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Abstract - This paper mainly deals with the strong solvability of the initial-boundary value problem for the Darcy-Boussinesq equations in two or three dimensional bounded domains. First, the existence is proved by the Galerkin method. Then, via a maximum principle, the uniqueness theorem and a stability criterion are derived.

1. PRELIMINARIES

Let Ω be an open connected bounded set in $\mathbb{R}^n$ (n=2 or 3) locally located on one side of the boundary $\partial\Omega$, which is a lipschitz manifold composed of a finite number of connected components; let real $\theta > 0$ be fixed and let us denote with Q the cylinder $\Omega \times (0,\theta)$.

With the assumptions and approximations which are frequently used for the thermal convection in a homogeneous porous medium saturated with an incompressible fluid, the governing evolution system of the Darcy-Boussinesq equations for the filtration velocity u, the pressure p and the temperature T may be written in a non-dimensional form as:

$$\operatorname{div} u=0 \text{ in } Q \tag{1.1}$$

$$\gamma u'+u= -\nabla p+aTe \quad \text{in } Q \tag{1.2}$$

$$T' + u \nabla T= \Delta T \quad \text{in } Q \tag{1.3}$$

where $\gamma > 0$ and $a > 0$ (the Rayleigh number) are two fixed real numbers, while e is the fixed versor of the gravitational acceleration. As we are in the non-dimensional case, we can assume that Ω can be included in a n-cube of edge-length 1.

Let Γ_0 be a subset of $\partial\Omega$ with positive surface measure and let us denote with $\Gamma_1 = \partial\Omega \setminus \Gamma_0$; the boundary conditions are:

$$u \cdot \nu = 0 \quad \text{on } \partial\Omega \times (0,\theta) \tag{1.4}$$

$$\frac{\partial T}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \times (0,\theta) \tag{1.5}$$

$$T = \tau \quad \text{on } \Gamma_0 \times (0,\theta) \tag{1.6}$$

where ν denotes the unit ontward normal on $\partial\Omega$ and $\tau = \tau(x,t)$ is the most important datum of our problem.

Also, u and T have to obey in some sense to the initial conditions:

$$u(0) = u^0 \quad \text{in } \Omega \tag{1.7}$$

$$T(0) = T^0 \quad \text{in } \Omega \tag{1.8}$$

Let us denote with

$$H = \left\{ u \in L^2(\Omega) \;\middle|\; \operatorname{div} u = 0 \text{ in } \Omega,\ u\cdot\nu = 0 \text{ on } \partial\Omega \right\} \tag{1.9}$$

the closure of

$$\mathcal{H}(\Omega) = \left\{ u \in \mathcal{D}(\Omega) \;\middle|\; \operatorname{div} u = 0 \text{ in } \Omega \right\}$$

in $L^2(\Omega)$, and with V the closure in $H^1(\Omega)$ of

$$\mathcal{V}(\Omega) = \left\{ S \in \mathcal{E}(\bar{\Omega}) \;\middle|\; S = 0 \text{ on } \Gamma_0 \right\}$$

We assume that Γ_0 is sufficiently smooth as

$$V = \left\{ S \in H^1(\Omega) \;\middle|\; S = 0 \text{ on } \Gamma_0 \right\} \tag{1.10}$$

As usual the scalar products and norms in $L^2(\Omega)$, $H^m(\Omega)$ and $H^1_0(\Omega)$ are respectively denoted by

$$(u,v) = \int_\Omega u \cdot v \;, \quad |u| = (u,u)^{1/2}$$

$$((u,v))_m = \sum_{|i| \leq m} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_j} \right), \quad \|u\|_m = ((u,u))_m^{1/2}$$

$$((u,v)) = (\nabla u, \nabla v), \quad \|u\| = ((u,u))^{1/2}$$

and the norm in $L^p(\Omega)$ $(p \neq 2)$ by $|\ |_p$. We agree to use the same notations for the scalar products and norms in the corresponding vector spaces, i.e. $\mathbf{L}^2(\Omega) = \left[L^2(\Omega)\right]^n$ and so on.

Reminding here the Friedrichs' inequality (see [1])

$$|s| \leq c_0 \|s\| \qquad (\forall)\ s \in V \tag{1.11}$$

we remark that $\|\ \|$ is a norm on V, also.

