two simulations

$$f\varrho \xrightarrow[a]{} f_1 \xleftarrow[b]{} \varrho f' ,$$

therefore

$$(\alpha) \quad \left[\begin{array}{c|c} A\varrho & Ba_{in} \\ \hline C\varrho & Da_{in} \end{array}\right] = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline a_{out}C_1 & a_{out}D_1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} \varrho A' & \varrho B'b_{in} \\ \hline C' & D'b_{in} \end{array}\right] = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline b_{out}C_1 & b_{out}D_1 \end{array}\right].$$

From $A\varrho = \varrho A'$ and (R4) in T follows that $A^{*}\varrho = \varrho A'^{*}$.

We shall define a flowchart $\bar{f}$ such that

$$f^{*}\varrho \xrightarrow[a]{} \bar{f} \xleftarrow[b]{} \varrho f'^{*} ,$$

namely

$$\bar{f} = e_1\left[\begin{array}{c|c} A^*\rho & A^*B_1 \\ \hline C_1A'^* & C_1A'^*B'b_{in} \cup D_1 \end{array}\right].$$

Indeed, using the equalities from (α) we prove

$$f^*\rho = e\left[\begin{array}{c|c} A^*\rho & A^*B \\ \hline CA^*\rho & CA^*B \cup D \end{array}\right] \xrightarrow[a]{} \bar{f} \text{ as follows}$$

A) $A^*\rho = A^*\rho$;

B) $A^*B\rho_{in} = A^*B_1$;

C) $CA^*\rho = C\rho A'^* = a_{out}C_1A'^*$;

D) $(CA^*B \cup D)a_{in} = CA^*\rho B'b_{in} \cup Da_{in} = a_{out}(C_1A'^*B'b_{in} \cup D_1)$,

and

$$\rho f'^* = e'\left[\begin{array}{c|c} \rho A'^* & \rho A'^*B' \\ \hline C'A'^* & C'A'^*B' \cup D' \end{array}\right] \xrightarrow[b]{} \bar{f} \text{ as follows}$$

A) $\rho A'^* = A^*\rho$;

B) $\rho A'^*B'b_{in} = A^*\rho B'b_{in} = A^*B_1$;

C) $C'A'^* = b_{out}C_1A'^*$;

D) $(C'A'^*B' \cup D')b_{in} = b_{out}C_1A'^*B'b_{in} \cup D'b_{in} = b_{out}(C_1A'^*B'b_{in} \cup D_1)$. □

By the proof from Appendix A we have the following

Corollary 5.4. $RF1_{\Sigma,M_{\{0,1\}}}$ is a repetitive reticulum. □

Open Problem. Is $RF1_{\Sigma,T}$ a repetitive reticulum eachtime when T is a repetitive reticulum?

Appendix A

Proposition. For every Σ, the simulation relation in $RFl_{\Sigma, M_{\{0,1\}}}$ has the confluent property.

Proof. The proof is based on the construction of fibered coproducts in the category: objects = sets, morphisms = relations.

Suppose $f \xrightarrow[\rho]{} f_1$, $f \xrightarrow[\tau]{} f_2$. We want to construct a flowchart $\bar{f}$ and two relations a, b such that $f_1 \xrightarrow[a]{} \bar{f} \xleftarrow[b]{} f_2$. We call a 0-1 column vector $\alpha \in M_{\{0,1\}}(|e|,1)$ a σ-onesorted vector for $e \in \Sigma^*$ if it has a 1 on a place i only if $e_i = \sigma$. For $\bar{e}$ we take, for indices all pair (α,β) such that

there is a $\sigma \in \Sigma$ such that α is a σ-onesorted vector for e_1, β is a σ-onesorted vector for e_2 and $\rho\alpha = \tau\beta$,

and put $\bar{e}_{(\alpha,\beta)} = \sigma$. In addition, a and b are given by

$$a\, y^{|\bar{e}|}_{(\alpha,\beta)} = \alpha \quad \text{and} \quad b\, y^{|\bar{e}|}_{(\alpha,\beta)} = \beta .$$

Now $\rho a = \tau b$ holds, since for every (α,β) in $\bar{e}$

$$\rho\, a\, y^{|\bar{e}|}_{(\alpha,\beta)} = \rho\alpha = \tau\beta = \tau b\, y^{|\bar{e}|}_{(\alpha,\beta)} .$$

Lemma. If $X \in M_{\{0,1\}}(r_{out}(e_1),1)$ and $Y \in M_{\{0,1\}}(r_{out}(e_2),1)$ are such that $\rho_{out}X = \tau_{out}Y$ then there exists $Z \in M_{\{0,1\}}(r_{out}(e),1)$ for which $a_{out}Z = X$ and $b_{out}Z = Y$.

Proof of lemma. X,Y may be not onesorted vectors, but if X_σ, Y_σ are their restrictions to the sort σ, then $X = \bigcup_{\sigma\in\Sigma} X_\sigma$, $Y = \bigcup_{\sigma\in\Sigma} Y_\sigma$ and the relations $\rho_{out}X = \tau_{out}Y$ also hold, as ρ,τ do not change the sorts. So, for every $\sigma \in \Sigma$, (X_σ, Y_σ) is an index.

If for every $\sigma\in\Sigma$, $r_{out}(\sigma) = 1$, namely $\rho_{out} = \rho$ and so on, then we take

$$x^{|\bar{e}|}_{(\alpha,\beta)} Z = 1 \text{ iff for one } \sigma\in\Sigma , (\alpha,\beta) = (X_\sigma, Y_\sigma) .$$

Clearly $aZ = \bigcup_{\sigma\in\Sigma} X_\sigma = X$ and $bZ = Y$.

In the general case we make the above construction for every output of a σ and put the results together. □

As $\rho_{out} [C_1 \quad D_1 a_{in}] = \tau_{out} [C_2 \quad D_2 b_{in}]$, making use of the above lemma for each column, one can find $[\bar{C} \quad \bar{D}]$ such that

$$[C_1 \quad D_1 a_{in}] = a_{out}[\bar{C} \quad \bar{D}] \text{ and } [C_2 \quad D_2 b_{in}] = b_{out}[\bar{C} \quad \bar{D}] .$$

Now one can easy see that

$$\bar{f} = \left[\begin{array}{c|c} A & B\rho_{in} a_{in} \\ \hline \bar{C} & \bar{D} \end{array}\right]$$

is a common simulation of f_1 and f_2 . □

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