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ISSN 0250 3638

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PREPRINT SERIES IN MATHEMATICS

No.34/1985

BUCURESTI

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Juny 1985

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On the Normal Bundle to Abelian Surfaces Embedded in $\mathbb{P}^4(\mathbb{C})$

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Nicolae MANOLACHE

In this note one computes the cohomology of the normal bundle N_X (and its twists) to any abelian surface X in $\mathbb{P}^4(\mathbb{C})$ and one shows that N_X is simple. As a by-product we reobtain the results of Decker about the smoothness of the irreducible component of the moduli scheme $M(-1,4)$ of rank 2 stable vector bundles on $\mathbb{P}^4$ with $c_1=-1, c_2=4$, along the orbit of the Horrocks-Mumford bundle by the action of $SL_5(\mathbb{C})$ (cf. [2], [3]).

As is very well known, an algebraic torus of dimension 1 can be embedded in $\mathbb{P}^2$. By a result of A. Van de Ven, an algebraic torus of dimension $n\geqslant 3$ cannot be embedded in $\mathbb{P}^{2n}(\mathbb{C})$, while, for $n=2$, Horrocks and Mumford showed in [6] that there are certain abelian surfaces which can be embedded in $\mathbb{P}^4$ and that there is a vector bundle E of rank 2 on $\mathbb{P}^4$ such that any abelian surface in $\mathbb{P}^4$ is projectively equivalent to the zero set of a section of E. Thus, we assume $X=V(s)=$"zero set of a section $s\in\Gamma(E)$ " and we have an exact sequence :

$$(*) \qquad o \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow I_X(5) \longrightarrow o$$

We recall briefly, after [6] the construction of the vector bundle E, which is invariant to the action of $N_H < SL_5(C)$, N_H being the normalizer of the Heisenberg group H of order 5. Namely, take $V=\mathrm{Map}(\mathbb{Z}_5,\mathbb{C})$, $\varepsilon=\exp(2\pi i/5)$ and $\sigma,\tau\in SL_5(C)$ given in $\mathrm{Aut}_{\mathbb{C}}(V)$ by $(\sigma x)(k)=x(k+1)$, $(\tau x)(k)=\varepsilon^k x(k)$, for $x\in V, k\in\mathbb{Z}_5$; then H is the group generated by σ,τ and it is realised as an extension

$$1 \longrightarrow \mu_5 \longrightarrow H \longrightarrow \mathbb{Z}_5\times\mathbb{Z}_5 \longrightarrow 1$$

where μ_5 is the group of 5th roots of 1. One shows that N_H is a semidirect product $H \rtimes SL_2(Z_5)$. Let θ be the generator of the Galois group of $\mathbb{Q}(\varepsilon)$ over $\mathbb{Q}$, which acts on H by $\theta(\varepsilon)=\varepsilon^2$ and take $V_i=\theta^i V$ (namely the action of H which is the composition $H \xrightarrow{\theta^i} H \longrightarrow Aut(V)$). Observe that θ^2 ="taking duals". Then $V=V_o, V_1, V_2, V_3$ together with the 25 representations of the abelian group $Z_5 \times Z_5$ are all the irreducible representations of H; in fact V_i are also irreducible representations of N_H. Considering $\mathbb{P}^4$ as the space of lines in V, H acts naturally on $O(1)$ and $\Gamma(O(1))=V^*=V_2$. The exterior algebra $\Lambda^{\circ}(O(1)\otimes V)$, together with the multiplication by the element $\partial \in \Gamma(O(1)\otimes V) \simeq Hom_C(V,V)$ which correspond to id_V, gives the Koszul complex $\mathcal{K}^{\circ}$:

$$o \longrightarrow O \longrightarrow O(1)\otimes V \longrightarrow O(2)\otimes \Lambda^2 V \longrightarrow O(3)\otimes \Lambda^3 V \longrightarrow O(4)\otimes \Lambda^4 V \longrightarrow O(5)\otimes \Lambda^5 V \longrightarrow o \ .$$