We start the study with the following hypothesis:

$$\tau \in H^1(0,\theta;\ H^{3/2}(\partial\Omega)) \tag{1.12}$$

$$u^0 \in H \cap \mathbf{H}^1(\Omega) \tag{1.13}$$

$$T^0 \in H^2(\Omega) \quad \text{with } (T^0 - \tau(0)) \in V \tag{1.14}$$

Remark 1.1. From (1.12) it follows that τ is a.e. equal to a function of $C^0([0,\theta],\ H^{3/2}(\partial\Omega))$ and hence $\tau(0)$ in (1.14) makes sense. □

Lemma 1.1. For any $h > 0$ there exists an element

$$\tau_h \in H^1(0,\theta;\ H^2(\Omega)) \tag{1.15}$$

satisfying a.e. in $(0,\theta)$

$$\tau_h = \tau \quad \text{on} \quad \partial\Omega \tag{1.16}$$

$$|s \nabla \tau_h| \leq h \|s\| \qquad (\forall)\ s \in V \tag{1.17}$$

Proof. The construction of τ_h is the same as in the steady case, which can be found in [2]. The only new point is that the vector-valued function $L(\tau)_{(t)} := L(\tau(t))$ (where $L: H^{3/2}(\partial\Omega) \to H^2(\Omega)$ is the linear and continuous "lifting of trace" operator) has the property

$$L(\tau) \in H^1(0,\theta;\ H^2(\Omega)) \tag{1.18}$$

But noticing that

$$(L(\tau))' = L(\tau'),$$

the property (1.18) is a straight consequence of (1.12). □

Keeping the right to choose the parameter $h > 0$ later in a proper way, we introduce

$$S = T - \tau_h \tag{1.19}$$

Thus the system (1.1)-(1.8) becomes:

$$\operatorname{div} u = 0 \qquad \text{in } Q \tag{1.20}$$

$$u' + u = -\nabla p + a(S + \tau_h)e \quad \text{in } Q \tag{1.21}$$

$$\gamma S' - \Delta S + u\nabla(S+\tau_h) = (\Delta\tau_h - \gamma\tau_h') \text{ in } \Omega \quad (1.22)$$

$$u \cdot \nu = 0 \quad \text{on} \quad \partial\Omega \times (0,\theta) \quad (1.23)$$

$$\frac{\partial S}{\partial\nu} = -\frac{\partial\tau_h}{\partial\nu} \quad \text{on } \Gamma_1 \times (0,\theta) \quad (1.24)$$

$$S = 0 \quad \text{on} \quad \Gamma_0 \quad (1.25)$$

$$u(0) = u^0 \in H \cap H^1(\Omega) \quad (1.26)$$

$$S(0) = S^0 := T^0 - \tau_h(0) \in V \cap H^2(\Omega) \quad (1.27)$$

2. THE WEAK SOLUTIONS

Let us suppose that u,p and S are smooth solutions of (1.20)-(1.27); then, making the dual product of (1.21) and (1.22) with $v \in \mathcal{H}(\Omega)$ and $T \in \mathcal{V}(\Omega)$, one can easily obtain:

$$\frac{d}{dt}(u,v) + (u,v) = a\,(S+\tau_h,\ e\cdot v) \quad (2.1)$$

$$\gamma \frac{d}{dt}(S+\tau_h,T) + ((S+\tau_h,T)) + (u\nabla(S+\tau_h),T) = 0 \quad (2.2)$$

Taking in account that $\mathcal{H}(\Omega)$ and $\mathcal{V}(\Omega)$ are dense in H and respectively in V, the relations (2.1)-(2.2) suggest the following variational formulation of the problem (1.20)-(1.27):

<u>Problem 2.1</u>. Find $u \in L^\infty(0,\theta;H \cap H^1(\Omega))$ and $S \in L^\infty(0,\theta;L^2(\Omega)) \cap L^2(0,\theta;V)$ satisying the initial conditions (1.26)-(1.27) and the equations (2.1)-(2.2) for any $v \in H$, respectively $T \in V$.

<u>Theorem 2.1</u>. The Problem 2.1 has at least one solution.

<u>Proof</u>. In order to prove the existence of the generalized solutions of the Problem 2.1 we employ the Galerkin method. Let

$$\{v_j\}_{j\in\mathbb{N}} \subseteq \mathcal{H}(\Omega) \quad \text{and} \quad \{T_j\}_{j\in\mathbb{N}} \subseteq \mathcal{V}(\Omega) \quad (2.3)$$

be two sequences which are free and total in H, respectively in V. For each $i \in \mathbb{N}$, denoting with H_i the space spanned by the set $\{v_j \mid j=1,\dots i\}$ and with V_i the space spanned by the set

$\{T_j \mid j=1,\dots i\}$, we define an approximate solution of the Problem 2.1 by

$$u_i(t) = \sum_{j=1}^{i} u_{ij}(t)v_j \quad \text{and} \quad S_i(t) = \sum_{j=1}^{i} S_{ij}(t)T_j$$

satisfying for any $k \in \{1,\dots i\}$ the system

$$(u_i', v_k) + (u_i, v_k) = a(S_i + \tau_h, e \cdot v_k) \tag{2.4}$$

$$\gamma(S_i', T_k) + ((S_i, T_k)) + (u_i \nabla(S_i + \tau_h)T_k = $$

$$= -\gamma(\tau_h', T_k) - (\nabla\tau_h, \nabla T_k) \tag{2.5}$$

$$u_i(0) = u_i^0 \tag{2.6}$$

$$S_i(0) = S_i^0 \tag{2.7}$$

where $u_i^0 \in H_i$ and $S_i^0 \in V_i$ are choose such that

$$u_i^0 \rightarrow u^o \text{ strongly in } H \tag{2.8}$$

$$S_i^0 \rightarrow S^0 \text{ strongly in } V \tag{2.9}$$

For any $k \in \{1,\dots i\}$, let us write (2.4)-(2.7) in terms of its effective unknowns, that is $u_{ij}(t)$ and $S_{ij}(t)$:

$$\sum_{j=1}^{i} (v_j, v_k)\, u_{ij}' + \sum_{j=1}^{i} (v_j, v_k) u_{ij} =$$

$$= a \sum_{j=1}^{i} (T_j, ev_k) S_{ij} + a(\tau_h, e \cdot v_k) \tag{2.10}$$

$$\gamma \sum_{j=1}^{i} (T_j, T_k) S_{ij}' + \sum_{j=1}^{i} ((T_j, T_k)) S_{ij} + \sum_{j=1}^{i} (v_j \nabla \tau_h, T_k) u_{ij}$$
$$+ \sum_{j=1}^{i} \sum_{l=1}^{i} (v_j \nabla T_l, T_k) u_{ij} S_{il} = -\gamma(\tau_h', T_k) -$$
$$-(\nabla\tau_h, \nabla T_k) \tag{2.11}$$

$$u_{ik}(0) = u_{ik}^0 := \text{the k-th component in } H_i \text{ of } u_i^0 \tag{2.12}$$

$$S_{ik}(0) = S_{ik}^0 := \text{the k-th component in } V_i \text{ of } S_i^0 \tag{2.13}$$

As the (2i x 2i) matrix

$$\begin{pmatrix} (v_j, v_k) & 0 \\ 0 & (T_j, T_k) \end{pmatrix}$$

is obviously non-singular, by inverting it (2.10)-(2.11) takes the canonical form of an ordinary differential system with (at least) continuous coefficients. Then there exists a local maximal solution of (2.10)-

-(2.13) defined on some interval $[0,\theta_i[$. For $t\in[0,\theta_i[$, we multiply (2.10)-

-(2.11) by $u_{ik}(t)$, respectively $S_{ik}(t)$, and sum correspondingly these equations over k from 1 to i; it follows

$$(u_i', u_i) + (u_i, u_i) = a(S_i + \tau_h, e\cdot u_i)$$

$$\gamma(S_i', S_i)+((S_i, S_i))+(u_i\nabla \tau_h, S_i)=$$
$$=-\gamma(\tau_h', S_i)-(\nabla\tau_h, \nabla S_i)$$

which yield the next estimations:

$$\frac{1}{2}\frac{d}{dt}|u_i|^2+|u_i|^2\leqslant a(|S_i|+|\tau_h|)|u_i|$$

$$\frac{\gamma}{2}\frac{d}{dt}|S_i|^2+\|S_i\|^2\leqslant h\|S_i\|\cdot|u_i|+\gamma|\tau_h'||S_i|+|\nabla\tau_h|\cdot\|S_i\|$$

where we have used property (1.17) of τ_h. With the Friedrichs' inequality (1.11), the preceding estimations become

$$\frac{d}{dt}|u_i|^2+|u_i|^2\leqslant 2a^2C_0^2\|S_i\|^2+2a^2|\tau_h|^2 \tag{2.14}$$

$$\gamma\frac{d}{dt}|S_i|^2+\|S_i\|^2\leqslant h\|S_i\|^2+h|u_i|^2+ + 2\gamma^2C_0^2|\tau_h'|^2+2|\nabla\tau_h|^2 \tag{2.15}$$

Multiplying (2.14) with (2 h) and adding it to (2.15) we receive

$$2h\frac{d}{dt}|u_i|^2+\gamma\frac{d}{dt}|S_i|^2+h|u_i|^2+(1-h-4a^2C_0^2h)\|S_i\|^2\leqslant$$

$$\leqslant 4a^2h|\tau_h|^2+2\gamma^2C_0^2|\tau_h'|^2+2|\nabla\tau_h|^2=:G(t)$$

Choosing $h<(2+8a^2C_0^2)^{-1}$ and integrating from 0 to some $s\in[0,\theta_i[$ it gives

$$2h|u_i(t)|^2+\gamma|S_i(t)|^2+h\int_0^s|u_i(t)|^2dt+ + \frac{1}{2}\int_0^s\|S_i(t)\|^2dt\leqslant 2h|u_i^0|^2+\gamma|S_i^0|^2+\int_0^s G(t)dt \tag{2.16}$$

As $\{u_i^0\}$ and $\{S_i^0\}$ are bounded and $\int_0^s G(t)\,dt\leqslant\int_0^\theta G(t)dt<\infty$

from (2.16) it follows that

$$\{u_i\} \text{ is bounded in } L^\infty(0,\theta;H) \tag{2.17}$$

$$\{S_i\} \text{ is bounded in } L^\infty(0,\theta;L^2(\Omega)) \tag{2.18}$$

In particular this yields $\theta_i=\theta$ for any i and hence (u_i, S_i) is a global solution of (2.4)-(2.7).