The image of ∂ in $O(k)\otimes \Lambda^k V$ is $\Lambda^{k-1}\mathcal{T}$, where $\mathcal{T}$ is the tangent bundle to $\mathbb{P}^4$, and the symmetric pairing $\mathcal{K}^i \otimes \mathcal{K}^{5-i} \longrightarrow O(5)$ given by the exterior product induces the natural pairing $\Lambda^i \mathcal{T} \otimes \Lambda^{4-i}\mathcal{T} \to O(5)$ compatible with the action of $SL_5(\mathbb{C})$. Further, one shows that $W:=Hom_H(V_1, \Lambda^2 V)$ is an irreducible representation of $N_H/H \simeq SL_2(Z_5)$; W is unimodular of degree 2, so that it has an invariant skew symmetric pairing, unique up to a factor. Using this pairing and maps from the Koszul complex, one builds a self-dual monad :

$$O(2)\otimes V_1 \xrightarrow{p} \Lambda^2 \mathcal{T} \otimes W \xrightarrow{q} O(3)\otimes V_3$$

(i.e. p is an injection of vector bundles, $q \simeq p^*(5)$ and $qp=o$). The bundle $E:=ker(q)/im(p)$ is the Horrocks-Mumford bundle (see also [1] , [7] for the explicit description, without representation theory, of the maps p,q). In [6] it is computed also the cohomology of $E(n)$, using the symmetries of E. One shows that $\Gamma(E)$ generates E and then that a general section in $\Gamma(E)$ has smooth zero set. Then it must be an abelian surface.

As $\Gamma(E)$ is an irreducible N_H-module of degree 4, the algebraic tori $X=V(s)$ are not invariant by the action of N_H. In order to compute the cohomology of the normal bundle N_X (and of its twists $N_X(n)$) to $X=V(s)$ it is convenient to use the group $G<N_H$ generazed by H and the element in $SL_5(C)$ denoted ι in [6], which extends to V the reflection map $x \to (-x)$ on X. In Aut(V) ι is given by $(\iota x)(k)=x(-k)$ and its image $\bar{\iota}$ in $SL_2(\mathbb{Z}_5)$ is (-identity). Then G is a semidirect product $G \simeq H \rtimes Z_2$ and we shall write $(\alpha,m,n)\iota^k$ ($\alpha\in\mu_5$, $m,n\in\mathbb{Z}_5$, $k=0,1$) for $\varepsilon^{2mn}\sigma^m\tau^n\iota^k$.

It is a standard exercise to compute the character table of G

The character table of G

$\{\alpha\}$	$C_{m,n}$	C_α	
1	1	1	I
$5\theta^i(\alpha)$	0	$\theta^i(\alpha)$	V_i
1	1	-1	S
2	$\varepsilon^{sn+tm}+\varepsilon^{-sn-tm}$	0	$Z_{s,t}$
$5\theta^i(\alpha)$	0	$-\theta^i(\alpha)$	$V_i^{\#}$

where $\{\alpha\}$ is the class containing only the central element $\alpha\in\mu_5$, $C_{m,n}=\{(\alpha,m,n),(\alpha,-m,-n)\mid\alpha\in\mu_5\}$ (hence $\#C_{m,n}=10$ and there are 12 different $C_{m,n}$) and $C_\alpha=\{(\alpha,m,n)\iota \mid m,n\in\mathbb{Z}_5$ (hence $\#C = 25$ and there are 5 classes C_α).

We shall denote by Z the direct sum of all twelve $Z_{s,t}$ and we have the following formulae:

$$V_i\otimes V_i = 3V_{i+1}\oplus 2V^{\#}_{i+1},\ V_i\otimes V_{i+1}=3V_{i+3}\oplus 2V^{\#}_{i+3},\ V_i\otimes V_{i+2}=I\oplus Z$$

$$V_i\otimes S = V_i^{\#},\ V_i\otimes Z = 12V_i\oplus 12V_i^{\#}$$

$$S\otimes S = I,\ S\otimes Z = Z$$

$$Z\otimes Z = 12I\oplus 12S\oplus 23Z$$

which can be established easily from the character table.We need also :

$$\Lambda^2 V = 2V_1^{\#} \ , \ \Lambda^3 V = 2V_3^{\#} \ , \ \Lambda^4 V = V_2 \ .$$