Also, by puting $s=\theta$ in (2.16) we see that

$$\{S_i\} \text{ is bounded in } L^2(0,\theta;V) \tag{2.19}$$

From (2.17)-(2.19) we learn that there exist $u\in L^\infty(0,\theta;H)$ and $S\in L^\infty(0,\theta;L^2(\Omega))\cap L^2(0,\theta;V)$ for which, passing some subsequence still denoted with $\{\ \}_i$, we have

$$u_i \longrightarrow u \quad \text{weakly star in } L^\infty(0,\theta;H) \tag{2.20}$$

$$S_i \longrightarrow S \quad \text{weakly star in } L^\infty(0,\theta;L^2(\Omega)) \tag{2.21}$$

$$S_i \longrightarrow S \quad \text{weakly in } L^2(0,\theta;V) \tag{2.22}$$

Let Ψ be a real function defined on $[0,\theta]$ of class C^1 with $\Psi(\theta)=0$. Multiplying (2.4) and (2.5) with Ψ and integrating on $[0,\theta]$ we obtain

$$\Psi(0)(u_i^0, v_k) - \int_0^\theta (u_i, v_k)\Psi' dt + \int_0^\theta (u_i,v_k)\Psi dt=$$

$$= a\int_0^\theta (S_i+\tau_h, ev_k)\Psi dt \tag{2.23}$$

$$\gamma\Psi(0)(S_i^0, T_k)-\gamma\int_0^\theta (S_i,T_k)\Psi' dt +\gamma\int_0^\theta (\tau_h', T_k)\Psi dt$$

$$+\int_0^\theta ((S_i,T_k))\Psi dt+\int_0^\theta (\nabla\tau_h, \nabla T_k)\Psi dt=$$

$$=-\int_0^\theta (u_i \nabla(S_i+\tau_h),T_k)\Psi dt \tag{2.24}$$

In the light of the convergences (2.8)-(2.9) and (2.20)-(2.22) we can try to pass (2.23) and (2.24) to the limit. All the terms converge easily except the right hand side of (2.24):

$$-\int_0^\theta (u_i \nabla(S_i+\tau_h),T_k)\Psi dt =$$

$$=\int_0^\theta (u_i \nabla T_k, S_i+\tau_h)\Psi dt$$

But the injection of V in $L^2(\Omega)$ being compacte from (2.22) we find also

$$S_i \longrightarrow S \text{ strongly in } L^2(0,\theta; L^2(\Omega)) \tag{2.25}$$

and with (2.20) it implies

$$-\int_0^\theta (u_i \nabla(S_i+\tau_h),T_k)\Psi dt \longrightarrow \int_0^\theta (u\nabla T_k, S+\tau_h)\Psi dt =$$

$$=-\int_0^\theta (u\nabla(S+\tau_h),T_k)\Psi dt$$

Consequently from (2.23) and (2.24) we get at the

$$\psi(0)(u^0,v_k) - \int_0^\theta (u,v_k)\psi' dt + \int_0^\theta (u,v_k)\psi\, dt =$$
$$= a\int_0^\theta (S+\tau_h, ev_k)\psi\, dt \tag{2.26}$$
$$\gamma\psi(0)(S^0,T_k) - \gamma\int_0^\theta (S,T_k)\psi' dt + \gamma\int_0^\theta (\tau_h',T_k)\psi\, dt +$$
$$+\int_0^\theta ((S,T_k))\psi\, dt + \int_0^\theta (\nabla\tau_h, \nabla T_k)\psi\, dt =$$
$$= -\int_0^\theta (u\nabla(S+\tau_h), T_k)\psi\, dt \tag{2.27}$$

All the terms in (2.26) are continuous in H with respect to the v_k-argument. Thus, choosing $\psi \in \mathcal{D}((0,\theta))$ we remark that (2.26) becomes exactly (2.1) for any $v \in H$, in the distribution sense. Moreover, it is easy to see that

$$\frac{d}{dt}(u,v) = -(u,v) + a(S+\tau_h, e\cdot v) \in L^2((0,\theta)),$$

$$(\forall)\, v \in H;$$

This means $u' \in L^2(0,\theta;H)$ and

$$(u'+u-a(S+\tau_h)e, v) = 0 \qquad (\forall)\ v \in H$$

As the orthogonal complement of H in $L^2(\Omega)$ is

$$H^\perp = \left\{ w \in L^2(\Omega) \,\middle|\, (\exists) p \in H^1(\Omega) \text{ such that } w = \nabla p \right\}$$

(for a proof see [3]), there exists $p(t) \in H^1(\Omega)$ such that

$$u'+u-a(S+\tau_h)e = \nabla p \tag{2.28}$$

If we choose $\psi \in C^1([0,\theta])$ with $\psi(\theta)=0$ and $\psi(0) \neq 0$, multiply (2.1) with it and integrate on $[0,\theta]$, then we obtain

$$\psi(0)(u(0),v) - \int_0^\theta (u,v)\psi' dt + \int_0^\theta (u,v)\psi\, dt =$$
$$= a\int_0^\theta (S+\tau_h, e\cdot v)\psi\, dt \tag{2.29}$$

Thus (1.26) follows by subtracting (2.29) from (2.26); as we have proved that $u \in H^1(0,\theta;H(\Omega))$ we see that, at least, $u \in C^0(0,\theta;H(\Omega))$ and hence (1.26) makes sense. Moreover, from (2.28) we find that $v(t) := \exp(t)\ \mathrm{rot}\ u(t)$ verify the problem

$$\begin{cases} v' = -ae \times \nabla(S+\tau_h) \in L^2(0,\theta;L^2(\Omega)) & (\times \text{ denotes the vectorial product}) \\ v(0) = \mathrm{rot}\ u^0 \in L^2(\Omega) \end{cases}$$

which obviously has a unique global solution in $L^\infty(0,\theta;L^2(\Omega))$. This implies $\mathrm{rot}\ u \in L^\infty(0,\theta;L^2(\Omega))$ and as the space

$$\left\{ u\in L^2(\Omega) \,\middle|\, \operatorname{div} u\in L^2(\Omega),\ \operatorname{rot} u\in L^2(\Omega),\ u\cdot\nu=0 \text{ on } \partial\Omega \right\}$$

is isomorphic to $H^1(\Omega)$ (see [4]), it follows $u\in L^\infty(0,\theta;H^1(\Omega))$.

This last result enables us to pass to the limit over $k\to\infty$ in (2.27), all the terms being continuous in V with respect to the T_k-argument; then choosing $\psi\in\mathcal{D}((0,\theta))$ we see that (2.27) is equivalent with (2.2) for any $T\in V$, in the distribution sense.

Finally we chose again $\psi\in C^1([0,\theta])$ with $\psi(\theta)=0$ and $\psi(0)\neq 0$. Multiplying (2.2) with it and integrating on $[0,\theta]$ we obtain

$$\gamma\psi(0)(S(0),T)-\gamma\int_0^\theta(S,T)\psi' dt+\gamma\int_0^\theta(\tau_h',T)\psi dt+$$
$$+\int_0^\theta((S,T))\psi dt+\int_0^\theta(\nabla\tau_h,\nabla T)\psi dt=$$
$$=-\int_0^\theta(u\nabla(S+\tau_h),T)\psi dt \tag{2.30}$$

and (1.27) is given by the substraction of (2.30) from (2.27).

In order to precise the sense of (2.4), let us consider for any (u,S), solution of the Problem 2.1, the function defined a.e. in $(0,\theta)$ by

$$\langle A(S),T\rangle=((S+\tau_h,T)),$$
$$\langle B(u,S),T\rangle=(u\nabla(S+\tau_h),T)\quad,(\forall)T\in V,$$

where $\langle,\rangle$ denotes the duality product between V and V'. It is easy to check that $A(S)\in L^2(0,\theta;V')$ and $B(u,S)\in L^1(0,\theta;V')$. Therefore from (2.2) it follows

$$\gamma\frac{d}{dt}(S,T)=\langle -A(S)-B(u,S)-\gamma\tau_h',T\rangle\in L^1((0,\theta))$$

and this means $S'\in L^1(0,\theta;V')$ and

$$\gamma S'=-A(S)-B(u,S)-\gamma\tau_h' \tag{2.31}$$

it the weak generalized sense. Moreover S is a.e. equal to a function of $C^0([0,\theta];V')$ and this is the way in which we understand (1.27). □

3. THE STRONG SOLUTION

In this section we impose an additional condition concerning the differentiability of the datum $\mathcal{E}$ with respect to the time variable; thus we replace (1.12) by

$$\mathcal{E} \in H^2(0,\theta;H^{3/2}(\partial\Omega)) \tag{3.1}$$

It implies correspondigly that $\mathcal{E}_h$ given by Lemma 1.1 has the property

$$\mathcal{E}_h \in H^2(0,\theta;H^2(\Omega)) \tag{3.2}$$

We have also to remark from the very beginning that u_i^0, from (2.6), and S_i^0, from (2.7), can be choosen in such a way that, besides (2.8)-(2.9), they satisfy

$$u_i^0 \longrightarrow u^0 \text{ strongly in } H \cap H^1(\Omega) \tag{3.3}$$

$$S_i^0 \longrightarrow S^0 \text{ strongly in } V \cap H^2(\Omega) \tag{3.4}$$

We will show here how these assumption can be justified:

The Banach space $W=V\cap H^2(\Omega)$ with its own topology given by the norm $|\ |_W = \|\ \| + \|\ \|_2$ is separable with $\mathcal{V}(\Omega)$ dense, that is there exists a sequence $\{w_j\}_{j\in\mathbb{N}} \subseteq \mathcal{V}(\Omega)$ which is free and total in W; W is a linear closed subspace of the Hilbert space V and let us denote with $W^{\perp}$ its orthogonal in V. As V endowed the norm $\|\ \|$ is also separable with $\mathcal{V}(\Omega)$ dense, there exists a sequence $\{z_j\}_{j\in\mathbb{N}} \subseteq \mathcal{V}(\Omega)$ which is free and total in $W^{\perp}$. Thus the sequence $\{T_j\}_{j\in\mathbb{N}} \subseteq \mathcal{V}(\Omega)$ defined by $T_{2j-1}=z_j$ and $T_{2j}=w_j$, $j\geqslant 1$, is free and total in V; we might think that this is exactly the sequence $\{T_j\}_{j\in\mathbb{N}}$ in (2.3). Finally we see that S_o^i can be the orthogonal projection of $S^0 \in V\cap H^2(\Omega)$ onto V_i, the space spanned by $\{T_j \mid j=1,\dots i\}$.