The symmetric power representations will be computed by the well-known formula :

$$S^i V = S^{i-1}V \otimes \Lambda^1 V - S^{i-2}V \otimes \Lambda^2 V + S^{i-3}V \otimes \Lambda^3 V - S^{i-4}V \otimes \Lambda^4 V + S^{i-5}V \ .$$

For instance: $S^0V=I$, $S^1V=V$, $S^2V=3V_1$, $S^3V=5V_3 \oplus 2V_3^{\#}$, $S^4V=10V_2 \oplus 4V_2^{\#}$, $S^5V=6I \oplus 5Z$, $S^6V=26V \oplus 16V^{\#}$, $S^7V=38V_1 \oplus 28V_1^{\#}$,etc.

Then the groups of cohomology of $E(n)$,as G-modules,are given by the following table,as we can see restricting to G the results of [6] :

Table of $H^i(E(n))$

n	H^0	H^1	H^2	H^3	H^4
$-n$ $(n \geq 11)$	o	o	o	o	$\theta^2 A_{n-10}$
-1o	o	o	o	2S	4I
-9	o	o	o	$2V_2^{\#}$	o
-8	o	o	o	$2V_3^{\#}$	o
-7	o	o	o	V_1	o
-6	o	o	o	o	o
-5	o	o	2S	o	o
-4	o	o	o	o	o
-3	o	V_3	o	o	o
-2	o	$2V_1^{\#}$	o	o	o
-1	o	$2V^{\#}$	o	o	o
o	4I	2S	o	o	o
n $(n>1)$	A_n	o	o	o	o

where $A_n = S^nV_2 \otimes 2S - S^{n+1}V_2 \otimes 2V^{\#} + S^{n+2}V_2 \otimes 3V_1 - S^{n+3}V_2 \otimes V_3$.

In particular, $\dim H^o(E(n)) = \dfrac{(n+4)(n+6)(n^2+10n+1)}{12}$.

From here and the exact sequence $(*)$,using also the duality on X,we have the cohomology of $I_X(n)$ and $O_X(n)$:

For $n \geq 6$ $H^3(I_X(-n)) = S^nV - \theta^2 A_{n-5} + S^{n-5}V$, $H^4(I_X(-n)) = S^{n-5}V$

$H^3(I_X(-5)) = 3I \oplus 2S \oplus 5Z$, $H^4(I_X(-5)) = I$, $H^3(I_X(-4)) = 10V_2 \oplus 6V_2^{\#}$.

$H^3(I_X(-3)) = 5V_3 \oplus 4V_3^{\#}$, $H^3(I_X(-2)) = 4V_1$, $H^3(I_X(-1)) = V$,

$H^2(I_X) = 2S$, $H^3(I_X) = I$, $H^1(I_X(2)) = V_3$, $H^2(I_X(3)) = 2V_1^{\#}$, $H^2(I_X(4)) = 2V^{\#}$,

$H^o(I_X(5)) = 3I$, $H^1(I_X(5)) = 2S$, $H^o(I_X(n)) = A_{n-5} - S^{n-5}V_2$,for $n \geq 6$. All the other groups vanish.

The cohomology of $O_X(n)$ is known from general results (cf.[8]), but as G-modules we obtain:

$H^o(O_X) = I$, $H^1(O_X) = 2S$, $H^2(O_X) = I$, $H^o(O_X(1)) = V_2$, $H^o(O_X(2)) = 4V_3$,

$H^o(O_X(3)) = 5V_1 \oplus 4V_1^{\#}$, $H^o(O_X(4)) = 10V \oplus 6V^{\#}$, $H^o(O_X(5)) = 3I \oplus 2S \oplus 5Z$,

and for $n \geq 6$: $H^o(O_X(n)) = S^nV_2 - A_{n-5} + S^{n-5}V_2$.

Then $H^2(O_X(-n))$ are G-dual to $H^o(O_X(n))$ and $H^1(O_X(n)) = o$ for $n \neq o$.

Note that $H^o(O_X(n))$ contains :

for $n=5k$ only components I, S, Z

for $n=5k+1$ only $V_2, V_2^{\#}$

for $n=5k+2$ only $V_3, V_3^{\#}$

for $n=5k+3$ only $V_1, V_1^{\#}$

for $n=5k+4$ only $V, V^{\#}$.

Now,denote by T the restriction to X of the tangent bundle of $\mathbb{P}^4$,by N the normal bundle of X in $\mathbb{P}^4$.Observing that the tangent bundle of X is $O_X \otimes 2S$,we have the exact sequences :

(1) $\quad o \longrightarrow O_X \longrightarrow O_X(1) \otimes V \longrightarrow T \longrightarrow o$

(2) $\quad o \longrightarrow O_X \otimes 2S \longrightarrow T \longrightarrow N \longrightarrow o$

From the exact sequences $(1) \otimes O_X(n)$ and $(2) \otimes O_X(n)$ we obtain, for $n \geq 1$: $H^o(\mathcal{T}(n)) = H^o(O_X(n+1)) \otimes V - H^o(O_X(n))$, $H^1(\mathcal{T}(n)) = o$, $H^2(\mathcal{T}(n)) = o$ and then $H^o(N(n)) = H^o(\mathcal{T}(n)) - H^o(O_X(n)) \otimes 2S$,

$H^1(N(n))=o, H^2(N(n))=o$. By duality on X we have $H^o(N(-n))=o$, $H^1(N(-n))=o, H^2(N(-n))= \theta^2 H^o(O_X(n-5))$, for $n \geqslant 6$.

The exact sequence of cohomology of (1) gives :
$H^o(T)=2S\oplus Z$, $H^1(T)=I$, $H^2(T)=o$; from (2) we obtain the exact sequence :

(A) $o \to 2S \to 2S\oplus Z \to H^o(N) \to 4I \to I \to H^1(N) \to 2S \to o \to H^2N \to o$

so that $H^2(N)=o$, and by duality $H^o(N(-5))=o$. We have also the following inequalities, whose meaning is evident :

$$3I\oplus Z \leqslant H^o(N) \leqslant 4I\oplus Z \quad , \quad 2S \leqslant H^1(N) \leqslant I\oplus 2S \ .$$

To decide the "I-content" we proceed as follows : consider the exact sequence $(*)\otimes E(-5)$:

$$o \to E(-5) \to \mathcal{E} \to I_X\otimes E \to o$$

where $\mathcal{E} = E\otimes E^\vee = E\otimes E(-5)$ is the bundle of local endomorphisms of E. From here, knowing that E is simple (because it is stable), we obtain the exact sequence of cohomology :

(B) $o \to H^1(\mathcal{E}) \to H^1(I_X\otimes E) \to 2S \to H^2(\mathcal{E}) \to H^2(I_X\otimes E) \to o$

and from the exact sequence

$$o \to I_X\otimes E \to E \to N \to o$$

we obtain :

(C) $o \to I \to 4I \to H^o(N) \to H^1(I_X\otimes E) \to 2S \to H^1(N) \to H^2(I_X\otimes E) \to o$

Recall now that, for a vector bundle F given by a monad $A \xrightarrow{a} B \xrightarrow{b} C$ we have $H^2(\mathcal{E}ndF) \simeq coker(d)$, where $d:Hom(A,B)\oplus Hom(B,C) \to$ $\to Hom(A,C)$ is defined by $d(u,v)=bu+va$, if certain groups of cohomology vanish (cf.[7], lemma 4.1.7). Since we are in the conditions of this lemma (easy to verify!), $H^2(\mathcal{E})$ is a quotient of $Hom(O(2)\otimes V_1 \ , O(3)\otimes V_3) \simeq V_3\otimes V_3\otimes V_2$, and it is an N_H-module, d being compatible with the action of N_H. We have, using the table of tensor products for G-modules $V_3\otimes V_3\otimes V_2 = 3I\oplus 2S\oplus 5Z$. In fact it is not difficult to make the computations over N_H, using [6] .