In the following we shall prove that under the hypothesis (3.1) the weak solutions of §2 are strong.

<u>Theorem 3.1.</u> If (u,S) is a solution of Problem 2.1, then

$u' \in L^{\infty}(0,\theta;H)$ and $S' \in L^{\infty}(0,\theta;L^2(\Omega)) \cap L^2(0,\theta;V)$.

Proof. We derive from (2.4)-(2.5) in particular

$$(u_i'(0), v_k) + (u_i^0, v_k) = a(S_i^0 + \tau_h(0), e\cdot v_k) \tag{3.5}$$

$$\gamma(S_i'(0), T_k) - (\Delta S_i^0 + \Delta\tau_h(0), T_k) + \gamma(\tau_h'(0), T_k) = -(u_i^0 \nabla(S_i^0 + \tau_h(0)), T_k) \tag{3.6}$$

Multiplying (3.5) and (3.6) with $u_{ik}'(0)$ and respectively $S_{ik}'(0)$ and adding over k we get

$$|u_i'(0)| \leqslant |u_i^0| + a|S_i^0 + \tau_h(0)|$$

$$\gamma|S_i'(0)| \leqslant |\Delta S_i^0 + \Delta\tau_h(0)| + \gamma|\tau_h'(0)| + C\|u_i^0\|_1 \|S_i^0 + \tau_h(0$$

Taking in account (3.3)-(3.4) we see that

$$|u_i'(0)| \leqslant C_1 \text{ and } |S_i'(0)| \leqslant C_2 \tag{3.7}$$

Let us differentiate the system (2.4)-(2.5); we obtain

$$(u''_i, v_k) + (u_i', v_k) = a(S_i' + \tau_h', e\cdot v_k) \tag{3.8}$$

$$(S''_i, T_k) + ((S_i', T_k)) + (u_i' \nabla(S_i + \tau_h), T_k) + + (u_i \nabla(S_i' + \tau_h'), T_k) = -\gamma(\tau_h'', T_k) - (\nabla\tau_h', \nabla T_k) \tag{3.9}$$

Multiplying (3.8) and (3.9) with $u_{ik}'(t)$, respectively $S_{ik}'(t)$, and making the sum over k from 1 to i, we find

$$\frac{1}{2}\frac{d}{dt}|u_i'|^2 + |u_i'|^2 = a(S_i' + \tau_h', eu_i') \tag{3.10}$$

$$\frac{\gamma}{2}\frac{d}{dt}|S_i'|^2 + \|S_i'\|^2 + (u_i' \nabla(S_i + \tau_h), S_i') + (u_i \nabla\tau_h', S_i') = -\gamma(\tau_h'', S_i') - (\nabla\tau_h', \nabla S_i') \tag{3.11}$$

Remarking that

$$2(u_i' \nabla S_i, S_i') = (u_i', \nabla(S_i^2)') = 0$$

from (3.10) and (3.11) we receive the following estimations:

$$\frac{d}{dt}|u_i'|^2 + |u_i'|^2 \leqslant 2a^2C_0^2\|S_i'\|^2 + 2a^2|\tau_h'|^2 \tag{3.12}$$

$$\gamma\frac{d}{dt}|S_i'|^2 + \|S_i'\|^2 \leqslant h\|S_i'\|^2 + h|u_i'|^2 + 3\gamma^2C_0^2|\tau_h''|^2 + +3|\nabla\tau_h'|^2 + 3|\tau_h'|_{\infty}^2|u_i|^2 \tag{3.13}$$

where C_0 is given by (1.11). In the light of (2.17), (3.2) and (3.7), the estimations (3.12)-(3.13) are similar to (2.14)-(2.15). Therefore we obtain analogously

$$\{u_i'\} \text{ bounded in } L^{\infty}(0,\theta;H)$$
$$\{S_i'\} \text{ bounded in } L^{\infty}(0,\theta;L^2(\Omega))$$
$$\{S_i'\} \text{ bounded in } L^2(0,\theta;V)$$

and there exist $u^* \in L^{\infty}(0,\theta;H)$ and $S^* \in L^{\infty}(0,\theta;L^2(\Omega)) \cap L^2(0,\theta;V)$ for which, passing just in case to a subsequence

$$u_i' \rightharpoonup u^* \text{ weakly star in } L^{\infty}(0,\theta;H)$$

$$S_i' \rightharpoonup S^* \text{ weakly star in } L^{\infty}(0,\theta;L^2(\Omega))$$

$$S_i' \rightharpoonup S^* \text{ weakly in } L^2(0,\theta;V)$$

But u_i' and S_i' converge to u', respectively S', in the distribution sense and thus the proof is completed. $\square$

The following weak maximum principle is formulated in terms of inequality in the sense of $H^1(\Omega)$. That's why we start by recalling this notion and some propositions, following [5].