By lemma 2.4. ii) in loc.cit. we have $V_2 \otimes V_3 = (\theta W \oplus U) \otimes V_1$,and using the isomorphism $V_1 \otimes V_3 \simeq I \oplus Z$,one obtains $V_3 \otimes V_3 \otimes V_2 \simeq \theta W \oplus U \oplus 5Z$,the decomposition in irreducible representations.The restriction of θW to G is 2S and that of U is 3I.

We show that $H^1(N) = 2S$,hence $H^o(N) = 3I \oplus Z$, $H^2(\mathcal{C}) = \theta W$, $H^1(\mathcal{C}) = Z$. For,if $H^1(N)$ would contain an I then,by (C) , $H^2(I_X \otimes E)$ must contain it,hence by (B), $H^2(\mathcal{C})$ must also contain it. But $H^2(\mathcal{C})$ is a N_H-module and as such it should consist of θW, U or Z pieces,so that if,as G-module $H^2(\mathcal{C})$ contains one I it contains U as N_H-module hence 3I as G-module,absurd.

Thus we have showed $H^o(N) = 3I \oplus Z$,$H^1(N) = 2S$, $H^2(N) = o$ and, by duality : $H^o(N(-5)) = o$,$H^1(N(-5)) = 2S$,$H^2(N(-5)) = 3I \oplus Z$.

Take now the cohomology of $(1) \otimes O_X(-1)$ and $(2) \otimes O_X(-1)$:

$$o \to V \to H^o(T(-1)) \to o \to 2V \to H^1(T(-1)) \to V \to V \to H^2(T(-1)) \to o$$

$$o \to H^o(T(-1)) \to H^o(N(-1)) \to o \to H^1(T(-1)) \to H^1(N(-1)) \to 2V^{\#} \to$$
$$\to H^2(T(-1)) \to H^2(N(-1)) \to o$$

Since $H^2(N(-1))$ is dual to $H^o(N(-4))$ we consider also the cohomology of $(1) \otimes O_X(-4)$,$(2) \otimes O_X(-4)$. One obtains firstly $H^o(T(-4)) = o$, secondly $H^o(N(-4)) = o$,and then $H^2(N(-1)) = o$.Then $H^2(T(-1))$ is a quotient both of V and of $2V^{\#}$,so that it must be zero.It follows $H^1(T(-1)) = 2V^{\#}$,$H^o(N(-1)) = V$,$H^1(N(-1)) = 4V^{\#}$.

Take now the cohomology of (1),(2) tensored with $O_X(-2)$ and $O_X(-3)$.One obtains $H^o(T(-2)) = o$,$H^o(N(-2)) = o$,$H^o(T(-3)) = o$,$H^o(N(-3)) = o$ and the exact sequences :

$$o \to H^1(T(-2)) \to H^2(O_X(-2)) \to H^2(O_X(-1)) \otimes V \to H^2(T(-2)) \to o$$

$$o \to H^1(T(-2)) \to H^1(N(-2)) \to 8V_1^{\#} \to H^2(T(-2)) \to o$$

$$o \to H^1(T(-3)) \to 5V_3 \oplus 4V_3^{\#} \to 12V_3 \oplus 8V_3^{\#} \to H^2(T(-3)) \to o$$

$$o \to H^1(T(-3)) \to H^1(N(-3)) \to 8V_3 \oplus 1oV_3^{\#} \to H^2(T(-3)) \to o$$

It follows $H^2(T(-2)) = 2V_1^{\#}$, $H^1(T(-2)) = V_1$,$H^1(N(-2)) = V_1 \oplus 6V_1^{\#}$ .By duality, $H^1(N(-3)) = V_3 \oplus 6V_3^{\#}$.

Proposition 1. In the above assumptions, the normal bundle N to $X \subset \mathbb{P}^4$ is simple and its cohomology is given by the following table

n	H^0	H^1	H^2
$-n$ $(n \geq 6)$	o	o	$\theta^2 C_{n-5}$
-5	o	$2S$	$3I \oplus Z$
-4	o	$4V_2^{\#}$	V_2
-3	o	$V_3 \oplus 6V_3^{\#}$	o
-2	o	$V_1 \oplus 6V_1^{\#}$	o
-1	V	$4V^{\#}$	o
o	$3I \oplus Z$	$2S$	o
n $(n \geq 1)$	C_n	0	o

where $C_n = H^0(\mathcal{O}_X(n+1)) \otimes V - H^0(\mathcal{O}_X(n)) - H^0(\mathcal{O}_X(n)) \otimes 2S$

Proof. We have only to show that N is simple. For this, consider (1) and (2) tensored by $N(-5)$:

$$o \longrightarrow N(-5) \longrightarrow N(-4) \otimes V \longrightarrow T \otimes N(-5) \longrightarrow o$$

$$o \longrightarrow N(-5) \otimes 2S \longrightarrow T \otimes N(-5) \longrightarrow \mathcal{E}nd(N) \longrightarrow o$$

and take the cohomology. As $H^2(N(-4) \otimes V) = I \oplus Z$ it follows that $H^2(T \otimes N(-5))$ contains at most one I and none S, so that $H^2(\mathcal{E}ndN)$ and its dual $H^0(\mathcal{E}ndN)$ contain precisely one I (N admits at least the multiplication by a scalar as an endomorphism) and no S. But we have

$$o \longrightarrow H^0(T \otimes N(-5)) \longrightarrow H^0(\mathcal{E}ndN) \longrightarrow 4I \longrightarrow \ldots$$

$$o \longrightarrow H^0(T \otimes N(-5)) \longrightarrow 2S \longrightarrow \ldots$$

so that it remains $H^0(\mathcal{E}ndN) = I$, $H^2(\mathcal{E}ndN) = I$ and then $H^1(\mathcal{E}ndN) = 8S \oplus 6Z$.

□

Proposition 2. The irreducible component of the moduli space $M(-1,4)$, of stable rank 2 vector bundles on $\mathbb{P}^4$ with $c_1=-1, c_2=4$, which contains the Horrocks-Mumford bundle E has dimension 24 and is smooth along the orbit of E by $SL_5(\mathbb{C})$. This orbit has also dimension 24.

Proof. The family $\mathcal{A}$ of abelian surfaces in $\mathbb{P}^4$ has dimension 27, because the moduli space of them has dimension 3 (cf. [6], Theorem 6.1) and only finitely many automorphisms of an abelian variety are restrictions of automorphisms of the ambient projective space (cf. [4], p.326). For an abelian surface Y in $\mathcal{A}$ consider the vector bundles F obtained by the method of Serre, as extensions (cf. [5], [7])

$$o \longrightarrow O \longrightarrow F \longrightarrow I_Y(5) \longrightarrow o$$

corresponding to elements ξ locally generating $Ext^1(I_Y(5),O) \simeq H^o(O_Y)$, and denote by $\mathcal{F}$ this family of vector bundles. In fact $\mathcal{F}$ consists of the orbit of the Horrocks-Mumford bundle by the action of $SL_5(C)$, as any abelian surface in $\mathbb{P}^4$ is projectively equivalent to the zero set of a section $s \in \Gamma(E)$ and $Ext^1(I_Y(5),O)=\mathbb{C}$. We have $h^o(F)=4$ for a F in $\mathcal{F}$, hence the family $\mathcal{F}$ has dimension = $\dim \mathcal{A} + h^o(O_Y) - h^o(F) = 24$. As $H^1(\mathcal{E}nd(F))$ is naturally isomorphic to the tangent space of the moduli scheme $M(-1,4)$ in the point corresponding to F (cf. [5]) and $h^1(\mathcal{E}nd(F))=24$ for F in $\mathcal{F}$, it follows that $M(-1,4)$ is smooth of dimension 24 in the points of $\mathcal{F}$. (Note that we made no distinction between vector bundles with $c_1=-1, c_2=4$ and those with $c_1=5, c_2=1o$, when we refered to the moduli spaces, because they differ only by a twist with $O(3)$.) $\square$

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