Let $u \in H^1(\Omega)$ and $E \subseteq \overline{\Omega}$; we say that u is nonnegative on E in the sense of $H^1(\Omega)$, or briefly, $u \geq 0$ on E in $H^1(\Omega)$, if there exists a sequence $u_n \in W_{\infty}^{(1)}(\Omega)$ such that $u_n(x) \geq 0$ for $x \in E$ and $u_n \to u$ in $H^1(\Omega)$. Let $v \in H^1(\Omega)$; naturally, we say that $u \leq v$ on E in $H^1(\Omega)$ if $v-u \geq 0$ on E in $H^1(\Omega)$. As v may be a constant, we define

$$\sup_E u = \inf \{ m \in \mathbb{R} \mid u \leq m \text{ on } E \text{ in } H^1(\Omega) \}.$$

Also, for any $x \in \Omega$, we say that $u(x) > 0$ in (the sense of) $H^1(\Omega)$ if there exist $B(x)$, a neighbourhood of x, and φ, a function from $\overset{o}{W}{}_{\infty}^{(1)}(B(x))$ with $\varphi \geq 0$ and $\varphi(x) > 0$, such that $u \geq \varphi$ on $B(x)$ in $H^1(\Omega)$. Let us remark that the set $\{x \in \Omega \mid u(x) > 0 \text{ in } H^1(\Omega)\}$ is open.

<u>Proposition 3.1</u>. If $u \geq 0$ on E in $H^1(\Omega)$, then $u \geq 0$ a.e. on E.

Proposition 3.2. If $\sup_{\partial\Omega} u < \infty$ then for any $M \geq \sup_{\partial\Omega} u$ we have

$$\max\{u-M,0\} \in H_0^1(\Omega) \quad \text{and}$$
$$\max\{u-M,0\} \geq 0 \text{ on } \Omega \text{ in } H^1(\Omega).$$

Proposition 3.3. Let $u \in W_p^{(1)}(\Omega)$ $(p \geq 1)$; then $v=\max\{u,0\} \in W_p^{(1)}(\Omega)$ and we have in the sense of distributions

$$\nabla v = \begin{cases} \nabla u \text{ in } \{x \in \Omega \mid u(x) > 0 \quad \text{in } H^1(\Omega)\} \\ 0 \quad \text{otherwise} \end{cases}$$

Now we are able to prove our main results: the maximum principle and based on it, the uniqueness.

Theorem 3.2. If (u,S) is a solution of Problem 2.1., then $S \in L^\infty(Q)$ with

$$|S + \mathcal{E}_h|_{L^\infty(Q)} \leq C(\mathcal{E},T^0) \tag{3.14}$$

where $C(\mathcal{E},T^0) = \max\left\{|\mathcal{E}|_{L^\infty(0,\theta;L^\infty(\partial\Omega))}, |T^0|_\infty\right\}$

Proof. The property (1.16) implies:

$$S(t)+\mathcal{E}_h(t) = \mathcal{E}(t) \text{ on } \Gamma_0 \text{ a.e. in } (0,\theta).$$

As $S \in L^2(0,\theta;V)$ from Proposition 3.2 we have

$$R(t)=\max\{S(t)+\mathcal{E}_h(t)-C(\mathcal{E},T^0),0\} \in L^2(0,\theta;V).$$

Apealing to Proposition 3.3. we get also

$$\nabla R = \begin{cases} \nabla(S+\mathcal{E}_h) & \text{when } R \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Noticing, via Theorem 3.1, that $R' \in L^\infty(0,\theta;L^2(\Omega))$ and hence

$$\frac{1}{2}\frac{d}{dt}|R|^2 = (R',R) = \begin{cases} (S'+\mathcal{E}_h',R) & \text{when } R \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and taking the scalar product of (2.31) with $R(t)$, we find

$$0 = \frac{\gamma}{2}\frac{d}{dt}|R|^2 + \|R\|^2 + (u\nabla R,R) = \frac{\gamma}{2}\frac{d}{dt}|R|^2 + \|R\|^2 \tag{3.15}$$

$$\frac{d}{dt}|R|^2 \leqslant 0$$

and by integration on $(0,t)$, for some t, it implies

$$|R(t)|^2 \leqslant |R(0)|^2 \quad \text{a.e. on } (0,\theta) \tag{3.16}$$

As $S(0)+\tau_h(0)=T^0$ in Ω it follows that $R(0)=0$, and (3.16) gives $|R(t)|=0$. Recalling (3.15) we see that $\|R(t)\|=0$, that is

$$S(t)+\tau_h(t) \leqslant C(\tau,T^0) \text{ on } \Omega \text{ in } H^1(\Omega), \text{ a.e. in } (0,\theta).$$

According to Proposition 3.1 this implies

$$S(t) + \tau_h(t) \leqslant C(\tau,T^0) \qquad \text{a.e. in } Q$$

Analogously, with $R(t)=\min\{S(t)+\tau(t)+C(\tau,T^0),0\}$ we get

$$S(t)+\tau_h(t) \geqslant -C(\tau,T^0) \qquad \text{a.e. in } Q$$

Thus (3.14) is proved and $S\in L^\infty(Q)$ because

$$\tau_h \in H^2(0,\theta;H^2(\Omega)) \subseteq L^\infty(Q). \qquad \square$$

Let (u_1, S_1) and (u_2, S_2) be two solutions of the Problem 2.1 corresponding to the initial data (u_1^0, T_1^0), respectively (u_2^0, T_2^0). Denoting with

$$u=u_1-u_2 \qquad , \; S=S_1-S_2$$

we obtain from (2.28) and (2.31) by subtraction

$$u'+u-aSe= \nabla(p_1-p_2) \tag{3.17}$$

$$\gamma S'+A(S) + B(u,S_1+\tau_h)+B(u_2,S) =0 \tag{3.18}$$

Taking the duality product of (3.17)-(3.18) with u, respectively S, we get

$$\frac{1}{2}\frac{d}{dt}|u|^2 + |u|^2 = (aSe,u)$$

$$\frac{\gamma}{2}\frac{d}{dt}|S|^2 + \|S\|^2 = (u\nabla S, S_i+\tau_h) \quad (i=1 \text{ or } 2)$$

which yield the following estimations

$$\frac{d}{dt}|u|^2 + 2|u|^2 \leqslant 2a|S|\,|u| \leqslant |u|^2 + a^2|S|^2 \tag{3.19}$$

$$\gamma\frac{d}{dt}|S|^2 + 2\|S\|^2 \leqslant 2K_0|u|\,\|S\| \leqslant \|S\|^2 + K_0^2|u|^2 \tag{3.20}$$

where $K_0=\min\{C(\tau,T_1^0),\, C(\tau,T_2^0)\}$. Using also the Friedrichs' inequality (1.11) from (3.19)-(3.20) we receive

$$\frac{d}{dt}\begin{pmatrix} |u|^2 \\ |s|^2 \end{pmatrix} \leqslant \mathcal{M} \begin{pmatrix} |u|^2 \\ |s|^2 \end{pmatrix} \tag{3.21}$$

where $\mathcal{M}$ is the following (2 x 2) matrix

$$\mathcal{M} = \begin{pmatrix} -1 & a^2 \\ C_0^2\, \gamma^{-1} & -K_0^{-2}\gamma^{-1} \end{pmatrix}$$

Integrating (3.21) from 0 to some t, we are lead to

$$\begin{pmatrix} |u(t)|^2 \\ |s(t)|^2 \end{pmatrix} \leqslant \exp\,(\mathcal{M}t) \begin{pmatrix} |u_1^0 - u_2^0|^2 \\ |T_1^0 - T_2^0|^2 \end{pmatrix} \tag{3.22}$$

In particular, with $(u_1^0, T_1^0) = (u_2^0, T_2^0)$ in (3.22), we have proved:

<u>Theorem 3.3</u>. The Problem 2.1 has a unique solution.

Finally, the relation (3.22) permit us to establish also a stability result ~~also~~, that is

<u>Theorem 3.4</u>. Any perturbation of the initial data (1.7)-(1.8) decreases exponentially in time if

$$\max\left\{ |\tilde{\sigma}|_{L^\infty(0,\theta;L^\infty(\partial\Omega))},\ \min\left\{ |T_1^0|_\infty, |T_2^0|_\infty \right\}\right\} < a^{-1} C_0 \tag{3.23}$$

<u>Proof.</u> If condition (3.23) is satisfied, then $\mathcal{M}$ has distinct eigenvalues wich are also strictly negative and the proof is completed using classical results on spectral decompositio

<u>Remark 3.1</u>. The relation (3.23) is equivalent to the unicity criterion (see [2]), in the steady case. $\square$

REFERENCES

[1] J.NEČAS, Les méthodes directes en théorie des équations elliptiques, Chap. 1 § 1, Ed.Acad.Prague (1967).

[2] D.POLIŠEVSKI, Weak Continuity in Convection Problems, to appear in J.Math.Anal.Appl.

[3] R. TEMAM, Navier-Stokes Equations, Chap.1 § 1, North Holland Amsterdam (1977)

[4] G. DUVAUT and J.L.LIONS, Inequalities in Mechanics and Phisics, Chap. 7 § 6, Springer-Verlag (1976)

[5] D. KINDERLEHRER and G. STAMPACCHIA, An Introduction to Variational Inequalities and Their Applications, Chap.2, Appendix B, Academic Press (1980)

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